Journal of Computational and Applied Mathematics 235 (2011) 1755–1762

Contents lists available at ScienceDirect

Journal of Computational and Applied Mathematics

journal homepage: www.elsevier.com/locate/cam

Growth factors of pivoting strategies associated with Neville elimination

Pedro Alonso [a,*], Jorge Delgado [b], Rafael Gallego [a], Juan Manuel Peña [b]

[a] *Departamento de Matemáticas, Universidad de Oviedo, Spain*

[b] *Departamento de Matemática Aplicada, Universidad de Zaragoza, Spain*

ARTICLE INFO

Article history:
Received 4 August 2009
Received in revised form 26 October 2009

MSC:
65F05
65G05

Keywords:
Neville elimination
Pivoting
Growth factor
Total positivity

ABSTRACT

Neville elimination is a direct method for solving linear systems. Several pivoting strategies for Neville elimination, including pairwise pivoting, are analyzed. Bounds for two different kinds of growth factors are provided. Finally, an approximation of the average normalized growth factor associated with several pivoting strategies is computed and analyzed using random matrices.

1. Introduction

The usual direct method for solving a linear system of equations $Ax = b$ is Gaussian elimination. Neville elimination is an alternative procedure to transform a square matrix A into an upper triangular matrix U, preferable for some classes of matrices and when using pivoting strategies in parallel implementations.

Neville elimination is especially useful with totally positive matrices, sign-regular matrices and other related types of matrices appearing in applications (see [1–4]).

In addition, some studies prove the high performance computing of Neville elimination for any nonsingular matrix (see [5]). In [6] the backward error of Neville elimination has also been analyzed. In [7] we give a sufficient condition ensuring the convergence of iterative refinement using Neville elimination for a system $Ax = b$ with A any nonsingular matrix in $\mathbb{R}^{n\times n}$, and apply it to the totally positive case. Other applications and a study of the stability have been presented in [8].

In general, Gaussian elimination and, in particular, Gaussian elimination with partial pivoting is the usual direct method for solving linear systems. However, as recalled in [9], in a serial implementation of Gaussian elimination with partial pivoting, the exchange of rows is recorded as an index permutation. In contrast, in parallel computations, physical exchange of elements of the matrix between processors will actually be required to implement that pivoting strategy, and so the communication cost associated with this exchange may be a limiting factor in the global efficiency of the parallel implementation, particularly if bidirectional communication is not feasible. On the other hand, pairwise pivoting strategies (which can be closely related to Neville elimination) were considered for parallel implementations (see [10–12]) and were already mentioned in the earliest days of scientific computing by Wilkinson in [13]. They also increase the accuracy of the exponential matrix computation of the algorithms of [14,15].

An error analysis of a variant of pairwise pivoting closely related to Gaussian elimination was presented in [10]. Numerical experiments on the growth factor with random matrices using neighbor pivoting (this variant of pairwise pivoting is closely

* Corresponding author. Tel.: +34 985 18 21 28.
E-mail addresses: palonso@uniovi.es (P. Alonso), jorgedel@unizar.es (J. Delgado), rgallego@uniovi.es (R. Gallego), jmpena@uniovi.es (J.M. Peña).

0377-0427/$ – see front matter
doi:10.1016/j.cam.2009.11.012

related to Neville elimination) were presented in [12]. The growth factor is an indicator of the numerical stability of a numerical algorithm, measuring the size of intermediate and final quantities relative to initial data (see [16,17]).

In this paper, several definitions of growth factors for Neville elimination are compared. Theoretical bounds for the growth factors of several pivoting strategies related to Neville elimination are also provided.

In the final section, we analyze numerical experiments for the average normalized growth factor (see [18,12]) for random $n\times n$ matrices. We use Neville elimination with partial pivoting and the other mentioned pivoting strategies related to Neville elimination, which present better behavior than partial pivoting despite satisfying similar theoretical bounds. Those results are also compared with some strategies related to Gaussian elimination.

2. Neville elimination

Neville elimination is an alternative procedure to Gaussian elimination for making zeros in a column of a matrix by adding to each row a multiple of the previous one. If A is an $n \times n$ matrix, it consists of at most $n-1$ successive major steps, resulting in a sequence of matrices as follows:

$$A = A^{(1)} \to \widetilde{A}^{(1)} \to A^{(2)} \to \widetilde{A}^{(2)} \to \cdots \to A^{(n)} = \widetilde{A}^{(n)} = U, \tag{1}$$

where U is an upper triangular matrix.

