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ORIGINAL ARTICLE

On Negative Conglomerability

Enrique Miranda[1] · **Marco Zaffalon**[2]

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Abstract

We focus on the notion of negative conglomerability. This is far less known than its counterpart, conglomerability. Both relate to the combination of conditional and unconditional information, with the latter taking in particular a foundational role in the special case of infinite partitions of the possibility space. The two notions look superficially very similar and are even equivalent in the case of precise probabilistic models. In the present paper, we do a thorough technical study of their relations with other main concepts in the literature, such as marginal extension and dilation, both in the precise and imprecise case. Moreover, we discuss why they are somewhat surprisingly different from the prescriptive point of view, in that conglomerability has a rationality stance that its negative counterpart has not.

Keywords Desirability · Imprecise probability · Conglomerability · Sets of desirable gambles · Coherent lower previsions · Credal sets

1 Introduction

Savage's *sure thing principle* [17, Section 2.7] is well known to be widely valid; in its simplest form, it states that if we prefer action a to b both conditionally to an event and conditional to its complement, then we should unconditionally prefer a to b. For instance, if Alice goes jogging no matter whether the weather forecast gives rain or not, we should deduce that she would be going also without knowing the forecast. Now say that Bob instead jogs neither when the forecast gives rain nor when it does not. Should we deduce that he would not go jogging in case he does not know the forecast? The latter situation looks very similar to the first; we could interpret it as a kind of 'negative' sure thing principle.

✉ Enrique Miranda
mirandaenrique@uniovi.es

Marco Zaffalon
zaffalon@idsia.ch

1 Department of Statistics and Operations Research, University of Oviedo, Oviedo, Spain

2 IDSIA, Lugano, Switzerland

The importance of the sure thing principle in probability stems in particular from its foundational role with regard to conditioning, and hence to statistics—at least from a Bayesian perspective. In fact, as we have detailed elsewhere [27], the sure thing principle essentially coincides with a type of 'conditional' additivity when we consider the desirability of lotteries (we call them gambles), which is the focus of this paper. Saying that a certain real-valued gamble f is desirable, means that one is willing to accept it. Now, if we denote by B the indicator function of an event, the product Bf, which is equal to f on B and is zero elsewhere, represents the contingent gamble that is called off if B does not occur; similarly for the complementary event B^c. Consequently, the sure thing principle in this setting states that if Bf and $B^c f$ are both desirable, then $f = Bf + B^c f$ must be desirable too. (Let us remark that the theory of desirable gambles [2, 9, 15, 24] is a generalisation of de Finetti's theory of probability to imprecisely specified probabilities and to non-Archimedeanity. Therefore the results presented in this paper hold under very broad conditions that encompass, for instance, the cases of lower and upper expectations, Bayesian robustness, and the special cases of lower and upper probabilities.)

Note that the desirability of f above can be made to follow from the bare application of the finite additivity axiom of probability; and yet the same is no longer true for infinite partitions of conditioning events. In this case, such a (conditional) additivity may have to be imposed, and it takes the name of 'conglomerability' [4, 5, 18]. As argued in [24], conglomerability gives solid foundations to the notion of conditional probability in the general case: it can for instance be formally linked to countable additivity [24, Section 6.9] [18]; and it maintains its special role in conditioning also for finite partitions whenever we generalise probability by dropping the axiom of finite additivity [12, Section 2.2]. It is then particularly interesting that conglomerability can be justified as a rationality requirement on its own via arguments of temporal coherence (this has been discussed for the first time in [25]; we provide a reworked and simplified justification in Sect. 7 of the present paper).

From this perspective, it appears very natural to turn our attention to the negative version of conglomerability, to see whether it has anything equally meaningful to say. It is the condition that regards f non-desirable in case for all the events B in a partition $\mathcal{B}$, Bf is non-desirable. Some questions of obvious interest are: is this 'negative conglomerability' widely applicable as its positive counterpart? Should it be regarded as a rationality requirement? What are its implications and relations with other notions in the literature?

Our investigation will eventually lead to both positive and negative answers to those questions. We shall see for example that there appears to be no ground to give negative conglomerability a rationality status. And yet, negative conglomerability will automatically be granted in very many cases of pragmatic interest: in particular, when we build our models through hierarchical information, in situations where we traditionally apply the law of total probability. Negative conglomerability will also play a peculiar role in conditioning in that it will turn out to be sufficient to prevent a probabilistic model from dilating. 'Dilation' [22] is a definite increase of uncertainty due to conditioning, irrespectively of the specific event one conditions on. Albeit reasonable in many cases, it may be an undesirable property of a probabilistic model in specific situations. In those cases, negative conglomerability can come to the rescue as a well-defined condition on which to rely to have the problem addressed.

We shall start our investigation by introducing some preliminary concepts of desirability theory in Sect. 2. In Sect. 3 we introduce conglomerability and its negative counterpart, and we detail their relations with some prominent notions in the literature, such as that of marginal extension ([8], [24, Section 6.7]). In Sect. 4 we study possible negatively conglomerable extensions of sets that are not such. In Sect. 5 we reconsider the previous results for the case where our assessments of desirability lead to precise probability models. We detail the conflict between negative conglomerability and dilation [22] in Sect. 6. The rationality status of negative conglomerability in studied in Sect. 7. Concluding considerations are reported in Sect. 8.

2 Preliminary Concepts

Let us recall the basic aspects of the theory of coherent lower previsions; we refer to [23, 24] for more details.

