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Solitons, breathers and rogue waves of the Yajima–Oikawa-Newell long wave–short wave system

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ABSTRACT

In this paper, we consider the recently-introduced Yajima–Oikawa–Newell (YON) system describing the nonlinear resonant interaction between a long wave and a short wave. It extends and generalises the Yajima–Oikawa (YO) and the Newell (N) systems, which can be obtained from the YON system for special choices of the two non-rescalable, arbitrary parameters that it features. Remarkably, for any choice of these latter constants, the YON system is integrable, in the sense of possessing a Lax pair. New families of solutions, including the bright and dark multi-solitons, as well as the breathers and the higher-order rogue waves are systematically derived by means of the τ-function reduction technique for the two-component KP and the KP-Toda hierarchies. In particular, we show that the condition that the wave parameters have to satisfy for the rogue wave solution to exist coincides with the prediction based on the stability spectra for base-band instability of the plane wave solutions. Several examples from each family of solutions are given in closed form, along with a discussion of their main properties and behaviours.

1. Introduction

In this paper, we study the recently-introduced Yajima–Oikawa–Newell (YON) system [1,2], modelling the resonant interaction between a short wave S and a long wave L:

$$\mathrm{i}S_t + S_{xx} + (\mathrm{i}\alpha L_x + \alpha^2 L^2 - \beta L - 2\alpha|S|^2)S = 0\,, \tag{1a}$$

$$L_t = 2(|S|^2)_x\,. \tag{1b}$$

Here, S is a complex variable representing the complex amplitude of the short wave, L is a real variable representing the amplitude of the coupled long wave envelope, subscripts denote partial differentiation, and α and β are two arbitrary, non-rescalable, real

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parameters. Remarkably, system (1) is integrable – in the sense of possessing a Lax pair [1] – for *any* choice of the parameters α and β. In particular, it is an integrable generalisation of both the Yajima–Oikawa (YO) model [3,4],

$$\mathrm{i}S_t + S_{xx} - LS = 0\,, \tag{2a}$$

$$L_t = 2(|S|^2)_x\,, \tag{2b}$$

which is obtained by setting $\alpha = 0$ and $\beta = 1$ in (1), and the Newell (N) model [5,6],

$$\mathrm{i}S_t + S_{xx} + (\mathrm{i}L_x + L^2 - 2\sigma|S|^2)\,S = 0\,, \tag{3a}$$

$$L_t = 2\sigma(|S|^2)_x\,, \quad \sigma^2 = 1\,, \tag{3b}$$

which is obtained by setting $\alpha = \sigma$ and $\beta = 0$ and by substituting the field L with $\sigma\, L$ in (1), and where the parameter σ acts as a sign that splits the system in two different cases, similarly to the focusing and defocusing regimes of the nonlinear Schrödinger (NLS) equation. The Yajima–Oikawa system has seen numerous applications in different fields of physics, such as sonic-Langmuir waves [3], capillary–gravity waves [4], water waves [7,8] and, most remarkably, in multiscale analysis whenever one combines a branch of optical nature with one of mechanical nature [9–12]. Being a generalisation of it, the YON system has the potential to be useful for modelling more general cases, although research on this front is still in progress. In [2], the authors provide an argument for the YON system playing a role in modelling stratified fluids (see also [13,14]), or short wave-long wave coupling in the presence of tension contrasting gravity and a short wave length comparable to the fluid's depth. Finally, the YON system can be regarded as a reduction of an auxiliary system used to study the Yajima–Oikawa system in [15], although some of its properties, such as stability, do not translate fully through this reduction.

In this work, we construct and study families of soliton and rogue-wave solutions of the YON model (1). Multiple methods have been employed throughout the years to generate solutions of integrable systems. Some of the most prominent of these take advantage of the Lax pair formulation, such as the inverse scattering method [16–20], and Bäcklund [21,22] or Darboux transformations [23–25]. However, the Hirota bilinear method [26–29] has two important features that distinguish it from the previous ones: it is not a spectral method, as it does not need a Lax pair formulation for the system in order to apply it (albeit its applicability for systems that are not Lax-integrable is unclear [30–32]); and it is not analytical in its nature, but algebraic.

The specific approach we will employ in the present work is the method of τ-functions [33–35], which allows one to use the Kadomtsev–Petviashvili (KP) equation [36] and the discrete KP (dKP) equation, also known as Hirota–Miwa (HM) equation [29,37] to rewrite the corresponding bilinear forms as elements of the KP hierarchy. Many systems have been studied through this approach in the last two decades, notably including several multicomponent systems. In particular, it has been successfully applied to obtain solutions of both the Yajima–Oikawa [38] and Newell [39] systems, although it had not yet been applied to the new YON system. Some periodic, rational, bright and dark soliton, and peakon solutions of the YON system have been previously obtained via an Ansatz approach in [2]; some bright and dark (anti-dark, grey, black) soliton solutions, as well as some breather solutions have also been derived via a traditional Hirota approach [40]. The method that we adopt in the present work will allow us to generate systematically and study in greater depth families of solutions encompassing all the previously known solutions, as well as new families of solutions (especially of rogue waves) not known previously in the literature. Moreover, we show that rogue wave regimes do coincide with the predictions made in [1] based on the system's stability spectra (see [41]), analogously to what was shown in [42] for the vector NLS system (see also [43]), as the onset of rogue waves is closely related to the stability behaviour of plane waves, with base-band instability being identified as one of the main ingredients (see [44,45]).

Finally, it is worth remarking that other extensions the Yajima–Oikawa system exist, including extensions as vector [46] and matrix [47] systems. Also, some additional kinds of solutions appear for Yajima–Oikawa that have not yet seen a translation into the YON system, including periodic-background solutions [48].

In Section 2 we introduce the direct Hirota bilinearisation of the YON system, which we use to generate the general N-bright-soliton solutions of the system, along with the properties of these solutions, such as soliton amplitude and velocity, and phase shift resulting from the collision of solitons. Furthermore, to illustrate the technique, we rederive the general bright soliton solution through the method of τ-functions.

In Section 3 we use the method of τ-functions to relate the bilinear form of the system to the KP-Toda hierarchy to generate the general N-dark-soliton solution of the system, for which we also compute its corresponding properties.

In Section 4 we employ the same formalism to obtain the general N-breather solution of the system, along with the range of parameters that enable its existence.

In Section 5 we take advantage of the breather solutions to explicitly write the general N-rogue-wave solution of the system, both in differential form and using elementary Schur polynomials.

2. Bright solitons

We start our investigation of the solutions of the YON system (1) by deriving the bright soliton solutions, understood as solitons propagating on a zero background. We will first use the traditional Hirota bilinear method to generate the general N-bright-soliton solution, and then repeat the construction with the τ-functions method, to better illustrate how the latter works and allow the reader to better understand its employment in the more complicated cases of the dark soliton, breather, and rogue wave solutions.

Before starting with the Hirota bilinear method, it is convenient to replace L with L/α and $\beta = 2\alpha\delta$, so that we can rewrite (1) as

$$\mathrm{i}S_t + S_{xx} + (\mathrm{i}L_x + L^2 - 2\delta L - 2\alpha|S|^2)S = 0\,, \tag{4a}$$

$$L_t = 2\alpha(|S|^2)_x\,. \tag{4b}$$

We will bilinearise (4) by introducing the variable transformations

$$L = \mathrm{i}\left(\log\frac{f^*}{f}\right)_x\,, \qquad S = \frac{g}{f}\,, \tag{5}$$

where f and g are complex functions and f^* denotes the complex conjugate of f. The rationale behind the form of L is to ensure the resulting quantity is real, plus balancing the natural degree of derivatives.

The application of (5) into (4b) entails

$$\mathrm{i}\left(\log\frac{f^*}{f}\right)_{tx} = 2\alpha\left(\frac{gg^*}{ff^*}\right)_x\,. \tag{6}$$

By integrating with respect to x, we get

$$\mathrm{i}\left(\log\frac{f^*}{f}\right)_t = 2\alpha\frac{gg^*}{ff^*} + C_1\,, \tag{7}$$

where C_1 is an arbitrary integration constant. That takes us to

$$\mathrm{i}D_t f\cdot f^* = -2\alpha gg^* - C_1 ff^*\,, \tag{8}$$

where we have made use of the Hirota bilinear operator defined by [27]

$$D_x^n D_t^m(a\cdot b) = \left(\frac{\partial}{\partial x} - \frac{\partial}{\partial x'}\right)^n\left(\frac{\partial}{\partial t} - \frac{\partial}{\partial t'}\right)^m a(x,t)b(x',t')\Big|_{x=x',t=t'}\,, \tag{9}$$

along with its antisymmetry and the property

$$\frac{\partial}{\partial x}\log\frac{a}{b} = \frac{D_x a\cdot b}{ab} \tag{10}$$

for arbitrary differentiable functions a and b. By setting $C_1 = 0$ in (8), we get the first of our bilinear equations,

$$\mathrm{i}D_t f\cdot f^* = -2\alpha|g|^2\,. \tag{11}$$

In a similar way, by substituting (5) into (4a) and after some tedious computations, we obtain the relation

$$\frac{(\mathrm{i}D_t + D_x^2)g\cdot f}{f^2} - \frac{g}{f}\,\frac{(D_x^2 - 2\mathrm{i}\delta D_x)f\cdot f^* + 2\alpha gg^*}{ff^*} = 0\,. \tag{12}$$

By decoupling (12), we obtain two additional bilinear equations,

$$(\mathrm{i}D_t + D_x^2)g\cdot f = 0\,, \tag{13a}$$

$$(D_x^2 - 2\mathrm{i}\delta D_x)f\cdot f^* + 2\alpha|g|^2 = 0\,. \tag{13b}$$

We can use the previous bilinear Eq. (11) to rewrite (13b) as

$$\mathrm{i}D_t f\cdot f^* = (D_x^2 - 2\mathrm{i}\delta D_x)f\cdot f^*\,. \tag{14}$$

Summarising, the YON system transforms into a system of three Hirota bilinear equations:

$$(\mathrm{i}D_t + D_x^2)g\cdot f = 0\,, \tag{15a}$$

$$\mathrm{i}D_t f\cdot f^* = (D_x^2 - 2\mathrm{i}\delta D_x)f\cdot f^*\,, \tag{15b}$$

$$\mathrm{i}D_t f\cdot f^* = -2\alpha|g|^2\,. \tag{15c}$$

2.1. Traditional Hirota construction of bright solitons

Following Hirota's procedure, we can set the variables f and g in the form

$$f = 1 + \epsilon^2 f_2 + \epsilon^4 f_4 + \cdots = 1 + \sum_{n=1}^{\infty}\epsilon^{2n}f_{2n}\,, \tag{16}$$

$$g = \epsilon g_1 + \epsilon^3 g_3 + \epsilon^5 g_5 + \cdots = \sum_{n=1}^{\infty}\epsilon^{2n-1}g_{2n-1}\,, \tag{17}$$

where f_i, g_i are arbitrary functions and ϵ is a formal parameter of expansion for f and g. We then substitute the formal series (16) into our bilinear Eqs. (15). In order to obtain the bright-soliton solution we can assume

$$f_4 = f_6 = \cdots = 0\,, \qquad g_3 = g_5 = \cdots = 0\,. \tag{18}$$

After this choices, to the lowest order in ϵ our bilinear Eqs. (15) yield

$$(\mathrm{i}D_t + D_x^2)g_1 \cdot 1 = 0\,, \tag{19a}$$
$$(D_x^2 - 2\mathrm{i}\delta D_x)f_2 \cdot 1 + (D_x^2 - 2\mathrm{i}\delta D_x)1 \cdot f_2^* = -2\alpha g_1 g_1^*\,, \tag{19b}$$
$$\mathrm{i}D_t f_2 \cdot 1 + \mathrm{i}D_t 1 \cdot f_2^* = -2\alpha g_1 g_1^*\,. \tag{19c}$$

From (19a) we get

$$\mathrm{i}\frac{\partial g_1}{\partial t} + \frac{\partial^2 g_1}{\partial x^2} = 0\,, \tag{20}$$

which gives us a solution

$$g_1 = \gamma_1 e^{\xi_1}\,, \qquad \xi_1 = k_1 x + \mathrm{i}k_1^2 t + \xi_{1,0}\,, \tag{21}$$

with k_1 the corresponding wave number, $\xi_{1,0}$ an arbitrary initial phase (corresponding to a translation in space and time) and γ_1 an arbitrary complex parameter.