Each $\widetilde{A}^{(t)}$ is obtained by a reordering of the rows of the matrix $A^{(t)}$ using an adequate pivoting strategy, so that the rows with a zero entry in column t are the final rows and

$$\widetilde{a}_{it}^{(t)} = 0, \quad i \ge t \Rightarrow \widetilde{a}_{ht}^{(t)} = 0, \quad \forall h \ge i. \tag{2}$$

The matrix $A^{(t+1)}$ is obtained from $\widetilde{A}^{(t)}$ making zeros in the column t below the main diagonal by adding an adequate multiple of the ith row to the $(i+1)$th, for $i = n-1, n-2, \ldots, t$, according to the following formula

$$a_{ij}^{(t+1)} = \begin{cases} \widetilde{a}_{ij}^{(t)}, & \text{if } 1 \le i \le t, \\ \widetilde{a}_{ij}^{(t)} - \dfrac{\widetilde{a}_{it}^{(t)}}{\widetilde{a}_{i-1,t}^{(t)}} \widetilde{a}_{i-1,j}^{(t)}, & \text{if } t+1 \le i \le n \text{ and } \widetilde{a}_{i-1,t}^{(t)} \ne 0, \\ \widetilde{a}_{ij}^{(t)}, & \text{if } t+1 \le i \le n \text{ and } \widetilde{a}_{i-1,t}^{(t)} = 0, \end{cases} \tag{3}$$

for all $j \in \{1, 2, \ldots, n\}$.

If A is nonsingular, then $A^{(t)}$ has zeros below its main diagonal in the first $t-1$ columns. Notice that in the end $A^{(n)} = \widetilde{A}^{(n)} = U$, and that when no row exchanges are needed, then $A^{(t)} = \widetilde{A}^{(t)}$ for all t. The element $p_{ij} = \widetilde{a}_{ij}^{(j)}$, $1 \le j \le i \le n$, is called the (i, j) pivot of Neville elimination of A and the number

$$m_{ij} = \begin{cases} \dfrac{\widetilde{a}_{ij}^{(j)}}{\widetilde{a}_{i-1,j}^{(j)}} = \dfrac{p_{ij}}{p_{i-1,j}}, & \text{if } \widetilde{a}_{i-1,j}^{(j)} \ne 0, \\ 0, & \text{if } \widetilde{a}_{i-1,j}^{(j)} = 0 \ (\Rightarrow \widetilde{a}_{ij}^{(j)} = 0), \end{cases} \tag{4}$$

the (i, j) multiplier.

The computational cost of Neville elimination coincides with that of Gaussian elimination.

We consider now the important case in which Neville elimination can be performed without row exchanges.

Following the notation of [19] matrices satisfying this condition will be referred to as matrices satisfying the *WR condition*. Denote by $E_i(\alpha)$, $2 \le i \le n$, the bidiagonal lower triangular matrix that coincides with the identity matrix except in the element at the position place $(i, i-1)$, which is α instead of 0.

A nonsingular matrix of order n satisfying the WR condition can be factored as follows

$$A = E_n(m_{n1})[E_{n-1}(m_{n-1,1})E_n(m_{n2})] \cdots [E_2(m_{21})E_3(m_{32}) \cdots E_n(m_{n,n-1})]U, \tag{5}$$

where m_{ij} is the (i, j) multiplier of Neville elimination of A (see p. 117 of [2]).

Denoting by

$$F_i := E_{n-(i-1)}(m_{n-(i-1),1})E_{n-(i-2)}(m_{n-(i-2),2}) \cdots E_n(m_{ni}). \tag{6}$$

Neville elimination of a matrix A satisfying the WR condition, performed by subdiagonals, can be described in the following form

$$A^{[1]} = A; \qquad F_{t-1}A^{[t]} = A^{[t-1]}, \quad 2 \le t \le n-1, \tag{7}$$

with F_{t-1} defined by (6) and $A^{[t]} = (a_{ij}^{[t]})_{1\le i,j,\le n}$ with

$$\begin{aligned} a_{ij}^{[t]} &:= a_{ij}^{[t-1]}, \quad 1 \le j \le n,\ 1 \le i \le n-t+1, \\ a_{ij}^{[t]} &:= a_{ij}^{[t-1]} - m_{i,i-n+t}a_{i-1,j}^{[t-1]}, \quad n-t+2 \le i \le n,\ i-n+t \le j \le n. \end{aligned} \tag{8}$$

Recall that this elimination process is just a reordering of the one explained at the beginning of this section, with the same multipliers. This means that Neville elimination of A can be done by subdiagonals: first we make zero the $(n, 1)$ entry, then the $(n-1, 1)$ and $(n, 2)$, and so on, finishing by making zeros the $(2, 1)$, $(3, 2), \ldots, (n, n-1)$ entries, as sketched in the following example

$$\begin{pmatrix} \times & \times & \times & \times & \times \\ ⑦ & \times & \times & \times & \times \\ ④ & ⑧ & \times & \times & \times \\ ② & ⑤ & ⑨ & \times & \times \\ ① & ③ & ⑥ & ⑩ & \times \end{pmatrix}. \tag{9}$$