Consider a possibility space Ω. A *gamble* $f : \Omega \to \mathbb{R}$ is a bounded real-valued function on Ω. We denote by $\mathcal{L}(\Omega)$ the set of all the gambles on Ω and by $\mathcal{L}^+(\Omega) := \{f \in \mathcal{L}(\Omega) : 0 \neq f \geq 0\}$ the subset of the *positive gambles*. We denote these sets also by $\mathcal{L}$ and $\mathcal{L}^+$, respectively, when there is no ambiguity about the space involved. Negative gambles are defined by $\mathcal{L}^- := -\mathcal{L}^+$, and we shall also use $\mathcal{L}_0^- := \mathcal{L}^- \cup \{0\}$ and $\mathcal{L}_< = \{f : \sup f < 0\}$. Events are denoted by capital letters such as $A, B, C \subseteq \Omega$. We shall identify an event A with its indicator function I_A, whence disjunctions $(A \cap B)$ will be represented by products (AB). As a consequence, the product Bf is equal to f on B and zero elsewhere. It is interpreted as a conditional gamble: one that is called off if B does not occur. Finally, given a partition $\mathcal{B}$ of Ω, a gamble is said to be $\mathcal{B}$-*measurable* when it is constant on the elements of $\mathcal{B}$; we shall denote by $\mathcal{L}_{\mathcal{B}}$ the set of $\mathcal{B}$-measurable gambles.

2.1 Desirability

The traditional approach to coherence in Williams-Walley's theory assumes that the scale in which the rewards, represented by gambles, are measured is linear [24, Section 2.2]. This implies that the gambles whose desirability is implied by those from a given set $\mathcal{D}$ are those in its *conic hull*:

$$\text{posi}(\mathcal{D}) := \left\{ \sum_{j=1}^{r} \lambda_j f_j : f_j \in \mathcal{D}, \lambda_j > 0, r \geq 1 \right\}.$$

Then a set of desirable gambles $\mathcal{D} \subseteq \mathcal{L}$ is called (Williams-)*coherent* if and only if the following conditions hold:

D1. $\mathcal{L}^+ \subseteq \mathcal{D}$ [Accepting Partial Gains];
D2. $0 \notin \mathcal{D}$ [Avoiding Status Quo];

D3. $f, g \in \mathcal{D} \Rightarrow f + g \in \mathcal{D}$ [Additivity];
D4. $f \in \mathcal{D}, \lambda > 0 \Rightarrow \lambda f \in \mathcal{D}$ [Positive Homogeneity].

This is equivalent to requiring that $\mathrm{posi}(\mathcal{D} \cup \mathcal{L}^+) = \mathcal{D}$ and $\mathcal{D} \cap \mathcal{L}_0^- = \emptyset$. When we regard the theory of desirability from a logical perspective, posi corresponds to the deductive closure; D1 to the tautologies and D2 to the status quo—which combined with the other axioms defines the contradictions, i.e., $\mathcal{L}^-$. In particular, we say that a set of gambles $\mathcal{D}$ is a coherent set of *strictly desirable* gambles when it satisfies axioms D1–D4 and moreover

$$(\forall f \in \mathcal{D} \setminus \mathcal{L}^+)(\exists \epsilon > 0)\ f - \epsilon \in \mathcal{D}.$$

More generally, for any set of gambles $\mathcal{K}$, if there is a coherent set of desirable gambles that includes $\mathcal{K}$ then there is a smallest such superset, and it is called the *natural extension* of $\mathcal{K}$. It is given by

$$\mathcal{E}_{\mathcal{K}} := \mathrm{posi}(\mathcal{K} \cup \mathcal{L}^+). \tag{1}$$

It also follows that $\mathcal{K}$ has a coherent superset iff $\mathcal{E}_{\mathcal{K}}$ is coherent, and that $\mathcal{K}$ is coherent iff $\mathcal{K} = \mathcal{E}_{\mathcal{K}}$ and $0 \notin \mathcal{K}$. We say that $\mathcal{K}$ *avoids sure loss* when $\mathcal{E}_{\mathcal{K}}$ is coherent, and that it *incurs a sure loss* otherwise.

For a deeper account of *desirability*, we refer to [2, 9, 15] and [24, Section 3.7].

2.2 Lower Previsions

Any coherent set of desirable gambles $\mathcal{D}$ induces a lower prevision and a conditional lower prevision by means of the formulae

$$\underline{P}(f) := \sup\{\mu : f - \mu \in \mathcal{D}\} \tag{2}$$

and

$$\underline{P}(f|B) := \sup\{\mu : B(f - \mu) \in \mathcal{D}\} \tag{3}$$

for any gamble f and any non-empty conditioning event $\emptyset \neq B \subseteq \Omega$. If we now consider a partition $\mathcal{B}$, the conditional lower prevision $\underline{P}(\cdot|\mathcal{B})$ is defined on a gamble f as the $\mathcal{B}$-measurable gamble

$$\underline{P}(f|\mathcal{B}) := \sum_{B \in \mathcal{B}} B\,\underline{P}(f|B).$$

A consequence of the above formulae is that $\mathcal{L}^+ \cup \{f : \underline{P}(f) > 0\} \subseteq \mathcal{D} \subseteq \{f : \underline{P}(f) \geq 0\}$.

The coherence of $\mathcal{D}$ implies that the lower prevision $\underline{P}$ it induces by Eq. (2) is also *coherent*, meaning that it satisfies

COH1 $\underline{P}(f) \geq \inf f$;

COH2 $\underline{P}(\lambda f) = \lambda \underline{P}(f)$;
COH3 $\underline{P}(f+g) \geq \underline{P}(f) + \underline{P}(g)$

for any gambles $f, g \in \mathcal{L}$ and any positive real number λ. The correspondence between coherent sets of desirable gambles and coherent lower previsions in Eq. (2) is many-to-one: the same coherent lower prevision can be induced by more than one coherent set of desirable gambles. There is however only one such set that is a coherent set of strictly desirable gambles, and it is given by

$$\mathcal{L}^+ \cup \{f : \underline{P}(f) > 0\}. \tag{4}$$

On the other hand, the closure of any coherent set of desirable gambles associated with $\underline{P}$ is given by

$$\{f : \underline{P}(f) \geq 0\}. \tag{5}$$

This is called the set of *almost-desirable gambles* associated with $\underline{P}$, and we shall denote it by $\overline{\mathcal{D}}_{\underline{P}}$. Note that it is not coherent, because it includes for instance the zero gamble.