From (19b) and (19c) we get

$$\frac{\partial^2 f_2}{\partial x^2} - 2\mathrm{i}\delta\frac{\partial f_2}{\partial x} + \frac{\partial^2 f_2^*}{\partial x^2} + 2\mathrm{i}\delta\frac{\partial f_2^*}{\partial x} = -2\alpha g_1 g_1^*\,, \tag{22a}$$
$$\mathrm{i}\frac{\partial f_2}{\partial t} - \mathrm{i}\frac{\partial f_2^*}{\partial t} = -2\alpha g_1 g_1^*\,. \tag{22b}$$

We can plug the expression for g_1 obtained above into (22) to get

$$\frac{\partial^2 f_2}{\partial x^2} - 2\mathrm{i}\delta\frac{\partial f_2}{\partial x} + \frac{\partial^2 f_2^*}{\partial x^2} + 2\mathrm{i}\delta\frac{\partial f_2^*}{\partial x} = -2\alpha|\gamma_1|^2 e^{\xi_1+\xi_1^*}\,, \tag{23a}$$
$$\mathrm{i}\frac{\partial f_2}{\partial t} - \mathrm{i}\frac{\partial f_2^*}{\partial t} = -2\alpha|\gamma_1|^2 e^{\xi_1+\xi_1^*}\,. \tag{23b}$$

For these equations we can try an Ansatz

$$f_2 = A_2 e^{\xi_1+\xi_1^*}\,, \qquad f_2^* = A_2^* e^{\xi_1+\xi_1^*}\,, \tag{24}$$

and after substituting we get that

$$A_2 = \frac{2\alpha|\gamma_1|^2(\mathrm{i}\delta + k_1^*)}{(k_1+k_1^*)^2(k_1-k_1^*)}\,. \tag{25}$$

Putting everything together, for a single soliton we can write it as

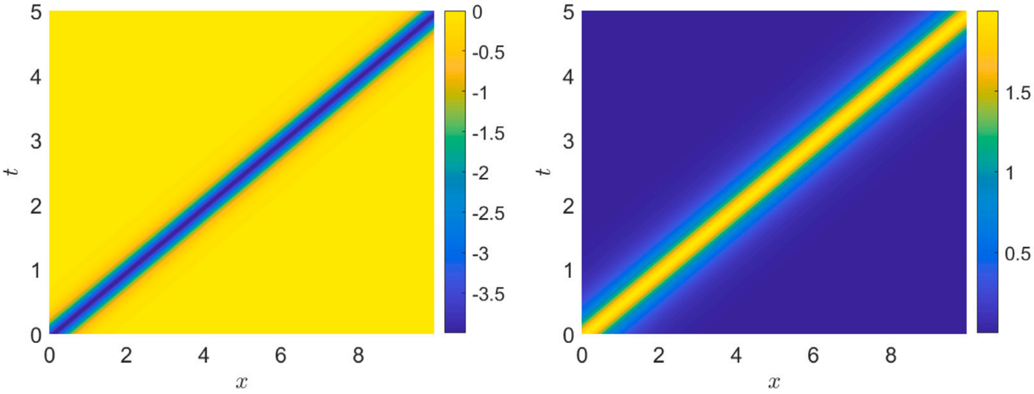
$$f = 1 + \frac{2\alpha|\gamma_1|^2(\mathrm{i}\delta + k_1^*)}{(k_1+k_1^*)^2(k_1-k_1^*)}e^{\xi_1+\xi_1^*} = \begin{vmatrix} \frac{\mathrm{i}\delta + k_1^*}{k_1+k_1^*}e^{\xi_1+\xi_1^*} & 1 \\ -1 & -\frac{2\alpha|\gamma_1|^2}{k_1^{*2}-k_1^2} \end{vmatrix}\,, \tag{26a}$$

$$f^* = 1 + \frac{2\alpha|\gamma_1|^2(\mathrm{i}\delta - k_1)}{(k_1+k_1^*)^2(k_1-k_1^*)}e^{\xi_1+\xi_1^*} = \begin{vmatrix} \frac{\mathrm{i}\delta - k_1}{k_1+k_1^*}e^{\xi_1+\xi_1^*} & 1 \\ -1 & -\frac{2\alpha|\gamma_1|^2}{k_1^{*2}-k_1^2} \end{vmatrix}\,, \tag{26b}$$

$$g = \gamma_1 e^{\xi_1} = \begin{vmatrix} \frac{\mathrm{i}\delta + k_1^*}{k_1+k_1^*}e^{\xi_1+\xi_1^*} & 1 & e^{\xi_1} \\ -1 & -\frac{2\alpha|\gamma_1|^2}{k_1^{*2}-k_1^2} & 0 \\ 0 & -\gamma_1 & 0 \end{vmatrix}\,, \qquad \xi_1 = k_1 x + \mathrm{i}k_1^2 t + \xi_{1,0}\,. \tag{26c}$$

Upon getting back to the S and L variables through the change (5), the formulae above provide the one-soliton solution. For example, for the short wave we have

$$S = \frac{g}{f} = \frac{\gamma_1 e^{\xi_1}}{1 + \frac{2\alpha|\gamma_1|^2(\mathrm{i}\delta+k_1^*)}{(k_1+k_1^*)^2(k_1-k_1^*)}e^{\xi_1+\xi_1^*}} \tag{27}$$



(a) Long wave L. (b) Short wave $|S|$.

Fig. 1. 1-bright-soliton solution with $\alpha = 1,\ \beta = 2,\ k_1 = 2 + i,\ \gamma_1 = 2 + i$.

and

$$|S|^2 = \frac{gg^*}{ff^*} = \frac{|\gamma|^2 e^{\xi_1+\xi_1^*}}{\left(1 + \frac{2\alpha|\gamma_1|^2(\mathrm{i}\delta+k_1^*)}{(k_1+k_1^*)^2(k_1-k_1^*)} e^{\xi_1+\xi_1^*}\right)\left(1 + \frac{2\alpha|\gamma_1|^2(\mathrm{i}\delta-k_1)}{(k_1+k_1^*)^2(k_1-k_1^*)} e^{\xi_1+\xi_1^*}\right)}. \tag{28}$$

We can denote $k_1 = k_1^{(r)}+\mathrm{i}k_1^{(i)}$ and get back to the original variables in (1) to write explicitly the general form of the one-bright-soliton solution as

$$S(x,t) = \frac{8e^{\mathrm{i}(k_1^{*2}t+k_1^*x)}k_1^{(i)}\left(k_1^{(r)}\right)^2\gamma_1}{8e^{4k_1^{(i)}k_1^{(r)}t}k_1^{(i)}\left(k_1^{(r)}\right)^2 + |\gamma_1|^2 e^{2k_1^{(r)}x}[\beta - 2(k_1^{(i)} + \mathrm{i}k_1^{(r)})\alpha]}, \tag{29a}$$

$$L(x,t) = -\frac{64e^{2k_1^{(r)}(2k_1^{(i)}t+x)}k_1^{(i)}\left(k_1^{(r)}\right)^4|\gamma_1|^2}{64e^{8k_1^{(i)}k_1^{(r)}t}\left(k_1^{(i)}\right)^2\left(k_1^{(r)}\right)^4 - 16e^{2k_1^{(r)}(2k_1^{(i)}t+x)}k_1^{(i)}\left(k_1^{(r)}\right)^2|\gamma_1|^2(2k_1^{(i)}\alpha - \beta) + |\gamma_1|^4 e^{4k_1^{(r)}x}(4\alpha^2|k_1|^2 - 4\alpha\beta k_1^{(i)} + \beta^2)}, \tag{29b}$$

where we have set $\xi_{1,0} = 0$ without loss of generality, as it can be absorbed by γ_1 (see Fig. 1).

Both S and L are solitons moving with velocity

$$V = 2k_1^{(i)}, \tag{30}$$

so that $L(x,0) = L(x + Vt, t)$ for every x and t (and the same applies for $|S(x,t)|$).

When $t = 0$, both $|S|$ and $|L|$ have their maximum at

$$x_{\max} = \frac{1}{4k_1^{(r)}}\log\left(\frac{64\left(k_1^{(i)}\right)^2\left(k_1^{(r)}\right)^4}{|\gamma_1|^4(4\alpha^2|k_1|^2 - 4\alpha\beta k_1^{(i)} + \beta^2)}\right), \tag{31}$$

with the soliton in L having an amplitude

$$A_L = \frac{4\left(k_1^{(r)}\right)^2}{-\mathrm{sgn}(k_1^{(i)})\sqrt{4\alpha^2|k_1|^2 - 4\alpha\beta k_1^{(i)} + \beta^2} + (2\alpha k_1^{(i)} - \beta)}, \tag{32}$$

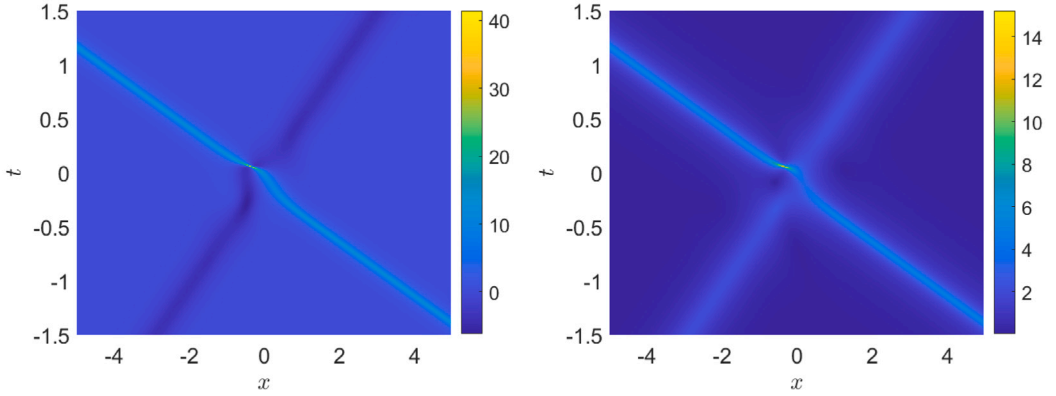
where $\mathrm{sgn}(k_1^{(i)})$ denotes the sign of $k_1^{(i)}$, while the amplitude of $|S|$ satisfies

$$A_S^2 = -k_1^{(i)}A_L \tag{33}$$

as a direct consequence of the property

$$\frac{L(x,t)}{|S(x,t)|^2} = -\frac{1}{k_1^{(i)}}. \tag{34}$$

Note that all the formulae above reduce nicely to the Newell case $\beta = 0$ for every value of the parameters. However, for the Yajima–Oikawa case $\alpha = 0$, the amplitude A_L in (32) diverges whenever $k_1^{(i)} < 0$. That, together with the expression for the velocity (30) indicates that Yajima–Oikawa admits only bright solitons travelling to the right direction.



(a) Long wave L. (b) Short wave $|S|$.

Fig. 2. 2-bright-soliton solution with $\alpha = 1$, $\beta = 2$, $k_1 = 2 + i$, $k_2 = 1 - 2i$, $\gamma_1 = 2 + i$, $\gamma_2 = 1 + 2i$.

One can check that the bright-soliton solutions obtained in [2] are subcases of the Hirota soliton for the special choice $k_1^{(i)} = \beta/2\alpha$, so that the velocity of the soliton is exactly $V = \beta/\alpha$. Note that this reduction does not work well for the Yajima–Oikawa case $\alpha = 0$, which is not covered by [2].

Through a similar process, we can compute the two-bright-soliton solutions by assuming f_4 and g_3 are also nonzero in (16). By doing that, one ends up with the expressions

$$f = \begin{vmatrix} \frac{i\delta+k_1^*}{k_1+k_1^*}e^{\xi_1+\xi_1^*} & \frac{i\delta+k_2^*}{k_1+k_2^*}e^{\xi_1+\xi_2^*} & 1 & 0 \\ \frac{i\delta+k_1^*}{k_2+k_1^*}e^{\xi_2+\xi_1^*} & \frac{i\delta+k_2^*}{k_2+k_2^*}e^{\xi_2+\xi_2^*} & 0 & 1 \\ -1 & 0 & -\frac{2\alpha|\gamma_1|^2}{k_1^{*2}-k_1^2} & -\frac{2\alpha\gamma_1^*\gamma_2}{k_1^{*2}-k_2^2} \\ 0 & -1 & -\frac{2\alpha\gamma_2^*\gamma_1}{k_2^{*2}-k_1^2} & -\frac{2\alpha|\gamma_2|^2}{k_2^{*2}-k_2^2} \end{vmatrix}, \tag{35a}$$

$$f^* = \begin{vmatrix} \frac{i\delta-k_1}{k_1+k_1^*}e^{\xi_1+\xi_1^*} & \frac{i\delta-k_2}{k_1+k_2^*}e^{\xi_1+\xi_2^*} & 1 & 0 \\ \frac{i\delta-k_1}{k_2+k_1^*}e^{\xi_2+\xi_1^*} & \frac{i\delta-k_2}{k_2+k_2^*}e^{\xi_2+\xi_2^*} & 0 & 1 \\ -1 & 0 & -\frac{2\alpha|\gamma_1|^2}{k_1^{*2}-k_1^2} & -\frac{2\alpha\gamma_1^*\gamma_2}{k_1^{*2}-k_2^2} \\ 0 & -1 & -\frac{2\alpha\gamma_2^*\gamma_1}{k_2^{*2}-k_1^2} & -\frac{2\alpha|\gamma_2|^2}{k_2^{*2}-k_2^2} \end{vmatrix}, \tag{35b}$$

$$g = \begin{vmatrix} \frac{i\delta+k_1^*}{k_1+k_1^*}e^{\xi_1+\xi_1^*} & \frac{i\delta+k_2^*}{k_1+k_2^*}e^{\xi_1+\xi_2^*} & 1 & 0 & e^{\xi_1} \\ \frac{i\delta+k_1^*}{k_2+k_1^*}e^{\xi_2+\xi_1^*} & \frac{i\delta+k_2^*}{k_2+k_2^*}e^{\xi_2+\xi_2^*} & 0 & 1 & e^{\xi_2} \\ -1 & 0 & -\frac{2\alpha|\gamma_1|^2}{k_1^{*2}-k_1^2} & -\frac{2\alpha\gamma_1^*\gamma_2}{k_1^{*2}-k_2^2} & 0 \\ 0 & -1 & -\frac{2\alpha\gamma_2^*\gamma_1}{k_2^{*2}-k_1^2} & -\frac{2\alpha|\gamma_2|^2}{k_2^{*2}-k_2^2} & 0 \\ 0 & 0 & -\gamma_1 & -\gamma_2 & 0 \end{vmatrix}, \tag{35c}$$

where $\xi_j = k_j x + ik_j^2 t + \xi_{j,0}$ for $j = 1, 2$, k_1 and k_2 are the wave numbers, $\xi_{1,0}$ and $\xi_{2,0}$ are the initial phases, and γ_1 and γ_2 are two arbitrary complex parameters (see Fig. 2).