Now we present some pivoting strategies for Neville elimination, using row exchanges with similar purposes to those of the pivoting strategies for Gaussian elimination. Recall that $\widetilde{A}^{(t)}$ is obtained by a reordering of the rows of matrix $A^{(t)}$ by an adequate pivoting strategy with a criterion for the choice of the pivots p_{ij}. Gaussian elimination with partial pivoting chooses the pivots so that all multipliers have absolute value not greater than 1. With a similar purpose, one can define Neville elimination with partial pivoting.We interchange the rows of $A^{(t)}$ so that $\widetilde{A}^{(t)}$ satisfies $|\widetilde{a}^{(t)}_{tt}| \geq |\widetilde{a}^{(t)}_{t+1,t}| \geq \cdots \geq |\widetilde{a}^{(t)}_{nt}|$ and from $\widetilde{A}^{(t)}$ we construct $A^{(t+1)}$ as in (3).

Other pivoting strategies used in both Gaussian and Neville elimination are called pairwise pivoting. These are suitable for implementations on parallel computers and reduce considerably the communication cost (see [9–12]).

Pairwise pivoting for Neville elimination is done as follows: to produce a zero at position (n, t) in $A^{(t)}$, one compares the element $a^{(t)}_{nt}$ with $a^{(t)}_{n-1,t}$. If $|a^{(t)}_{nt}| > |a^{(t)}_{n-1,t}|$ the corresponding rows are exchanged, in order that $|m_{nt}| \leq 1$. Then update the elements of row n. This process is repeated with $a^{(t)}_{it}$ and $a^{(t)}_{i-1,t}$, for $n-1 \geq i \geq t+1$. Observe that this elimination can also be performed by subdiagonals.

If Neville elimination by subdiagonals with pairwise pivoting is considered, the process is carried out as follows: if $|a^{[1]}_{n1}| > |a^{[1]}_{n-1,1}|$, then rows n and $n-1$ are interchanged, then row n is updated. The process is repeated with elements $a^{[2]}_{n-1,1}$ and $a^{[2]}_{n-2,1}$ and then a zero is made at position $(n-1, 1)$. The same procedure is performed on elements $a^{[2]}_{n2}$ and $a^{[3]}_{n-1,2}$ and so on, following the order showed in (9).

3. Wilkinson growth factor

In this section we analyze the growth factor introduced by Wilkinson and, for some pivoting strategies for Neville elimination, we prove that the same bound known for Gaussian elimination with partial pivoting holds. In the case of Neville elimination of an $n \times n$ matrix associated with the sequence of matrices given by (1), with or without pivoting strategies, this growth factor can be defined as

$$\rho_n(A) := \frac{\max\limits_{i,j,k} |a^{(k)}_{ij}|}{\max\limits_{i,j} |a_{ij}|}, \tag{10}$$

where the matrices $A^{(k)} = (a^{(k)}_{ij})_{1\leq i,j\leq n}$ are the intermediate matrices of the elimination process as in (1).

The following result can be applied to any pivoting strategy for Neville elimination such that $|m_{ik}| \leq 1$ for all i, k. In particular, it can be applied to Neville elimination with partial pivoting and to pairwise pivoting.

Theorem 1. *Let $A = (a_{ij})_{1\leq i,j\leq n}$ be a nonsingular matrix. Let us perform Neville elimination with a pivoting strategy such that all the multipliers have absolute value less than or equal to 1. Then the sequence of matrices $A^{(t)}$ given in (1) satisfies $|a^{(t+1)}_{ij}| \leq 2^t \max_{i,j} |a_{ij}|$ and the growth factor defined by (10) satisfies $\rho_n(A) \leq 2^{n-1}$.*

Proof. By (3), the elements $a^{(t+1)}_{ij} \neq a^{(t)}_{ij}$ satisfy $a^{(t+1)}_{ij} = \widetilde{a}^{(t)}_{ij} - m_{it}\widetilde{a}^{(t)}_{i-1,j}$, where m_{it} is the multiplier defined in (4). Then, taking absolute values, one has that

$$|a^{(t+1)}_{ij}| \leq |\widetilde{a}^{(t)}_{ij}| + |m_{it}||\widetilde{a}^{(t)}_{i-1,j}| \leq |\widetilde{a}^{(t)}_{ij}| + |\widetilde{a}^{(t)}_{i-1,j}|,$$

and so

$$|a^{(t+1)}_{ij}| \leq 2 \max_{i,j} |a^{(t)}_{ij}|. \tag{11}$$

Using (11), and iterating the previous process, we obtain the desired inequality:

$$|a^{(t+1)}_{ij}| \leq 2 \max_{i,j} |a^{(t)}_{ij}| \leq 2^2 \max_{i,j} |a^{(t-1)}_{ij}| \leq \cdots \leq 2^t \max_{i,j} |a^{(1)}_{ij}| = 2^t \max_{i,j} |a_{ij}|. \tag{12}$$