Similarly, the conditional lower prevision $\underline{P}(\cdot|\mathcal{B})$ is *separately coherent*, meaning that

SCOH1 $\underline{P}(f|B) \geq \inf_B f$;
SCOH2 $\underline{P}(\lambda f|B) = \lambda \underline{P}(f|B)$;
SCOH3 $\underline{P}(f+g|B) \geq \underline{P}(f|B) + \underline{P}(g|B)$

for any gambles $f, g \in \mathcal{L}$, any positive real number λ and any event $B \in \mathcal{B}$. A given conditional lower prevision $\underline{P}(\cdot|\mathcal{B})$ transforms any gamble f into a gamble $G(f|\mathcal{B})$ by means of the formula

$$G(f|\mathcal{B}) := f - \underline{P}(f|\mathcal{B}) = \sum_B B(f - \underline{P}(f|B)) := \sum_B G(f|B).$$

Both $\underline{P}$ and $\underline{P}(\cdot|\mathcal{B})$ satisfy what we shall refer to as *constant additivity*: for any gamble f and any real number μ

$$\underline{P}(f+\mu) = \underline{P}(f) + \mu \quad \text{and} \quad \underline{P}(f+\mu|\mathcal{B}) = \underline{P}(f|\mathcal{B}) + \mu.$$

2.3 Linear Previsions and Sensitivity Analysis Interpretation

When a coherent lower prevision $\underline{P}$ with domain $\mathcal{L}$ satisfies condition COH3 with equality, i.e., when $\underline{P}(f+g) = \underline{P}(f) + \underline{P}(g)$ for any pair of gambles f, g, it is called a *linear prevision*. It corresponds to the expectation operator with respect to its restriction to events, which is a finitely additive probability. Moreover, a coherent lower prevision is in a one-to-one correspondence with a closed (in the weak* topology) and convex set of linear previsions, namely

$$\mathcal{M}(\underline{P}) := \{P \text{ linear prevision} : (\forall f \in \mathcal{L})\ P(f) \geq \underline{P}(f)\}.$$

A sufficient condition for a coherent set of desirable gambles $\mathcal{D}$ to induce a linear prevision is that it is *maximal*, meaning that

$$(\forall f \neq 0)\ f \notin \mathcal{D} \Rightarrow -f \in \mathcal{D}.$$

Maximal sets of gambles allow us to give a strong belief structure to coherent sets of desirable gambles: any coherent set of desirable gambles is equal to the intersection of its maximal supersets.

3 Conglomerability and Negative Conglomerability

3.1 Conglomerability

Although our focus in this paper is on the meaning and implications of negative conglomerability, we shall start our analysis by briefly investigating what entails for a coherent set of desirable gambles to be conglomerable.

Definition 1 *(Conglomerability)* Given a partition $\mathcal{B}$ of Ω, we say that a coherent set of desirable gambles $\mathcal{D}$ is $\mathcal{B}$-conglomerable if and only if

$$(Bf \in \mathcal{D} \cup \{0\}\ \forall B \in \mathcal{B}) \Rightarrow f \in \mathcal{D}$$

for all $f \neq 0$. If $\mathcal{D}$ is $\mathcal{B}$-conglomerable for all partitions $\mathcal{B}$ of Ω it is called *fully conglomerable*.

As we said in Sect. 2, the rationality criteria of coherence for sets of desirable gambles impose that if two gambles f, g represent acceptable transactions for a subject, they should be disposed to take the two transactions at the same time, that is, they should also find the gamble $f + g$ desirable. While a similar condition is not imposed for infinite sums of desirable gambles, conglomerability points out to a particular scenario when such a desirability assessment may be reasonable: when no two of the gambles to be added are active at the same time, or, in other words, when they are only non-zero on disjoint events of the possibility space. Conglomerability is one of the main points of disagreement between Walley's and de Finetti's behavioural approach to probability [24, Section 6.8], and allows to go from finitely additive models in the direction of countably additive ones. We refer to [10, 11, 13, 18, 19, 21] for some papers investigating in detail this concept.

Let $\mathcal{D}$ be a coherent set of desirable gambles and let $\mathcal{B}$ be a partition of Ω. Let us denote by $\underline{P}$, $\underline{P}(\cdot|\mathcal{B})$ the unconditional and conditional lower previsions it induces by means of Eqs. (2) and (3). Consider the following conditions:

Co1. $(\forall f \in \mathcal{L})\ \underline{P}(f|\mathcal{B}) \in \mathcal{D} \Rightarrow f \in \mathcal{D}$.
Co2. $\mathcal{D}$ is $\mathcal{B}$-conglomerable.
Co3. $(\forall f \in \mathcal{L})\ \underline{P}(G(f|\mathcal{B})) \geq 0$.

Co4. $(\forall f \in \mathcal{L})\ \underline{P}(f) \geq \underline{P}(\underline{P}(f|\mathcal{B}))$.

The connections between these conditions are summarised in the following proposition:

Proposition 1 (a) *Co1, Co2 $\Rightarrow$ Co3 $\Leftrightarrow$ Co4.*
(b) *If $\mathcal{D}$ is a set of strictly desirable gambles, then Co1 $\Rightarrow$ Co2 $\Leftrightarrow$ Co3 $\Leftrightarrow$ Co4.*
(c) *If Ω is finite, then Co2–Co4 always hold.*

Proof (a) We begin by establishing the equivalence between Co3 and Co4. To prove that Co3 implies Co4, note that the super-additivity COH3 of the coherent lower prevision $\underline{P}$ implies that for any gamble f,

$$\underline{P}(f) = \underline{P}(G(f|\mathcal{B}) + \underline{P}(f|\mathcal{B})) \geq \underline{P}(G(f|\mathcal{B})) + \underline{P}(\underline{P}(f|\mathcal{B})) \geq \underline{P}(\underline{P}(f|\mathcal{B})),$$

where the last inequality follows from Co3.
Conversely, if Co4 holds and there is a gamble f such that $\underline{P}(G(f|\mathcal{B})) < 0$, then given $g := G(f|\mathcal{B})$,

$$\underline{P}(g|\mathcal{B}) = \sum_B B\underline{P}(g|B) = \sum_B B\underline{P}(G(f|\mathcal{B})|B) = \sum_B B(\underline{P}(f|B) - \underline{P}(f|B)) = 0,$$