One can study the phase shift via the changes of variable $X = x + 2k_1^{(i)}$ and $X = x + 2k_2^{(i)}$, where $k_1^{(i)}$ and $k_2^{(i)}$ are the imaginary parts of k_1 and k_2, in order to make "stationary" the first and second soliton, respectively. That way one can make t go to $\pm\infty$ to collapse it into a one-soliton solution, and then study the difference in phase between the two asymptotic one-soliton solutions. The phase shift φ_{12} for the first soliton in the 2-soliton solution then takes the form

$$\varphi_{12} = \chi \frac{1}{k_1^{(r)}} \log\left[\frac{\left[\left(k_1^{(i)} - k_2^{(i)}\right)^2 + \left(k_1^{(r)} - k_2^{(r)}\right)^2\right]\left[\left(k_1^{(i)} + k_2^{(i)}\right)^2 + \left(k_1^{(r)} + k_2^{(r)}\right)^2\right]}{\left[\left(k_1^{(i)} - k_2^{(i)}\right)^2 + \left(k_1^{(r)} + k_2^{(r)}\right)^2\right]\left[\left(k_1^{(i)} + k_2^{(i)}\right)^2 + \left(k_1^{(r)} - k_2^{(r)}\right)^2\right]}\right]$$

$$= \chi \frac{2}{k_1 + k_1^*} \log \left[\frac{(k_1^2 - k_2^2)(k_1^{*2} - k_2^{*2})}{(k_1^{*2} - k_2^2)(k_1^2 - k_2^{*2})} \right] , \tag{36a}$$

where $k_1 = k_1^{(r)} + \mathrm{i}k_1^{(i)}$, $k_2 = k_2^{(r)} + \mathrm{i}k_2^{(i)}$, and the sign χ is

$$\chi = \mathrm{sgn}[k_2^{(r)}(k_2^{(i)} - k_1^{(i)})] . \tag{36b}$$

The corresponding shift φ_{21} for the second soliton can be obtained by simply swapping the subindices 1 and 2 in the formulae above. Note the formulae for the phase shifts do not depend either on the parameters α and δ in (4) or on the complex parameters $\gamma_{1,2}$.

Further solitons can be added to the solution by taking more nonzero terms in f and g. The resulting f and g have similar forms as above, by extending the size of the determinant by 2 for each soliton added, so the N-soliton solution has the form

$$f = \begin{vmatrix} A & I_N \\ -I_N & B \end{vmatrix} , \tag{37a}$$

$$g = \begin{vmatrix} A & I_N & c_\xi \\ -I_N & B & 0_{N\times 1} \\ 0_{1\times N} & r_\gamma & 0 \end{vmatrix} , \tag{37b}$$

where I_N denotes the $N \times N$ identity matrix, $0_{i\times j}$ denotes the $i \times j$ matrix of zeroes, and the $N \times N$ matrices A and B are defined by

$$A_{ij} = \frac{\mathrm{i}\delta + k_j^*}{k_i + k_j^*} e^{\xi_i + \xi_j^*} , \qquad B_{ij} = -\frac{2\alpha\gamma_i^*\gamma_j}{k_i^{*2} - k_j^2} , \tag{38a}$$

where, as before, $\xi_j = k_j x + \mathrm{i}k_j^2 t + \xi_{j,0}$, the k_i are the wave numbers, the $\xi_{j,0}$ are the initial phases, the γ_j are arbitrary complex parameters, the column vector c_ξ satisfies $(c_\xi)_i = e^{\xi_i}$, $i = 1, \ldots, N$, and the row vector r_γ satisfies $(r_\gamma)_i = -\gamma_i$.

The soliton solutions above coincide with the ones obtained for the Newell system in [39] when setting $\beta = 0$ and $\alpha = 1$, and with the solutions for Yajima–Oikawa obtained in [38] upon setting $\alpha = 0$ and $\beta = 1$. Similarly to the 1-soliton case, in the reduction $\alpha = 0$ solitons are only allowed to travel in the right direction.

2.2. τ-Functions construction of bright solitons

In what follows, we will derive the general N-bright-soliton solution via the KP hierarchy reduction method. We start with τ-functions expressed by Gram determinants in two-component KP hierarchy

$$F_{n,m} = \begin{vmatrix} A(n) & I_N \\ -I_N & B(m) \end{vmatrix} , \qquad \bar{F}_{n,m} = \begin{vmatrix} A'(n) & I_N \\ -I_N & B(m) \end{vmatrix} , \tag{39a}$$

$$G_{n,m} = \begin{vmatrix} A(n) & I_N & \Phi_n^T \\ -I_N & B(m) & 0_{N\times 1} \\ 0_{1\times N} & -\bar{\Psi}_m^{(k)} & 0 \end{vmatrix} , \qquad H_{n,m} = \begin{vmatrix} A(n) & I_N & 0_{N\times 1} \\ -I_N & B(m) & \Psi_m^{(k)T} \\ -\bar{\Phi}_n & 0_{1\times N} & 0 \end{vmatrix} , \tag{39b}$$

Here n, m are integers, the block matrices $A(n)$, $A'(n)$, $B(m)$ and the row vectors Φ_n, $\bar{\Phi}_n$ Ψ_m, $\bar{\Psi}_m$ are defined by

$$A(n) = \left(a_{ij}(n)\right)_{1\le i,j\le N} , \qquad A'(n) = \left(a'_{ij}(n)\right)_{1\le i,j\le N} , \tag{40a}$$

$$B(m) = \left(b_{ij}(m)\right)_{1\le i,j\le N} , \tag{40b}$$

$$\Phi_n = \left(\phi_1(n), \ldots, \phi_N(n)\right) , \qquad \bar{\Phi}_n = \left(\bar{\phi}_1(n), \ldots, \bar{\phi}_N(n)\right) , \tag{40c}$$

$$\Psi_m = \left(\psi_1(m), \ldots, \psi_N(m)\right) , \qquad \bar{\Psi}_m = \left(\bar{\psi}_1(m), \ldots, \bar{\psi}_N(m)\right) , \tag{40d}$$

where $a_{ij}(n)$, $a'_{ij}(n)$, $\phi_i(n)$, $\bar{\phi}_i(n)$, $\psi_i(m)$ and $\bar{\psi}_i(m)$ are defined as

$$\phi_i(n) = (p_i)^n e^{\xi_i} , \qquad \bar{\phi}_j(n) = (-\bar{p}_j)^{-n} e^{\bar{\xi}_j} , \tag{41a}$$

$$\psi_i(m) = \left(q_i\right)^m e^{\eta_i} , \qquad \bar{\psi}_j(m) = \left(-\bar{q}_j\right)^{-m} e^{\bar{\eta}_j} . \tag{41b}$$

$$a_{ij}(n) = \frac{\bar{p}_j - \mu}{p_i + \bar{p}_j} \left(-\frac{p_i}{\bar{p}_j}\right)^n e^{\xi_i + \bar{\xi}_j} , \qquad a'_{ij}(n) = -\frac{p_i + \mu}{p_i + \bar{p}_j} \left(-\frac{p_i}{\bar{p}_j}\right)^n e^{\xi_i + \bar{\xi}_j} , \tag{41c}$$

$$b_{ij}(m) = \frac{\nu}{q_i + \bar{q}_j} \left(-\frac{q_i}{\bar{q}_j}\right)^m e^{\eta_i + \bar{\eta}_j} , \tag{41d}$$

with

$$\xi_i = p_i x_1 + p_i^2 x_2 + \xi_{i,0} , \qquad \bar{\xi}_j = \bar{p}_j x_1 - \bar{p}_j^2 x_2 + \bar{\xi}_{j,0} , \tag{42a}$$

$$\eta_i = q_i y_1 + \eta_{i,0} , \qquad \bar{\eta}_j = \bar{q}_j y_1 + \bar{\eta}_{j,0} , \tag{42b}$$

where $\mu, \nu, p_i, \bar{p}_j, \xi_{i,0}, \bar{\xi}_{j,0}, q_i, \bar{q}_j, \eta_{i,0}, \bar{\eta}_{j,0}$ $(i, j = 1, \dots, N)$ are free complex parameters. The τ-functions defined above satisfy the following bilinear equations in the two-component KP hierarchy [49–51].

$$(D_{x_2} - D_{x_1}^2) G_{n,m} \cdot F_{n,m} = 0, \tag{43a}$$

$$(D_{x_1}^2 + D_{x_2} + 2\mu D_{x_1}) F_{n,m} \cdot \bar{F}_{n,m} = 0, \tag{43b}$$

$$D_{y_1} F_{n,m} \cdot \bar{F}_{n,m} + \nu G_{n,m} H_{n,m} = 0. \tag{43c}$$

Now we start the reduction process. First, we carry out the dimension reduction. By performing row and column operations, two τ-functions $F_{n,m}$ and $\bar{F}_{n,m}$ can be rewritten as

$$F_{n,m} = \begin{vmatrix} \tilde{A}_{n,m} & I_N \\ -I_N & \tilde{B}_{n,m} \end{vmatrix}, \qquad \bar{F}_{n,m} = \begin{vmatrix} \tilde{A}'_{n,m} & I_N \\ -I_N & \tilde{B}_{n,m} \end{vmatrix} \tag{44}$$

where $\tilde{A}_{n,m}$, $\tilde{A}'_{n,m}$ and $\tilde{B}_{n,m}$ are $N \times N$ matrices whose elements defined by

$$\tilde{a}_{ij}(n) = \frac{\bar{p}_j - \mu}{p_i + \bar{p}_j} \left(-\frac{p_i}{\bar{p}_j} \right)^n, \qquad \tilde{a}'_{ij}(n) = -\frac{p_i + \mu}{p_i + \bar{p}_j} \left(-\frac{p_i}{\bar{p}_j} \right)^n, \tag{45a}$$

$$\tilde{b}_{ij}(m) = \frac{\nu}{q_i + \bar{q}_j} \left(-\frac{q_i}{\bar{q}_j} \right)^m e^{\zeta_i + \bar{\zeta}_j}, \tag{45b}$$

with

$$\zeta_i = \eta_i + \bar{\xi}_i = q_i y_1 + \bar{p}_i x_1 - \bar{p}_i^2 x_2 + \eta_{i,0} + \bar{\xi}_{i,0}, \tag{45c}$$

$$\bar{\zeta}_j = \bar{\eta}_j + \xi_j = \bar{q}_j y_1 + p_j x_1 + p_j^2 x_2 + \bar{\eta}_{j,0} + \xi_{j,0}. \tag{45d}$$

Imposing the reduction conditions

$$q_i = -\frac{\bar{p}_i^2}{2}, \qquad \bar{q}_i = \frac{p_i^2}{2}, \tag{46}$$

the following relations hold

$$\partial_{x_2} F_{n,m} = 2\partial_{y_1} F_{n,m}, \qquad \partial_{x_2} \bar{F}_{n,m} = 2\partial_{y_1} \bar{F}_{n,m}, \tag{47}$$

which implies

$$D_{x_2} F_{n,m} \cdot \bar{F}_{n,m} = -2\nu G_{n,m} H_{n,m}. \tag{48}$$

Next, we carry out the complex conjugate reduction. Note that the determinant defining $H_{n,m}$ remains unchanged upon replacing the block matrix $A(n)$ by $A'(n)$. Additionally, since by means of (48) we have removed every derivative with respect to y_1, we can treat it as a constant. Assume that x_1, ν are real, μ, x_2, y_1 are purely imaginary, and

$$\bar{p}_i = p_i^*, \quad \bar{\xi}_{i,0} = \xi_{i,0}^*, \quad \bar{\eta}_{i,0} = \eta_{i,0}^*. \tag{49}$$

Then, one can check that

$$a_{ij}^*(n) = -a'_{ji}(-n), \quad b_{ij}^*(m) = -b_{ji}(-m), \tag{50}$$

which implies

$$F_{n,m}^* = \bar{F}_{-n,-m}, \quad G_{n,m}^* = (-1)^{1-n} H_{-n,-m}. \tag{51}$$

Therefore, by setting $n = m = 0$, $\mu = -\mathrm{i}\delta$, $\nu = \alpha$ and applying the variable transformations

$$x_1 = x, \quad x_2 = \mathrm{i}t, \quad \text{i.e.,} \quad \partial_{x_1} = \partial_x, \quad \partial_{x_2} = -\mathrm{i}\partial_t, \tag{52}$$

and

$$F_{0,0} = f, \quad \bar{F}_{0,0} = f^*, \quad G_{0,0} = g, \quad H_{0,0} = -g^*, \tag{53}$$

the bilinear Eqs. (43a), (48) and (43b) are then transformed into the target ones (15) exactly.

Consequently, we can let $e^{\eta_i} = \gamma_i^*$, $e^{\bar{\eta}_i} = \gamma_i$, $p_i = k_i$ for $i = 1, \dots, N$ and redefine $A(0) = A$, $B(0) = B$, $\Phi_0 = c_\xi$, $\bar{\Phi}_0 = c_\xi^*$, $\bar{\Psi}_0 = -r_\gamma$, $\Psi_0 = -r_\gamma^*$, so that the N-bright-soliton solution to the YON system (4) can take the form of the following theorem, which coincides with the previous formulae (37).

Theorem 1. *The YON system (4) possesses the N-bright-soliton solution (5), where the determinants f, f^* and g are given by*

$$f = \begin{vmatrix} A & I_N \\ -I_N & B \end{vmatrix}, \qquad g = \begin{vmatrix} A & I_N & c_\xi \\ -I_N & B & 0_{N\times 1} \\ 0_{1\times N} & r_\gamma & 0 \end{vmatrix},$$

where I_N is the $N \times N$ identity matrix, $0_{i\times j}$ is the $i \times j$ matrix of zeroes, A and B are $N \times N$ matrices whose entries are

$$a_{ij} = \frac{k_j^* + \mathrm{i}\delta}{k_i + k_j^*} \mathrm{e}^{\xi_i+\xi_j^*}, \qquad b_{ij} = -\frac{2\alpha\gamma_i^*\gamma_j}{k_i^{*2} - k_j^2}.$$

The matrices c_ξ and r_γ are defined by

$$c_\xi = (\mathrm{e}^{\xi_1}, \dots, \mathrm{e}^{\xi_N}), \qquad r_\gamma = (-\gamma_1, \dots, -\gamma_N),$$

with $\xi_i = k_i x + \mathrm{i}k_i^2 t + \xi_{i,0}$. The parameters k_i, $\xi_{i,0}$ and γ_i for $i = 1, \dots, N$ are arbitrary complex constants.