For the bound of $\rho_n(A)$:

$$\rho_n(A) = \frac{\max\limits_{i,j,k} |a^{(k)}_{ij}|}{\max\limits_{i,j} |a_{ij}|} \leq \frac{2^{n-1} \max\limits_{i,j} |a_{ij}|}{\max\limits_{i,j} |a_{ij}|} = 2^{n-1}. \quad \square$$

Remark. Taking $t = i - 1$ in (12) and taking into account that the ith row of $\widetilde{A}^{(i)}$ coincides with the ith row of U, one obtains

$$|\widetilde{a}_{ij}^{(i)}| = |a_{ij}^{(n)}| = |u_{ij}| \leq 2^{i-1} \max_{i,j} |a_{ij}|. \tag{13}$$

If Neville elimination is performed by subdiagonals, then the corresponding Wilkinson growth factor can be defined as

$$\rho_n(A) := \frac{\max_{i,j,k} |a_{ij}^{[k]}|}{\max_{i,j} |a_{ij}|}, \tag{14}$$

where the matrices $A^{[k]} = (a_{ij}^{[k]})_{1\leq i,j\leq n}$ are the intermediate matrices obtained from (8): the elements $a_{ij}^{[k]} \neq a_{ij}^{[k-1]}$ satisfy $a_{ij}^{[k]} = a_{ij}^{[k-1]} - m_{i,i-n+k} a_{i-1,j}^{[k-1]}$. If we assume that $|m_{it}| \leq 1, \forall i, t$, one obtains a bound similar to (12):

$$|a_{ij}^{[t]}| \leq 2^{t-1} \max_{i,j} |a_{ij}^{[1]}| = 2^{t-1} \max_{i,j} |a_{ij}|. \tag{15}$$

From which one concludes that, under the previous hypothesis, the Wilkinson growth factor for Neville elimination by subdiagonals given by (14) also satisfies $\rho_n(A) \leq 2^{n-1}$.

4. Growth factor associated with a triangular decomposition

In the previous remark we stated that, for Neville elimination with several pivoting strategies, the elements of the resulting upper triangular matrix $U = (u_{ij})_{1\leq i,j\leq n}$ obtained after the elimination process (1) satisfy $|u_{ij}| \leq 2^{i-1} \max_{i,j} |a_{ij}|$ for all $1 \leq i \leq j \leq n$, the same bound as for Gaussian elimination with partial pivoting. However this last strategy has a better behavior in the numerical experiments of the following section and the results of this section provide a possible explanation of this fact. They use, when it is possible, the triangular decomposition LU associated with the pivoting strategy. In particular, the size of L and of the growth factor are closely related:

$$\rho_n^{\mathrm{LU}}(A) := \frac{\| |L||U| \|_\infty}{\|A\|_\infty}, \tag{16}$$

where $|M|$ denotes the matrix whose entries are the absolute values of the corresponding entries of the matrix M.

In contrast to row pivoting strategies (i.e., pivoting strategies that exchange rows) for Gaussian elimination, for row pivoting strategies for Neville elimination it is not always possible to find a permutation matrix P such that performing Neville elimination on A with the pivoting strategy coincides with performing Neville elimination without row exchanges on $PA\ (= LU)$. However, assuming this property will allow us to show that the fact that all multipliers are bounded above by 1 does not guarantee for Neville elimination the bounds $|L| \leq 1$ and for the growth factor $\rho_n^{\mathrm{LU}}(A)$ of (16) unlike Gaussian elimination with partial pivoting. This explains that the pivoting strategies mentioned in the previous section present worse stability properties than Gaussian elimination with partial pivoting in spite of presenting the same theoretical bound for the Wilkinson growth factor.

We say that Neville elimination with a pivoting strategy leads to a triangular factorization LU when there exists a permutation matrix P such that $PA = LU$, where L is a lower triangular matrix with unit diagonal and $U = (u_{ij})_{1\leq i,j\leq n}$ is the resulting upper triangular matrix obtained after the elimination process in (1).

The next result provides an upper bound for $|L|$ which will be used to derive upper bounds for the growth factor defined by (16).