meaning that $\underline{P}(\underline{P}(g|\mathcal{B})) > \underline{P}(g)$, a contradiction with Co4.
To prove that Co2 implies Co3, note that for any gamble f and any $\epsilon > 0$ the gamble $G(f|B) + \epsilon B$ belongs to $\mathcal{D}$ by Eq. (3); applying Co2 we deduce that $G(f|\mathcal{B}) + \varepsilon$ belongs to $\mathcal{D}$, whence $\underline{P}(G(f|\mathcal{B}) + \epsilon) = \underline{P}(G(f|\mathcal{B})) + \epsilon \geq 0$. Since this holds for any $\epsilon > 0$, we conclude that $\underline{P}(G(f|\mathcal{B})) \geq 0$.
Finally, to establish that Co1 implies Co4, assume ex-absurdo the existence of a gamble f such that $\underline{P}(f) < \underline{P}(\underline{P}(f|\mathcal{B}))$. By constant additivity, there exists some real number μ such that $\underline{P}(f + \mu) < 0 < \underline{P}(\underline{P}(f + \mu|\mathcal{B}))$, and this means that $\underline{P}(f + \mu|\mathcal{B})$ belongs to $\mathcal{D}$ while $f + \mu$ does not, a contradiction.
(b) Let us show that under strict desirability Co4 implies Co2. Consider a gamble $f \neq 0$ such that $Bf \in \mathcal{D} \cup \{0\}$ for any $B \in \mathcal{B}$, and define $\mathcal{B}^* := \{B \in \mathcal{B} : Bf \notin \mathcal{L}^+ \cup \{0\}\}$. If $\mathcal{B}^* = \emptyset$, then $f \in \mathcal{L}^+ \subseteq \mathcal{D}$. Otherwise, for any $B \in \mathcal{B}^*$ there exists some $\epsilon > 0$ such that $Bf - \epsilon \in \mathcal{D}$. This means on the one hand that

$$0 < \underline{P}(Bf) \leq \underline{P}(B \sup f) = \sup f \underline{P}(B) \Rightarrow \underline{P}(B) > 0,$$

and also that

$$\underline{P}(f|B) = \sup\{\mu : B(f - \mu) \in \mathcal{D}\} \geq \epsilon > 0, \text{ since } B(f - \epsilon) \geq Bf - \epsilon \in \mathcal{D}.$$

From this we deduce that, picking any $B \in \mathcal{B}^*$,

$$\underline{P}(\underline{P}(f|\mathcal{B})) \geq \underline{P}(B\underline{P}(f|B)) > 0,$$

and then Co4 implies that $\underline{P}(f) > 0$ from which we deduce that $f \in \mathcal{D}$.
(c) It suffices to note that in the finite case Co2 follows from the coherence of $\mathcal{D}$, and to apply (a). □

Figure 1 illustrates the result.

For completeness, we provide counterexamples showing that no further implication holds:

Example 1 (Co2 $\not\Rightarrow$ Co1 under strict desirability) Let $\Omega := \mathbb{N}$, $B_n := \{2n-1, 2n\}$ and $\mathcal{B} := \{B_n : n \in \mathbb{N}\}$. Let P be the σ-additive probability measure given by

$$P(\{1\}) = P(\{2\}) = 0, (\forall n \geq 2)\ P(\{2n-1\}) = P(\{2n\}) = \frac{1}{2^n}.$$

Let $\mathcal{D}$ be the coherent set of strictly desirable gambles associated with P by means of Eq. (4). If we consider the gamble f given by

$$(\forall n \in \mathbb{N})\ f(2n-1) = 1, (\forall n \geq 2)\ f(2n) = -1, f(2) = 1,$$

we obtain that $P(f|\mathcal{B}) = I_{B_1} \in \mathcal{L}^+ \subseteq \mathcal{D}$, but $P(f) = 0$, whence $f \notin \mathcal{D}$. ♦

As a consequence, we deduce that also Co3, Co4 $\not\Rightarrow$ Co1.

Example 2 (Co1 $\not\Rightarrow$ Co2) Consider $\Omega := \mathbb{N}$, $B_n := \{2n-1, 2n\}$, $\mathcal{B} := \{B_n : n \in \mathbb{N}\}$, and let P be the σ-additive probability satisfying

$$(\forall n)\ P(\{2n-1\}) = P(\{2n\}) = \frac{1}{2^{n+1}}.$$

Let $\mathcal{K} := \{f : P(f) > 0\} \cup \{I_{\{2n:n\in\mathbb{N}\}} - I_{\{2n-1:n\in\mathbb{N}\}}\} \cup \{I_{2n-1} - I_{2n} : n \in \mathbb{N}\}$, and let $\mathcal{D}$ be the natural extension of $\mathcal{K}$, given by Eq. (1).

- $\mathcal{D}$ is coherent: we only need to show that $0 \notin \mathcal{D}$. By definition of the natural extension, any gamble g of $\mathcal{D}$ can be expressed as

$$g = \lambda_0 f + \sum_{i=1}^{n} \lambda_i f_i + \lambda(I_{\{2n:n\in\mathbb{N}\}} - I_{\{2n-1:n\in\mathbb{N}\}}), \tag{6}$$

where $\lambda_0, \lambda_1, \ldots, \lambda_n, \lambda \geq 0$, with not all of them 0, and $P(f) > 0$, $f_i \in \{I_{2n-1} - I_{2n} : n \in \mathbb{N}\}$ for all i.
Assume ex-absurdo that $0 \in \mathcal{D}$ and consider its expression as in Eq. (6). Since

Fig. 1 Implications between conditions Co1–Co4 for arbitrary coherent sets of gambles (left) and for coherent sets of strictly desirable gambles (right)

$P(f_i) = 0 = P(I_{\{2n:n\in\mathbb{N}\}} - I_{\{2n-1:n\in\mathbb{N}\}})$ for any $f_i \in \{I_{2n-1} - I_{2n} : n \in \mathbb{N}\}$, it follows that it should be $\lambda_0 = 0$. On the other hand, both when $\lambda > 0$ or when $\lambda = 0$ we deduce the existence of some natural number n where the right-hand side is strictly positive, a contradiction.

