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Young diagrams, Brauer algebras, and bubbling geometries

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Abstract: We study the 1/4 BPS geometries corresponding to the 1/4 BPS operators of the dual gauge theory side, in $\mathcal{N} = 4$ SYM. By analyzing asymptotic structure and flux integration of the geometries, we present a mapping between droplet configurations arising from the geometries and Young diagrams of the Brauer algebra. In particular, the integer k classifying the operators in the Brauer basis is mapped to the mixing between the two angular directions.

Keywords: Gauge-gravity correspondence, AdS-CFT Correspondence

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1 Introduction

In this paper we study a particular problem in the context of the gauge/gravity correspondence [1–3]. A family of 1/2 BPS geometries were found and they are dual to a family of 1/2 BPS operators of $\mathcal{N}$=4 SYM, as described in [4–6]. On the gauge side, they can be described by Schur polynomial operators or Young diagrams. They can also be described by wavefunctions of multi-body system. A droplet space on the gravity side were found and the geometries dual to the corresponding operators on the gauge side were mapped [6]. On the gravity side, the geometries can be described by the phase space of multi-body system and Young diagrams. These spacetime geometries are nontrivial quantum states on the gravity side.

There are also 1/4 BPS operators and 1/8 BPS operators that are also dual to corresponding 1/4 BPS geometries and 1/8 BPS geometries. A general family of 1/4 BPS, 1/8 BPS geometries corresponding to two-charge and three-charge geometries were given in [9–15], pertaining to their corresponding sectors. The conditions that the S^3 shrinks or S^1 shrinks smoothly were analyzed in details in e.g. [10, 11, 16].

The careful analysis of the geometries of the gravity side shows that the condition on the droplet space which characterizes the regular geometries encodes the condition for having globally well-defined spacetime geometries, as emphasized by [6]. In this paper we also analyze these conditions and characterize the geometries.

Meanwhile, gauge invariant operators which are dual to geometries have scaling dimension of order N^2, which means that handling huge combinatoric factors arising from summing up non-planar diagrams is inevitable. A new observation is that the problem can be handled systematically with the help of group theory. Following the earlier work [4], some bases for local gauge invariant operators with two R-charges were given in [17–19]. These bases are labelled by Young diagrams, and the calculations can be performed efficiently by representation theory. See [26–28, 30], for example.

In this paper we study the relation between the droplet space of the two-charge geometries or 1/4 BPS geometries and the dual two-charge operators with the various bases. In particular we find the relation between the droplet space and the basis built using elements of the Brauer algebra given in [17]. A class of the BPS operators were obtained at weak coupling in [29], in which they are labelled by two Young diagrams. We note that the size of each Young diagram is determined by the R-charge of the fields and an integer. We will focus on how the Young diagrams show up from the bubbling geometries. Other bases may also be related to the droplet picture, since these bases can be related by transformations from each other. There is also a droplet description from other method on the gauge side by [7].

The Young diagrams are also convenient for describing additional excitations on these states. In particular, starting from a large dimension BPS operator labelled by Young diagrams, one can modify the operators by replacing some fields with other fields or multiplying some other fields. The presence of those other fields in the operators make the states to be non-BPS and we can describe those states as new excitations on the BPS states. See also related discussions on those viewpoints, including e.g. [8, 34–36, 43].

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The organization of this paper is as follows. In section 2 and 3, we introduce the general metric and flux. In section 4, we study the flux integration on the droplet space. In section 5, we analyze metric functions and mixings of metric components, for general droplet configurations. In section 6, we study the large R behavior of the mixings of the metric components. In section 7, we analyze the configurations on the droplet space and Young diagrams. In section 8, we discuss more about the operators handled by the Brauer algebra. In section 9, we analyze the large r asympotics of the geometry. Finally, in section 10, we briefly discuss our results and conclusions. We also include several appendices.

2 Metric and ansatz

We analyze two-charge geometries with J_1, J_2 of two U(1) global symmetries inside SO(6). General family of solutions have been studied in [9–16]. They have SO(4)×SO(2) symmetry. The geometries have been studied from various perspectives, see also e.g. [12]–[15]. It was found [10] that on the droplet space, the S^3 shrinks smoothly, or the S^1 shrinks smoothly, see also e.g. [11, 16].

They can be written via a Kähler potential $K(z_i, \bar{z}_i; y)$. We have the 1/4 BPS geometry in the form,

$$ds^2 = -h^{-2}(dt+\omega)^2 + h^2\left(dy^2 + \frac{2\partial_i\bar{\partial}_j K}{Z+\frac{1}{2}}dz_i d\bar{z}_j\right) + ye^G d\Omega_3^2 + ye^{-G}d\psi^2, \tag{2.1}$$

and see appendix A.

We will later analyze asymptotic structure of the geometries. In order to perform the analysis, it is convenient to make a change of coordinates

$$\begin{aligned} y &= r\cos\theta_1, \\ z_1 &= R_1(r)\sin\theta_1\cos\theta_2 e^{i\phi_1}, \\ z_2 &= R_2(r)\sin\theta_1\sin\theta_2 e^{i\phi_2}. \end{aligned} \tag{2.2}$$

We also denote $\mu_1 = \sin\theta_1\cos\theta_2$, $\mu_2 = \sin\theta_1\sin\theta_2$, $\mu_3 = \cos\theta_1$ and $r_i = R_i(r)\mu_i$, $i = 1, 2$. We often use $R^2 = r_1^2 + r_2^2$. We also define

$$S_{ij} = \frac{2e^{i(\phi_i-\phi_j)}\partial_i\partial_{\bar{j}}K}{Z+1/2}. \tag{2.3}$$

In the following of this section, we mainly focus on the case that $K = K(r_1, r_2, y)$. This means that the S_{ij} is symmetric: $S_{ij} = S_{ji}$. Under this condition, with the shift of the angular variables $\phi_i \to \phi_i - t$, the geometries can be expressed by

$$\begin{aligned} ds^2 =& -h^{-2}\left(1 + h_{ab}M_aM_b - S_t\right)dt^2 + h^2\left(\mu_3^2 + S_{11}\mu_1^2T_1^2 + S_{22}\mu_2^2T_2^2 + 2S_{12}\mu_1\mu_2T_1T_2\right)dr^2 \\ &+2h^2(S_{11}R_1T_1\mu_1 + S_{12}T_2\mu_2R_1 - \mu_1 r)d\mu_1 dr + 2h^2(S_{22}R_2T_2\mu_2 + S_{12}T_1\mu_1R_2 - \mu_2 r)d\mu_2 dr \\ &+\sqrt{\Delta}r^2 d\Omega_3^2 + \frac{1}{\sqrt{\Delta}}\left(d\mu_3^2 + H_1 d\mu_1^2 + H_2 d\mu_2^2\right) + 2h^2\left(S_{12}R_1R_2 - \frac{\mu_1\mu_2}{\Delta}\right)d\mu_1 d\mu_2 \\ &+\frac{\mu_3^2}{\sqrt{\Delta}}d\psi^2 + h^{-2}h_{ij}(d\phi_i + M_i dt)(d\phi_j + M_j dt), \end{aligned} \tag{2.4}$$

where we have defined $T_i = dR_i/dr$. We present the details of this calculation in appendix B.

The metric functions are defined as follows,

$$\Delta = \frac{\mu_3^2}{r^2}\frac{1+2Z}{1-2Z}, \tag{2.5}$$

$$h^{-2} = \frac{r^2\Delta + \mu_3^2}{\sqrt{\Delta}}, \tag{2.6}$$

$$H_i = \sqrt{\Delta}h^2\left(S_{ii}R_i^2 - \frac{\mu_i^2}{\Delta}\right). \tag{2.7}$$

The mixing between time and angles, which will play an important role to determine the angular momenta of the geometries, is given by

$$M_1 = -1 + \frac{-S_2\omega_{\phi_1} + N_{12}\omega_{\phi_2}}{S_1S_2 - N_{12}^2}, \tag{2.8}$$

$$M_2 = -1 + \frac{-S_1\omega_{\phi_2} + N_{12}\omega_{\phi_1}}{S_1S_2 - N_{12}^2}. \tag{2.9}$$

The functions in the angular part are

$$h_{11} = S_1, \quad h_{22} = S_2, \quad h_{12} = N_{12}, \tag{2.10}$$

and the function in time is

$$S_t = S_1 + S_2 + 2N_{12} + 2\omega_{\phi_1} + 2\omega_{\phi_2}, \tag{2.11}$$

where

$$S_i = h^4 S_{ii} r_i^2 - \omega_{\phi_i}^2, \qquad i = 1, 2, \tag{2.12}$$

$$N_{12} = h^4 S_{12} r_1 r_2 - \omega_{\phi_1}\omega_{\phi_2}. \tag{2.13}$$

The $AdS_5 \times S^5$ can be recovered by plugging $\Delta = 1$ and $R_1 = R_2 = \sqrt{r^2 + r_0^2}$ in (2.4):

$$ds^2 = -\left(1+r^2\right)dt^2 + \frac{1}{1+r^2}dr^2 + r^2 d\Omega_3^2 + \sum_{i=1}^{3}\left(d\mu_i^2 + \mu_i^2 d\phi_i^2\right), \tag{2.14}$$

where the above are written in unit $r_0 = 1$, and we have renamed $\psi = \phi_3$.

3 Flux in general form

We start by writing out the metric, in particular by expanding out the fibration over the time direction:

$$\begin{aligned} ds_{10}^2 = &-h^{-2}\left(dt + \frac{1}{y}\left(\bar{\partial}_i\partial_y K\right)d\bar{z}^i - \frac{1}{y}\left(\partial_i\partial_y K\right)dz^i\right)^2 \\ &+h^2\left(dy^2 + \frac{2}{Z+\frac{1}{2}}\partial_i\bar{\partial}_j K dz^i d\bar{z}^j\right) + y\left(e^G d\Omega_3^2 + e^{-G}d\psi^2\right), \end{aligned} \tag{3.1}$$

and $Z = \frac{1}{2}\tanh G = -\frac{1}{2} y\partial_y(\frac{1}{y}\partial_y K)$. The five form, is then given by

$$F_5 = \left(-d\left(y^2 e^{2G}(dt+\omega)\right) - y^2 d\omega + 2i\partial_i\bar{\partial}_j K dz^i d\bar{z}^j\right) \wedge d\Omega_3 + dual. \tag{3.2}$$

There are different types of components. We can split them into two types of components. The five form here may be written as

$$F_5 = F_2 \wedge d\Omega_3 + F_4 \wedge d\psi. \tag{3.3}$$

The various components of F_2 are

$$\begin{aligned} F_2 =& \partial_y\left(y^2 e^{2G}\right) dt\wedge dy + \partial_i\left(y^2e^{2G}\right) dt\wedge dz^i + \bar{\partial}_i\left(y^2 e^{2G}\right) dt \wedge d\bar{z}^i \\ &+ \left(iy(e^{2G}+1)\partial_i Z - \frac{i}{2y}\partial_y\left(y^2e^{2G}\right)\partial_i\partial_y K\right) dz^i\wedge dy \\ &- \left(iy(e^{2G}+1)\bar{\partial}_i Z - \frac{i}{2y}\partial_y\left(y^2e^{2G}\right)\bar{\partial}_i\partial_y K\right) d\bar{z}^i\wedge dy \\ &+ \frac{1}{2}iy\left(\partial_i\left(e^{2G}\right)\partial_j\partial_y K dz^i\wedge dz^j - \bar{\partial}_i\left(e^{2G}\right)\bar{\partial}_j\partial_y K d\bar{z}^i\wedge d\bar{z}^j\right) \\ &+ \left(2i\partial_i\bar{\partial}_j K - iy\left((e^{2G}+1)\left(\partial_i\bar{\partial}_j\partial_y K\right) + \frac{1}{2}\left(\partial_i\left(e^{2G}\right)\bar{\partial}_j\partial_y K + \bar{\partial}_j\left(e^{2G}\right)\partial_i\partial_y K\right)\right)\right) dz^i\wedge d\bar{z}^j. \end{aligned} \tag{3.4}$$

These components are multiplied by $d\Omega_3$.

We write the full dual field strength for the five form as [20]:

$$\begin{aligned} F_4 =& \frac{1}{(1+2Z)^2}\left(y\partial_j\bar{\partial}_k K\partial_y\left(\frac{1}{y^2}\partial_y\partial_i K\partial_y\bar{\partial}_l K\right) - \frac{1-4Z^2+2y\partial_y Z}{y^2}\partial_j\bar{\partial}_l K\partial_i\bar{\partial}_k K\right) dz^i\wedge dz^j\wedge d\bar{z}^k\wedge d\bar{z}^l \\ &+ \frac{iy}{8(1+2Z)^2}\wedge\left(\partial_i\bar{\partial}_j K\partial_y\left(\frac{1}{y}\partial_y\bar{\partial}_k K\right) dt\wedge dz^i\wedge d\bar{z}^j\wedge d\bar{z}^k - \partial_j\bar{\partial}_i K\partial_y\left(\frac{1}{y}\partial_y\partial_k K\right) dt\wedge dz^j\wedge d\bar{z}^k\wedge d\bar{z}^i\right) \\ &+ \frac{(1-2Z)}{2y^2(1+2Z)}\left(\partial_i\bar{\partial}_k K\left(\frac{1}{8}\left(\frac{1-2Z}{2y}\partial_j\left(y^2e^{2G}\right) - \partial_j\partial_y K\right) + \frac{2y\epsilon^{ef}\epsilon^{gl}\partial_f\bar{\partial}_l K\partial_y\partial_e K\partial_y\partial_j\bar{\partial}_g K}{(1-2Z)det\left(\partial_i\bar{\partial}_j K\right)}\right) dz^i\wedge dz^j\wedge d\bar{z}^k\wedge dy\right. \\ &- \left.\partial_i\bar{\partial}_k K\left(\frac{1}{8}\left(\frac{1-2Z}{2y}\bar{\partial}_j\left(y^2e^{2G}\right) - \bar{\partial}_j\partial_y K\right) + \frac{2y\epsilon^{ef}\epsilon^{gl}\partial_l\bar{\partial}_f K\partial_y\bar{\partial}_e K\partial_y\partial_g\bar{\partial}_j K}{(1-2Z)det\left(\partial_i\bar{\partial}_j K\right)}\right) dz^i\wedge d\bar{z}^j\wedge d\bar{z}^k\wedge dy\right) \\ &+ \frac{i(1-2Z)}{y(1+2Z)}\left(\frac{\partial_i\bar{\partial}_j K}{8} + \frac{2y}{1-2Z}\left(\frac{\partial_y\partial_i\bar{\partial}_j K}{16} + \frac{\epsilon^{ek}\epsilon^{lf}\partial_k\bar{\partial}_l K\partial_i\bar{\partial}_j K\partial_y\partial_e\bar{\partial}_f K}{det\left(\partial_i\bar{\partial}_j K\right)}\right)\right) dt\wedge dz^i\wedge d\bar{z}^j\wedge dy \end{aligned} \tag{3.5}$$

where these should be multiplied by $d\psi$.