Lemma 1. *Let A be a nonsingular $n \times n$ matrix and assume that Neville elimination with a pivoting strategy leads to a triangular factorization LU and that all the multipliers have absolute value less than or equal to* 1. *Then the lower triangular matrix $L = (l_{ij})_{1\leq i,j\leq n}$ satisfies $|L| \leq B$, where $B = (b_{ij})_{1\leq i,j\leq n}$ satisfies the following properties:*

(i) *The elements b_{ij} satisfy the recurrence relation:*

$$b_{ij} = b_{i-1,j-1} + b_{i-1,j}, \quad \forall\, i, j, \text{ where } b_{0j} = b_{i0} = 0,\ 1 \leq i, j \leq n \text{ and } b_{00} = 1.$$

(ii) *Moreover, b_{ij} is given by*

$$b_{ij} = \begin{cases} 0 & \text{if } i < j \\ \binom{i-1}{j-1} & \text{if } i \geq j. \end{cases} \tag{17}$$

Proof. Instead of the LU factorization of A given by (5), the hypotheses provide a similar factorization of PA, where P is a permutation matrix, satisfying the following properties: $PA = LU$ with $L = F_1F_2 \ldots F_{n-1}$ and $U = A^{[n]}$, where, for all $k = 1 \ldots, n-1$, F_k is given by (6).

From this,

$$|L| \leq |F_1||F_2| \ldots |F_{n-1}|,$$

and taking into account the explicit expression of F_k and our hypothesis on the multipliers ($|m_{ij}| \leq 1$), one infers that $|L| \leq B$ where

$$B = \begin{pmatrix} 1 & & & & \\ 0 & 1 & & & \\ & \ddots & \ddots & & \\ & & 0 & 1 & \\ & & & 1 & 1 \end{pmatrix} \begin{pmatrix} 1 & & & & \\ 0 & 1 & & & \\ & \ddots & \ddots & & \\ & & 1 & 1 & \\ & & & 1 & 1 \end{pmatrix} \cdots \begin{pmatrix} 1 & & & & \\ 1 & 1 & & & \\ & \ddots & \ddots & & \\ & & 1 & 1 & \\ & & & 1 & 1 \end{pmatrix}. \quad (18)$$

Thus, the entries of B provide an upper bound for the corresponding entries of $|L|$, and in the particular case when $m_{ij} = 1,\ \forall i, j$, one has $L = F_1F_2 \dots F_{n-1} = B$.

(i) Let B_i $(1 \leq i \leq n-1)$ be the $n \times n$ matrix corresponding to the matrix of the ith place of the right hand side of (18), that is, $B = B_1B_2 \dots B_{n-1}$.

Let $B^{(k)} = (b_{ij}^{(k)})_{1\leq i,j\leq n}$ be the matrix $B^{(k)} = B_1B_2 \dots B_k$. Clearly, $B^{(k)}$ is a lower triangular matrix with $b_{ii}^{(k)} = 1,\ \forall i, k$. Besides, the first $n-k-1$ columns of $B^{(k)}$ coincide with those of the $n \times n$ identity matrix.

Let us prove by induction on k that

$$b_{ij}^{(k)} = b_{i-1,j-1}^{(k)} + b_{i-1,j}^{(k)}, \quad 1 \leq i \leq n,\ n-k \leq j \leq n, \quad (19)$$

defining $b_{i0} = b_{0j} = 0$, with $1 \leq i, j \leq n$ and $b_{00} = 1$. Notice that $B^{(k)}$ can be obtained from $B^{(k-1)}$ adding to each of the columns $n-k, n-k+1, \dots, n-1$ the next one.

Since $B^{(k)}$ is lower triangular with unit diagonal, (19) is obvious $\forall k$ for $i \leq j$, and in particular in the last column of $B^{(k)}$. So, we have only to prove that (19) holds for $i > j$ with $j = n-k, n-k+1, \dots, n-1$, which is obvious for $k = 1$.

For $k = 1$ it is obvious. Assume that (19) holds for $k-1$ and let us prove it for k. If $n-k \leq j \leq n-1$, then

$$b_{ij}^{(k)} = b_{ij}^{(k-1)} + b_{i,j+1}^{(k-1)}, \quad \forall i \quad (20)$$

and by the induction hypothesis, for $i < n$, one gets

$$b_{ij}^{(k)} = b_{i+1,j+1}^{(k-1)}, \quad n-k \leq j \leq n-1. \quad (21)$$

If $n-k < j \leq n-1$ and $i < n$, then, from (21), $b_{ij}^{(k-1)} = b_{i-1,j-1}^{(k)}$, $b_{i,j+1}^{(k-1)} = b_{i-1,j}^{(k)}$, and substituting in (20), one gets (19).