3. Dark solitons

In the previous section we derived the bright soliton solutions. Now we will construct dark soliton solutions (by which we mean soliton solutions on a constant non-zero background). In order to do so, we will have to modify the change of variables we used for the bilinearisation. In the bright case, we just wrote a change of variable on a zero background, however now we need to explicitly account for the plane wave

$$S = \rho e^{\mathrm{i}[qx-(q^2+2\rho^2)t]}, \qquad L = \ell\,. \tag{55}$$

where ρ and ℓ are the amplitudes of S and L in the plane wave and q is the wave number of S. To account for it, we can modify our previous change of variables (5) into

$$S = \rho\frac{g}{f} e^{\mathrm{i}[qx-(q^2+2\rho^2)t]}, \qquad L = \ell + \mathrm{i}\left(\log\frac{f^*}{f}\right)_x. \tag{56}$$

by which, we get the following bilinear equations

$$(\mathrm{i}D_t + 2\mathrm{i}qD_x + D_x^2)g\cdot f = 0\,, \tag{57a}$$

$$\mathrm{i}D_t f\cdot f^* = [D_x^2 - 2\mathrm{i}(\delta-\ell)D_x]f\cdot f^*\,, \tag{57b}$$

$$\mathrm{i}D_t f\cdot f^* = 2\alpha\rho^2(|f|^2 - |g|^2)\,. \tag{57c}$$

From (4b) we get

$$\mathrm{i}\left(\log\frac{f^*}{f}\right)_{xt} = 2\alpha\rho^2\left(\frac{gg^*}{ff^*}\right)_x. \tag{58}$$

By integrating it with respect to x we get

$$\mathrm{i}\left(\log\frac{f^*}{f}\right)_t = 2\alpha\rho^2\frac{gg^*}{ff^*} + C_1\,, \tag{59}$$

entailing

$$\mathrm{i}D_t f\cdot f^* = -2\alpha\rho^2 gg^* - C_1 ff^*\,. \tag{60}$$

In this case we will set $C_1 = -2\alpha\rho^2$ to obtain

$$\mathrm{i}D_t f\cdot f^* = 2\alpha\rho^2(|f|^2 - |g|^2)\,, \tag{61}$$

which will be one of our bilinear equations. Introducing the change of variables in (4b), we have,

$$\begin{aligned}&\mathrm{i}\left(\frac{g}{f}\right)_t + (q^2+2\alpha\rho^2-\ell^2+2\delta\ell)\frac{g}{f} + 2\mathrm{i}q\left(\frac{g}{f}\right)_x + \left(\frac{g}{f}\right)_{xx} - \frac{g}{f}\left(\log\frac{f^*}{f}\right)_{xx}\\ &\qquad + \frac{g}{f}\left[\ell + \mathrm{i}\left(\log\frac{f^*}{f}\right)_x\right]^2 - 2\mathrm{i}\delta\frac{g}{f}\left[\ell+\mathrm{i}\left(\log\frac{f^*}{f}\right)_x\right] - 2\alpha\rho^2\frac{gg^*}{ff^*}\frac{g}{f} = 0\,,\end{aligned} \tag{62a}$$

entailing

$$\frac{(\mathrm{i}D_t + 2\mathrm{i}qD_x + D_x^2 + 2\alpha\rho^2)g\cdot f}{f^2} - \frac{g}{f}\frac{(D_x^2 - 2\mathrm{i}(\delta-\ell)D_x)f\cdot f^* + 2\alpha\rho^2 gg^*}{ff^*} = 0\,. \tag{62b}$$

By decoupling (62b), we obtain the bilinear equations

$$(\mathrm{i}D_t + 2\mathrm{i}qD_x + D_x^2)g\cdot f = 0\,, \tag{63a}$$

$$(D_x^2 - 2\mathrm{i}(\delta-\ell)D_x)f\cdot f^* = 2\alpha\rho^2(|f|^2-|g|^2)\,. \tag{63b}$$

We can use the previous bilinear Eq. (61) to rewrite the latter as

$$\mathrm{i}D_t f\cdot f^* = [D_x^2 - 2\mathrm{i}(\delta-\ell)D_x]f\cdot f^*\,. \tag{64}$$

Summarising, by using the change of variable (56) we were able to rewrite the YON system as

$$(\mathrm{i}D_t + 2\mathrm{i}qD_x + D_x^2)g \cdot f = 0\,, \tag{65a}$$

$$\mathrm{i}D_t f \cdot f^* = [D_x^2 - 2\mathrm{i}(\delta - \ell)D_x]f \cdot f^*\,, \tag{65b}$$

$$\mathrm{i}D_t f \cdot f^* = 2\alpha\rho^2(|f|^2 - |g|^2)\,. \tag{65c}$$

To derive dark soliton solutions, we start with the following bilinear equations in the KP-Toda hierarchy

$$(D_{x_2} - 2aD_{x_1} - D_{x_1}^2)\tau_{n,h+1} \cdot \tau_{n,h} = 0\,, \tag{66a}$$

$$(D_{x_2} - 2bD_{x_1} + D_{x_1}^2)\tau_{n,h} \cdot \tau_{n+1,h} = 0\,, \tag{66b}$$

$$[(a-b)D_{x_{-1}} + 1]\tau_{n,h} \cdot \tau_{n+1,h} = \tau_{n,h+1}\tau_{n+1,h-1}\,, \tag{66c}$$

with n and h arbitrary integers, which admit a Gram-type solution [50,51].

$$\tau_{n,h} = |m_{ij}^{n,h}|_{1\le i,j\le N}\,, \tag{67a}$$

with

$$m_{ij}^{n,h} = \delta_{ij} + \frac{\mathrm{i}(k_i - b)}{k_i + \bar{k}_j}\left(-\frac{k_i - b}{\bar{k}_j + b}\right)^n \left(-\frac{k_i - a}{\bar{k}_j + a}\right)^h e^{\xi_i + \bar{\xi}_j}\,, \tag{67b}$$

where

$$\xi_i = \frac{1}{k_i - a}x_{-1} + k_i x_1 + k_i^2 x_2 + \xi_{i,0}\,, \quad \bar{\xi}_i = \frac{1}{\bar{k}_i + a}x_{-1} + \bar{k}_i x_1 - \bar{k}_i^2 x_2 + \bar{\xi}_{i,0}\,, \tag{68}$$

and a, b, k_i $\bar{k}_i$, $\xi_{i,0}$ and $\bar{\xi}_{i,0}$ are arbitrary complex parameters. By imposing the constraint condition

$$\frac{1}{k_i - a} + \frac{1}{\bar{k}_i + a} = \frac{1}{2\alpha(a-b)\rho^2}(k_i^2 - \bar{k}_i^2)\,, \tag{69}$$

which we can rewrite as

$$\frac{2\alpha(a-b)\rho^2}{(k_i - a)(\bar{k}_i + a)} = k_i - \bar{k}_i\,, \tag{70}$$

to the N-soliton solutions, then the τ-functions satisfy

$$\left((a-b)D_{x_{-1}} - \frac{1}{2\alpha\rho^2}D_{x_2}\right)\tau_{n,h} = C_1\tau_{n,h}\,, \tag{71}$$

where C_1 is an arbitrary constant. By introducing (71) into (66c) and setting $C_1 = 0$, we get

$$(D_{x_2} + 2\alpha\rho^2)\tau_{n,h} \cdot \tau_{n+1,h} = 2\alpha\rho^2\tau_{n,h+1}\tau_{n+1,h-1}\,. \tag{72}$$

After adding this constraint, we can set $n = -1$ and $h = 0$ in the three KP bilinear equations to get

$$(D_{x_2} - 2aD_{x_1} - D_{x_1}^2)\tau_{-1,1} \cdot \tau_{-1,0} = 0\,, \tag{73a}$$

$$(D_{x_2} - 2bD_{x_1} + D_{x_1}^2)\tau_{-1,0} \cdot \tau_{0,0} = 0\,, \tag{73b}$$

$$(D_{x_2} + 2\alpha\rho^2)\tau_{-1,0} \cdot \tau_{0,0} = 2\alpha\rho^2\tau_{-1,1}\tau_{0,-1}\,. \tag{73c}$$

By taking $a = \mathrm{i}q$ and $b = \mathrm{i}(\delta - \ell)$ being purely imaginary and $x_2 = \mathrm{i}t$, as well as choosing $\bar{k}_i = k_i^*$ and $\bar{\xi}_{i,0} = \xi_{i,0}^*$, it can be readily checked that $\tau_{n,k} = \tau_{-n-1,-k}^*$ and $\bar{\xi}_i = \xi_i^*$. Hence, by introducing

$$f = \tau_{-1,0}\,, \qquad g = \tau_{-1,1}\,, \tag{74}$$

it follows that

$$f^* = \tau_{0,0}\,, \qquad g^* = \tau_{0,-1}\,. \tag{75}$$

With that, the bilinear equations above become

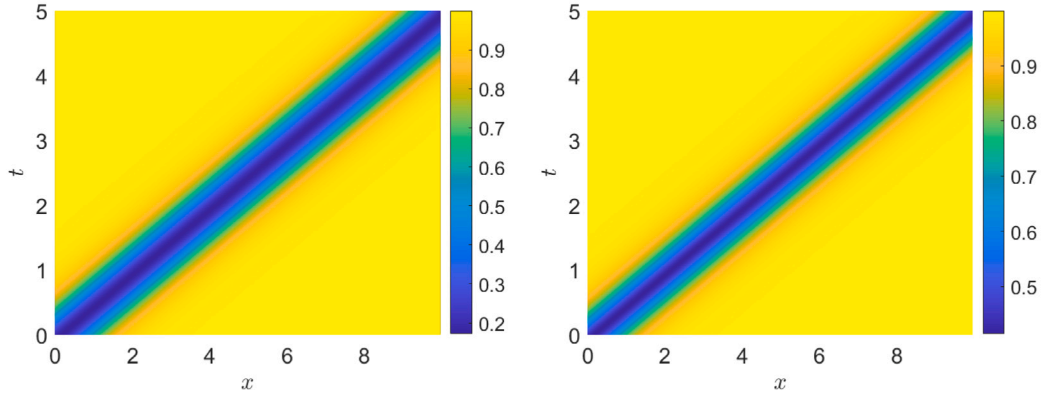
$$(\mathrm{i}D_t + 2\mathrm{i}qD_x + D_x^2)g \cdot f = 0\,, \tag{76a}$$

$$\mathrm{i}D_t f \cdot f^* = (D_x^2 - 2\mathrm{i}(\delta - \ell)D_x)f \cdot f^*\,, \tag{76b}$$

$$\mathrm{i}D_t f \cdot f^* = 2\alpha\rho^2(|f|^2 - |g|^2)\,, \tag{76c}$$

which are exactly the Eqs. (57) that we obtained for the YON system.

That means that we can adapt the solutions (67) as solutions of the YON system through the following theorem.



(a) Long wave L. (b) Short wave $|S|$.

Fig. 3. 1-dark-soliton solution with $\alpha=-1$, $\delta=3$, $k_1=1+i$, $\rho=1$, $\ell=1$, $q=1$.

Theorem 2. *The YON system (4) possesses the N-dark-soliton solution (56), where the determinants f and g are given by*

$$f=\left|\delta_{ij}-\frac{\mathrm{i}k_j^*-\delta+\ell}{k_i+k_j^*}e^{\xi_i+\xi_j^*}\right|_{N\times N}, \tag{77a}$$

$$g=\left|\delta_{ij}+\frac{\mathrm{i}k_j^*-\delta+\ell}{k_i+k_j^*}\,\frac{k_i-\mathrm{i}q}{k_i^*+\mathrm{i}q}e^{\xi_i+\xi_j^*}\right|_{N\times N}, \tag{77b}$$

$$\xi_i=k_ix+\mathrm{i}k_i^2t+\xi_{i,0}, \tag{77c}$$

where k_j are the wave numbers of the solitons, $\xi_{i,0}$ are the initial phases, and the parameters are subject to the constraint

$$\frac{2\mathrm{i}\alpha(q-\delta+\ell)\rho^2}{|k_j-\mathrm{i}q|^2}=k_j-k_j^*. \tag{78}$$

Furthermore, if we set $k_j=k_j^{(r)}+\mathrm{i}k_j^{(i)}$, the constraint can be rewritten as

$$k_j^{(r)}=\pm\left(\frac{\alpha(q-\delta+\ell)\rho^2}{k_j^{(i)}}-\left(k_j^{(i)}-q\right)^2\right)^{\frac{1}{2}}. \tag{79}$$

The 1-dark-soliton solution is then given by

$$f=1-\frac{\mathrm{i}k_1^*-\delta+\ell}{k_1+k_1^*}e^{\xi_1+\xi_1^*}, \tag{80a}$$

$$g=1+\frac{\mathrm{i}k_1^*-\delta+\ell}{k_1+k_1^*}\,\frac{k_1-\mathrm{i}q}{k_1^*+\mathrm{i}q}e^{\xi_1+\xi_1^*}, \tag{80b}$$

$$\xi_1=k_1x+\mathrm{i}k_1^2t+\xi_{1,0}, \tag{80c}$$

where k_1 is the (complex) wave number of the soliton and $\xi_{1,0}$ is an initial phase, and where the parameters must satisfy the constraint condition

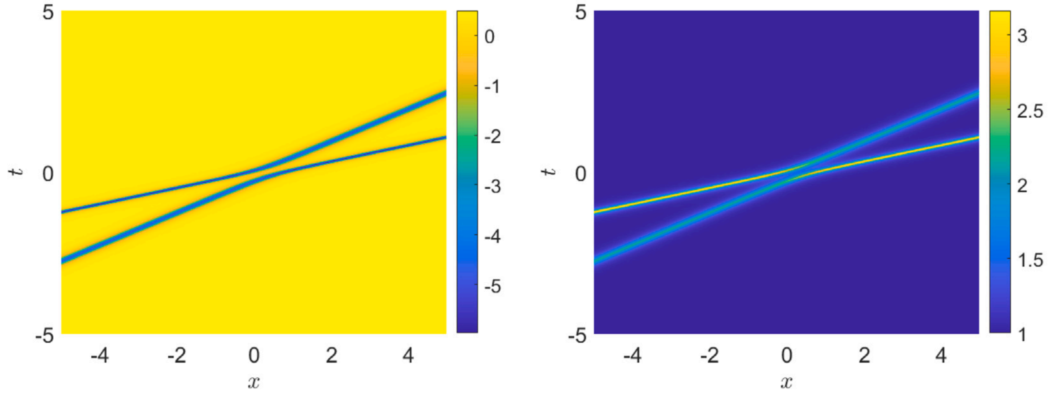
$$\frac{2\mathrm{i}\alpha(q-\delta+\ell)\rho^2}{|k_1-\mathrm{i}q|^2}=k_1-k_1^*. \tag{81}$$

By taking $k_1=k_1^{(r)}+\mathrm{i}k_1^{(i)}$, we can rewrite the constraint condition as

$$k_1^{(r)}=\pm\left(\frac{\alpha(q-\delta+\ell)\rho^2}{k_1^{(i)}}-\left(k_1^{(i)}-q\right)^2\right)^{\frac{1}{2}}. \tag{82}$$

As with the bright case, the dark solitons move with a velocity $V=2k_j^{(i)}$, that is, they satisfy $L(x,t)=L(x+2k_j^{(i)}t,0)$ and $|S(x,t)|=|S(x+2k_j^{(i)}t,0)|$ (see Figs. 3 and 4).