4 Flux integration

We focus on the droplet space on which the S^3 or S^1 vanishes. These occur at $y = 0$. The droplet space is divided into two droplet regions, with one region where $Z = -\frac{1}{2}$ and the S^3 vanishes, and another region where $Z = \frac{1}{2}$ and the S^1 vanishes. The flux integration at $y = 0$ involves several different situations, depending on different types of droplets.

We look at the first term in the expression (3.5). We can find the $Z = -\frac{1}{2}$ small y behavior of this term and find this term near $y = 0$, with $Z = -\frac{1}{2}$ droplet region,

$$-dz^1\wedge d\bar{z}^1\wedge dz^2\wedge d\bar{z}^2\wedge d\psi. \tag{4.1}$$

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The flux integration in $Z=-\frac{1}{2}$ is

$$\int_{M_4\times S^1} F_5 = \int_{M_4\times S^1} dz^1\wedge dz^2\wedge d\bar{z}^1\wedge d\bar{z}^2\wedge d\psi = 2\pi\int_{M_4} dz^1\wedge dz^2\wedge d\bar{z}^1\wedge d\bar{z}^2, \tag{4.2}$$

with

$$\frac{1}{8\pi^3 l_p^4}\int_{M_4} dz^1\wedge dz^2\wedge d\bar{z}^1\wedge d\bar{z}^2 = N_i, \tag{4.3}$$

which are quantized, due to the quantization of the F_5 flux. N_i is the flux quantum number in each region. The volume of ψ is $V_{S^1}=2\pi$.

For example, for $AdS_5\times S^5$, when $Z=-\frac{1}{2}$, we have $r=0$, so $z_1=r_0\sin\theta_1\cos\theta_2 e^{i\phi_1}$, $z_2= r_0\sin\theta_1\sin\theta_2 e^{i\phi_2}$, and

$$\frac{1}{4}\int_{M_4} dz^1\wedge dz^2\wedge d\bar{z}^1\wedge d\bar{z}^2 = \frac{\pi^2}{2}r_0^4, \tag{4.4}$$

$$\frac{1}{8\pi^3 l_p^4}\int_{M_4} dz^1\wedge dz^2\wedge d\bar{z}^1\wedge d\bar{z}^2 = \frac{r_0^4}{4\pi l_p^4} = N, \tag{4.5}$$

where for the ground state, M_4 is the compact region bounded by $|z_1|^2+|z_2|^2=r_0^2$. We have $dz^1\wedge d\bar{z}^1\wedge dz^2\wedge d\bar{z}^2=-4dx_1\wedge dx_2\wedge dx_3\wedge dx_4$, where $z_1=x_1+ix_2$, $z_2=x_3+ix_4$. We may view this as a 4d droplet space.

The flux quantum number would map to the lengths N_i of the vertical edges of the Young diagram operators. Here i denotes the different $Z=-\frac{1}{2}$ regions at $y=0$. Compact $Z=-\frac{1}{2}$ droplets correspond to AdS_5 asymptotics, while non-compact $Z=-\frac{1}{2}$ droplets could give rise to other asymptotics.

Now we look at the flux integration in $Z=\frac{1}{2}$ region. On the other hand, at $y=0$, with $Z=\frac{1}{2}$ droplet region, the F_5 has components

$$2i\partial_i\bar{\partial}_j K_0 dz^i\wedge d\bar{z}^j\wedge d\Omega_3, \tag{4.6}$$

where K_0 is defined in (A.5) in the appendix, so we have the flux integrations

$$\int_{D_2\times S^3} F_5 = \int_{D_2} 4\pi^2 i\partial_i\bar{\partial}_j K_0 dz^i\wedge d\bar{z}^j, \tag{4.7}$$

$$\int_{\tilde{D}_2\times S^3} F_5 = \int_{\tilde{D}_2} 4\pi^2 i\partial_i\bar{\partial}_j K_0 dz^i\wedge d\bar{z}^j, \tag{4.8}$$

where $V_{S^3}=2\pi^2$, and D_2, $\tilde{D}_2$ are two dimensional domains. We have that

$$\frac{1}{(2\pi)^2 l_p^4}\int_{D_2} i\partial_i\bar{\partial}_j K_0 dz^i\wedge d\bar{z}^j = m_i, \tag{4.9}$$

$$\frac{1}{(2\pi)^2 l_p^4}\int_{\tilde{D}_2} i\partial_i\bar{\partial}_j K_0 dz^i\wedge d\bar{z}^j = \tilde{m}_i. \tag{4.10}$$

Here, we have non-contractible two-cycles, so we have several situations.

For $AdS_5\times S^5$,

$$K_0=\frac{1}{2}aR^2-\frac{1}{2}q\log(aR^2). \tag{4.11}$$

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In that case there is no such two-cycles. We have $\partial_1\bar{\partial}_1 K_0|_{z_2=0} = \frac{1}{2}a, \partial_2\bar{\partial}_2 K_0|_{z_1=0} = \frac{1}{2}a$, so the flux components along the z_1 plane where $z_2 = 0$ is constant, and the flux components along the z_2 plane where $z_1 = 0$ is also constant.

One of the simplest situations is that there are many $Z = -\frac{1}{2}$ regions on the z_1 plane and z_2 plane. We can form the two-cycles between the $Z = -\frac{1}{2}$ regions on the z_1 plane for D_2, and on the z_2 plane for $\tilde{D}_2$. They are the white strips in figure 1. Now we consider very thin $Z = -\frac{1}{2}$ regions on top of the $Z = \frac{1}{2}$ background. See figure 1 for an example of thin $Z = -\frac{1}{2}$ regions. In this case, on the $Z = \frac{1}{2}$ background, away from the very thin $Z = -\frac{1}{2}$ regions, $\partial_1\bar{\partial}_1 K_0|_{z_2=0}$, $\partial_2\bar{\partial}_2 K_0|_{z_1=0}$ are approximately given by those of the AdS expression e.g. (4.11).

We now consider the D_2 domain of $Z = \frac{1}{2}$ on z_1 plane at $z_2 = 0$, surrounded by $Z = -\frac{1}{2}$ regions, and we have

$$\frac{1}{(2\pi)^2 l_p^4} \int_{D_2} i\partial_1\bar{\partial}_1 K_0 dz^1 \wedge d\bar{z}^1 = m_i. \tag{4.12}$$

Similarly $\tilde{D}_2$ is the domain of $Z = \frac{1}{2}$ on z_2 plane at $z_1 = 0$, surrounded by $Z = -\frac{1}{2}$ regions, and we have

$$\frac{1}{(2\pi)^2 l_p^4} \int_{\tilde{D}_2} i\partial_2\bar{\partial}_2 K_0 dz^2 \wedge d\bar{z}^2 = \tilde{m}_i. \tag{4.13}$$

According to the BPS operators in [29] expressed by the Brauer basis in [17], the operators can be labelled by two Young diagrams. These operators are briefly summarized in section 8. We may identify the flux quantum numbers m_i with the size of the horizontal edges m_i of the first Young diagram, and identify the flux quantum number $\tilde{m}_i$ with the size of the horizontal edges $\tilde{m}_i$ of the second Young diagram. The two Young diagrams have total number of boxes $m - k$ and $n - k$ respectively,

$$\sum_i \sum_{i'=1}^{i} m_{i'} N_i = m - k, \tag{4.14}$$

$$\sum_i \sum_{i'=1}^{i} \tilde{m}_{i'} \tilde{N}_i = n - k. \tag{4.15}$$

The flux integrations at $y = 0$ show how the edges of Young diagrams would be mapped, according to above discussions.

Following the earlier works, we further identify the appropriate variables and functions to characterize the geometries, and focused on the droplet space which are divided into two droplet regions, with $Z = -\frac{1}{2}$ where the S^3 vanishes and with $Z = \frac{1}{2}$ where the S^1 vanishes. The flux quantum numbers are determined by the flux integrals. The dimension of the operator is $m + n$. Note that these are the flux quantization in the small y region. These are not the same as the flux quantization in the large r region.

K_0 plays an important role in these flux integrations that involve the two-cycles in the $Z = \frac{1}{2}$ regions. K_0 can be solved by the coupled equations (A.7) of K_0, K_1 for these regions. Figure 1 shows a plot of K_0 where there are several thin $Z = -\frac{1}{2}$ strips.

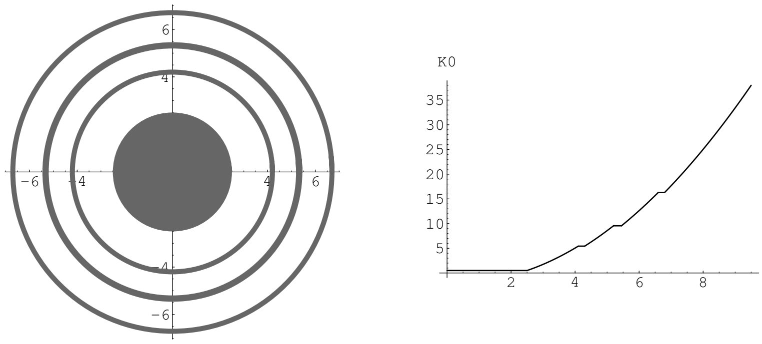

Figure 1. On the left side is the droplet configuration on the z_1 plane. On the right side is the plot of K_0 on the z_1 plane along the radial axis. Similarly for the z_2 plane. There are thin $Z = -\frac{1}{2}$ regions. In the limit that the thin black strips go to zero, the function goes to that of the AdS expression.

5 Droplet space, metric functions and N_{12}, M_1, M_2

5.1 Metric functions

We now study the metric functions in the geometry as well as the metric components mixing time and angles, as well as components mixing the angles. The expressions in this section are valid for all possible ranges of z_i, $\bar{z}_i$. In a different section 6, we will study their large R behaviors, in the large R region. We denote $R^2 = r_1^2 + r_2^2$, and $r_1 = |z_1|\,, r_2 = |z_2|\,.$

We expand K in powers of y^2. The entire solutions to K are determined by the functions K_0, K_1, K_2, see appendix A. The equations for K are in appendix A.

In this section we study exact expressions in all possible ranges of z_i, $\bar{z}_i$, but for small y, since we consider K_0, K_1, K_2 in the series expansion in powers of y^2.

Near $Z = 1/2$, we have the expansion

$$K = -\frac{1}{4}y^2 \log y^2 + K_0 + y^2 K_1 + y^4 K_2 + y^6 K_3 + O(y^8), \tag{5.1}$$

$$Z = \frac{1}{2} - 4y^2 K_2 - 12 y^4 K_3 + O(y^6). \tag{5.2}$$

These expressions will be used in the later derivations in this section.

We also use

$$\omega_{\phi_i} = -\frac{1}{2y}\partial_y (r_i \partial_{r_i} K) = -r_i \partial_{r_i} K_1 - 2y^2 r_i \partial_{r_i} K_2 + O(y^4). \tag{5.3}$$

The Δ can be expanded as

$$\begin{aligned} \Delta &= \frac{\mu_3^2}{r^2}\frac{1+2Z}{1-2Z} \\ &= \frac{\mu_3^2}{r^2}\frac{1 - 4y^2 K_2 - 12 y^4 K_3 + O(y^6)}{4y^2 K_2 + 12 y^4 K_3 + O(y^6)} \\ &= \frac{1}{4r^4 K_2}\left(1 - 4y^2 K_2 - 3y^2 \frac{K_3}{K_2} + O(y^4)\right). \end{aligned} \tag{5.4}$$

Note that $y = r\mu_3$.

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Around $y = 0$ with $Z = \frac{1}{2}$, which is approached by $\mu_3 = 0$, we may expand it as

$$Z = \frac{1}{2} \frac{1 - \frac{r_0^2 \mu_3^2}{r^2 \Delta}}{1 + \frac{r_0^2 \mu_3^2}{r^2 \Delta}} = \frac{1}{2} - \frac{r_0^2 y^2}{r^4 \Delta} + \left(\frac{r_0^2 y^2}{r^4 \Delta} \right)^2 + O(y^6). \tag{5.5}$$

Comparing (5.2), (5.5), we have

$$K_2 = \frac{r_0^2}{4 r^4 \Delta} + O(y^2), \tag{5.6}$$

$$h^2 = \sqrt{4K_2} + O(y^2). \tag{5.7}$$

5.2 Mixing components

We study particularly the important functions M_1, M_2, N_{12} which are the mixing between time and the angles, and the mixing between the angles themselves.

We focus on the components of the geometry,

$$\frac{h_{ij}}{h^2}(d\phi_i + M_i dt)(d\phi_j + M_j dt). \tag{5.8}$$

In particular, there is also a mixing term between the angles,

$$2\frac{N_{12}}{h^2}(d\phi_1 + M_1 dt)(d\phi_2 + M_2 dt), \tag{5.9}$$

where $h_{12} = N_{12}$.

M_1, M_2 account for the mixing between t and ϕ_1, ϕ_2 respectively. The function N_{12} account for the mixing of angles ϕ_1 and ϕ_2.

Using the small y expansions presented in section 5.1 and $h^2 = (1/4 - Z^2)/y^2$, one can expand the S_i, N_{12} in section 2 as

$$\begin{aligned} S_i &= h^4 S_{ii} r_i^2 - \omega_{\phi_i}^2 \\ &= 2K_2 \partial_{t_i}^2 K_0 - (\partial_{t_i} K_1)^2 + O(y^2) \\ &= s_i + O(y^2) \end{aligned} \tag{5.10}$$

and

$$\begin{aligned} N_{12} &= h^4 S_{12} r_1 r_2 - \omega_{\phi_1} \omega_{\phi_2} \\ &= 2K_2 \partial_{t_1} \partial_{t_2} K_0 - (\partial_{t_1} K_1)(\partial_{t_2} K_1) + O(y^2) \\ &= n_{12} + O(y^2), \end{aligned} \tag{5.11}$$

where

$$\begin{aligned} s_1 &= 2K_2 \partial_{t_1}^2 K_0 - (\partial_{t_1} K_1)^2 \\ s_2 &= 2K_2 \partial_{t_2}^2 K_0 - (\partial_{t_2} K_1)^2 \\ n_{12} &= 2K_2 \partial_{t_1} \partial_{t_2} K_0 - (\partial_{t_1} K_1)(\partial_{t_2} K_1) \end{aligned} \tag{5.12}$$

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have been defined. We also have introduced $t_i = \log r_i$.[1] It is interesting to note that only the 2nd derivatives of K_0, and 1st derivatives of K_1, and no derivative of K_2 appears in the expression.

Hence the mixing functions M_1 and M_2 in (2.8), (2.9) can be expanded as

$$M_1 = -1 + \frac{s_2 \partial_{t_1} K_1 - n_{12} \partial_{t_2} K_1}{s_1 s_2 - n_{12}^2} + O(y^2), \tag{5.13}$$

$$M_2 = -1 + \frac{s_1 \partial_{t_2} K_1 - n_{12} \partial_{t_1} K_1}{s_1 s_2 - n_{12}^2} + O(y^2). \tag{5.14}$$

These are exact expressions for any value of r_1, r_2. The special case of large $R^2 (= r_1^2 + r_2^2)$ will be discussed in section 6.