If $j = n-k$ and $i < n$, since the first $n-k$ columns of $B^{(k-1)}$ coincide with those of the identity matrix, $b_{ij}^{(k-1)} = 0$, and in (20) one has $b_{ij}^{(k)} = b_{i,j+1}^{(k-1)}$, which combined with (21), gives $b_{ij}^{(k)} = b_{i-1,j}^{(k)} = b_{i-1,j}^{(k)} + b_{i-1,j-1}^{(k)}$, since $b_{i-1,j-1}^{(k)} = 0$. Hence (19) holds.

Finally, if $i = n$ and $n-k \leq j \leq n-1$, applying (21) for $i = n-1$ one gets $b_{n,j+1}^{(k-1)} = b_{n-1,j}^{(k)}$, $b_{n,j}^{(k-1)} = b_{n-1,j-1}^{(k)}$, and by (20) $b_{n,j}^{(k)} = b_{n-1,j-1}^{(k)} + b_{n-1,j}^{(k)}$, and we are done.

(ii) We use proof by induction on the rows i of B for $i = 1, \dots, n$. The inductive statement clearly holds for the first row. Assume that it holds for $1, \dots, i-1$ and let us prove it for i.

If $1 < j < i$, by (i) we have $b_{ij} = \binom{i-2}{j-2} + \binom{i-2}{j-1} = \binom{i-1}{j-1}$.

If $j = 1$, then $b_{i1} = b_{i-1,1} = \binom{i-2}{0} = 1 = \binom{i-1}{0}$. Finally, if $i = j$, $b_{ii} = 1$, and the proof is finished. □

Remark. Taking into account (ii) in the previous lemma, observe that the lower triangular matrix $|L| = (|l_{ij}|)_{1\leq i,j\leq n}$ has its maximal entries at positions $(n, \frac{n}{2})$ and $(n, \frac{n}{2}+1)$ if n is even and at position $(n, \frac{n+1}{2})$ if n is odd. Using this fact, assertion ii) in the previous lemma, and that the entries of the ith row of $|U|$ satisfy (13), we can derive an upper bound for all elements of $|L||U|$. This bound has its largest value for the (n, n)-entry of $|L||U|$, which turns out to be 3^{n-1}. For the case of Gaussian elimination with partial pivoting the largest value for the upper bound for all elements of $|L||U|$ also correspond to the (n, n)-entry, but in this case this bound is $2^n - 1$. Furthermore, under the hypotheses of Lemma 1, it can be proved for all i, j that $|l_{ij}| \leq 2^{n-2}$ whereas for Gauss elimination with partial pivoting one has $|l_{ij}| \leq 1$. As a result, the upper bound for the growth factor $\rho_n^{\text{LU}}(A)$ is larger for Neville elimination, as we will remark later.

The following result gives a bound for $\rho_n^{\text{LU}}(A)$ for Neville elimination.

Theorem 2. *Let A be a nonsingular $n \times n$ matrix and assume that Neville elimination with a pivoting strategy leads to a triangular factorization LU and that all the multipliers have absolute value less than or equal to 1. Then the growth factor defined by (16) satisfies*

$$\rho_n^{LU}(A) \leq 3^{n-2}(n+2). \quad (22)$$

Proof. By (13) and Lemma 1 we have $|L||U| \leq \max_{i,j} |a_{ij}|BC$, where the elements of B are defined in (17) and C is an $n \times n$ upper triangular matrix whose nonzero elements are given by $c_{ij} = 2^{i-1}$, $i \leq j$. On the other hand, we have that $\|A\|_\infty^{-1} \leq (\max_{ij} |a_{ij}|)^{-1}$ so that $\rho_n^{\text{LU}}(A) \leq \|BC\|_\infty$. The infinity norm of BC is given by the sum of the elements of its last row, which amounts to

$$\|BC\|_\infty = \sum_{i=0}^{n-1} \binom{n-1}{i} (n-i)2^i,$$

which can be rewritten as

$$\sum_{i=0}^{n-1} \binom{n-1}{i} (n-i)2^i = n\sum_{i=0}^{n-1} \binom{n-1}{i} 2^i - \sum_{i=1}^{n-1} \binom{n-1}{i} i2^i.$$

For the first term, one has

$$n\sum_{i=0}^{n-1} \binom{n-1}{i} 2^i = n(2+1)^{n-1} = n3^{n-1}.$$

To compute the second term, notice that $i\binom{n-1}{i} = (n-1)\binom{n-2}{i-1}$, so that

$$\sum_{i=1}^{n-1} \binom{n-1}{i} i2^i = (n-1)\sum_{i=1}^{n-1} \binom{n-2}{i-1} 2^i = 2(n-1)\sum_{i=1}^{n-1} \binom{n-2}{i-1} 2^{i-1}$$
$$= 2(n-1)\sum_{i=0}^{n-2} \binom{n-2}{i} 2^i = 2(n-1)3^{n-2}.$$

Hence $\rho_n^{\mathrm{LU}}(A) \le n3^{n-1} - 2(n-1)3^{n-2} = 3^{n-2}(n+2)$. □

The growth factor $\rho_n^{\mathrm{LU}}(A)$ for Gaussian elimination with partial pivoting can be computed in the same way as for Neville elimination in Theorem 2. The infinity norm of the upper bound of $|L||U|$ also corresponds to the sum of the entries of the nth row, and is equal to $2^{n+1} - n - 2$, which is less than the bound of (22).