- $\mathcal{D}$ does not satisfy Co2, since $I_{2n-1} - I_{2n} \in \mathcal{D}$ for every $n \in \mathbb{N}$ but their sum $I_{\{2n-1:n\in\mathbb{N}\}} - I_{\{2n:n\in\mathbb{N}\}}$ does not.
- By construction, $\mathcal{D}$ induces the linear prevision P and the conditional linear prevision $P(\cdot|B_n)$ associated with the uniform distribution for every $B_n \in \mathbb{N}$. It holds that $P = P(P(\cdot|\mathcal{B}))$, whence Co4 holds. To prove that in fact Co1 also holds, let g be such that $P(g|\mathcal{B}) \in \mathcal{D}$. Applying Eq. (6), we observe that if $\lambda_0 = 0$ then the combination $\sum_{i=1}^{n} \lambda_i f_i + \lambda(I_{\{2n:n\in\mathbb{N}\}} - I_{\{2n-1:n\in\mathbb{N}\}})$ does not produce a $\mathcal{B}$-measurable gamble. As a consequence,

$$P(g) = P(P(g|\mathcal{B})) = \lambda_0 P(f) > 0,$$

using again Eq. (6). We conclude that $g \in \mathcal{D}$. ♦

As a consequence, we deduce that also Co3, Co4 $\not\Rightarrow$ Co1.

Example 3 (Co1 may not hold even when Ω is finite) Consider $\Omega := \{1, 2, 3, 4\}$ and let $\mathcal{D}$ be the set of strictly desirable gambles associated with the probability P with mass function $(0, 0, 0.5, 0.5)$. The gamble f given by $f(1) = f(2) = f(3) = 1$, $f(4) = -1$ satisfies that $P(f) = 0$ and $f \notin \mathcal{L}^+$, whence $f \notin \mathcal{D}$.

On the other hand, if we consider $B := \{1, 2\}$ and the partition $\mathcal{B} := \{B, B^c\}$ we obtain that $P(f|B) = 1$ and $P(f|B^c) = 0$. This implies that $P(f|\mathcal{B}) = I_B \in \mathcal{D}$ and therefore that Co1 does not hold. ♦

Finally, to establish that condition Co4 is not trivial, in that it does not follow from the coherence of $\mathcal{D}$, we refer to [13, Example 2].

3.2 Negative Conglomerability

We shift now our attention to the main focus of this paper: negative conglomerability.

Definition 2 (*Negative conglomerability*) Given a partition $\mathcal{B}$ of Ω, we say that a coherent set of desirable gambles $\mathcal{D}$ is *negatively $\mathcal{B}$-conglomerable* if and only if

$$(Bf \notin \mathcal{D}\ \forall B \in \mathcal{B}) \Rightarrow f \notin \mathcal{D}.$$

for all $f \in \mathcal{L}$. If $\mathcal{D}$ is negatively $\mathcal{B}$-conglomerable for all partitions $\mathcal{B}$ of Ω it is called *fully negatively conglomerable*.

If we compare this definition with Definition 1, we observe that we need not be explicit about f or Bf being different from 0, since the 0 gamble is not desirable by assumption.

In analogy with the previous subsection, we shall first of all study its connection with the dual negative conditions of Co1–Co4. We consider thus:

Co5 $(\forall f \in \mathcal{L})\ \underline{P}(f|\mathcal{B}) \notin \mathcal{D} \Rightarrow f \notin \mathcal{D}$.
Co6 $\mathcal{D}$ is negatively $\mathcal{B}$-conglomerable.
Co7 $(\forall f \in \mathcal{L})\ \underline{P}(G(f|\mathcal{B})) \leq 0$.
Co8 $(\forall f \in \mathcal{L})\ \underline{P}(f) \leq \underline{P}(\underline{P}(f|\mathcal{B}))$.

The following proposition summarises the implications between them:

Proposition 2 (a) *In general, Co5 $\Rightarrow$ Co8 $\Rightarrow$ Co7 and Co5 $\Rightarrow$ Co6 $\Rightarrow$ Co7.*
(b) *When $\mathcal{D}$ is a set of strictly desirable gambles, Co5 $\Rightarrow$ Co8 $\Rightarrow$ Co6 $\Leftrightarrow$ Co7.*

Proof (a) We begin by establishing that Co5 $\Rightarrow$ Co8. Ex-absurdo, if there were some gamble f such that $\underline{P}(f) > \underline{P}(\underline{P}(f|\mathcal{B}))$ then by constant additivity there would be a real number μ such that $\underline{P}(f+\mu) > 0 > \underline{P}(\underline{P}(f+\mu|\mathcal{B}))$, whence $f+\mu \in \mathcal{D}$ while $\underline{P}(f+\mu|\mathcal{B})$ does not, a contradiction with Co5.
To prove that Co5 $\Rightarrow$ Co6, consider a gamble f such that $Bf \notin \mathcal{D}$ for any $B \in \mathcal{B}$. This means that $\underline{P}(f|B) \leq 0$, whence the gamble $\underline{P}(f|\mathcal{B})$ is non-positive and as a consequence it cannot belong to $\mathcal{D}$. Applying Co5, we conclude that $f \notin \mathcal{D}$.
Next we prove that Co6 $\Rightarrow$ Co7. Observe that for any gamble f, any $\epsilon > 0$ and any $B \in \mathcal{B}$ it holds that $G(f|B) - \epsilon B \notin \mathcal{D}$. Applying Co6, we deduce that $G(f|\mathcal{B}) - \epsilon \notin \mathcal{D}$ and as a consequence $\underline{P}(G(f|\mathcal{B}) - \epsilon) \leq 0$. Applying constant additivity, we deduce that $\underline{P}(G(f|\mathcal{B})) \leq \epsilon$ and since this holds for every $\epsilon > 0$ we conclude that $\underline{P}(G(f|\mathcal{B})) \leq 0$.
Finally, we prove that Co8 $\Rightarrow$ Co7. For this, note that