The mixing function between the angles will be particularly important in our analysis to make a connection with the gauge theory. In general cases, it is nonzero, $n_{12} \neq 0$. But there are special cases when $n_{12} = 0$. In this case, the M_i becomes simpler. The special case that $n_{12} = 0$ correspond to defining N_i:

$$N_i = M_i|_{n_{12}=0}. \tag{5.15}$$

We find that

$$N_1 = -1 + \frac{\partial_{t_1} K_1}{s_1} + O(y^2), \tag{5.16}$$

$$N_2 = -1 + \frac{\partial_{t_2} K_1}{s_2} + O(y^2). \tag{5.17}$$

6 Droplet space and analysis of mixing components

We have presented general expressions for small y expansions, in section 5. The geometries are parameterized by y and $z_i, \bar{z}_i$. We denote $R^2 = r_1^2 + r_2^2$, where $r_1 = |z_1|, r_2 = |z_2|$. Now we analyze these expressions in the large R region, by expansions in $1/R^2$. We find solutions in series expansion in powers of $1/R^2$. We will analyze the behavior of the mixing components N_{12}, M_1, M_2 in large R.

The Kähler potential is given by the Monge-Ampere equation (A.4), and in the small y expansion, the equation gives a set of equations for K_0 and K_1 [16],

$$(\partial_{t_1}\partial_{t_1} K_0)(\partial_{t_2}\partial_{t_2} K_1) + (\partial_{t_1}\partial_{t_1} K_1)(\partial_{t_2}\partial_{t_2} K_0) - 2(\partial_{t_1}\partial_{t_2} K_0)(\partial_{t_1}\partial_{t_2} K_1) = 0, \tag{6.1}$$

$$(\partial_{t_1}\partial_{t_1} K_0)(\partial_{t_2}\partial_{t_2} K_0) - (\partial_{t_1}\partial_{t_2} K_0)^2 = \frac{4}{e} e^{2t_1 + 2t_2} e^{2K_1}, \tag{6.2}$$

where $t_i = \log r_i$.

[1] It would be convenient to use

$$e^{i(\phi_i - \phi_j)} \partial_i \partial_{\bar{j}} = \frac{1}{4 r_i r_j} \partial_{t_i} \partial_{t_j}.$$

This set of the equations have some rescaling transformations:

$$K_0 \to qK_0, \tag{6.3}$$
$$t_i \to \xi_i t_i + b_i \qquad (r_i \to e^{b_i}(r_i)^{\xi_i}), \tag{6.4}$$
$$K_1 \to K_1 + \log q - \sum_i \left((\xi_i - 1)t_i + b_i + \frac{1}{2}\log \xi_i^2 \right). \tag{6.5}$$

Overall constant shifts in K_0 are not important because they appear with derivatives in the metric. The $\xi_i = -1$ transformation is the inversion transformation in [16], which exchanges small r_i with large r_i.

The most general form of solutions representing the $AdS_5 \times S^5$ geometry, in a region with $Z = 1/2$, is[2]

$$K_0 = \frac{1}{2}(a(sr_1^m + r_2^n)) - \frac{q}{2}\log(a(sr_1^m + r_2^n)). \tag{6.6}$$

Using the rescaling transforms, this may be brought to the simplest form $K_0 = \frac{1}{2}R^2 - \frac{1}{2}\log R^2$, where $R^2 = r_1^2 + r_2^2$. In our analysis, parameters a, q will be reserved to account for the possible scaling transformation for R (or $z_i, \bar{z}_i$) and K.

For example, we can have the rescaling transformations:

$$K_0 \to qK_0, \tag{6.7}$$
$$\frac{q}{a}R^2 \to R^2, \tag{6.8}$$
$$sr_1^2 \to r_1^2, \tag{6.9}$$

together with constant shifts of K_1, and rescaling of y.

We are interested in more general solutions. We will first consider from large R point of view. In the large R, we have that, $r^2 = aR^2(1 + O(1/r^2)) + O(y^2)$, so in the region where R is large, we have that r is also large.

We denote $R^2 = r_1^2 + r_2^2$. Note that aR^2, ar_1^2, ar_2^2 often appear in combinations in their products, due to that $\sqrt{a}$ rescales the r_i coordinates, as in (6.8).

We find a family of expressions that satisfy the set of the equations, as follows:
K_0 is given by

$$K_0 = \frac{1}{2}aR^2 - \frac{1}{2}q\log(aR^2) + K_0^{(1)}, \tag{6.10}$$
$$\partial_{t_1}^2 K_0 = 2ar_1^2\left(1 - \frac{qr_2^2}{aR^4} + O(1/r^4)\right), \tag{6.11}$$
$$\partial_{t_2}^2 K_0 = 2ar_2^2\left(1 - \frac{qr_1^2}{aR^4} + O(1/r^4)\right), \tag{6.12}$$
$$\partial_{t_1}\partial_{t_2} K_0 = \frac{2qr_1^2 r_2^2}{R^4}\left(1 + \frac{q\alpha_0}{aR^2} + O(1/r^4)\right), \tag{6.13}$$

[2]If we consider the following one,

$$K_0 = \frac{1}{2}(a(r_1^2 + r_2^2)) - \frac{q}{2}\log(a(sr_1^2 + r_2^2)),$$

the equations require $s = 1$.

and K_1 is

$$K_1 = \frac{1}{2}\log\left(\frac{a\left(aR^2-q\right)}{R^2}\right)+\frac{1}{2}+K_1^{(1)}, \tag{6.14}$$

$$\partial_{t_1}K_1 = \frac{qr_1^2}{aR^4}\left(1+\frac{q}{aR^2}(1+\kappa_1)+O(1/r^4)\right), \tag{6.15}$$

$$\partial_{t_2}K_1 = \frac{qr_2^2}{aR^4}\left(1+\frac{q}{aR^2}(1+\kappa_2)+O(1/r^4)\right), \tag{6.16}$$

and K_2 is

$$K_2 = \frac{q}{4(aR^2-q)^2}+K_2^{(1)} \tag{6.17}$$

$$= \frac{q}{4a^2R^4}\left(1+\frac{q}{aR^2}(2+\alpha_2)+O(1/r^4)\right). \tag{6.18}$$

The expressions without $K_0^{(1)}, K_1^{(1)}, K_2^{(1)}$ is the solution for AdS (see [16] for other related analysis). $K_0^{(1)}, K_1^{(1)}, K_2^{(1)}$ are deviations from AdS. The α_0, κ_1, κ_2, α_2 are the effect of turning on $K_0^{(1)}, K_1^{(1)}, K_2^{(1)}$. In the second lines in (6.13), (6.15), (6.16), (6.18), the large r expansion with the effect of $K_0^{(1)}, K_1^{(1)}, K_2^{(1)}$ are given.

From the differential equations, we find a family of $K^{(1)}$,

$$K_0^{(1)} = \frac{q^2}{a}\left(\frac{dr_1^2+er_2^2}{R^4}+O(1/r^4)\right), \tag{6.19}$$

$$K_1^{(1)} = -\frac{q^2}{a^2R^2}\left(\frac{(d-e)(r_1^2-r_2^2)}{R^4}+O(1/r^4)\right), \tag{6.20}$$

$$K_2^{(1)} = \frac{q^2}{4a^3R^6}\left(c-2(d+e)+\frac{6(d-e)(r_1^2-r_2^2)}{R^2}\right)+\frac{q^2}{4a^2R^4}O(1/r^4). \tag{6.21}$$

We note that three parameters c,d,e have come in, and they will be identified with the three parameters m, n, k of Young diagrams in (8.2).

These solutions give that in (6.13), (6.15), (6.16), (6.18),

$$\begin{aligned}
\alpha_0 &= 2(d+e)+\frac{6(d-e)(r_1^2-r_2^2)}{R^2},\\
\alpha_2 &= \alpha_0+c-4(d+e),\\
\kappa_1 &= \alpha_0-4d,\\
\kappa_2 &= \alpha_0-4e.
\end{aligned} \tag{6.22}$$

Now we evaluate the N_{12}, M_1, M_2 in the large r region, using the general expressions (5.11), (5.13), (5.14), as follows:

$$\begin{aligned} N_{12} &= 2K_2\partial_{t_1}\partial_{t_2}K_0 - \partial_{t_1}K_1\partial_{t_2}K_1 + O(y^2) \\ &= \frac{r_1^2 r_2^2}{a^2 R^8}\frac{q^3}{aR^2}(\alpha_0+\alpha_2-\kappa_1-\kappa_2) + O(1/r^8) + O(y^2) \\ &= \frac{q^3\mu_1^2\mu_2^2}{r^6}k + O(1/r^8) + O(y^2), \end{aligned} \tag{6.23}$$

$$\begin{aligned} M_1 &= -1 + \frac{s_2\partial_{t_1}K_1 - n_{12}\partial_{t_2}K_1}{s_1 s_2 - n_{12}^2} + O(y^2) \\ &= \frac{q}{aR^2}(-\alpha_2+\kappa_1) + O(1/r^4) + O(y^2) \\ &= -\frac{qm}{r^2} + O(1/r^4) + O(y^2), \end{aligned} \tag{6.24}$$

and

$$\begin{aligned} M_2 &= -1 + \frac{s_1\partial_{t_2}K_1 - n_{12}\partial_{t_1}K_1}{s_1 s_2 - n_{12}^2} + O(y^2) \\ &= \frac{q}{aR^2}(-\alpha_2+\kappa_2) + O(1/r^4) + O(y^2) \\ &= -\frac{qn}{r^2} + O(1/r^4) + O(y^2), \end{aligned} \tag{6.25}$$

where we identify $r^2 = aR^2(1+O(1/r^2)) + O(y^2)$ and $\frac{1}{r^2} = \frac{1}{aR^2}(1+O(1/r^2)) + O(y^2)$. We are making expansions in power series of y^2 and $1/r^2$. More details of the above derivations are in appendix C.

In the expression of M_1, M_2, we have identified

$$q(\alpha_2-\kappa_1) = qm = q_1, \tag{6.26}$$

$$q(\alpha_2-\kappa_2) = qn = q_2. \tag{6.27}$$

In the large r, we have the quantization of electric charges by

$$A^i = M_i dt, \tag{6.28}$$

$$F^1_{(2)} = dM_1 dt = dr dt\left(\frac{2q}{r^3}m + O(1/r^5)\right), \tag{6.29}$$

$$F^2_{(2)} = dM_2 dt = dr dt\left(\frac{2q}{r^3}n + O(1/r^5)\right), \tag{6.30}$$

$$\frac{1}{4\pi^2}\int *F^1_{(2)} = m, \tag{6.31}$$

$$\frac{1}{4\pi^2}\int *F^2_{(2)} = n, \tag{6.32}$$

where in our coordinates $g_{tt}g_{rr} = q$ in large r, and where m, n are the two charges. Note that terms corresponding to $O(1/r^4)$ in (6.24), (6.25) will not contribute to the integral.

In the expression of N_{12}, we have identified

$$\alpha_0+\alpha_2-\kappa_1-\kappa_2 = k. \tag{6.33}$$

The two angles are the angles in the z_1 plane and z_2 plane respectively.

This k will be identified with the k parameter in the Brauer algebra representation. In other words, for nonzero k,

$$N_{12} = k\frac{\mu_1^2\mu_2^2 q^3}{r^6} + O(1/r^8). \tag{6.34}$$

It appears at order $1/r^6$ with coefficients k. For the special case $k = 0$, it may appear at only order $1/r^8$. See appendix C for more details. The difference between M_i and N_i is order k/r^4 for nonzero k, and $1/r^6$ for zero k.

Plugging these relations (6.26), (6.27), (6.33) into the solutions (6.22), we find

$$\begin{aligned} 4e &= k - m = k - q_1/q, \\ 4d &= k - n = k - q_2/q, \\ c &= k. \end{aligned} \tag{6.35}$$

Note that in this family of solutions, the coefficients $e \leqslant 0, d \leqslant 0$ amount to that $k - m \leqslant 0, k - n \leqslant 0$. Therefore the sign property of e, d, which are either negative or zero, imposes the constraints $k \leqslant m, n$, or equivalently $k \leqslant \min(m, n)$.

In the y expansions, we have, from (6.22),

$$q\alpha_0 = qk + 3q_1\mu_1^2 + 3q_2\mu_2^2 - 2q_1 - 2q_2, \tag{6.36}$$
$$q\alpha_2 = 3q_1\mu_1^2 + 3q_2\mu_2^2 - q_1 - q_2, \tag{6.37}$$
$$q\kappa_1 = 3q_1\mu_1^2 + 3q_2\mu_2^2 - 2q_1 - q_2, \tag{6.38}$$
$$q\kappa_2 = 3q_1\mu_1^2 + 3q_2\mu_2^2 - q_1 - 2q_2, \tag{6.39}$$

where we used that $\mu_1^2 + \mu_2^2 = 1 - y^2/r^2 = 1 - O(y^2)$.

There are several equivalent and alternative ways of writing these variables, see appendix C for more details. For example,

$$qk = q\alpha_0 - q\alpha_2 + q_1 + q_2 = q\alpha_0 + q\alpha_2 - q\kappa_1 - q\kappa_2, \tag{6.40}$$

which will be frequently used in the derivations.

7 Droplets and Young diagram operators

Now we turn to the analysis of the droplet configurations. The solutions in section 6 are the large R expressions that result from the droplet configurations. The solution in section 6 are in the large R region of the full solution in all the ranges of z_i, $\bar{z}_i$. In this section, we also provide further duality relation with the operators labelled by Brauer algebra.