The phase shift for the 2-dark-soliton solution can be written explicitly by denoting $k_1=k_1^{(r)}+\mathrm{i}k_1^{(i)}$ and $k_2=k_2^{(r)}+\mathrm{i}k_2^{(i)}$ and proceeding as in the bright case, that is, moving with the velocity of one of the solitons to make it stationary, so that for $t\to\pm\infty$



(a) Long wave L. (b) Short wave $|S|$.

Fig. 4. 2-dark-soliton solution with $\alpha = 2$, $\delta = -3$, $k_1 = \sqrt{2} + 2\mathrm{i}$, $k_2 = \sqrt{6} + \mathrm{i}$, $\rho = 1$, $\ell = 1$, $q = 1$.

the solution collapses into a one-soliton solution. For the first soliton, the phase shift reads

$$\begin{aligned}\varphi_{12} &= \frac{1}{2k_1^{(r)}} \log\left(\frac{\left(k_1^{(i)} - k_2^{(i)}\right)^2 + \left(k_1^{(r)} + k_2^{(r)}\right)^2}{\left(k_1^{(i)} - k_2^{(i)}\right)^2 + \left(k_1^{(r)} - k_2^{(r)}\right)^2}\right) \\ &= \frac{1}{k_1 + k_1^*} \log\left(\frac{(k_1^* + k_2)(k_1 + k_2^*)}{(k_1 - k_2)(k_1^* - k_2^*)}\right).\end{aligned} \tag{83}$$

The formula for the phase shift φ_{21} of the second soliton can be written by simply exchanging indices 1 and 2 in (83). Note the formulae for the phase shifts do not depend either on the parameters α and δ in (4) or on the background constants ρ, q and ℓ.

4. Breathers and rogue waves

To derive the breather solutions, the target bilinear Eqs. (57) and variable transformation (56) remain the same, but we will start with slightly different τ-functions in the extended KP hierarchy [51,52]

$$\tau_{n,h} = |m_{ij}^{n,h}|_{1\le i,j\le N}, \tag{84a}$$

where

$$m_{ij}^{n,h} = \sum_{p,r=1}^{2} \frac{c_{ip}\bar{c}_{jr}\mathrm{i}(k_{ip} - b)}{k_{ip} + \bar{k}_{jr}} \left(-\frac{k_{ip} - b}{\bar{k}_{jr} + b}\right)^n \left(-\frac{k_{ip} - a}{\bar{k}_{jr} + a}\right)^h e^{\xi_{ip} + \bar{\xi}_{jr}}, \tag{84b}$$

with

$$\xi_{ip} = \frac{1}{k_{ip} - a} x_{-1} + k_{ip} x_1 + k_{ip}^2 x_2 + \xi_{ip,0}, \tag{85a}$$

$$\bar{\xi}_{jr} = \frac{1}{\bar{k}_{jr} + a} x_{-1} + \bar{k}_{jr} x_1 - \bar{k}_{jr}^2 x_2 + \bar{\xi}_{jr,0}, \tag{85b}$$

and k_{ip}, k_{jr}, $\bar{k}_{ip}$, $\bar{k}_{jr}$, c_{ip}, $\bar{c}_{jr}$, $\xi_{ip,0}$, $\bar{\xi}_{jr,0}$ arbitrary complex parameters. It can be shown above tau functions satisfy the same set of bilinear Eqs. (66).

Moreover, if we impose the constraint conditions

$$\frac{1}{k_{i1} - a} - \frac{1}{k_{i2} - a} = -\frac{1}{2\alpha(a - b)\rho^2}(k_{i1}^2 - k_{i2}^2), \tag{86}$$

$$\frac{1}{\bar{k}_{i1} + a} - \frac{1}{\bar{k}_{i2} + a} = -\frac{1}{2\alpha(a - b)\rho^2}(k_{i1}^2 - k_{i2}^2), \tag{87}$$

which we can rewrite as

$$\frac{2\alpha(a - b)\rho^2}{(k_{i1} - a)(k_{i2} - a)} = k_{i1} + k_{i2}, \tag{88}$$

$$-\frac{2\alpha(a - b)\rho^2}{(\bar{k}_{i1} + a)(\bar{k}_{i2} + a)} = k_{i1} + k_{i2}, \tag{89}$$

then the τ-functions satisfy

$$(a-b)\partial_{x_{-1}}\tau_{n,h}=\frac{1}{2\alpha\rho^2}\partial_{x_2}\tau_{n,h}\,, \tag{90}$$

so that (66c) becomes

$$(D_{x_2}+2\alpha\rho^2)\tau_{n,h}\cdot\tau_{n+1,h}=2\alpha\rho^2\tau_{n,h+1}\tau_{n+1,h-1}\,. \tag{91}$$

In order to satisfy the complex conjugate condition, we choose x_1, x_{-1} to be real, x_2, a and b purely imaginary, and we let $\bar{c}_{jr}=c^*_{jr}$, $\bar{k}_{jr}=k^*_{jr}$, $\bar{\xi}_{jr,0}=\xi^*_{jr,0}$, for $j=1,\dots,N$, $r=1,2$. Then it can be easily shown that

$$\left(m_{ij}^{n,h}\right)^*=m_{ji}^{-n-1,-h},\quad \tau^*_{n,h}=\tau_{-n-1,-h}. \tag{92}$$

By using that relation, if we define

$$f=\tau_{-1,0}\,,\qquad g=\tau_{-1,1}\,, \tag{93}$$

then we have that

$$f^*=\tau_{0,0}\,,\qquad g^*=\tau_{0,-1}\,. \tag{94}$$

which coincides with the transformations (74)–(75). Hence, we can proceed as in the dark soliton case by setting $x_2=\mathrm{i}t$, $a=\mathrm{i}q$ and $b=\mathrm{i}(\delta-\ell)$ to arrive at the same set of bilinear Eqs. (57). With that, we complete the reduction process and can present the N-breather solution through the following theorem.

Theorem 3. *The YON system (1) admits the N-breather solution (56), where the determinants f and g are given by*

$$f=\left|\sum_{p,r=1}^{2}\frac{a_{ip}a^*_{jr}[-\mathrm{i}k^*_{jr}+\delta-\ell]}{k_{ip}+k^*_{jr}}e^{\xi_{ip}+\xi^*_{jr}}\right|_{N\times N}, \tag{95a}$$

$$g=\left|\sum_{p,r=1}^{2}\frac{a_{ip}a^*_{jr}[-\mathrm{i}k^*_{jr}+\delta-\ell]}{k_{ip}+k^*_{jr}}\,\frac{\mathrm{i}q-k_{ip}}{\mathrm{i}q+k_{jr}}e^{\xi_{ip}+\xi^*_{jr}}\right|_{N\times N}, \tag{95b}$$

$$\xi_{ip}=k_{ip}x+\mathrm{i}k_{ip}^2t+\xi_{ip,0}\,. \tag{95c}$$

The constraint condition becomes

$$\frac{2\alpha(q-\delta+\ell)\rho^2}{(k_{i1}-\mathrm{i}q)(k_{i2}-\mathrm{i}q)}=\mathrm{i}(k_{i1}+k_{i2})\,,\qquad i=1,\dots,N,. \tag{96}$$

If we further define k_{ip} as

$$k_{ip}=k_{ip}^{(r)}+\mathrm{i}k_{ip}^{(i)},\qquad i=1,\dots,N,\quad p=1,2, \tag{97}$$

then it can be shown that every breather is localised along the direction of the line

$$\left(k_{i1}^{(r)}-k_{i2}^{(r)}\right)x+2\left(k_{i1}^{(r)}k_{i1}^{(i)}+k_{i2}^{(r)}k_{i2}^{(i)}\right)t=0\,, \tag{98a}$$

and periodic along the direction of the line

$$\left(k_{i1}^{(i)}-k_{i2}^{(i)}\right)x+\left(k_{i1}^{(r)2}-k_{i2}^{(r)2}-k_{i1}^{(i)2}+k_{i2}^{(i)2}\right)t=0\,. \tag{98b}$$

We close this section by showing an example of a 1-breather solution for the simple case where we set all the $c_{1p}=1,\xi_{1p,0}=0$. In that case, the τ-functions are given by

$$f=\frac{e^{\zeta_1+\zeta_1^*}}{k_{11}+k_{11}^*}\frac{k_{11}^*+\mathrm{i}(\delta-\ell)}{k_{12}^*+\mathrm{i}(\delta-\ell)}+\frac{e^{\zeta_1}}{k_{11}+k_{12}^*}\frac{k_{11}^*+\mathrm{i}(\delta-\ell)}{k_{12}^*+\mathrm{i}(\delta-\ell)}+\frac{e^{\zeta_1^*}}{k_{12}+k_{11}^*}+\frac{1}{k_{12}+k_{12}^*}\,, \tag{99a}$$

$$g=\frac{\Theta_1(\Theta_1^*)^{-1}e^{\zeta_i+\zeta_j^*}}{k_{11}+k_{11}^*}\frac{k_{11}^*+\mathrm{i}(\delta-\ell)}{k_{12}^*+\mathrm{i}(\delta-\ell)}+\frac{\Theta_1e^{\zeta_1}}{k_{11}+k_{12}^*}\frac{k_{11}^*+\mathrm{i}(\delta-\ell)}{k_{12}^*+\mathrm{i}(\delta-\ell)}+\frac{(\Theta_1^*)^{-1}e^{\zeta_1^*}}{k_{12}+k_{11}^*}+\frac{1}{k_{12}+k_{12}^*}\,, \tag{99b}$$

with

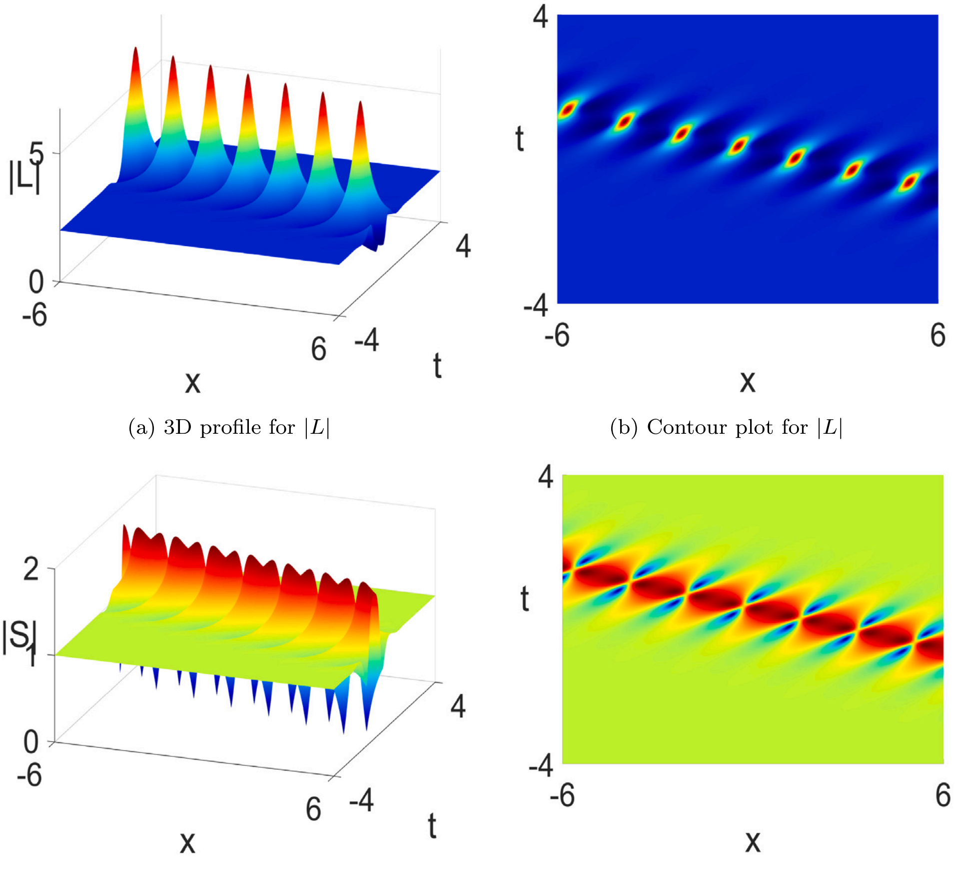
$$\Theta_1=\frac{k_{11}-\mathrm{i}q}{k_{12}-\mathrm{i}q}, \tag{99c}$$

and

$$\zeta_1=(k_{11}-k_{12})x+(k_{11}^2-k_{12}^2)\mathrm{i}t\,, \tag{99d}$$

in which these parameters need to satisfy the constraint condition (96) with $i=1$. One typical example of breather solution is illustrated in Fig. 5.

The τ-functions of the 2-breather solution can be obtained by taking $N=2$ in Theorem 3, but these formulae are too tedious and we will omit them for the purpose of this paper. An example is illustrated in Fig. 6 with selected parameters.



(a) 3D profile for $|L|$ (b) Contour plot for $|L|$

(c) 3D profile for $|S|$ (d) Contour plot for $|S|$

Fig. 5. 1-breather solution for the parameters $\alpha = 1, q = 3, \ell = 2, \delta = 1, \rho = 1, k_{11} = 1 - 0.5\mathrm{i}, k_{12} = 0.45647 + 2.0878\mathrm{i}$.