$$\underline{P}(G(f|\mathcal{B})) + \underline{P}(\underline{P}(f|\mathcal{B})) \leq \underline{P}(f) \leq \underline{P}(\underline{P}(f|\mathcal{B})) \Rightarrow \underline{P}(G(f|\mathcal{B})) \leq 0$$

for any gamble f, where the second inequality in the chain above follows from Co8.
(b) Let us establish that Co7 $\Rightarrow$ Co6 under strict desirability. Consider a gamble f such that $Bf \notin \mathcal{D}$ for any $B \in \mathcal{B}$. This implies in particular that $f \notin \mathcal{L}^+$. Then $\underline{P}(f|B) \leq 0$ for all B, whence $G(f|\mathcal{B}) \geq f$. Applying monotonicity and Co7, we obtain $\underline{P}(f) \leq \underline{P}(G(f|\mathcal{B})) \leq 0$. But then since we have already established that $f \notin \mathcal{L}^+$, we conclude that $f \notin \mathcal{D}$. $\square$

Figure 2 illustrates the result.

Again, let us give examples showing that no additional implication holds between these conditions.

Example 4 (*Co8 $\not\Rightarrow$ Co6*) Let $\Omega := \{1, 2, 3, 4\}$, $B := \{1, 2\}$, $\mathcal{B} := \{B, B^c\}$ and

$$\mathcal{D} := \left\{ f : \sum_i f(i) > 0 \text{ or } \sum_i f(i) = 0, \min\{f(1), f(3)\} > 0 \right\}.$$

Fig. 2 Implications between conditions Co5–Co8 for arbitrary coherent sets of gambles (left) and for coherent sets of strictly desirable gambles (right)

If we apply Eq. (2) we obtain that $\mathcal{D}$ induces the linear prevision P associated with the uniform distribution, which satisfies $P = P(P(\cdot|\mathcal{B}))$. Thus, condition Co8 holds. However, the gamble $f = (1, -1, 1, -1)$ belongs to $\mathcal{D}$ even if $Bf, B^c f \notin \mathcal{D}$. Therefore, $\mathcal{D}$ does not satisfy Co6. ♦

Note that this example, together with Proposition 2, allows us to conclude also that Co8 $\not\Rightarrow$ Co5 and that Co7 $\not\Rightarrow$ Co6.

Example 5 (*Co6 $\not\Rightarrow$ Co8*) Let $\Omega := \{1, 2, 3, 4\}$. For any $A \subseteq \Omega$ with $|A| = 2$, we denote by P_A the linear prevision given by $P_A(f) := \sum_{\omega \in A} 0.5 f(\omega)$. Let $\mathcal{D}_A := \{f : P_A(f) > 0\} \cup \mathcal{L}^+$ denote the set of strictly desirable gambles associated with P_A by means of Eq. (4), and define $\mathcal{D} := \cap_{|A|=2} \mathcal{D}_A$. Then $\mathcal{D}$ is a coherent set of strictly desirable gambles.

Consider the gamble f given by $f(i) := i$ for $i = 1, 2, 3, 4$, and let $B := \{1, 2\}$, $\mathcal{B} := \{B, B^c\}$. Then $\underline{P}(f) = 1.5$. On the other hand, for any $\mu > 1$ the gamble $B(f - \mu) = (1 - \mu, 2 - \mu, 0, 0) \notin \mathcal{D}$, while if $\mu < 1$ it is $B(f - \mu) \in \mathcal{L}^+ \subseteq \mathcal{D}$; thus, $\underline{P}(f|B) = 1$. Similarly, $\underline{P}(f|B^c) = 3$ and therefore $\underline{P}(\underline{P}(f|\mathcal{B})) = 1$. Thus, Co8 does not hold.

To see that on the other hand $\mathcal{D}$ is negatively $\mathcal{B}$-conglomerable, consider a gamble g such that $Bg, B^c g \notin \mathcal{D}$. Then it cannot be $Bg \in \mathcal{L}^+$, so either $Bg = 0$ or there is some $\omega \in B$ such that $g(\omega) < 0$. Similarly, since $B^c g \notin \mathcal{L}^+$ then either $B^c g = 0$ or there is some $\omega' \in B^c$ such that $g(\omega') < 0$. From this we deduce that either $g = 0$ (and then $g \notin \mathcal{D}$) or we can find an event A of cardinality two such that $P_A(g) < 0$, leading to $g \notin \mathcal{D}$. ♦

If we consider this example together with Proposition 2 we deduce that Co6 $\not\Rightarrow$ Co5 and that Co7 $\not\Rightarrow$ Co8.

With respect to the remaining implications in the right-hand side of Fig. 2, note that Example 5 already involves strictly desirable gambles. Let us then establish that Co8 $\not\Rightarrow$ Co5 under strict desirability:

Example 6 Let $\Omega := \{1, 2, 3, 4\}$, $B := \{1, 2\}$, $\mathcal{B} := \{B, B^c\}$. Consider the coherent lower prevision $\underline{P}$ given by

$$\underline{P}(f) := 0.5 \min\{f(1), f(2)\} + 0.5 \min\{f(3), f(4)\}$$

and let $\mathcal{D}$ be its associated set of strictly desirable gambles. Note that $\underline{P}(f|B) = \min\{f(1), f(2)\}$ and $\underline{P}(f|B^c) = \min\{f(3), f(4)\}$. It follows that $\underline{P}(f) = \underline{P}(\underline{P}(f|\mathcal{B}))$ for any gamble f, and Co8 holds. On the other hand, for the gamble $f := (1, 0, 1, 0) \in \mathcal{L}^+ \subseteq \mathcal{D}$, we have that $\underline{P}(f|\mathcal{B}) = 0 \notin \mathcal{D}$. Thus, Co5 does not hold. ♦

These examples also tell us that no additional implication holds when Ω is finite.