We have that from section 6,

$$K_0 = \frac{1}{2}aR^2 - \frac{q}{2}\log(aR^2) + \frac{q^2}{a}\left(\frac{(d+e)}{2R^2} + \frac{(d-e)(r_1^2 - r_2^2)}{2R^4} + O(1/r^4)\right) \tag{7.1}$$

in large R, and where

$$-4d = n - k, \tag{7.2}$$
$$-4e = m - k. \tag{7.3}$$

As in section 4, e.g. (4.3), the flux quantization requires that the total droplet volume to be quantized,

$$\int_{\mathcal{D}} d^2z_1'd^2z_2' = \int_{\mathcal{D}(\emptyset)} d^2z_1'd^2z_2' = 2\pi^3 l_p^4 N \tag{7.4}$$

where $\mathcal{D}$ is the total $Z = -\frac{1}{2}$ droplets. Here we use the notation that $d^2z_1d^2z_2 = dx_1dx_2dx_3dx_4$ for convenience. $\mathcal{D}(\emptyset)$ is the $Z = -\frac{1}{2}$ configuration such that there is no any finite $Z = \frac{1}{2}$ domains or $Z = \frac{1}{2}$ bubbles. In the large R, it was shown [16] that, if we expand

$$K_0 = \frac{1}{2}a(|z_1|^2 + |z_2|^2) + \widetilde{K}_0, \qquad K_1 = \frac{1}{2} + \widetilde{K}_1, \tag{7.5}$$

$$\frac{1}{a}(\partial_1\partial_{\bar{1}}\widetilde{K}_0 + \partial_2\partial_{\bar{2}}\widetilde{K}_0) = \widetilde{K}_1, \tag{7.6}$$

$$\partial_1\partial_{\bar{1}}\widetilde{K}_1 + \partial_2\partial_{\bar{2}}\widetilde{K}_1 = 0. \tag{7.7}$$

This means that a general solution to $\widetilde{K}_0$ is $-\frac{1}{2}\log(a\,|z_1 - z_1'|^2 + a\,|z_2 - z_2'|^2)$ with $(z_1', \bar{z}_1', z_2', \bar{z}_2')$ arbitrary, therefore we can approximately expand K_0 as

$$\begin{aligned} K_0 = \frac{1}{2}aR^2 - \frac{q}{2}\log(aR^2) - \frac{q}{4\pi^3 l_p^4 N}\int_{\mathcal{D}} d^2z_1'd^2z_2' \log(a\,|z_1 - z_1'|^2 + a\,|z_2 - z_2'|^2) \\ + \frac{q}{4\pi^3 l_p^4 N}\int_{\mathcal{D}(\emptyset)} d^2z_1'd^2z_2' \log(a\,|z_1 - z_1'|^2 + a\,|z_2 - z_2'|^2) + O(1/R^4). \end{aligned} \tag{7.8}$$

Both (7.1), (7.8) satisfy the equations (7.6), (7.7). The droplet configuration in $\mathcal{D}$ gives non-trivially the information of d, e in the large R, when comparing two expressions (7.1), (7.8).

Suppose we consider configurations that only depend on r_1, r_2, and in the case when d and e are equal, where there are more symmetry in the droplet configuration, we can expand (7.8),

$$K_0 = \frac{1}{2}aR^2 - \frac{q}{2}\log(aR^2) - \frac{q(M_{11}^{[2]} - M_{11,\emptyset}^{[2]})}{4\pi^3 l_p^4 N R^2} - \frac{q(M_{22}^{[2]} - M_{22,,\emptyset}^{[2]})}{4\pi^3 l_p^4 N R^2} + O(1/R^4) \tag{7.9}$$

where

$$M_{11}^{[2]} - M_{11,\emptyset}^{[2]} = \int_{\mathcal{D}} d^2z_1'd^2z_2'\,|z_1'|^2 - \int_{\mathcal{D}(\emptyset)} d^2z_1'd^2z_2'\,|z_1'|^2, \tag{7.10}$$

$$M_{22}^{[2]} - M_{22,\emptyset}^{[2]} = \int_{\mathcal{D}} d^2z_1'd^2z_2'\,|z_2'|^2 - \int_{\mathcal{D}(\emptyset)} d^2z_1'd^2z_2'\,|z_2'|^2, \tag{7.11}$$

are the second moments along the axis perpendicular to z_1 plane and z_2 plane respectively, subtracted from those of the configuration without any finite $Z = \frac{1}{2}$ domains. The crossing terms $z_i\bar{z}_i'$ cancels due to symmetric configuration in z_1 plane and in z_2 plane.

We read off the coefficients that

$$\frac{1}{2\pi^3 l_p^4 N}(M_{11}^{[2]} - M_{11,\emptyset}^{[2]}) + \frac{1}{2\pi^3 l_p^4 N}(M_{22}^{[2]} - M_{22,\emptyset}^{[2]}) \tag{7.12}$$

$$= \frac{1}{4}\frac{q}{a}[(n-k) + (m-k)] = -\frac{q}{a}(d+e). \tag{7.13}$$

If we use the convention $\frac{q}{a} = r_0^2 = (4\pi l_p^4 N)^{1/2}$ e.g. as from (4.5), and in the unit $r_0 = 1$,

$$(M_{11}^{[2]} - M_{11,\emptyset}^{[2]}) + (M_{22}^{[2]} - M_{22,\emptyset}^{[2]}) = \frac{\pi^2}{8}[(n-k) + (m-k)]. \tag{7.14}$$

We then identify the second moments in two directions,

$$(M_{11}^{[2]} - M_{11,\emptyset}^{[2]}) = \frac{\pi^2}{8}(n-k), \tag{7.15}$$

$$(M_{22}^{[2]} - M_{22,\emptyset}^{[2]}) = \frac{\pi^2}{8}(m-k), \tag{7.16}$$

where we are in the unit $r_0 = 1$.

Note that we also have the relation, from section 4,

$$\sum_i \sum_{i'=1}^{i} m_{i'} N_i = m - k, \tag{7.17}$$

$$\sum_i \sum_{i'=1}^{i} \tilde{m}_{i'} \tilde{N}_i = n - k. \tag{7.18}$$

The change in the potential K_0 is negative, when increasing the second moments. It is analogous to the change of the potential from $-\log R^2$ to $-\log(R^2 + \delta^2)$, due to increased second moments. So the signs of d, e are negative or zero in the above expression, that is

$$d \leqslant 0, \quad e \leqslant 0. \tag{7.19}$$

This sign property can be understood as that the potential in (7.8) will decrease when the $Z = -\frac{1}{2}$ droplets in the droplet space are more outwards. The droplet second moments are increased from those of the configuration when there is no any finite $Z = \frac{1}{2}$ domains. The condition (7.19) is consistent with

$$(M_{11}^{[2]} - M_{11,\emptyset}^{[2]}) \geqslant 0, \;\; (M_{22}^{[2]} - M_{22,\emptyset}^{[2]}) \geqslant 0. \tag{7.20}$$

This amounts to

$$m - k \geqslant 0, \quad n - k \geqslant 0 \tag{7.21}$$

or equivalently

$$k \leqslant \min(m, n) \tag{7.22}$$

from the gravity side.

The dual operator has dimension $m+n$, and the two Young diagrams have $m-k$ and $n-k$ boxes respectively. The dimension is larger than the total number of boxes by the amount $2k$. The k parameter is identified in the last section as in (6.23), (6.33) as due to the mixing of two angular directions. The two angles are the angles in the z_1 plane and z_2 plane respectively. We see that $2k$ measures the energy excess over the total number of boxes, and is accounted for by the mixing of two angular directions in the situation described here.

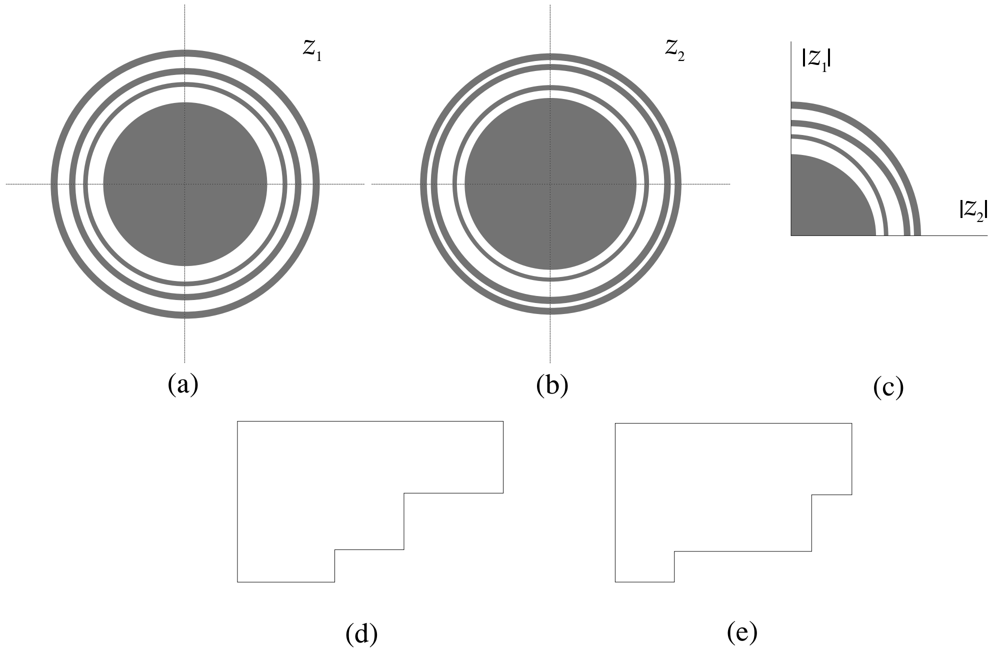

Figure 2. Illustration of the mapping between Young diagrams and the droplet configurations. $Z = -\frac{1}{2}$ are regions where S^3 vanishes and are drawn in black, while $Z = \frac{1}{2}$ are regions where S^1 vanishes and are drawn in white. In (c), there are black droplets and white droplets in the $|z_1|, |z_2|$ quadrant. Figure (c) determines (a,b), where (c) is projected onto z_1, z_2 planes. The white regions of (a),(b) map to the horizontal edges of (d),(e). The black regions of (c) map to the vertical edges of (d),(e). The outward directions in the droplet planes correspond to the upper-right directions along the edges of the Young diagrams (d),(e). The flux quantum numbers in corresponding regions map to the lengths of the edges of the Young diagrams (d),(e). This is an illustration with the example of relatively few number of circles. More general configurations involve many more circles in z_1, z_2 planes. The Young diagrams are filled with boxes which are not shown in the illustration.

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The total number of boxes match the calculation from the flux quantum numbers, as $m-k$ and $n-k$ for two two-planes.

The Brauer algebra provides two Young diagram representations, and each correspond to droplet configurations in z_1 and z_2 planes respectively. For example, for configurations that have $2l_1$ concentric circles in the droplet plane z_1, this configuration maps to the $2l_1$ edges of the first Young diagram γ_+, and the flux quantization numbers on each droplet region map to the lengths of the edges of the Young diagram γ_+. There are also $2l_2$ concentric circles in the droplet plane z_2, and this configuration maps to the $2l_2$ edges of the second Young diagram γ_-, and the flux quantization numbers on each droplet region map to the lengths of the edges of the Young diagram γ_-. See figure 2. These two Young diagrams are depicted in figures 2(d), 2(e) in the example.

We can characterize the droplet configuration by three diagrams, for those depending only on r_1, r_2. The first two diagrams are concentric ring patterns of alternative $Z = \frac{1}{2}$ and $Z = -\frac{1}{2}$ regions in z_1 planes and in z_2 planes, see figures 2(a), 2(b) for example. The third diagram is a diagram in the (r_1, r_2) quadrant, see figure 2(c). We first map the corners in two Young diagrams to ordered points along r_1, r_2 axis respectively in figure 2(c). Then we draw diagrams connecting ordered points along r_1 axis to those along r_2 axis, dividing regions in (r_1, r_2) space into $Z = -\frac{1}{2}$ (depicted in black) and $Z = \frac{1}{2}$ (depicted in white).

We can also draw more complicated droplets. In the gravity description, when we draw the lines, there are extra possibilities for possible lines to go in the middle region of (r_1, r_2) space. These may correspond to other operators that are superpositions of the Brauer basis.

For those configurations that depend not only on r_1, r_2, they could be the superpositions of the Young diagram operators in the above discussions. For example, one can add small ripples on any boundaries of the droplets. See also related discussions, e.g. [21]–[24]. The configurations that correspond to ripples on the droplet boundaries, can be considered as the superpositions of the Young diagram operators in the above discussions.

In [10], 1/2 BPS geometries were uplifted into the systems of 1/4 and 1/8 BPS geometries, and there are disconnected droplets with various topologies in 4d or 6d droplet space respectively, and the topology change transitions occurred in [6, 25] uplift to the topology change transitions in 4d and 6d. In general, in the 4d droplet space we study here, the topology change transition happens commonly.

The condition (7.22) is consistent with the range of k in Brauer algebra representation. The droplet information are encoded in the Young diagrams. There is correspondence between the pair of Young diagrams and the concentric droplet configuration in the two complex planes. The operators labelled by the Young diagrams of Brauer algebra give a family of globally well-defined spacetime geometries. Other bases may also be related to the droplet picture, since they can be related by transformations, and it might be interesting to see how k is produced in other bases. The system of geometries are dual to the system of the corresponding operators.

8 Gauge invariant operators by Brauer algebra

Now we will briefly summarize the operators based on the Brauer algebra [17, 28]. See also related discussion in [37, 38].

We take two complex fields X, Y out of three complex fields and consider gauge invariant operators constructed from m Xs and n Ys. The $\mathrm{U}(N)$ gauge group is considered. The operators are conveniently expressed in the notation of [17, 28] by

$$O^{\gamma}_{A,ij}(X,Y) = tr_{m,n}\left(Q^{\gamma}_{A,ij} X^{\otimes m} \otimes (Y^T)^{\otimes n}\right), \tag{8.1}$$

where $Q^{\gamma}_{A,ij}$ is given by a linear combination of elements in the Brauer algebra. Irreducible representations of the Brauer algebra are denoted by γ, which are given by two Young diagrams:

$$\gamma = (\gamma_+, \gamma_-), \tag{8.2}$$

where γ_+ is a Young diagram with $m-k$ boxes and γ_- is a Young diagram with $n-k$ boxes. k is an integer satisfying $0 \leqslant k \leqslant \min(m,n)$. $A = (R, S)$ is an irreducible representation of

$S_m \times S_n$, labelled by two Young diagrams with m and n boxes. i, j are multiplicity indices with respect to the embedding of A into γ.

An advantage of this basis is that the free two-point functions are diagonal. We also note that non-planar corrections are fully taken into account.

Because the number of boxes in γ is characterized by k, it is convenient to classify the operators by the integer k.

The labels can be simplified when $k = 0$, because i, j are trivial, and we have $\gamma_+ = R$, $\gamma_- = S$. Hence the operators in $k = 0$ are labelled by two Young diagrams. Denoting $P_{R,S} = Q^{\gamma(k=0)}_{A,ij}$, the expression in (8.1) becomes

$$O_{R,S}(X,Y) = tr_{m,n}\left(P_{R,S}X^{\otimes m} \otimes (Y^T)^{\otimes n}\right). \tag{8.3}$$

In this sector, the operators have the nice expansion with respect to $1/N$, whose leading term is given by

$$O_{R,S}(X,Y) = O_R(X)O_S(Y) + \cdots . \tag{8.4}$$

Here $O_R(X)$ is the Schur polynomial built from the X fields labelled by a Young diagram R with m boxes, and $O_S(Y)$ is the Schur polynomial built from the Y fields labelled by a Young diagram S with n boxes.