5. Numerical experiments

Stability of Gaussian elimination with partial pivoting on average was analyzed through numerical experiments in [12]. In [18] Cortes and Peña considered the stability on average of other pivoting strategies for Gaussian elimination. Here, we consider the stability on average of Neville elimination without pivoting and with the row pivoting strategies presented in this paper. We compare our results with the stability on average of Gaussian elimination without pivoting and with partial pivoting. Following [18,12] we have generated randomly $n \times n$ matrices whose entries are independent samples of the standard normal distribution of mean 0 and variance 1. The sample size N diminishes as n increases in order to keep the computing time within reasonable bounds. A typical set of dimensions and sampled sizes are listed below:

Dimension n	2–64	128	256–512	1024–2048
Sample size N	1000	500	100	10

We also modify the definition of the growth factors $\rho_n(A)$ and $\rho_n^{\mathrm{LU}}(A)$. We consider in this section the average growth factor which is defined by

$$\widehat{\rho}(A) := \frac{\max_{i,j,k} |a_{ij}^{(k)}|}{\sigma_A},$$

where σ_A denotes the standard deviation of the distribution of the initial elements. For each dimension we compute the statistical mean of the average growth factors associated with a set of sample matrices.

We have computed the average growth factor for Gaussian elimination without pivoting (GE) and with partial pivoting (GEPP), and Neville elimination without pivoting (NE), with partial pivoting (NEPP) and with pairwise pivoting by columns (PWC) and by subdiagonals (PWS). The results are summarized in Table 1 and Fig. 1.

The results show that the average growth factor for NEPP is extremely large. In fact, it is much larger than that of NE. We have also observed that the average growth factor for NE is about two orders of magnitude larger than that for GE for the largest matrix dimensions. As expected, the least average growth factor corresponds to GEPP, although the average growth factor for Neville elimination with pairwise pivoting both by columns and by subdiagonals is very close to this one.

6. Conclusions

In this work several pivoting strategies for Neville elimination, including pairwise pivoting, have been studied.

Upper bounds for the Wilkinson growth factor ($\rho_n(A)$) have been obtained using suitable pivoting strategies. These bounds, for Neville elimination by both columns and subdiagonals, have been proved to be similar as that of Gaussian elimination: $\rho_n(A) \le 2^{n-1}$.

Table 1
Average growth factors for Gaussian and Neville eliminations for several matrix dimensions and several pivoting strategies as described in the text.

n	GE	NE	GEPP	NEPP	PWC	PWS
2	1.00×10^1	1.00×10^1	2.05	2.05	2.05	2.05
4	3.95×10^1	5.14×10^1	2.49	2.59	2.53	2.51
8	1.95×10^2	3.06×10^2	3.65	4.84	3.90	3.82
16	7.34×10^2	4.32×10^3	5.92	1.43×10^1	6.73	6.50
32	3.14×10^3	1.35×10^4	9.72	1.06×10^2	1.21×10^1	1.16×10^1
64	1.88×10^4	1.08×10^5	1.60×10^1	4.93×10^3	2.16×10^1	2.07×10^1
128	4.08×10^4	9.44×10^5	2.59×10^1	1.06×10^7	4.12×10^1	3.89×10^1
256	1.57×10^5	7.52×10^6	4.07×10^1	4.27×10^{13}	8.48×10^1	8.03×10^1
512	2.78×10^5	7.60×10^7	6.35×10^1	8.70×10^{26}	1.83×10^2	1.74×10^2
1024	7.65×10^5	5.18×10^8	9.27×10^1	1.23×10^{66}	4.77×10^2	4.50×10^2
2048	1.84×10^6	3.97×10^8	1.53×10^2	"overflow"	1.33×10^3	1.27×10^3

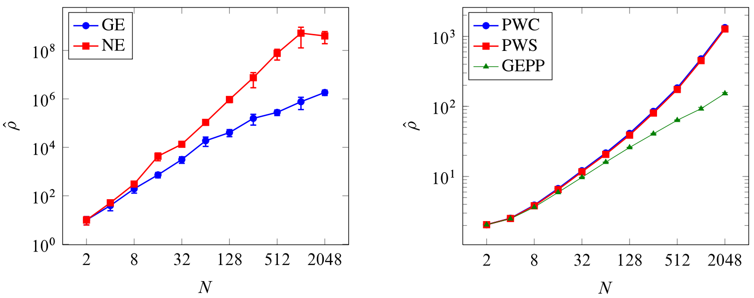

Fig. 1. Average growth factor as a function of the matrix dimension for Gaussian and Neville eliminations and several pivoting strategies. Error bars coming from the statistical deviation of the data are displayed: **(left)** Gaussian elimination (GE) and Neville elimination without pivoting (NE); **(right)** Gaussian elimination with partial pivoting (GEPP) and Neville elimination with pairwise pivoting by columns (PWC) and by subdiagonals (PWS).