If we compare Figs. 1 and 2, we find that some of the implications that hold in the case of conglomerability do not hold for the analogous negative conditions, and vice versa. We think it is instructive to deepen a bit into this issue, to understand better how coherence and conglomerability intertwine. Specifically:

- While the condition $\underline{P}(f|\mathcal{B}) \in \mathcal{D} \Rightarrow f \in \mathcal{D}$ does not imply the $\mathcal{B}$-conglomerability of $\mathcal{D}$, we observe that the negative condition $\underline{P}(f|\mathcal{B}) \notin \mathcal{D} \Rightarrow f \notin \mathcal{D}$ implies that $\mathcal{D}$ is negatively conglomerable. The reason for this lies in the geometry of coherent sets of desirable gambles: a non-conglomerable set $\mathcal{D}$ may still satisfy condition Co1 due to the boundary behaviour, and we indeed observe in Proposition 1 that the implication Co1 $\Rightarrow$ Co2 holds when $\mathcal{D}$ is a coherent set of strictly desirable gambles.
- It holds that $\underline{P} \geq \underline{P}(\underline{P}(\cdot|\mathcal{B}))$ is equivalent to $\underline{P}(G(\cdot|\mathcal{B})) \geq 0$, while only the implication $\underline{P} \leq \underline{P}(\underline{P}(\cdot|\mathcal{B})) \Rightarrow \underline{P}(G(\cdot|\mathcal{B})) \leq 0$ holds. This is because a coherent lower prevision is super-additive COH3 $\underline{P}(f+g) \geq \underline{P}(f)+\underline{P}(g)$, and this is used in the proof of Proposition 1 to establish that $\underline{P}(G(\cdot|\mathcal{B})) \geq 0 \Rightarrow \underline{P} \geq \underline{P}(\underline{P}(\cdot|\mathcal{B}))$. However, in order to prove the implication $\underline{P}(G(\cdot|\mathcal{B})) \leq 0 \Rightarrow \underline{P} \leq \underline{P}(\underline{P}(\cdot|\mathcal{B}))$ we would need $\underline{P}$ to satisfy a subadditivity condition that will not hold in general.
- The same explanation lies underneath another discrepancy: we obtain that if $\mathcal{D}$ is conglomerable then $\underline{P} \geq \underline{P}(\underline{P}(\cdot|\mathcal{B}))$, but the negative conglomerability of $\mathcal{D}$ does not imply that $\underline{P} \leq \underline{P}(\underline{P}(\cdot|\mathcal{B}))$. If we examine the proof of Proposition 1, the implication Co2 $\Rightarrow$ Co4 makes use of the implication Co3 $\Rightarrow$ Co4, which in turn requires the super-additivity of $\underline{P}$.

Remark 1 When $\mathcal{D}$ is a coherent set of strictly desirable gambles, it is not difficult to obtain an alternative characterisation of negative conglomerability in terms of the associated coherent lower prevision $\underline{P}$ induced by $\mathcal{D}$: it holds that $\mathcal{D}$ satisfies Co6 if and only if for any gamble f,

$$(\forall B \in \mathcal{B})\ \underline{P}(Bf) \leq 0,\ Bf \notin \mathcal{L}^+ \Rightarrow \underline{P}(f) \leq 0. \tag{7}$$

- To prove that Co6 implies (7), observe that if $\underline{P}(Bf) \leq 0$, $Bf \notin \mathcal{L}^+$ for every $B \in \mathcal{B}$ then $Bf \notin \mathcal{D}$ for every B, whence by Co6 $f \notin \mathcal{D}$ and therefore $\underline{P}(f) \leq 0$.
- To prove that (7) implies Co6, note that if $Bf \notin \mathcal{D}$ for every B then it must be $\underline{P}(Bf) \leq 0$ and $Bf \notin \mathcal{L}^+$. This second condition entails that $f \notin \mathcal{L}^+$, and since by (7) $\underline{P}(f) \leq 0$ we deduce that $f \notin \mathcal{D}$, establishing that Co6 holds.

Observe on the other hand that Co6 does *not* imply that

$$(\forall B \in \mathcal{B})\ \underline{P}(Bf) \leq 0 \Rightarrow \underline{P}(f) \leq 0.$$

A counterexample can be built considering the lower prevision $\underline{P}$ and the gamble f in Example 5: there, it holds that $\underline{P}(Bf) = \underline{P}(B^c f) = 0$ while $\underline{P}(f) = 1.5$. This means that the condition $Bf \notin \mathcal{L}^+$ cannot be omitted in Eq. (7). ♦

3.3 Negative Conglomerability and Marginal Extension

In the above discussion we have seen some connections between positive and negative conglomerability and lower previsions that can be expressed as marginal extensions. The intuition here is that the underlying meaning of negative conglomerability may make sense when our information is either of local (conditional on an event B) or global (about the events B that determine the local levels) nature. In this section we explore the connection in more detail.

Definition 3 (*Marginal extension*) We say that a coherent lower prevision $\underline{P}$ is a *marginal extension* when it can be expressed as $\underline{P} = \underline{P}(\underline{P}(\cdot|\mathcal{B}))$ for some separately coherent conditional lower prevision $\underline{P}(\cdot|\mathcal{B})$.

When the partition $\mathcal{B}$ is finite, any linear prevision P can be expressed as $P = P(P(\cdot|\mathcal{B}))$, where $P(\cdot|\mathcal{B})$ is derived from P by Bayes' rule; this is simply a reformulation of the law of total probability in terms of gambles. However, a coherent lower prevision need not be a marginal extension in general, as shown for instance in Example 5.

We begin by giving a characterisation of marginal extensions in terms of sets of almost desirable gambles; observe the similarities with conditions Co1 and Co5.

Proposition 3 *Let $\mathcal{D}$ be a coherent set of desirable gambles and let $\underline{P}$, $\underline{P}(\cdot|\mathcal{B})$ be the unconditional and conditional lower previsions induced by $\mathcal{D}$ by means of Eqs.* (2), (3). *Let $\overline{\mathcal{D}}_{\underline{P}}$ denote the set of almost desirable gambles associated with $\underline{P}$ by means of* (5).