In general, including the case $k = 0$, the leading term of the operators (8.1) looks schematically like

$$O^{\gamma}_{A,ij}(X,Y) \sim tr_{m,n}\left(\sigma C^k X^{\otimes m} \otimes (Y^T)^{\otimes n}\right) + \cdots , \tag{8.5}$$

where σ is an element in $S_m \times S_n$ and C is an operation contracting the upper index of an X and the upper index of a Y^T. Each term in the dots in the above expression (8.5) contains more contractions. In other words, k is the minimum number of contractions involved in the $Q^{\gamma}_{A,ij}$ of an operator. Therefore the k can be given the intuitive meaning that it measures the degree of the mixing between the two fields. For example, the $k = 0$ has no mixing in the sense of (8.4). The opposite case is the case k takes the maximum value. With the condition $m = n$, operators in $k = m = n$ are found to be expressed by operators of the combined matrix XY [38].

In the paper [29], a class of the 1/4 BPS operators were constructed by exploiting algebraic properties of the Brauer algebra, where it was shown that the operators $O_{R,S}(X,Y)$ and $\sum_{R,S,i} O^{\gamma\,(k\neq 0)}_{A,ii}(X,Y)$ are annihilated by the one-loop dilatation operator, for any m, n and N. Defining $P^{\gamma} = \sum_{A,i} Q^{\gamma}_{A,ii}$ for $k \neq 0$, the BPS operators are presented by

$$O^{\gamma}(X,Y) = tr_{m,n}(P^{\gamma}X^{\otimes m} \otimes (Y^T)^{\otimes n}) \tag{8.6}$$

for both $k = 0$ and $k \neq 0$. Note that P^{γ} is the projector associated with an irreducible representation γ of the Brauer algebra.

It is interesting to find that they are labelled by two Young diagrams with boxes whose total number depends on the integer k for fixed m and n. In $k = 0$, the number of boxes involved in a representation γ is equal to the sum of the R-charges. On the other hand, when k is non-zero, the Young diagrams $\gamma = (\gamma_+, \gamma_-)$ have a smaller number of boxes than the sum of the R-charges. The number of the deficit boxes in the γ is $2k$.

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When k is equal to n for $m \geqslant n$, the operators are labelled by a single Young diagram with $m-k$ boxes. A special case happens for $k=m$ with $m=n$. The operator does not have any Young diagrams, but this is different from the case $m=n=0$. For a given $m=n$, there is only one 1/4 BPS operator labelled by the trivial representation.

We now have a remark on a constraint for representations of the Brauer algebra. The representations γ have the constraint $c_1(\gamma_+)+c_1(\gamma_-)\leq N$, where c_1 denotes the length of the first column of the Young diagram. More explanations are provided in [17]. This constraint is consistent with the identifications in sections 4 and 7.

The Brauer basis can be used in various ways. One of the motivations of constructing the Brauer basis in [17] was to construct an operator describing a set of D-branes and anti-D-branes, which is realized as the $k=0$ sector. In the application of the Brauer basis to the $su(2)$ sector, which is relevant for the present work, the $k=0$ sector realizes natural operators dual to the objects with two fields. Furthermore the construction of the basis introduces the operators labelled by $k\neq 0$ as well. In the mapping proposed in this paper, the quantum number k has been given a meaning as the mixing between the two angular directions from the gravity point of view. The use of the Brauer algebra may be rephrased as the manifestation of such a good quantum number.

On the other hand, the BPS operators may be described by other bases diagonalizing free two-point functions, as in [18, 19], see also related discussions, [39–43]. Because other bases respect other quantum numbers, understanding a map between the two sides for other bases could also be useful for getting a complete duality of this sector.

9 Asymptotics of metric

In this section we analyze the large r asymptotics of the geometry. We start from the expression (2.4). The large r region includes the large R region of the droplet space. We first expand in small y, and have K_0, K_1, K_2. We then expand these functions in powers of $1/r^2$. In the large r, we have

$$\frac{r_1^2}{R^2}=\mu_1^2\left(1+O(1/r^2)\right), \qquad \frac{r_2^2}{R^2}=\mu_2^2\left(1+O(1/r^2)\right), \tag{9.1}$$

$$R_1=\frac{1}{\sqrt{a}}\sqrt{r^2+qC_1}\left(1+O(1/r^2)\right), \qquad R_2=\frac{1}{\sqrt{a}}\sqrt{r^2+qC_2}\left(1+O(1/r^2)\right). \tag{9.2}$$

Near $y=0$, but large R, from (5.4),

$$\begin{aligned}\Delta &= \frac{1}{4r^4K_2}+O(y^2)\\ &= \frac{a^2R^4}{r^4q}\left(1-\frac{q(2+\alpha_2)}{aR^2}+O(1/r^4)\right)+O(y^2)\\ &= \frac{1}{q}\left(1+\frac{q(2C_i\mu_i^2-2-\alpha_2)}{r^2}\right)+O(1/r^4)+O(y^2)\\ &= \frac{1}{q}\left(1+\frac{q(q_1+q_2-q_1\mu_1^2-q_2\mu_2^2)}{r^2}\right)+O(1/r^4)+O(y^2)\end{aligned} \tag{9.3}$$

where we have used (6.37) and have identified

$$qC_1 = q + q_1, \quad qC_2 = q + q_2. \tag{9.4}$$

We find that Δ has the same asymptotic form as the two-charge superstar.

The asymptotic form of the metric can be rewritten in the following form

$$ds^2 = \sqrt{\Delta} ds_1^2 + \frac{1}{\sqrt{\Delta}} ds_2^2 \tag{9.5}$$

analogous to the gauged supergravity ansatz. We have that

$$\begin{aligned} ds_1^2 = & -\left(r^2 + q - \frac{q^2(C_1\mu_1^2 + C_2\mu_2^2 - 1 + \alpha_2 - 2\kappa_1\mu_1^2 - 2\kappa_2\mu_2^2)}{r^2} + O(1/r^4)\right) dt^2 \\ & + \frac{q}{r^2}\left(1 - \frac{q(3C_1\mu_1^2 + 3C_2\mu_2^2 - \alpha_2 - 2)}{r^2} + O(1/r^4)\right) dr^2 + r^2 d\Omega_3^2 \end{aligned} \tag{9.6}$$

and

$$\begin{aligned} ds_2^2 = & \, d\mu_3^2 + \mu_3^2 d\psi^2 + H_1 \left[d\mu_1^2 + \mu_1^2 \left(1 + O(1/r^4)\right) (d\phi_1 + M_{\phi_1} dt)^2\right] \\ & + H_2 \left[d\mu_2^2 + \mu_2^2 \left(1 + O(1/r^4)\right) (d\phi_2 + M_{\phi_2} dt)^2\right] \\ & + O(1/r^5) d\mu_1 dr + O(1/r^5) d\mu_2 dr \\ & + 2\frac{q^2}{r^4}(C_1 + C_2 + \alpha_0 - \alpha_2 - 2)(1 + O(1/r^2))\mu_1\mu_2 d\mu_1 d\mu_2 \\ & + 2\frac{q^2\mu_1^2\mu_2^2}{r^4}(\alpha_2 + \alpha_0 - \kappa_1 - \kappa_2)(1 + O(1/r^2))(d\phi_1 + M_{\phi_1} dt)(d\phi_2 + M_{\phi_2} dt) \end{aligned} \tag{9.7}$$

where the M_i were calculated in the section 6,

$$M_1 = \frac{q}{r^2}(-\alpha_2 + \kappa_1) + O(1/r^4) + O(y^2), \tag{9.8}$$

$$M_2 = \frac{q}{r^2}(-\alpha_2 + \kappa_2) + O(1/r^4) + O(y^2). \tag{9.9}$$

See appendix D for detailed derivation.

When we use (see section 6 and appendix C, e.g. equation (6.37)),

$$q\kappa_1 = q\alpha_2 - q_1, \tag{9.10}$$

$$q\kappa_2 = q\alpha_2 - q_2, \tag{9.11}$$

$$q\alpha_2 = 3q_1\mu_1^2 + 3q_2\mu_2^2 - q_1 - q_2, \tag{9.12}$$

for the g_{tt} and g_{rr}, we get

$$\begin{aligned} ds_1^2 = & -\left(r^2 + 1 - \frac{q_1 + q_2}{r^2} + O(1/r^4)\right) dt^2 \\ & + \frac{1}{r^2}\left(1 - \frac{q_1 + q_2 + 1}{r^2} + O(1/r^4)\right) dr^2 + r^2 d\Omega_3^2 \end{aligned} \tag{9.13}$$

and we have rescaled $r^2 \to qr^2$. By a rescaling $r^2 \to qr^2$, q appears as an overall factor of the metric. The metric has the same asymptotic form as two-charge superstar [31, 32], [33].

When we use

$$qk = q\alpha_0 - q\alpha_2 + q_1 + q_2 = q\alpha_0 + q\alpha_2 - q\kappa_1 - q\kappa_2, \tag{9.14}$$

then the last two terms in ds_2^2 becomes

$$2\frac{\mu_1\mu_2}{r^4}kd\mu_1 d\mu_2 + 2\frac{\mu_1^2\mu_2^2}{r^4}k(d\phi_1 + M_{\phi_1}dt)(d\phi_2 + M_{\phi_2}dt), \tag{9.15}$$

and we have rescaled $r^2 \to qr^2$.

The asymptotic geometry is similar in form to the $\mathrm{U}(1)^3$ gauged supergravity ansatz. One of the differences is that we have additional mixing of μ_1, μ_2, and mixing of ϕ_1, ϕ_2, as in (9.7) or (9.15). These mixing terms correspond to the k parameter in Brauer algebra.

10 Discussions

We studied the characterization of the droplet configurations of the 1/4 BPS geometries which are dual to a family of 1/4 BPS operators with large dimensions in $\mathcal{N}$=4 SYM. We characterized the 4d droplet configurations underlying the 1/4 BPS geometries, following early works. The droplet space is enlarged from the 2d droplet space observed in the 1/2 BPS case. The droplet regions are divided into two regions, and we projected the droplet configuration into two complex planes. We map the concentric circle patterns in the z_1 plane and z_2 plane to two Young diagrams. We identify these two Young diagrams as the two Young diagrams in the operators of the Brauer basis [17, 29], which have total number of boxes $m-k$ and $n-k$. The flux quantum numbers on the droplets map to the edges of the Young diagrams and the radial directions in the two-planes correspond to the upper-right directions along the edges of the two Young diagrams.

We simplified the droplet configurations by projecting it to two two-planes, and draw three diagrams. The first two diagrams are black/white coloring on the two-planes, and the third diagram is the black/white coloring on the (r_1, r_2) space. We simplified droplet configurations in particular by the third diagram. These include general radially symmetric configurations in two two-planes.

There is also a droplet description from other method on the gauge side by [7]. We see more consistency suggesting it to be the 4d droplet space related to the multi-body system. There are some subtleties in this droplet space.

We studied more about the small y expansion of the geometries. In particular, K_0 can be viewed as a potential on the droplet space, and itself is determined by the total droplet configurations. We also performed the large R analysis in the droplet space, and find their connections with the large r asymptotics of the geometries. The large R in the droplet space encodes information of the large r asymptotics. These information are encoded in the K_0, K_1, K_2 in the small y expansion. The large R expansion captures the large r expansion. The asymptotics knows m, n, k, since $J_1 = (m-k)+k$, $J_2 = (n-k)+k$. The droplet configuration captures $m-k, n-k, k$ and the details of all the information of the shapes of two Young diagrams.

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We identified families of geometries that have mixing between two angular directions which are the two angles ϕ_1, ϕ_2 in the z_1 plane and z_2 plane. The geometries in the asymptotic region have mixing metric-component in $h_{\phi_1\phi_2}$ with a family of parameter k. An interesting observation was given that the parameter k have to satisfy $k \leqslant \min(m, n)$. We gave the interpretation that this parameter is identified with the k parameter in the Brauer algebra representations.

Performing a similar analysis for the 1/8 BPS sector would raise an interesting question. A proper basis using Brauer algebras has not been constructed to deal with 1/8 BPS operators. One may guess that three integers would be involved as the coefficients of the mixing among the three angular directions, generalizing the one integer k in the present case. Such analysis may give rise to a hint to apply Brauer algebra for gauge invariant operators involving more kinds of fields than two.

We mainly provided a mapping for the operators built from the projector of the Brauer algebra in (8.6). However, this above-mentioned class of expressions do not cover all types of 1/4 BPS operators. In the droplet picture, there are also other more complicated droplet configurations. It would be nice to understand other types of BPS operators from the Brauer algebra, as well as in other bases. The droplet configurations on gravity side would be helpful to get a complete list of the BPS operators manipulated by the Brauer algebra.

We can also study other excitations on these states. One can consider the supergravity field excitations on them. We can also consider strings excited on them. One can also see the emergence of the other local excitations on the geometries, e.g. [34–36].

We may view the non-BPS states as the excitation above the BPS states. Starting from these heavy BPS states, one can add additional non-BPS excitations on them. These can be done by modifying the operators by adding other fields or multiplying other fields. These studies will also provide another view on the physical meaning of the parameters of the bases.

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A Review of gravity ansatz

We review the geometry and the ansatz for the 1/4 BPS configurations. These backgrounds have an additional S^1 isometry compared with the 1/8 BPS geometries, and have a ten-

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dimensional solution of the form, in the conventions of [9–11],

$$
\begin{aligned}
ds_{10}^2 &= -h^{-2}(dt+\omega)^2 + h^2\left(\left(Z+\frac{1}{2}\right)^{-1} 2\partial_i\bar{\partial}_j K dz^i d\bar{z}^{\bar{j}} + dy^2\right) + y(e^G d\Omega_3^2 + e^{-G}(d\psi+\mathcal{A})^2),\\
F_5 &= \left\{-d[y^2 e^{2G}(dt+\omega)] - y^2(d\omega+\eta\mathcal{F}) + 2i\partial\bar{\partial}K\right\}\wedge d\Omega^3 + \text{dual},\\
h^{-2} &= 2y\cosh G,\\
Z &= \frac{1}{2}\tanh G = -\frac{1}{2}y\partial_y\left(\frac{1}{y}\partial_y K\right),\\
d\omega &= \frac{i}{2}d\left[\frac{1}{y}\partial_y(\bar{\partial}-\partial)K\right] = \frac{i}{y}(\partial_i\partial_{\bar{j}}\partial_y K dz_i d\bar{z}_{\bar{j}} + \partial_{\bar{i}}Z d\bar{z}_i dy - \partial_i Z dz_i dy),\\
2\eta\mathcal{F} &= -i\partial\bar{\partial}D.
\end{aligned}
\tag{A.1}
$$

$K = K(z_i, \bar{z}_i; y)$, where $i = 1, 2,$ is the Kähler potential for the 4d base, which also varies with the y direction. D can be set to a constant, if the fibration of the S^1 is a direct product. The volume of the 4d base is constrained by a Monge-Ampere equation, as well as an equation for function D,

$$\log\det h_{i\bar{j}} = \log\left(Z+\frac{1}{2}\right) + n\eta\log y + \frac{1}{y}(2-n\eta)\partial_y K + D(z_i, \bar{z}_{\bar{j}}), \tag{A.2}$$

$$(1+*_4)\partial\bar{\partial}D = \frac{4}{y^2}(1-n\eta)\partial\bar{\partial}K. \tag{A.3}$$

In other words,

$$\det\partial_i\partial_{\bar{j}}K = \left(Z+\frac{1}{2}\right) y^{n\eta} e^{\frac{1}{y}(2-n\eta)\partial_y K} e^D. \tag{A.4}$$

According to the analysis of [10, 11, 16] a family of geometries have the expansion from the droplet space as:

$$K = -\frac{1}{4}y^2\log(y^2) + K_0(z_i,\bar{z}_i) + y^2 K_1(z_i,\bar{z}_i) + (y^2)^2 K_2(z_i,\bar{z}_i) + \sum_{n\geqslant 3}(y^2)^n K_n(z_i,\bar{z}_i) \tag{A.5}$$

for $Z = \frac{1}{2}$ and

$$K = \frac{1}{4}y^2\log(y^2) + K_0(z_i,\bar{z}_i) + y^2 K_1(z_i,\bar{z}_i) + (y^2)^2 K_2(z_i,\bar{z}_i) + \sum_{n\geqslant 3}(y^2)^n K_n(z_i,\bar{z}_i) \tag{A.6}$$

for $Z = -\frac{1}{2}$, with $\partial_i\partial_{\bar{j}}K_0 = 0$. The last terms above correspond to expansion with higher order terms $(y^2)^n K_n(z_i,\bar{z}_i)$. The $K_0(z_i,\bar{z}_i)$ is a function defined on the droplet space.