5. Rogue wave solutions

In this section we will construct rogue wave solutions making use of the τ-functions machinery. In line with the scientific literature on exact solutions of integrable systems (and potentially with a little abuse of terminology), here we understand rogue waves as rationally-decaying, localised solutions, whose maximum does not propagate in space and time, and which appear as the limiting case for the breather solutions. In order to do so, we will first introduce a new object that we denote $m^{(n,h)}$ as

$$m^{(n,h)} = \frac{\mathrm{i}(k-b)}{k+\bar{k}} \left(-\frac{k-b}{\bar{k}+b}\right)^n \left(-\frac{k-a}{\bar{k}+a}\right)^h e^{\xi+\bar{\xi}}, \tag{100}$$

and

$$\xi = kx_1 + k^2 x_2 + \frac{1}{k-a} x_{-1} + \xi_0, \tag{101a}$$

$$\bar{\xi} = \bar{k}x_1 - \bar{k}^2 x_2 + \frac{1}{\bar{k}+a} x_{-1} + \bar{\xi}_0, \tag{101b}$$

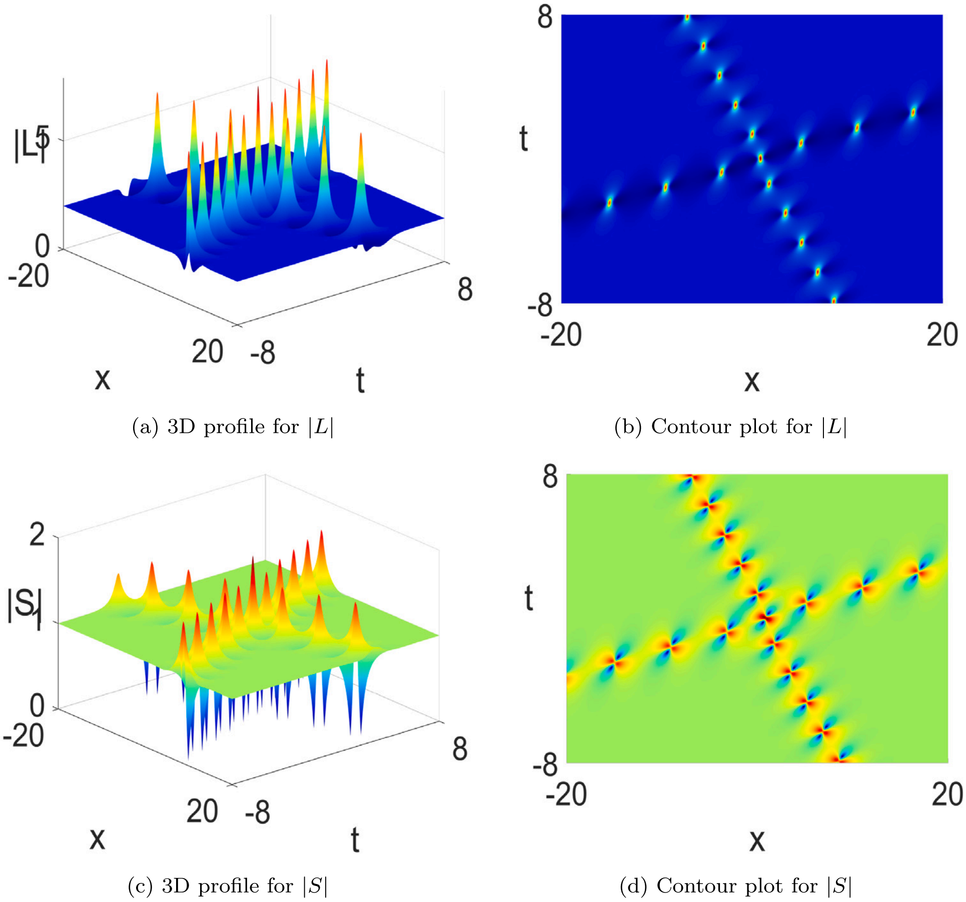
where $k, \bar{k}, \xi_0, \bar{\xi}_0, a$ and b are arbitrary complex parameters. With that, we can define the matrix elements of our Gram-type solution as

$$m_{ij}^{(n,h)} = \mathcal{A}_i \mathcal{B}_j m^{(n,h)}, \tag{102}$$

where $\mathcal{A}_i$ and $\mathcal{B}_j$ are differential operators with respect to k and $\bar{k}$ respectively,

$$\mathcal{A}_i = \frac{1}{i!}\left[f_1(k)\partial_k\right]^i, \quad \mathcal{B}_j = \frac{1}{j!}\left[f_2(\bar{k})\partial_{\bar{k}}\right]^j, \tag{103}$$

and $f_1(k)$, $f_2(\bar{k})$ are arbitrary functions that will be determined in Section 5.1 via the dimensional reduction process described in [53,54]. Since the operators $\mathcal{A}_i$ and $\mathcal{B}_j$ commute with the bilinear operators D_{x_1}, $D_{x_{-1}}$ and D_{x_2}, it can be easily seen that for any



(a) 3D profile for $|L|$ (b) Contour plot for $|L|$

(c) 3D profile for $|S|$ (d) Contour plot for $|S|$

Fig. 6. 2-breather solution for the parameters $\alpha = 1, q = 3, \ell = 2, \delta = 1, \rho = 1, k_{11} = 0.5 + \mathrm{i}, k_{21} = 1.4386 + 0.011416\mathrm{i}, k_{12} = 1 + \mathrm{i}, k_{22} = 0.6838 - 0.13355\mathrm{i}$.

permutation of the indices $(i_1, i_2, \dots, i_\nu, \dots, i_N; j_1, j_2, \dots, j_\kappa, \dots, j_N)$, the determinant

$$\tau_{n,h} = \det\left(m^{(n,h)}_{i_\nu, j_\kappa}\right)_{1\le\nu,\kappa\le N} \tag{104}$$

satisfies the set of bilinear Eqs. (66).

5.1. Rogue waves in differential formulation

Following the generalised dimensional reduction technique developed in [54], we introduce the linear differential operator

$$\mathcal{L} = (a-b)D_{x_{-1}} - \frac{1}{2\alpha\rho^2}D_{x_2}. \tag{105}$$

In the following section, we will show that the dimensional reduction condition

$$\mathcal{L}\tau_{n,h} = C\tau_{n,h}, \tag{106}$$

for some constant C, is satisfied.

It can be checked that

$$\mathcal{L}m^{(n,h)}_{ij} = \mathcal{A}_i\mathcal{B}_j\mathcal{L}m^{(n,h)} = \mathcal{A}_i\mathcal{B}_j\left[\mathcal{Q}_1(k) + \mathcal{Q}_2(\bar{k})\right]m^{(n,h)}, \tag{107}$$

where

$$\mathcal{Q}_1(k) = \frac{(a-b)}{k-a} + \frac{1}{2\alpha\rho^2}k^2, \qquad \mathcal{Q}_2(\bar{k}) = \frac{(a-b)}{\bar{k}+a} - \frac{1}{2\alpha\rho^2}\bar{k}^2. \tag{108}$$

Using the Leibniz rule, the operator relations [54]

$$\mathcal{A}_i \mathcal{Q}_1(k) = \sum_{p=0}^{i} \frac{1}{p!} \left[\left(f_1 \partial_k \right)^p \mathcal{Q}_1(k) \right] \mathcal{A}_{i-p}, \tag{109a}$$

$$\mathcal{B}_j \mathcal{Q}_2(\bar{k}) = \sum_{r=0}^{j} \frac{1}{r!} \left[\left(f_2 \partial_{\bar{k}} \right)^r \mathcal{Q}_2(\bar{k}) \right] \mathcal{B}_{j-r}, \tag{109b}$$

where both of them are to be understood as operators, hold, which entails

$$\mathcal{L} m_{ij}^{(n,h)} = \sum_{p=0}^{i} \frac{1}{p!} \left[\left(f_1 \partial_k \right)^p \mathcal{Q}_1(k) \right] m_{i-p,j}^{(n,h)} + \sum_{r=0}^{j} \frac{1}{r!} \left[\left(f_2 \partial_{\bar{k}} \right)^r \mathcal{Q}_2(\bar{k}) \right] m_{i,j-r}^{(n,h)}. \tag{110}$$

In order for the reduction process to be applied, we will assume that the algebraic equations

$$\mathcal{Q}_1'(k) = 0, \qquad \mathcal{Q}_2'(\bar{k}) = 0, \tag{111}$$

which are equivalent to the cubic equations

$$k(k-a)^2 + (a-b)\alpha\rho^2 = 0, \qquad \bar{k}(\bar{k}+a)^2 - (a-b)\alpha\rho^2 = 0, \tag{112}$$

have simple roots k_0 and $\bar{k}_0$, respectively.

Following the steps for the simple root case in [54], we will need to solve the differential equations

$$\left(f_1 \partial_k \right)^2 \mathcal{Q}_1(k) = \mathcal{Q}_1(k), \qquad \left(f_2 \partial_{\bar{k}} \right)^2 \mathcal{Q}_2(\bar{k}) = \mathcal{Q}_2(\bar{k}). \tag{113}$$

To this end, we assume

$$f_1(k) = \frac{\mathcal{W}_1(k)}{\mathcal{W}_1'(k)}, \qquad f_2(\bar{k}) = \frac{\mathcal{W}_2(\bar{k})}{\mathcal{W}_2'(\bar{k})}, \tag{114}$$

where the prime denotes derivation, for some functions $\mathcal{W}_1(k)$ and $\mathcal{W}_2(\bar{k})$, subject to the constraints $\mathcal{W}_1(k_0) = 1$ and $\mathcal{W}_2(\bar{k}_0) = 1$, which after some calculations implies that

$$\mathcal{W}_1(k) = \frac{\mathcal{Q}_1(k) \pm \sqrt{\mathcal{Q}_1^2(k) - \mathcal{Q}_1^2(k_0)}}{\mathcal{Q}_1(k_0)}, \tag{115a}$$

$$\mathcal{W}_2(\bar{k}) = \frac{\mathcal{Q}_2(\bar{k}) \pm \sqrt{\mathcal{Q}_2^2(\bar{k}) - \mathcal{Q}_2^2(\bar{k}_0)}}{\mathcal{Q}_2(\bar{k}_0)}, \tag{115b}$$

entailing

$$f_1(k) = \pm \frac{\sqrt{\mathcal{Q}_1^2(k) - \mathcal{Q}_1^2(k_0)}}{\mathcal{Q}_1'(k)}, \qquad f_2(\bar{k}) = \pm \frac{\sqrt{\mathcal{Q}_2^2(\bar{k}) - \mathcal{Q}_2^2(\bar{k}_0)}}{\mathcal{Q}_2'(\bar{k})}. \tag{116}$$

Since both signs yield equivalent rogue wave solutions, we choose the positive sign in the following derivation without loss of generality. From conditions (111) and (113), we obtain that

$$\mathcal{L} m_{ij}^{(n,h)} \Big|_{k=k_0, \bar{k}=\bar{k}_0} = \mathcal{Q}_1(k_0) \sum_{\substack{p=0,\\ p:even}}^{i} \frac{1}{p!} m_{i-p,j}^{(n,h)} \Big|_{k=k_0, \bar{k}=\bar{k}_0} + \mathcal{Q}_2(\bar{k}_0) \sum_{\substack{r=0,\\ r:even}}^{j} \frac{1}{r!} m_{i,j-r}^{(n,h)} \Big|_{k=k_0, \bar{k}=\bar{k}_0}, \tag{117}$$

as only the even elements in the sums are non-zero.

Since that implies that the matrix elements $m_{i,j}^{(n,h)}$ with even i and/or j are identical to those with $i+1$ or $j+1$, we will restrict the general determinant (104) to only odd indices, without loss of generality,

$$\tau_{n,h} = \det \left(m_{2i-1,2j-1}^{(n,h)} \Big|_{k=k_0, \bar{k}=\bar{k}_0} \right)_{1 \le i,j \le N}. \tag{118}$$

By using the contiguity relation (117) as in [55], we obtain

$$\mathcal{L} \tau_{n,h} = \left[\mathcal{Q}_1(k_0) + \mathcal{Q}_2(\bar{k}_0) \right] N \tau_{n,h}, \tag{119}$$

which implies that the τ-function (118) satisfies the dimensional reduction condition (106) with $C = [\mathcal{Q}_1(k_0) + \mathcal{Q}_2(\bar{k}_0)]N$. The complex conjugate reduction: $\tau_{n,k} = \tau_{-n-1,-k}^*$ can be realised by the same conditions: $a = \mathrm{i}q$ and $b = \mathrm{i}(\delta - \ell)$, $x_2 = \mathrm{i}t$ and $\bar{k} = k^*$.

We will further introduce expansions of $\xi_0(k)$ in terms of $\ln \mathcal{W}_1(k)$ and of $\bar{\xi}_0(\bar{k})$ in terms of $\ln \mathcal{W}_2(\bar{k})$, following the idea in [56],

$$\xi_0(k) = \sum_{r=1}^{\infty} \hat{a}_r \ln^r \mathcal{W}_1(k), \qquad \bar{\xi}_0(\bar{k}) = \sum_{r=1}^{\infty} \hat{a}_r^* \ln^r \mathcal{W}_2(\bar{k}), \tag{120}$$

where $\hat{a}_r$ are arbitrary complex constants.

Putting all the above together, we can construct general rogue wave solutions for the system through the following theorem.