Once the lower triangular matrix L is bounded, an upper bound of $3^{n-2}(n+2)$ for the growth factor $\rho_n^{LU}(A)$ has been computed. This bound is somewhat larger than that of the Gaussian case.

The numerical results show a better behavior of the experimental growth factors with respect to their theoretical bounds. A similar behavior was observed in [12] for Gaussian elimination with partial pivoting.

On the other hand, the numerical growth factors for pairwise pivoting (PWC and PWS) are much smaller than those of partial pivoting (NEPP).

Hence, Neville elimination with pairwise pivoting is a very efficient alternative to Gaussian elimination with partial pivoting.

Acknowledgements

This work has been partially supported by the Spanish Research Grant MTM2006-03388 and by MEC and FEDER Grant TIN2007-61273.

References

[1] J. Demmel, P. Koev, The accurate and efficient solution of a totally positive generalized Van der monde linear system, SIAM J. Matrix Anal. Appl. 27 (2005) 142–152.
[2] M. Gasca, C.A. Micchelli (Eds.), Total Positivity and its Applications, Kluwer Academic Publishers, Boston, 1996.
[3] M. Gasca, J.M. Peña, Total positivity and Neville elimination, Linear Algebra Appl. 165 (1992) 25–44.
[4] M. Gassó, J.R. Torregrosa, A totally positive factorization of rectangular matrices by the Neville elimination, SIAM J. Matrix Anal. Appl. 25 (2004) 986–994.
[5] P. Alonso, R. Cortina, I. Díaz, J. Ranilla, Neville elimination: A study of the efficiency using checkerboard partitioning, Linear Algebra Appl. 393 (2004) 3–14.
[6] P. Alonso, M. Gasca, J.M. Peña, Backward error analysis of Neville elimination, Appl. Numer. Math. 23 (1997) 193–204.
[7] P. Alonso, J. Delgado, R. Gallego, J.M. Peña, Iterative refinement for Neville elimination, Int. J. Comput. Math. 86 (2) (2009) 341–353.
[8] P. Alonso, J. Delgado, R. Gallego, J.M. Peña, Neville elimination: An efficient algorithm with application to Chemistry, J. Math. Chem. (in press).
[9] J.L. Mead, R.A. Renaut, B.D. Welfert, Stability of a pivoting strategy for parallel Gaussian elimination, BIT 41 (2001) 633–639.
[10] D.C. Sorensen, Analysis of pairwise pivoting in Gaussian elimination, IEEE Trans. Comput. C-34 (1985) 274–278.
[11] A. Tiskin, Communication-efficient parallel generic pairwise elimination, Future Gene. Comput. Syst. 23 (2007) 179–188.
[12] L.N. Trefethen, R.S. Schreiber, Average case stability of Gaussian elimination, SIAM J. Matrix Anal. Appl. 11 (1990) 335–360.
[13] J.H. Wilkinson, The Algebraic Eigenvalue Problem, Clarendon Press, Oxford, 1965.
[14] S. Natesan, J. Vigo-Aguiar, N. Ramanujam, A numerical algorithm for singular perturbation problems exhibiting weak boundary layers, Comput. Math. Appl. 45 (1–3) (2003) 469–479.
[15] J. Vigo-Aguiar, J. Martin-Vaquero, R. Criado, On the stability of exponential fitting BDF algorithms, J. Comput. Appl. Math. 175 (1) (2005) 183–194.

[16] N.J. Higham, Accuracy and Stability of Numerical Algorithms, SIAM, Philadelphia, 2002.
[17] A. George, K.D. Ikramov, Andrey B. Kucherov, On the growth factor in Gaussian elimination for generalized Higham matrices, Numer. Linear Algebra Appl. 9 (2002) 107–114.
[18] V. Cortés, J.M. Peña, Growth factor and expected growth factor of some pivoting strategies, J. Comput. Appl. Math. 202 (2007) 292–303.
[19] M. Gasca, J.M. Peña, A matricial description of Neville elimination with applications to total positivity, Linear Algebra Appl. 202 (1994) 33–45.