As was shown in [16], all other higher K_n, with $n \geqslant 3$, are expressed in terms of K_0, K_1, K_2, and thus the entire solutions to K are given uniquely by K_0, K_1, K_2.

K_0, K_1 are determined by the coupled equations on the $Z = \frac{1}{2}$ droplet region [16]:

$$
\begin{aligned}
(\partial_1\partial_{\bar{1}}K_0)(\partial_2\partial_{\bar{2}}K_1) + (\partial_1\partial_{\bar{1}}K_1)(\partial_2\partial_{\bar{2}}K_0) - 2(\partial_1\partial_{\bar{2}}K_0)(\partial_{\bar{1}}\partial_2 K_1) &= 0,\\
\det\partial_i\partial_{\bar{j}}K_0 &= \frac{1}{4e}e^{2K_1}.
\end{aligned}
\tag{A.7}
$$

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For the $AdS_5 \times S^5$,

$$K_0 = \begin{cases} \frac{1}{2}aR^2 - \frac{1}{2}ar_0^2\log(R^2/r_0^2), & R^2 \geqslant r_0^2 \\ \frac{1}{2}ar_0^2, & 0 \leqslant R^2 \leqslant r_0^2 \end{cases} \tag{A.8}$$

as analyzed in [16], where $Z = -\frac{1}{2}$ droplets are in $0 \leqslant R^2 \leqslant r_0^2$ and $Z = \frac{1}{2}$ droplets are in $R^2 \geqslant r_0^2$. Here, the K_0 is constant in $Z = -\frac{1}{2}$ droplets. In the above, $R^2 = |z_1|^2 + |z_2|^2$, in this appendix. The plot in figure $Z = -\frac{1}{2}$ strips; and in the limit when these three strips go to zero, it recovers the expression (A.8). One can also introduce an overall constant shift in K_0, since only its derivatives appear. One can also rescale the above expression by the rescaling transformations as in section 6.

B Derivation of the metric

In this appendix we derive the general form of the metric, using new variables in the section 2.

For the situation that the Kähler potential does not depend on the angular coordinates ϕ_1, ϕ_2, that $K = K(r_1, r_2, y)$, the metric (2.1) can be expressed by

$$\begin{aligned} ds^2 = & -h^{-2}\left(dt^2 + 2(\omega_{\phi_1}d\phi_1 + \omega_{\phi_2}d\phi_2)dt + \omega_{\phi_1}^2 d\phi_1^2 + \omega_{\phi_2}^2 d\phi_2^2 + 2\omega_{\phi_1}\omega_{\phi_2}d\phi_1 d\phi_2\right) \\ & +h^2\left(\mu_3^2 dr^2 + r^2 d\mu_3^2 + S_{11}R_1^2(d\mu_1^2 + \mu_1^2 d\phi_1^2) + S_{22}R_2^2(d\mu_2^2 + \mu_2^2 d\phi_2^2)\right. \\ & + \left(S_{11}\mu_1^2 T_1^2 + S_{22}\mu_2^2 T_2^2 + 2S_{12}\mu_1\mu_2 T_1 T_2\right) dr^2 \\ & +2\left(S_{11}R_1 T_1\mu_1 + S_{12}T_2\mu_2 R_1 - \mu_1 r\right) d\mu_1 dr \\ & +2\left(S_{22}R_2 T_2\mu_2 + S_{12}T_1\mu_1 R_2 - \mu_2 r\right) d\mu_2 dr \\ & \left.+2S_{12}R_1 R_2 d\mu_1 d\mu_2 + 2S_{12}R_1 R_2\mu_1\mu_2 d\phi_1 d\phi_2\right) \\ & +ye^G d\Omega_3^2 + ye^{-G}d\psi^2, \end{aligned} \tag{B.1}$$

where we have used $\omega = \omega_{\phi_1}d\phi_1 + \omega_{\phi_2}d\phi_2$ and $S_{ij} = S_{ji}$, which are the consequence of the assumption. We have defined $T_i = dR_i/dr$.

Performing the shift of the angular variables $\phi_i \to \phi_i - t$, $i = 1, 2$, the metric can be further written to be the form

$$\begin{aligned} ds^2 = & -h^{-2}\left(1 + h_{ab}M_a M_b - S_t\right) dt^2 + h^2\left(\mu_3^2 + S_{11}\mu_1^2 T_1^2 + S_{22}\mu_2^2 T_2^2 + 2S_{12}\mu_1\mu_2 T_1 T_2\right) dr^2 \\ & +2h^2(S_{11}R_1 T_1\mu_1 + S_{12}T_2\mu_2 R_1 - \mu_1 r)d\mu_1 dr + 2h^2(S_{22}R_2 T_2\mu_2 + S_{12}T_1\mu_1 R_2 - \mu_2 r)d\mu_2 dr \\ & +\sqrt{\Delta}r^2 d\Omega_3^2 + \frac{1}{\sqrt{\Delta}}\left(d\mu_3^2 + H_1 d\mu_1^2 + H_2 d\mu_2^2\right) + 2h^2\left(S_{12}R_1 R_2 - \frac{\mu_1\mu_2}{\Delta}\right) d\mu_1 d\mu_2 \\ & +\frac{\mu_3^2}{\sqrt{\Delta}}d\psi^2 + h^{-2}h_{ab}(d\phi_a + M_a dt)(d\phi_b + M_b dt). \end{aligned} \tag{B.2}$$

The more detailed computations are given below.

B.1 Metric functions

We introduce Δ by the equation

$$e^G = \frac{r\sqrt{\Delta}}{\mu_3}. \tag{B.3}$$

Then Z and h^{-2} may be expressed as

$$Z = \frac{1}{2}\tanh G = \frac{1}{2}\frac{r^2\Delta - \mu_3^2}{r^2\Delta + \mu_3^2}, \tag{B.4}$$

$$h^{-2} = 2y\cosh G = \frac{r^2\Delta + \mu_3^2}{\sqrt{\Delta}}. \tag{B.5}$$

Eq. (B.4) may be used to give Δ in terms of Z as

$$\Delta = \frac{\mu_3^2}{r^2}\frac{1+2Z}{1-2Z}. \tag{B.6}$$

B.2 Terms with $d\mu_i d\mu_j$

We first calculate the following,

$$\begin{aligned} h^2 r^2 d\mu_3^2 &= \frac{1}{\sqrt{\Delta}}\frac{r^2\Delta}{r^2\Delta + \mu_3^2} d\mu_3^2 \\ &= \frac{1}{\sqrt{\Delta}} d\mu_3^2 - \frac{1}{\sqrt{\Delta}}\frac{1}{r^2\Delta + \mu_3^2}(\mu_1 d\mu_1 + \mu_2 d\mu_2)^2 \\ &= \frac{1}{\sqrt{\Delta}} d\mu_3^2 - \frac{h^2}{\Delta}(\mu_1^2 d\mu_1^2 + \mu_2^2 d\mu_2^2 + 2\mu_1\mu_2 d\mu_1 d\mu_2). \end{aligned} \tag{B.7}$$

Using this, we have

$$\begin{aligned} &h^2\left(r^2 d\mu_3^2 + S_{11}R_1^2 d\mu_1^2 + S_{22}R_2^2 d\mu_2^2 + 2S_{12}R_1R_2 d\mu_1 d\mu_2\right) \\ &= \frac{1}{\sqrt{\Delta}}\left(d\mu_3^2 + H_1 d\mu_1^2 + H_2 d\mu_2^2\right) + 2h^2\left(S_{12}R_1R_2 - \frac{\mu_1\mu_2}{\Delta}\right) d\mu_1 d\mu_2, \end{aligned} \tag{B.8}$$

where we have defined

$$H_i = \sqrt{\Delta} h^2 \left(S_{ii}R_i^2 - \frac{\mu_i^2}{\Delta}\right) \tag{B.9}$$

for $i = 1, 2$. For two-charge superstar, this becomes the H_i of the superstar.

When the following condition is satisfied,

$$S_{12}R_1R_2 - \frac{\mu_1\mu_2}{\Delta} = 0, \tag{B.10}$$

the metric does not have the mixing term $d\mu_1 d\mu_2$. For superstar, this is the case.

B.3 Terms with $d\phi_i d\phi_j$

The relevant terms in the metric (B.1) are

$$\begin{aligned} &\sum_{i=1,2}(-h^{-2}2\omega_{\phi_i} dt d\phi_i - h^{-2}\omega_{\phi_i}^2 d\phi_i^2 + h^2 S_{ii} r_i^2 d\phi_i^2) \\ &-h^{-2}2\omega_{\phi_1}\omega_{\phi_2} d\phi_1 d\phi_2 + 2h^2 S_{12}R_1R_2\mu_1\mu_2 d\phi_1 d\phi_2 \\ &\quad = h^{-2}\left(S_1 d\phi_1^2 - 2\omega_{\phi_1} dt d\phi_1 + S_2 d\phi_2^2 - 2\omega_{\phi_2} dt d\phi_2 + 2N_{12} d\phi_1 d\phi_2\right), \end{aligned} \tag{B.11}$$

where we have defined

$$S_i = h^4 S_{ii} r_i^2 - \omega_{\phi_i}^2 \quad (i = 1, 2), \tag{B.12}$$

$$N_{12} = h^4 S_{12} r_1 r_2 - \omega_{\phi_1}\omega_{\phi_2}. \tag{B.13}$$

Making the shift $\phi_i \to \phi_i - t$, (B.11) becomes

$$\begin{aligned} & h^{-2} S_1 d\phi_1^2 - 2h^{-2}(S_1 + \omega_{\phi_1} + N_{12}) dt d\phi_1 \\ & + h^{-2} S_2 d\phi_2^2 - 2h^{-2}(S_2 + \omega_{\phi_2} + N_{12}) dt d\phi_2 + 2h^{-2} N_{12} d\phi_1 d\phi_2 \\ & + h^{-2}(S_1 + S_2 + 2\omega_{\phi_1} + 2\omega_{\phi_2} + 2N_{12}) dt^2 \\ & \quad = h^{-2}\left(S_1 d\phi_1^2 + 2S_{1t} dt d\phi_2 + S_2 d\phi_2^2 + 2S_{2t} dt d\phi_3 + 2N_{12} d\phi_1 d\phi_2 + S_t dt^2\right). \end{aligned} \tag{B.14}$$

where we have defined

$$\begin{aligned} S_{1t} &= -S_1 - \omega_{\phi_1} - N_{12}, \\ S_{2t} &= -S_2 - \omega_{\phi_2} - N_{12}, \\ S_t &= S_1 + S_2 + 2\omega_{\phi_1} + 2\omega_{\phi_2} + 2N_{12}. \end{aligned} \tag{B.15}$$

Finally, (B.14) can be written as

$$\begin{aligned} & S_1 d\phi_1^2 + 2S_{1t} dt d\phi_1 + S_2 d\phi_2^2 + 2S_{2t} dt d\phi_2 + 2N_{12} d\phi_1 d\phi_2 + S_t dt^2 \\ & \quad = h_{ij}(d\phi_i + M_i dt)(d\phi_j + M_j dt) - h_{ij} M_i M_j dt^2 + S_t dt^2, \end{aligned} \tag{B.16}$$

where

$$h_{11} = S_1, \quad h_{22} = S_2, \quad h_{12} = N_{12}, \tag{B.17}$$

and

$$\begin{aligned} \begin{pmatrix} M_1 \\ M_2 \end{pmatrix} &= \frac{1}{S_1 S_2 - N_{12}^2} \begin{pmatrix} S_2 S_{1t} - N_{12} S_{2t} \\ S_1 S_{2t} - N_{12} S_{1t} \end{pmatrix} \\ &= -1 + \frac{1}{S_1 S_2 - N_{12}^2} \begin{pmatrix} -S_2 \omega_{\phi_1} + N_{12}\omega_{\phi_2} \\ -S_1 \omega_{\phi_2} + N_{12}\omega_{\phi_1} \end{pmatrix}. \end{aligned} \tag{B.18}$$

When $h_{12} = N_{12} = 0$, the metric does not have the mixing term $d\phi_1 d\phi_2$, and M_i will get a simple expression. Defining $N_i = M_i|_{N_{12}=0}$, we have

$$\begin{aligned} N_i &= -1 - S_i^{-1}\omega_{\phi_i} \\ &= -1 - \frac{\omega_{\phi_i}}{h^4 S_{ii} r_i^2 - \omega_{\phi_i}^2}. \end{aligned} \tag{B.19}$$

For superstar, $N_{12} = 0$, and N_i are calculated

$$N_i = -\frac{q_i}{r^2 + q_i}. \tag{B.20}$$

See appendix D.3 for more details.

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B.4 Terms with $d\mu_i dr$

Here we will analyze the condition under which the mixing terms $d\mu_i dr$ vanish.

The mixing terms vanish when the following conditions are satisfied

$$
\begin{aligned}
S_{11}R_1T_1\mu_1 + S_{12}T_2\mu_2R_1 &= r\mu_1, \\
S_{22}R_2T_2\mu_2 + S_{12}T_1\mu_1R_2 &= r\mu_2,
\end{aligned}
\tag{B.21}
$$

One can show that these are satisfied for two-charge superstar with the help of the equations in appendix D.3.