Theorem 4. *The YON system (4) possesses the following rogue wave solutions*

$$S = \rho \frac{g}{f} e^{\mathrm{i}[qx-(q^2+2\alpha\rho^2+2\delta\ell-\ell^2)t]}, \qquad L = \ell + \mathrm{i}\left(\log \frac{f^*}{f}\right)_x, \tag{121}$$

where

$$f = \tau_{-1,0}, \qquad f^* = \tau_{0,0}, \qquad g = \tau_{-1,1}, \qquad g^* = \tau_{0,-1}, \tag{122}$$

and the elements in the determinant $\tau_{n,h} = \det\left(\tilde{m}^{(n,h)}_{2i-1,2j-1}\right)_{1\le i,j\le N}$ *are defined by*

$$\tilde{m}^{(n,h)}_{i,j} = \frac{\left[f_1(k)\partial_k\right]^i}{i!}\frac{\left[f_2(\bar{k})\partial_{\bar{k}}\right]^j}{j!} m^{(n,h)}\Big|_{k=k_0,\bar{k}=k_0^*}, \tag{123a}$$

$$m^{(n,h)} = \frac{\mathrm{i}(k-b)}{k+\bar{k}}\left(-\frac{k-b}{\bar{k}+b}\right)^n\left(-\frac{k-a}{\bar{k}+a}\right)^h e^{\Omega}, \tag{123b}$$

$$\Omega = (k+\bar{k})x + \left(k^2-\bar{k}^2\right)\mathrm{i}t + \sum_{r=1}^{\infty}\hat{a}_r \ln^r \mathcal{W}_1(k) + \sum_{r=1}^{\infty}\hat{a}_r^* \ln^r \mathcal{W}_2(\bar{k}), \tag{123c}$$

with k_0 *as defined in (112),* $b = \mathrm{i}(\delta-\ell)$ *and* $a = \mathrm{i}q$*. Furthermore,* q*,* ρ *and* ℓ *are arbitrary real parameters and* $\hat{a}_r$ *for* $r = 1, 2, \ldots$ *are arbitrary complex parameters.*

Note that even though, in principle, we choose infinitely many arbitrary parameters $\hat{a}_r$, only the first $2N-1$ of them enter the computation through the derivatives, since $\mathcal{W}_1(k_0) = \mathcal{W}_2(k_0^*) = 1$. The rationale behind the form of the elements inside the infinite sums is to cancel out $f_1(k)$ and $f_2(k)$ terms once the derivative is performed.

An important point is that the definition of k_0 via (112) is equivalent to the condition for the existence of rogue waves predicted in [1] via the analysis of the stability spectrum, hence supporting the general understanding that base-band instability of plane wave solutions plays a pivotal role in the onset of rogue waves.

5.2. Rogue waves by elementary Schur polynomials

In this subsection, the rogue wave solutions above will be rewritten more explicitly using elementary Schur polynomials. Following the technique in [54], the extended generator $\mathcal{G}$ of the differential operators $\left[f_1\partial_k\right]^i\left[f_2\partial_{\bar{k}}\right]^j$ is

$$\mathcal{G} = \sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\frac{\mu^i}{i!}\frac{\lambda^j}{j!}\left[f_1\partial_k\right]^i\left[f_2\partial_{\bar{k}}\right]^j, \tag{124}$$

which, thanks to the fact that

$$f_1(k) = \left(\frac{\partial \ln \mathcal{W}_1}{\partial k}\right)^{-1}, \qquad f_2(k) = \left(\frac{\partial \ln \mathcal{W}_2}{\partial k}\right)^{-1}, \tag{125}$$

due to (114), can be expressed in terms of $\mathcal{W}_1$ and $\mathcal{W}_2$ as

$$\mathcal{G} = \sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\frac{\mu^i}{i!}\frac{\lambda^j}{j!}\left[\partial_{\ln\mathcal{W}_1}\right]^i\left[\partial_{\ln\mathcal{W}_2}\right]^j = \exp(\mu\partial_{\ln\mathcal{W}_1} + \lambda\partial_{\ln\mathcal{W}_2}). \tag{126}$$

As stated in [55], it follows that for any function $F(\mathcal{W}_1, \mathcal{W}_2)$ we have that

$$\mathcal{G}F(\mathcal{W}_1,\mathcal{W}_2) = F(e^{\mu}\mathcal{W}_1, e^{\lambda}\mathcal{W}_2). \tag{127}$$

Applying the relation (127) to $m^{(n,h)}$ at $k = k_0, \bar{k} = \bar{k}_0$, one has

$$\begin{aligned}\mathcal{G}m^{(n,h)}\Big|_{k=k_0,\bar{k}=\bar{k}_0} &= \frac{\mathrm{i}(k(\mu)-b)(-1)^{n+h}}{k(\mu)+\bar{k}(\lambda)}\left[\frac{k(\mu)-a}{\bar{k}(\lambda)+a}\right]^h\left[\frac{k(\mu)-b}{\bar{k}(\lambda)+b}\right]^n \times \\ &\exp\left\{\left[k(\mu)+\bar{k}(\lambda)\right]x + \left[k^2(\mu)-\bar{k}^2(\lambda)\right]\mathrm{i}t + \sum_{r=1}^{\infty}\left[\hat{a}_r\mu^r + \hat{a}_r^*\lambda^r\right]\right\},\end{aligned} \tag{128}$$

where we have defined

$$k(\mu) \equiv k(\mathcal{W}_1)|_{\mathcal{W}_1=e^{\mu}}, \qquad \bar{k}(\lambda) \equiv \bar{k}(\mathcal{W}_2)|_{\mathcal{W}_2=e^{\lambda}}, \tag{129}$$

by inverting the functional dependence between k and $\mathcal{W}_1$ and between $\bar{k}$ and $\mathcal{W}_2$. Since

$$m^{(n,h)}\Big|_{k=k_0,\bar{k}=\bar{k}_0} = \frac{\mathrm{i}(k_0-b)(-1)^{n+h}}{k_0+\bar{k}_0}\left(\frac{k_0-a}{\bar{k}_0+a}\right)^h\left(\frac{k_0-b}{\bar{k}_0+b}\right)^n \exp\left[\left(k_0+\bar{k}_0\right)x + \left(k_0^2-\bar{k}_0^2\right)\mathrm{i}t\right], \tag{130}$$

we then obtain that

$$\frac{\mathcal{G}m^{(n,h)}}{m^{(n,h)}}\bigg|_{k=k_0,\bar{k}=\bar{k}_0} = \frac{k_0+\bar{k}_0}{k(\mu)+\bar{k}(\lambda)}\left[\frac{k(\mu)-a}{k_0-a}\right]^h\left[\frac{\bar{k}(\lambda)+a}{\bar{k}_0+a}\right]^{-h}\left[\frac{k(\mu)-b}{k_0-b}\right]^{n+1}\left[\frac{\bar{k}(\lambda)+b}{\bar{k}_0+b}\right]^{-n}\times$$
$$\exp\left\{\left[k(\mu)-k_0+\bar{k}(\lambda)-\bar{k}_0\right]x+\left[k^2(\mu)-k_0^2-\bar{k}^2(\lambda)+\bar{k}_0^2\right]\mathrm{i}t+\sum_{r=1}^{\infty}\left[\hat{a}_r\mu^r+\hat{a}_r^*\lambda^r\right]\right\}. \tag{131}$$

The next step will be to expand the r.h.s of the equation above in a power series in μ and λ. Using the techniques introduced in [54], the first term can be expressed as

$$\frac{k_0+\bar{k}_0}{k(\mu)+\bar{k}(\lambda)} = \sum_{\gamma=0}^{\infty}\left(\frac{k_1\bar{k}_1}{(k_0+\bar{k}_0)^2}\mu\lambda\right)^{\gamma}\exp\left(\sum_{r=1}^{\infty}(\gamma s_r-b_r)\mu^r+(\gamma s_r^*-b_r^*)\lambda^r\right), \tag{132}$$

where

$$k_1 = \frac{dk(\mu)}{d\mu}\bigg|_{\mu=0}, \qquad \bar{k}_1 = \frac{d\bar{k}(\lambda)}{d\lambda}\bigg|_{\lambda=0}. \tag{133}$$

The parameters s_r and b_r are defined as the expansion coefficients of μ^r and λ^r in the following series expansions:

$$\ln\left[\frac{k_0+\bar{k}_0}{k_1\mu}\,\frac{k(\mu)-k_0}{k(\mu)+\bar{k}_0}\right] = \sum_{r=1}^{\infty}s_r\mu^r, \qquad \ln\left[\frac{k(\mu)+\bar{k}_0}{k_0+\bar{k}_0}\right] = \sum_{r=1}^{\infty}b_r\mu^r, \tag{134a}$$

$$\ln\left[\frac{k_0+\bar{k}_0}{\bar{k}_1\lambda}\,\frac{\bar{k}(\lambda)-\bar{k}_0}{\bar{k}(\lambda)+k_0}\right] = \sum_{r=1}^{\infty}s_r^*\lambda^r, \qquad \ln\left[\frac{\bar{k}(\lambda)+k_0}{k_0+\bar{k}_0}\right] = \sum_{r=1}^{\infty}b_r^*\lambda^r, \tag{134b}$$

where we have chosen $\bar{k}(\lambda)=k^*(\mu)$ and, as a consequence, $\bar{k}_0=k_0^*$. Hence, $s_r^*=s_r$, $b_r^*=b_r$ and the expression (132) is simplified as

$$\frac{k_0+\bar{k}_0}{k(\mu)+\bar{k}(\lambda)} = \sum_{\gamma=0}^{\infty}\left(\frac{\mu\lambda}{4}\right)^{\gamma}\exp\left(\sum_{r=1}^{\infty}(\gamma s_r-b_r)(\mu^r+\lambda^r)\right). \tag{135}$$

On the other hand, by means of the additional choices $a=\mathrm{i}q$, $b=\mathrm{i}(\delta-l)$, we can perform the following expansions

$$k(\mu)-k_0 = \sum_{r=1}^{\infty}\varphi_r^{(1)}\mu^r, \qquad k^2(\mu)-k_0^2 = \sum_{r=1}^{\infty}\varphi_r^{(2)}\mu^r, \tag{136a}$$

$$\ln\frac{k(\mu)-a}{k_0-a} = \sum_{r=1}^{\infty}\varphi_r^{(3)}\mu^r, \qquad \ln\frac{k(\mu)-b}{k_0-b} = \sum_{r=1}^{\infty}\varphi_r^{(4)}\mu^r. \tag{136b}$$

With the help of (136), the remaining terms on the r.h.s. of (131) can be rewritten as

$$\exp\left\{\sum_{r=1}^{\infty}[\varphi_r^{(1)}x+\varphi_r^{(2)}\mathrm{i}t+h\varphi_r^{(3)}+(n+1)\varphi_r^{(4)}+\hat{a}_r]\mu^r+\sum_{r=1}^{\infty}[\varphi_r^{(1)*}x-\varphi_r^{(2)*}\mathrm{i}t-h\varphi_r^{(3)*}-n\varphi_r^{(4)*}+\hat{a}_r^*]\lambda^r\right\}. \tag{137}$$

Additionally, we will define

$$a_r = \frac{1}{2}\varphi_r^{(4)}+\hat{a}_r-b_r, \tag{138}$$

so that (131) can be rewritten as

$$\begin{aligned}\frac{\mathcal{G}m^{(n,h)}}{m^{(n,h)}}\bigg|_{k=k_0,\bar{k}=\bar{k}_0} &= \sum_{\gamma=0}^{\infty}\left(\frac{\mu\lambda}{4}\right)^{\gamma}\exp\left(\sum_{r=1}^{\infty}(x_r^+(n,h)+\gamma s_r)\mu^r+\sum_{r=1}^{\infty}(x_r^-(n,h)+\gamma s_r)\lambda^r\right)\\ &= \sum_{\gamma=0}^{\infty}\left(\frac{\mu\lambda}{4}\right)^{\gamma}\sum_{i=0}^{\infty}S_i(\boldsymbol{x}^+(n,h)+\gamma\boldsymbol{s})\mu^i\sum_{j=0}^{\infty}S_j(\boldsymbol{x}^-(n,h)+\gamma\boldsymbol{s})\lambda^j\\ &= \sum_{\gamma=0}^{\infty}\sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\frac{1}{4^{\gamma}}S_i(\boldsymbol{x}^+(n,h)+\gamma\boldsymbol{s})S_j(\boldsymbol{x}^-(n,h)+\gamma\boldsymbol{s})\mu^{i+\gamma}\lambda^{j+\gamma},\end{aligned} \tag{139}$$

where $x_r^{\pm}(n,h)$ are defined as

$$x_r^+(n,h) = \varphi_r^{(1)}x+\varphi_r^{(2)}\mathrm{i}t+h\varphi_r^{(3)}+\left(n+\frac{1}{2}\right)\varphi_r^{(4)}+a_r\,, \tag{140a}$$

$$x_r^-(n,h) = \varphi_r^{(1)*}x-\varphi_r^{(2)*}\mathrm{i}t-h\varphi_r^{(3)*}-\left(n+\frac{1}{2}\right)\varphi_r^{(4)*}+a_r^*\,, \tag{140b}$$

the infinite vectors $\boldsymbol{x}^{\pm}(n,h)$ are defined as $\boldsymbol{x}^{\pm}(n,h) = (x_r^{\pm}(n,h))$, and the elementary Schur polynomial $S_j(\boldsymbol{x})$ is defined via the generating function [57–59]

$$\sum_{j=0}^{\infty}S_j(\boldsymbol{x})\epsilon^j = \exp\left(\sum_{j=1}^{\infty}x_j\epsilon^j\right), \tag{141a}$$

or, equivalently, via the formula

$$S_j(\boldsymbol{x}) = \sum_{l_1+2l_2+\cdots+pl_p=j} \left(\prod_{i=1}^{p} \frac{x_i^{l_i}}{l_i!} \right), \tag{141b}$$

where the sum extends over all partitions of j of the given form. A more detailed derivation of the elementary Schur polynomials in the context of this computation can be found in [56].

By equating the coefficients of $\mu^i \lambda^j$ on both sides of (139) one can check that

$$\frac{\tilde{m}_{i,j}^{(n,h)}}{m^{(n,h)}|_{k=k_0,\bar{k}=\bar{k}_0}} = \sum_{\gamma=0}^{\min(i,j)} \frac{1}{4^\gamma} S_{i-\gamma}(\boldsymbol{x}^+(n,h)+\gamma \boldsymbol{s}) S_{j-\gamma}(\boldsymbol{x}^-(n,h)+\gamma \boldsymbol{s}), \tag{142}$$

where $\tilde{m}_{i,j}^{(n,h)}$ is the matrix element given in Theorem 4.

The final ingredient needed for the computation is the gauge freedom of the τ-functions, which ensures that the determinant of the matrix whose elements are given by (142),

$$\sigma_{n,h} = \frac{\tau_{n,h}}{\left(m^{(n,h)}|_{k=k_0,\bar{k}=\bar{k}_0} \right)^N}, \tag{143}$$

is also a rational solution to the YON system (4) of the form presented in Theorem 4. With that, we can take the determinant of (142) to present our rewritten solutions via the following theorem.

Theorem 5. *The YON system (4) possesses the following rogue wave solutions*

$$S = \rho \frac{g}{f} e^{\mathrm{i}[qx-(q^2+2\alpha\rho^2+2\delta\ell-\ell^2)t]}, \qquad L = \ell + \mathrm{i} \left(\log \frac{f^*}{f} \right)_x, \tag{144}$$

where

$$f = \sigma_{-1,0}, \qquad f^* = \sigma_{0,0}, \qquad g = \sigma_{-1,1}, \qquad g^* = \sigma_{0,-1}, \tag{145}$$

and the elements in the determinant $\sigma_{n,h} = \det \left(m_{2i-1,2j-1}^{(n,h)} \right)_{1\le i,j\le N}$ are given by

$$m_{i,j}^{(n,h)} = \sum_{\gamma=0}^{\min(i,j)} \frac{1}{4^\gamma} S_{i-\gamma}(\boldsymbol{x}^+(n,h)+\gamma \boldsymbol{s}) S_{j-\gamma}(\boldsymbol{x}^-(n,h)+\gamma \boldsymbol{s}), \tag{146}$$

with the infinite vectors $\boldsymbol{x}^{\pm}(n,h) = \left(x_1^{\pm}(n,h), x_2^{\pm}(n,h), \dots \right) \equiv \left(x_1^{\pm}, x_2^{\pm}, \dots \right)$ defined by (140), $\boldsymbol{s} = (s_1, s_2, \dots)$, and S_j denoting the jth elementary Schur polynomial defined by (141).

Using Theorem 5, we can obtain the first-order rogue wave solution (or fundamental rogue wave) by setting $N = 1$, which gives us the following τ-functions

$$f = m_{11}^{(-1,0)}, \qquad f^* = m_{11}^{(0,0)}, \qquad g = m_{11}^{(-1,1)} \tag{147a}$$

where the elements are determined by

$$m_{11}^{(n,h)} = x_1^+ x_1^- + c_0, \tag{147b}$$

with

$$c_0 = \frac{k_1 k_1^*}{(k_0+k_0^*)^2}. \tag{147c}$$

Fig. 7 illustrates the profile of a first-order rogue wave for a particular choice of parameters.

The second-order rogue wave solution is obtained from Theorem 5 with $N = 2$. In this case, the τ-functions f and g are given by

$$f = \begin{vmatrix} m_{11}^{(-1,0)} & m_{13}^{(-1,0)} \\ m_{31}^{(-1,0)} & m_{33}^{(-1,0)} \end{vmatrix}, \quad f^* = \begin{vmatrix} m_{11}^{(0,0)} & m_{13}^{(0,0)} \\ m_{31}^{(0,0)} & m_{33}^{(0,0)} \end{vmatrix}, \quad g = \begin{vmatrix} m_{11}^{(-1,1)} & m_{13}^{(-1,1)} \\ m_{31}^{(-1,1)} & m_{33}^{(-1,1)} \end{vmatrix}, \tag{148a}$$

where the elements are determined by

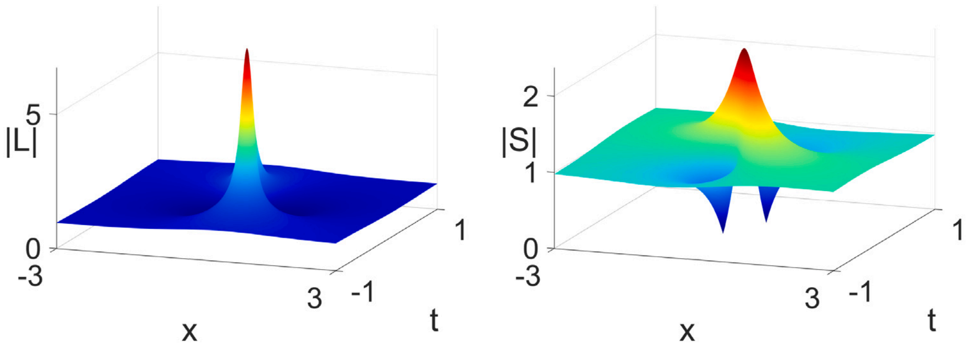
$$m_{11}^{(n,h)} = x_1^+ x_1^- + c_0, \qquad m_{13}^{(n,h)} = x_1^+ \hat{x}_1^- + \frac{c_0}{2} \hat{x}_2^-, \tag{148b}$$

$$m_{33}^{(n,h)} = \hat{x}_1^+ \hat{x}_1^- + \frac{c_0}{4} \hat{x}_2^+ \hat{x}_2^- + \frac{c_0}{4} \left(x_1^+ + 2s_1 \right) \left(x_1^- + 2s_1 \right) + \frac{c_0^2}{16}, \qquad m_{31}^{(n,h)} = x_1^- \hat{x}_1^+ + \frac{c_0}{2} \hat{x}_2^+, \tag{148c}$$

$$\hat{x}_1^+ = \frac{1}{6}(x_1^+)^3 + x_1^+ x_2^+ + x_3^+, \qquad \hat{x}_1^- = \frac{1}{6}(x_1^-)^3 + x_1^- x_2^- + x_3^-, \tag{148d}$$

$$\hat{x}_2^+ = \left(x_1^+ + s_1 \right)^2 + 2(s_2 + x_2^+), \qquad \hat{x}_2^- = \left(x_1^- + s_1 \right)^2 + 2(s_2 + x_2^-), \tag{148e}$$

with c_0 defined as above.



(a) 3D profile for $|L|$ (b) 3D profile for $|S|$

Fig. 7. First-order rogue wave with parameter values $\alpha = 1$, $\delta = -1$ ($\beta = -2$), $\rho = 1$, $q = 1$, $\ell = 1$, $k_0 = 1.1713 - 0.08728\mathrm{i}$, $k_1 = -0.88698 + 0.51361\mathrm{i}$, $a_1 = 0$.

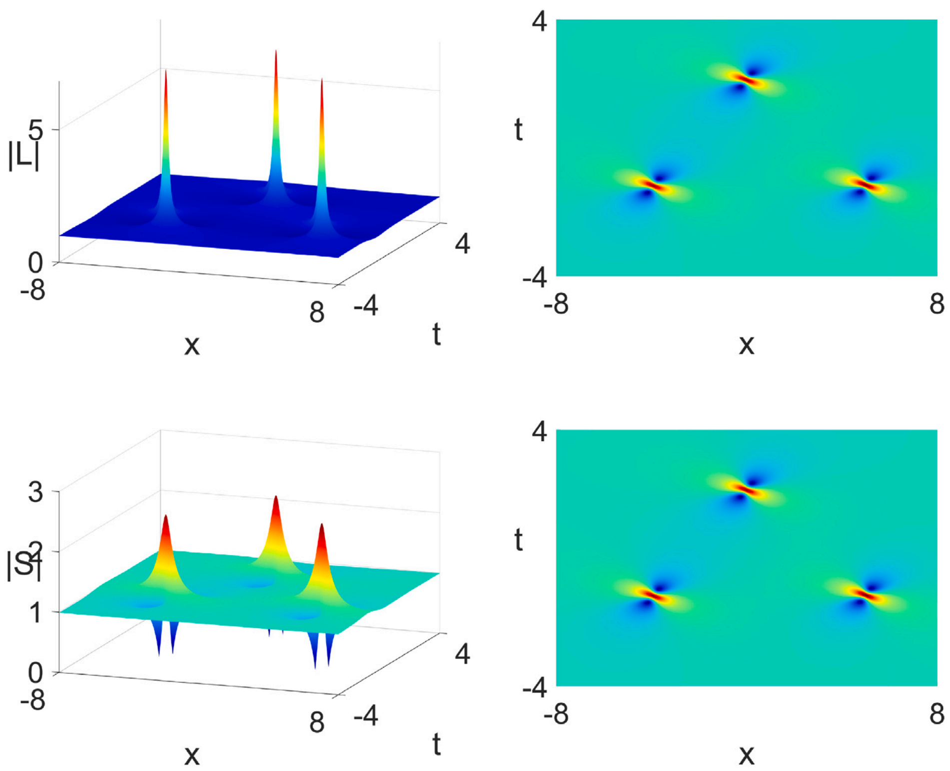


Fig. 8. Second-order rogue wave with parameter values $\alpha = 1$, $\delta = -1$ ($\beta = -2$), $\rho = 1$, $q = 1$, $\ell = 1$, $k_0 = 1.1713 - 0.08728\mathrm{i}$, $k_1 = -0.88698 + 0.51361\mathrm{i}$, $a_1 = 0$, $a_3 = 50$.

From these explicit expressions, we can see that each τ-function is a polynomial of degree six with respect to the variables x and t. Two representative examples of second-order rogue waves with different parameter values are illustrated in Figs. 8 and 9.

The first of them, Fig. 8, exhibits a second-order rogue wave consisting of three clearly separate fundamental rogue waves distributed in a triangular array. It can be observed that the geometric patterns for the second-order rogue wave of the YON system are similar to the ones obtained in [60,61]. The rigorous proof for this fact can be done very similarly to the one given in [62].

In the second solution, Fig. 9, the parameters take the same values as the ones we used for Fig. 8 except for the choice $a_3 = 0$. In this case, the individual rogue waves that make up the second-order solution coalesce into a single one, giving rise to what is typically termed a super rogue wave.

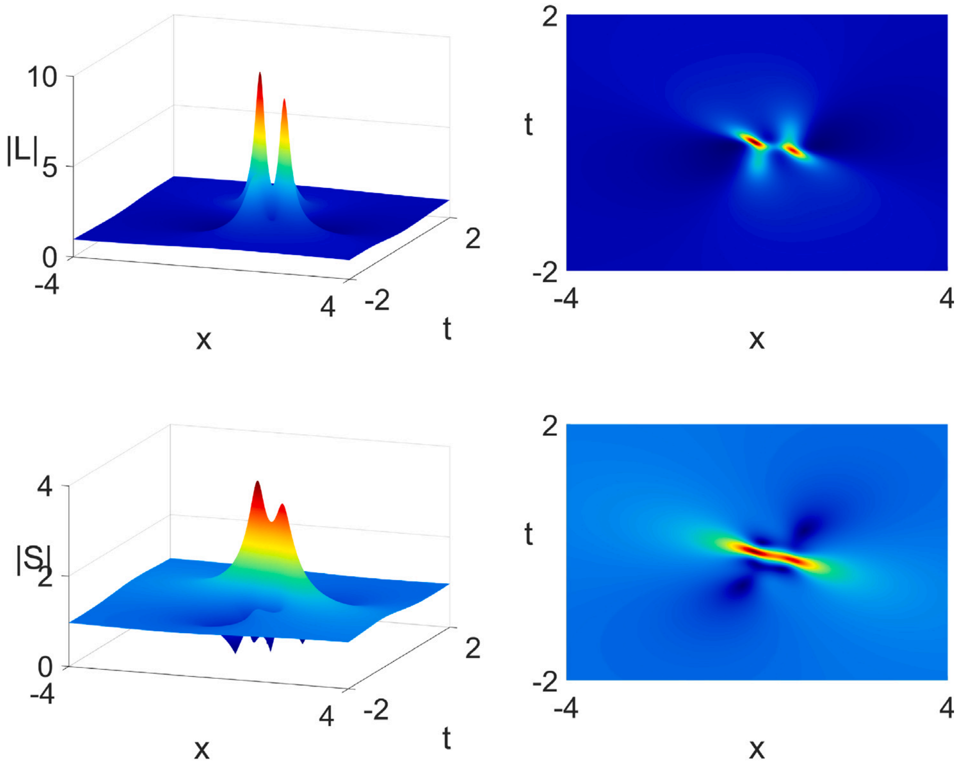


Fig. 9. Second-order rogue wave with parameter values $\alpha = 1$, $\delta = -1$ $(\beta = -2)$, $\rho = 1$, $q = 1$, $\ell = 1$, $k_0 = 1.1713 - 0.08728\mathrm{i}$, $k_1 = -0.88698 + 0.51361\mathrm{i}$, $a_1 = a_3 = 0$.

6. Conclusions

In this paper, we have derived new families of solutions for the YON system. We used both a traditional Hirota approach and the τ-function reduction technique for the two-component KP hierarchy in order to obtain general bright soliton solutions. We further employed the τ-function reduction for the KP-Toda hierarchy to obtain general families of dark soliton, breather and rogue wave solutions.

For each family of solutions we provided their main physical features and described their general behaviour.

The higher-order rogue wave solutions match the triangular pattern previously derived for other wave systems, such as the nonlinear Schrödinger equation [60]. When the individual fundamental rogue waves that make up the higher-order rogue wave coalesce into one, the resulting structure is a super rogue wave.

The constraints that the parameters must satisfy in order for the rogue wave solutions to exist coincide with the predictions presented in [1] making use of the base-band instability regimes obtained via the stability spectra of plane waves.

The problem of generating families of solutions for the YON system via spectral methods such as inverse scattering or Darboux-dressing remains open, and so does the problem of whether one can construct Bäcklund transformations for the YON system by means of its Hirota bilinear structure in the spirit of [63].

CRediT authorship contribution statement

Marcos Caso-Huerta: Writing – review & editing, Writing – original draft, Methodology, Investigation, Conceptualization. **Bao-Feng Feng:** Writing – review & editing, Writing – original draft, Methodology, Investigation, Conceptualization. **Sara Lombardo:** Writing – review & editing, Writing – original draft, Methodology, Investigation, Conceptualization. **Ken-ichi Maruno:** Writing – review & editing, Writing – original draft, Methodology, Investigation, Conceptualization. **Matteo Sommacal:** Writing – review & editing, Writing – original draft, Methodology, Investigation, Conceptualization.

Declaration of competing interest

The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

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Data availability

No data was used for the research described in the article.

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