The set of the equations can be summarized as

$$
\begin{pmatrix} S_{11}R_1\mu_1 & S_{12}R_1\mu_2 \\ S_{12}R_2\mu_1 & S_{22}R_2\mu_2 \end{pmatrix} \begin{pmatrix} T_1 \\ T_2 \end{pmatrix} = r \begin{pmatrix} \mu_1 \\ \mu_2 \end{pmatrix}
\tag{B.22}
$$

Note that the determinant of the matrix is $(S_{11}S_{22} - S_{12}^2)R_1R_2\mu_1\mu_2 = (\det S_{ij})r_1r_2$. Using $T_i = R_i'$,

$$
\begin{aligned}
\begin{pmatrix} R_1' \\ R_2' \end{pmatrix} &= \frac{r}{(\det S_{ij})r_1r_2} \begin{pmatrix} S_{22}R_2\mu_2 & -S_{12}R_1\mu_2 \\ -S_{12}R_2\mu_1 & S_{11}R_1\mu_1 \end{pmatrix} \begin{pmatrix} \mu_1 \\ \mu_2 \end{pmatrix} \\
&= \frac{r}{(\det S_{ij})r_1r_2} \begin{pmatrix} \mu_2(S_{22}R_2\mu_1 - S_{12}R_1\mu_2) \\ \mu_1(S_{11}R_1\mu_2 - S_{12}R_2\mu_1) \end{pmatrix} \\
&= \frac{r}{(\det S_{ij})} \begin{pmatrix} S_{22}R_1^{-1} - R_{12}R_2^{-1}\mu_2\mu_1^{-1} \\ S_{11}R_2^{-1} - R_{12}R_1^{-1}\mu_1\mu_2^{-1} \end{pmatrix}
\end{aligned}
\tag{B.23}
$$

It can be further simplified if we use (B.10), which is the equation for vanishing $d\mu_1 d\mu_2$ term. We then have

$$
\begin{aligned}
\begin{pmatrix} R_1' \\ R_2' \end{pmatrix} &= \frac{r}{(\det S_{ij})} \begin{pmatrix} S_{22}R_1^{-1} - \frac{1}{\Delta}R_1^{-1}R_2^{-2}\mu_2^2 \\ S_{11}R_2^{-1} - \frac{1}{\Delta}R_2^{-1}R_1^{-2}\mu_1^2 \end{pmatrix} \\
&= \frac{r}{(\det S_{ij})} \begin{pmatrix} \frac{1}{\sqrt{\Delta}}h^{-2}H_2R_1^{-1}R_2^{-2} \\ \frac{1}{\sqrt{\Delta}}h^{-2}H_1R_2^{-1}R_1^{-2} \end{pmatrix}.
\end{aligned}
\tag{B.24}
$$

where we have used the definition of H_i (B.9). Dividing the first equation by the second equation, we obtain

$$
(\log R_1^2)'H_1 = (\log R_2^2)'H_2
\tag{B.25}
$$

We have that for superstar, (B.25) is equal to $2rH_1H_2f^{-1}$.

C Derivation of the N_{12}, M_1, M_2 in large r

In this appendix, we present the details of the calculations to derive the asymptotic forms of the mixing functions N_{12}, M_1, M_2.

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Using (6.13), (6.15), (6.16), (6.18), the expressions (5.12) are evaluated as

$$
\begin{aligned}
n_{12} &= 2K_2\partial_{t_1}\partial_{t_2}K_0 - \partial_{t_1}K_1\partial_{t_2}K_1 \\
&= \frac{q}{2a^2R^4}\left(1+\frac{q}{aR^2}(2+\alpha_2)+O(1/r^4)\right)\frac{2qr_1^2r_2^2}{R^4}\left(1+\frac{q\alpha_0}{aR^2}+O(1/r^4)\right) \\
&\quad -r_1^2r_2^2\left(\frac{q}{aR^4}\right)^2\left(1+\frac{q}{aR^2}(1+\kappa_1)+O(1/r^4)\right)\left(1+\frac{q}{aR^2}(1+\kappa_2)+O(1/r^4)\right) \\
&= \frac{r_1^2r_2^2}{a^2R^8}\left(\frac{q^3}{aR^2}(\alpha_2+\alpha_0-\kappa_1-\kappa_2)+O(1/r^4)\right) \\
&= \frac{q^3r_1^2r_2^2}{a^3R^{10}}k+O(1/r^8),
\end{aligned}
\tag{C.1}
$$

and

$$
\begin{aligned}
s_1 &= 2K_2\partial_{t_1}^2K_0-(\partial_{t_1}K_1)^2 \\
&= \frac{q}{2a^2R^4}\left(1+\frac{q(\alpha_2+2)}{aR^2}\right)\left(2ar_1^2-\frac{2qr_1^2r_2^2}{R^4}+O(1/r^2)\right)-\left(\frac{qr_1^2}{aR^4}\right)^2\left(1+\frac{q}{aR^2}(1+\kappa_1)\right)^2 \\
&= \frac{qr_1^2}{aR^4}\left(1+\frac{q(\alpha_2+2)}{aR^2}-\frac{qr_2^2}{aR^4}-\frac{qr_1^2}{aR^4}+O(1/r^4)\right) \\
&= \frac{qr_1^2}{aR^4}\left(1+\frac{q(\alpha_2+1)}{aR^2}+O(1/r^4)\right), \\
s_2 &= \frac{qr_2^2}{aR^4}\left(1+\frac{q(\alpha_2+1)}{aR^2}+O(1/r^4)\right).
\end{aligned}
\tag{C.2}
$$

The mixing between time and the angles are given by M_i

$$
M_1 = -1+\frac{s_2\partial_{t_1}K_1-n_{12}\partial_{t_2}K_1}{s_1s_2-n_{12}^2}, \tag{C.3}
$$

$$
M_2 = -1+\frac{s_1\partial_{t_2}K_1-n_{12}\partial_{t_1}K_1}{s_1s_2-n_{12}^2}. \tag{C.4}
$$

Since $s_1, s_2, \partial_{t_1}K_1, \partial_{t_2}K_1 = O(1/r^2)$, and $n_{12}=O(k/r^6)$, the effect of n_{12}^2 in the denominator is to give at most $O(1/r^8)$ terms in M_1, M_2. So the following expressions are up to $O(1/r^8)$,

$$
M_1 = -1+\frac{\partial_{t_1}K_1}{s_1}-\frac{n_{12}\partial_{t_2}K_1}{s_1s_2}+O(1/r^8), \tag{C.5}
$$

$$
M_2 = -1+\frac{\partial_{t_2}K_1}{s_2}-\frac{n_{12}\partial_{t_1}K_1}{s_1s_2}+O(1/r^8), \tag{C.6}
$$

where the last terms will be evaluated as

$$
-\frac{n_{12}\partial_{t_2}K_1}{s_1s_2} = -\frac{q^2r_2^2}{a^2R^6}k+O(1/r^6), \tag{C.7}
$$

$$
-\frac{n_{12}\partial_{t_1}K_1}{s_1s_2} = -\frac{q^2r_1^2}{a^2R^6}k+O(1/r^6). \tag{C.8}
$$

In other words, the difference between M_i, N_i is

$$M_1 - N_1 = -\frac{n_{12}\partial_{t_2} K_1}{s_1 s_2} + O(1/r^8) = \left(-k\frac{q^2 r_2^2}{a^2 R^6} + O(1/r^6)\right) + O(1/r^8), \tag{C.9}$$

$$M_2 - N_2 = -\frac{n_{12}\partial_{t_1} K_1}{s_1 s_2} + O(1/r^8) = \left(-k\frac{q^2 r_1^2}{a^2 R^6} + O(1/r^6)\right) + O(1/r^8). \tag{C.10}$$

The $-k\frac{q^2 r_2^2}{a^2 R^6}$ is one of the $O(1/r^4)$ terms in N_1 and in M_1, but it is the only $O(1/r^4)$ term in $M_1 - N_1$. Therefore $M_i - N_i = O(k/r^4)$, for $k \neq 0$; and $M_i - N_i = O(1/r^6)$, for $k = 0$.

Now we calculate M_i in the large r,

$$\begin{aligned} M_1 &= -1 + \frac{\partial_{t_1} K_1}{s_1} + O(1/r^4) \\ &= -1 + \left(1 + \frac{q}{aR^2}(1+\kappa_1)\right)\left(1 + \frac{q(\alpha_2+1)}{aR^2} + O(1/r^4)\right)^{-1} + O(1/r^4) \\ &= \frac{q}{aR^2}\left(-\alpha_2 + \kappa_1\right) + O(1/r^4) \\ &= \frac{q}{r^2}\left(-\alpha_2 + \kappa_1\right) + O(1/r^4). \end{aligned} \tag{C.11}$$

Similarly

$$\begin{aligned} M_2 &= -1 + \frac{\partial_{t_2} K_1}{s_2} + O(1/r^4) \\ &= -1 + \left(1 + \frac{q}{aR^2}(1+\kappa_2)\right)\left(1 + \frac{q(\alpha_2+1)}{aR^2} + O(1/r^4)\right)^{-1} + O(1/r^4) \\ &= \frac{q}{aR^2}\left(-\alpha_2 + \kappa_2\right) + O(1/r^4) \\ &= \frac{q}{r^2}\left(-\alpha_2 + \kappa_2\right) + O(1/r^4). \end{aligned} \tag{C.12}$$

Now we focus on the family of solutions

$$K_0^{(1)} = \frac{q^2}{a}\left(\frac{dr_1^2 + er_2^2}{R^4} + \frac{fr_1 r_2}{R^4}\right). \tag{C.13}$$

1. when $f = 0$,

$$\begin{aligned} \kappa_1 &= \frac{4(e-d)}{R^2}(2r_2^2 - r_1^2), \\ \kappa_2 &= \frac{4(d-e)}{R^2}(2r_1^2 - r_2^2), \\ \alpha_0 &= \frac{4}{R^2}\left(d(2r_1^2 - r_2^2) + e(2r_2^2 - r_1^2)\right). \end{aligned} \tag{C.14}$$

   Note that $\kappa_1 = \kappa_2 = 0$, $\alpha_0 \neq 0$ when $d = e$.

2. when $d = e = 0$,

$$\begin{aligned} \kappa_1 &= -\frac{f}{r_1 r_2 R^4}(r_2^6 + 13r_1^2 r_2^4 - 33r_1^4 r_2^2 + 3r_1^6), \\ \kappa_2 &= -\frac{f}{r_1 r_2 R^4}(r_1^6 + 13r_2^2 r_1^4 - 33r_2^4 r_1^2 + 3r_2^6), \\ \alpha_0 &= -\frac{3f}{r_1 r_3 R^2}(r_1^4 - 6r_2^2 r_1^4 + r_2^4). \end{aligned} \tag{C.15}$$

Although it is possible to have the third parameter f, we think that it does not have a relevant physical meaning, and we have chosen $f = 0$. We have set therefore

$$f = 0. \tag{C.16}$$

These are the solutions presented in the text in (6.19)–(6.21), (6.22).

From the expressions of N_{12}, M_1, M_2,

$$\begin{aligned} q\,(\alpha_2 - \kappa_1) &= qm = q_1, \\ q\,(\alpha_2 - \kappa_2) &= qn = q_2, \\ \alpha_0 + \alpha_2 - \kappa_1 - \kappa_2 &= k. \end{aligned} \tag{C.17}$$

Plugging these conditions into the solutions (6.22), we find

$$\begin{aligned} 4e &= k - m = k - q_1/q, \\ 4d &= k - n = k - q_2/q, \\ c &= k. \end{aligned} \tag{C.18}$$

and we have $\kappa_1 - \kappa_2 = n - m$.

There are several equivalent and alternative ways of writing these variables. For example,

$$\begin{aligned} q\alpha_0 &= q\alpha_2 + qk - q_1 - q_2, \\ q\alpha_2 &= q\alpha_0 - qk + q_1 + q_2, \\ q\kappa_1 &= q\alpha_2 - q_1 = q\alpha_0 - qk + q_2, \\ q\kappa_2 &= q\alpha_2 - q_2 = q\alpha_0 - qk + q_1, \\ qk &= q\alpha_0 - q\alpha_2 + q_1 + q_2 = q\alpha_0 + q\alpha_2 - q\kappa_1 - q\kappa_2, \\ 2q\alpha_2 &= q\kappa_1 + q\kappa_2 + q_1 + q_2, \\ q_1 + q_2 &= q\alpha_2 - q\alpha_0 + qk = 2q\alpha_2 - q\kappa_1 - q\kappa_2, \\ q\kappa_1 - q\kappa_2 &= q_2 - q_1, \\ q\kappa_1 + q\kappa_2 &= q\alpha_0 + q\alpha_2 - qk = 2q\alpha_2 - q_1 - q_2. \end{aligned} \tag{C.19}$$

D Derivation of asymptotic metric

In this appendix, we present the details of the derivation of (9.6) and (9.7).

D.1 Metric functions

Here we collect equations which would be helpful to calculate the asymptotic form of the metric.

$$h^2 = \frac{1}{r^2\sqrt{\Delta}} + O(y^2), \tag{D.1}$$

$$\Delta = \frac{1}{4r^4 K_2} + O(y^2). \tag{D.2}$$

$$\begin{aligned}
K_2 &= \frac{q}{4a^2R^4}\left(1+\frac{q(\alpha_2+2)}{aR^2}\right)+O(1/r^8) \\
&= \frac{q}{4r^4}\left(1-2\frac{qC_1\mu_1^2+qC_2\mu_2^2}{r^2}\right)\left(1+\frac{q(\alpha_2+2)}{r^2}\right)+O(1/r^8) \\
&= \frac{q}{4r^4}\left(1+\frac{q(\alpha_2+2-2qC_1\mu_1^2-2qC_2\mu_2^2)}{r^2}\right)+O(1/r^8). \qquad \text{(D.3)}
\end{aligned}$$

$$R_1 = \frac{1}{\sqrt{a}}\sqrt{r^2+qC_1}\left(1+O(1/r^2)\right), \qquad R_2 = \frac{1}{\sqrt{a}}\sqrt{r^2+qC_2}\left(1+O(1/r^2)\right), \quad \text{(D.4)}$$

$$aR^2 = a(r_1^2+r_2^2) = r^2+qC_1\mu_1^2+qC_2\mu_2^2+O(1/r^2)-O(y^2). \qquad \text{(D.5)}$$

$$\begin{aligned}
S_{ii} &= 2\frac{\partial_i\partial_{\bar{i}}K}{Z+\frac{1}{2}} = \frac{1}{2}\frac{1}{r_i^2}\partial_{t_i}^2K+O(y^2) \\
&= a-\frac{q(R^2-r_i^2)}{R^4}+O(1/r^4)+O(y^2), \qquad \text{(D.6)}
\end{aligned}$$

$$\begin{aligned}
S_{12} &= \frac{1}{2}\frac{\partial_{r_1}\partial_{r_2}K}{Z+\frac{1}{2}} = \frac{1}{2}\partial_{r_1}\partial_{r_2}K_0+O(y^2) \\
&= \frac{qr_1r_2}{R^4}\left(1+\frac{q\alpha_0}{aR^2}+O(1/r^4)\right)+O(y^2). \qquad \text{(D.7)}
\end{aligned}$$

$$\begin{aligned}
s_i &= \frac{qr_i^2}{aR^4}\left(1+\frac{q(\alpha_2+1)}{aR^2}+O(1/r^4)\right), \\
h^2 &= \sqrt{4K_2}+O(y^2) = \frac{\sqrt{q}}{aR^2}\left(1+\frac{q(\alpha_2+2)}{2aR^2}\right)+O(1/r^6)+O(y^2). \qquad \text{(D.8)}
\end{aligned}$$

Comparing with (5.2), (5.5),

$$K_2 = \frac{r_0^2}{4r^4\Delta}. \qquad \text{(D.9)}$$

Because AdS is given by $\Delta = 1$, $K_{2(AdS)} = \frac{r_0^2}{4r^4}$. We also have,

$$K_2 = \frac{q}{4(aR^2-q)^2}, \qquad \text{(D.10)}$$

for AdS. So we have that for AdS, $C_1 = C_2 = 1$. Note that $R^2 = r_1^2 + r_2^2$.

D.2 Calculation of metric

The factor in front of dt^2 is now calculated. Taking account of $M_i = O\left(1/r^2\right)$ and $N_{12} = O\left(1/r^6\right)$, we have

$$\begin{aligned}
&h^{-2}\left(1+h_{ab}M_aM_b-S_t\right) \qquad \text{(D.11)} \\
&= h^{-2}\left(1+S_1M_1^2+S_2M_2^2+2N_{12}M_1M_2-S_1-S_2-2\omega_1-2\omega_2-2N_{12}\right) \\
&= h^{-2}\left(1-S_1-S_2-2\omega_1-2\omega_2+O(1/r^6)\right)
\end{aligned}$$

$$
\begin{aligned}
&= h^{-2}\left(1-\frac{qr_1^2}{aR^4}\left(1+\frac{q(\alpha_2+1)}{aR^2}\right)-\frac{qr_2^2}{aR^4}\left(1+\frac{q(\alpha_2+1)}{aR^2}\right)\right.\\
&\quad\left.+2\frac{qr_1^2}{aR^4}\left(1+\frac{q(1+\kappa_1)}{aR^2}\right)+2\frac{qr_2^2}{aR^4}\left(1+\frac{q(1+\kappa_2)}{aR^2}\right)+O(1/r^6)+O(y^2)\right)\\
&=\sqrt{\Delta}r^2\left(1+\frac{q}{aR^2}+\frac{q^2(1-\alpha_2)}{a^2R^4}+\frac{2q^2(r_1^2\kappa_1+r_2^2\kappa_2)}{a^2R^6}\right)+O(1/r^4)+O(y^2)\\
&=\sqrt{\Delta}\left(r^2+q-q^2\frac{C_1\mu_1^2+C_2\mu_2^2}{r^2}+\frac{q^2(1-\alpha_2)}{r^2}+\frac{2q^2(\kappa_1\mu_1^2+\kappa_2\mu_2^2)}{r^2}\right)+O(1/r^4)+O(y^2).
\end{aligned}
$$

We next calculate the factor in front of dr^2 in large r. We first show

$$
\begin{aligned}
&\mu_3^2+S_{11}\mu_1^2T_1^2+S_2\mu_2^2T_2^2+2S_{12}\mu_1\mu_2T_1T_2\\
&\quad=\mu_3^2+a\left(1-\frac{qr_2^2}{aR^4}\right)\mu_1^2T_1^2+a\left(1-\frac{qr_1^2}{aR^4}\right)\mu_2^2T_2^2+2\frac{qr_1r_2}{R^4}\mu_1\mu_2T_1T_2+O(1/r^4)\\
&\quad=\mu_3^2+a\mu_1^2T_1^2+a\mu_2^2T_2^2-\frac{q}{R^4}\left(r_1\mu_2T_2-r_2\mu_1T_1\right)^2+O(1/r^4)\\
&\quad=\mu_3^2+\mu_1^2\left(1-\frac{qC_1}{r^2}\right)+\mu_2^2\left(1-\frac{qC_2}{r^2}\right)+O(1/r^4)\\
&\quad=1-\frac{qC_1\mu_1^2+qC_2\mu_2^2}{r^2}+O(1/r^4).
\end{aligned}
\tag{D.12}
$$

With the help of (D.1), (D.2) and (D.3), we obtain

$$
\begin{aligned}
&h^2\left(\mu_3^2+S_{11}\mu_1^2T_1^2+S_{22}\mu_2^2T_2^2+2S_{12}\mu_1\mu_2T_1T_2\right)\\
&\quad=\sqrt{\Delta}4r^2K_2\left(1-\frac{qC_1\mu_1^2+qC_2\mu_2^2}{r^2}\right)+O(1/r^4)+O(y^2)\\
&\quad=\sqrt{\Delta}\frac{q}{r^2}\left(1+\frac{q(\alpha_2+2)-3q(C_1\mu_1^2+C_2\mu_2^2)}{r^2}\right)+O(1/r^4)+O(y^2).
\end{aligned}
\tag{D.13}
$$

The factor H_i is evaluated at large r,

$$
\begin{aligned}
H_1&=\sqrt{\Delta}h^2\left(S_{11}R_1^2-\frac{\mu_1^2}{\Delta}\right)\\
&=\frac{1}{r^2}\left(\left(a-\frac{qr_2^2}{R^4}\right)R_1^2-4r^4K_2\mu_1^2\right)+O(1/r^4)+O(y^2)\\
&=\frac{1}{r^2}\left(\left(a-\frac{qr_2^2}{R^4}\right)R_1^2-r^4\frac{q}{a^2R^4}\mu_1^2\right)+O(1/r^4)+O(y^2)\\
&=\frac{1}{r^2}\left(\left(1-\frac{q\mu_2^2}{r^2}\right)(r^2+qC_1)-q\mu_1^2\right)+O(1/r^4)+O(y^2)\\
&=1+\frac{q(C_1-1)}{r^2}+O(1/r^4)+O(y^2),
\end{aligned}
\tag{D.14}
$$

$$
H_2=1+\frac{q(C_2-1)}{r^2}+O(1/r^4)+O(y^2).
\tag{D.15}
$$

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We calculate the factor in front of $d\mu_1 d\mu_2$ in large r:

$$
\begin{aligned}
&2h^2\left(S_{12}R_1R_2-\frac{\mu_1\mu_2}{\Delta}\right)\\
&=\frac{2}{r^2\sqrt{\Delta}}\left(\frac{qr_1r_2}{R^4}\left(1+\frac{q\alpha_0}{aR^2}+O(1/r^4)\right)R_1R_2-\mu_1\mu_2 4r^4K_2\right)+O(y^2)\\
&=\frac{2}{r^2\sqrt{\Delta}}\left(\frac{q}{R^4}\left(1+\frac{q\alpha_0}{aR^2}+O(1/r^4)\right)R_1^2R_2^2-4r^4K_2\right)\mu_1\mu_2+O(y^2)\\
&=\frac{q}{a^2R^4}\frac{2}{r^2\sqrt{\Delta}}\left(a^2\left(1+\frac{q\alpha_0}{aR^2}\right)R_1^2R_2^2-r^4\left(1+\frac{q(\alpha_2+2)}{r^2}\right)+O(1)\right)\mu_1\mu_2+O(y^2)\\
&=\frac{2}{\sqrt{\Delta}}\frac{q^2}{r^4}\left(C_1+C_2+\alpha_0-\alpha_2-2\right)\mu_1\mu_2+O(1/r^6)+O(y^2).
\end{aligned}
\tag{D.16}
$$

In order to calculate the factor in front of $d\mu_i dr$, we calculate

$$
\begin{aligned}
&S_{11}R_1T_1\mu_1+S_{12}T_2\mu_2R_1\\
&\quad=\left(a-\frac{qr_2^2}{R^4}+O(1/r^4)\right)R_1T_1\mu_1+\left(\frac{qr_1r_2}{R^4}+O(1/r^4)\right)T_2\mu_2R_1+O(y^2)\\
&\quad=aR_1T_1\mu_1-\frac{qr_2^2}{R^4}R_1T_1\mu_1+\frac{qr_1r_2}{R^4}T_2\mu_2R_1+O(1/r^3)+O(y^2)\\
&\quad=aR_1T_1\mu_1+\frac{q\mu_1\mu_2^2}{R^4}(-R_1T_1R_2^2+R_2T_2R_1^2)+O(1/r^3)+O(y^2)\\
&\quad=r\mu_1+\frac{q\mu_1\mu_2^2}{R^4}\left(-\frac{r}{a}R_2^2+\frac{r}{a}R_1^2\right)+O(1/r^3)+O(y^2)\\
&\quad=r\mu_1+\frac{q^2\mu_1\mu_2^2}{r^3}\left(C_1-C_2\right)+O(1/r^3)+O(y^2).
\end{aligned}
\tag{D.17}
$$

Note that $R_1T_1=r/a+O(1/r^3)$. Therefore, we have

$$
\begin{aligned}
&h^2(S_{11}R_1T_1\mu_1+S_{12}T_2\mu_2R_1-r\mu_1)\\
&\quad=\frac{1}{r^2\sqrt{\Delta}}O(1/r^3)+O(y^2)\\
&\quad=\frac{1}{\sqrt{\Delta}}O(1/r^5)+O(y^2).
\end{aligned}
\tag{D.18}
$$

The functions in front of $d\phi_i d\phi_j$ will be evaluated. We first calculate the function in the diagonal part:

$$
\begin{aligned}
&h^{-2}S_i\\
&=\frac{H_i}{\sqrt{\Delta}}H_i^{-1}r^2\Delta s_i+O(y^2)\\
&=\frac{H_i}{\sqrt{\Delta}}\left(1-\frac{q(C_i-1)}{r^2}\right)r^2\frac{a^2R^4}{r^4q}\left(1-\frac{q(\alpha_2+2)}{aR^2}\right)\frac{qr_i^2}{aR^4}\left(1+\frac{q(\alpha_2+1)}{aR^2}\right)+O(1/r^4)+O(y^2)\\
&=\frac{H_i}{\sqrt{\Delta}}\frac{ar_i^2}{r^2}\left(1-\frac{q(C_i-1)}{r^2}\right)\left(1-\frac{q(\alpha_2+2)}{aR^2}\right)\left(1+\frac{q(\alpha_2+1)}{aR^2}\right)+O(1/r^4)+O(y^2)\\
&=\frac{H_i}{\sqrt{\Delta}}\mu_i^2\left(1+\frac{qC_i}{r^2}\right)\left(1-\frac{q(C_i-1)}{r^2}\right)\left(1-\frac{q(\alpha_2+2)}{r^2}\right)\left(1+\frac{q(\alpha_2+1)}{r^2}\right)+O(1/r^4)+O(y^2)\\
&=\frac{H_i}{\sqrt{\Delta}}\mu_i^2+O(1/r^4)+O(y^2).
\end{aligned}
\tag{D.19}
$$

We next calculate the mixing part:

$$
\begin{aligned}
&h^{-2}h_{12}\\
&=\frac{1}{\sqrt{\Delta}}\sqrt{\Delta}h^{-2}N_{12}\\
&=\frac{1}{\sqrt{\Delta}}\frac{1}{4r^2K_2}N_{12}+O(y^2)\\
&=\frac{1}{\sqrt{\Delta}}\frac{1}{r^2}\frac{a^2R^4}{q}\frac{r_1^2r_2^2}{a^2R^8}\frac{q^3}{aR^2}(\alpha_2+\alpha_0-\kappa_1-\kappa_2)+O(1/r^6)+O(y^2)\\
&=\frac{1}{\sqrt{\Delta}}\frac{1}{r^2}\frac{q^2r_2^2r_3^2}{aR^6}(\alpha_2+\alpha_0-\kappa_1-\kappa_2)+O(1/r^6)+O(y^2)\\
&=\frac{1}{\sqrt{\Delta}}\frac{q^2\mu_1^2\mu_2^2}{r^4}(\alpha_2+\alpha_0-\kappa_1-\kappa_2)+O(1/r^6)+O(y^2).
\end{aligned}
\tag{D.20}
$$

The metric in the angles is then expressed by

$$
\begin{aligned}
&\sqrt{\Delta}h^{-2}h_{ab}(d\phi_a+M_{\phi_a}dt)(d\phi_b+M_{\phi_b}dt)\\
&=H_1\left[\mu_1^2\left(1+O(1/r^4)\right)(d\phi_1+M_{\phi_1}dt)^2\right]\\
&\quad+H_2\left[\mu_2^2\left(1+O(1/r^4)\right)(d\phi_2+M_{\phi_2}dt)^2\right]\\
&\quad+\left(\frac{2q^2\mu_1^2\mu_2^2}{r^4}(\alpha_2+\alpha_0-\kappa_1-\kappa_2)+O(1/r^6)\right)(d\phi_1+M_{\phi_1}dt)(d\phi_2+M_{\phi_2}dt).
\end{aligned}
\tag{D.21}
$$

D.3 Related formulas

In this appendix, we will summarize some formulas. These formulas are also related to some useful expressions in [10]. (The conventions here are obtained from the conventions in [10] by $\Delta\to(H_2H_3)^{-2/3}\Delta$, the subscript change $(1,2,3\to3,1,2)$, $\rho_i\to R_i$.) We also summarize equations which correspond to the two-charge superstar.

$$
h^{-2}=\frac{r^2\Delta+\mu_3^2}{\sqrt{\Delta}},\tag{E.1}
$$

$$
\omega_{\phi_i}=-h^2\frac{\mu_i^2}{\sqrt{\Delta}},\quad(i=1,2)\tag{E.2}
$$

$$
S_{ij}=2e^{i(\phi_i-\phi_j)}\frac{\partial_i\bar{\partial}_jK}{Z+\frac{1}{2}}=\begin{cases}\frac{\mu_i^2+\sqrt{\Delta}h^{-2}H_i}{R_i^2\Delta}, & (i=j)\\ \frac{\mu_i\mu_j}{R_iR_j\Delta}, & (i\neq j)\end{cases}\tag{E.3}
$$

where $H_i=1+q_i/r^2$. R_1 and R_2 obey a set of differential equations:

$$
\begin{aligned}
T_1&=rR_1H_2f^{-1},\\
T_2&=rR_2H_1f^{-1},
\end{aligned}\tag{E.4}
$$

where $f=1+r^2H_1H_2$ and $T_i=dR_i/dr$.

For superstar, the S_i, N_{12} and M_i $(i=1,2)$ are evaluated as

$$
\begin{aligned}
S_i&=\frac{h^2\mu_i^2}{\sqrt{\Delta}}H_i,\qquad N_{12}=0,\\
M_i&=-1+H_i^{-1}=-\frac{q_i}{r^2+q_i}.
\end{aligned}\tag{E.5}
$$

Formulas for AdS are available by taking $\Delta = 1$ and $q_i = 0$. The differential equations (E.4) can be solved to give

$$R_1 = R_2 = \sqrt{r^2 + 1}. \tag{E.6}$$

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