(a) $\underline{P} \geq \underline{P}(\underline{P}(\cdot|\mathcal{B})) \Leftrightarrow (\forall f \in \mathcal{L})[\underline{P}(f|\mathcal{B}) \in \overline{\mathcal{D}}_{\underline{P}} \Rightarrow f \in \overline{\mathcal{D}}_{\underline{P}}]$.
(b) $\underline{P} \leq \underline{P}(\underline{P}(\cdot|\mathcal{B})) \Leftrightarrow (\forall f \in \mathcal{L})[\underline{P}(f|\mathcal{B}) \notin \overline{\mathcal{D}}_{\underline{P}} \Rightarrow f \notin \overline{\mathcal{D}}_{\underline{P}}]$.

As a consequence,

$$\underline{P} = \underline{P}(\underline{P}(\cdot|\mathcal{B})) \Leftrightarrow (\forall f \in \mathcal{L})[\underline{P}(f|\mathcal{B}) \in \overline{\mathcal{D}}_{\underline{P}} \Leftrightarrow f \in \overline{\mathcal{D}}_{\underline{P}}].$$

Proof The proof is based on the following equivalence, valid for any pair of coherent lower previsions $\underline{P}$, $\underline{Q}$:

$$\underline{P} \leq \underline{Q} \Leftrightarrow (\forall f \in \mathcal{L})[\underline{Q}(f) < 0 \Rightarrow \underline{P}(f) < 0]. \tag{8}$$

The direct implication from $\underline{P} \leq \underline{Q}$ is trivial, while the converse follows from the constant additivity property that is satisfied by coherent lower previsions. For instance, if there is ex-absurdo some gamble f such that $\underline{P}(f) > \underline{Q}(f)$, then given $0 < \epsilon < \underline{P}(f) - \underline{Q}(f)$ and $f' = f - \underline{Q}(f) - \epsilon$ it holds that $\underline{Q}(f') = -\epsilon < 0 < \underline{P}(f) - \underline{Q}(f) - \epsilon = \underline{P}(f')$.

(a) The condition on the right-hand side can be reformulated as $\underline{P}(f) < 0 \Rightarrow \underline{P}(\underline{P}(f|\mathcal{B})) < 0$, which by Eq. (8) (with the lower previsions $\underline{P}(\underline{P}(\cdot|\mathcal{B}))$, $\underline{P}$) is equivalent to $\underline{P} \geq \underline{P}(\underline{P}(\cdot|\mathcal{B}))$.
(b) The condition on the right-hand side can be reformulated as $\underline{P}(\underline{P}(f|\mathcal{B})) < 0 \Rightarrow \underline{P}(f) < 0$, which, again by Eq. (8) is equivalent to $\underline{P} \leq \underline{P}(\underline{P}(\cdot|\mathcal{B}))$.

The last equivalence is an immediate consequence of items (a) and (b). □

By putting together Propositions 1 and 2 we may derive necessary or sufficient conditions for the lower prevision induced by $\mathcal{D}$ to be a marginal extension:

Corollary 1 *Let $\mathcal{D}$ be a coherent set of desirable gambles and let $\underline{P}$, $\underline{P}(\cdot|\mathcal{B})$ be the unconditional and conditional lower previsions it induces by means of Eqs.* (2), (3).

(a) $(\forall f \in \mathcal{L})[\underline{P}(f|\mathcal{B}) \in \mathcal{D} \Leftrightarrow f \in \mathcal{D}]$ *implies that* $\underline{P} = \underline{P}(\underline{P}(\cdot|\mathcal{B}))$.
(b) *If* $\mathcal{D}$ *is a coherent set of strictly desirable gambles, then* $\underline{P} = \underline{P}(\underline{P}(\cdot|\mathcal{B}))$ *implies that* $\mathcal{D}$ *is conglomerable and negatively conglomerable.*

Proof (a) This is a consequence of the implications Co1 $\Rightarrow$ Co4 in Proposition 1(a) and Co5 $\Rightarrow$ Co8 in Proposition 2(a).
(b) This follows from the implications Co4 $\Rightarrow$ Co2 in Proposition 1(b) and Co8 $\Rightarrow$ Co6 in Proposition 2(b). □

Note that the necessary conditions in Corollary 1(b) are not sufficient in general, meaning that the lower prevision induced by a conglomerable and negatively conglomerable set of desirable gambles need not take the form of the marginal extension; a counterexample is Example 5.

On the other hand, Example 4 shows that negative conglomerability is not a necessary condition for $\underline{P}$ to be a marginal extension model if $\mathcal{D}$ is not a set of strictly desirable gambles. Note moreover that that example gives a linear prevision on a finitary context.

4 (Negative) Conglomerable Combination of Sets of Desirable Gambles

In this section, we shall explore the suitability of negative conglomerability as a structural assessment. As we mentioned in Sect. 2, given a set of desirable gambles $\mathcal{K}$, it is possible to determine the smallest coherent superset, when a coherent superset exists: it is given by its natural extension. Similarly, in [10, 13] we investigated how conglomerability can be incorporated as a structural assessment, in the sense of determining the smallest coherent and conglomerable set of gambles that includes a given one.

Inspired by our work in the previous section, we recall that one scenario where the *natural* and the *conglomerable natural* extensions are easy to determine is when the gambles in our initial assessment are either $\mathcal{B}$-measurable or conditional with respect to one of the elements of $\mathcal{B}$. Specifically, assume that for a given partition $\mathcal{B}$ of Ω we have a set $\mathcal{D}_{\mathcal{B}}$ of $\mathcal{B}$-measurable gambles that is coherent with respect to $\mathcal{L}_{\mathcal{B}}$, and also that for each $B \in \mathcal{B}$ we have a set of desirable gambles $\mathcal{D}(B)$ that is coherent with respect to $\mathcal{L}(B)$. Then the smallest coherent set of gambles $\mathcal{D}$ satisfying[1]

$$\begin{cases} \mathcal{D} \cap \mathcal{L}_{\mathcal{B}} = \mathcal{D}_{\mathcal{B}} \\ \mathcal{D} \cap \mathcal{L}(B) = \mathcal{D}(B) \ \forall B \in \mathcal{B} \end{cases} \tag{9}$$

is the natural extension

[1] We are making a slight abuse of notation throughout this section, in that we are identifying a gamble f on B with its extension to Ω given by $g(\omega) = f(\omega)$ if $\omega \in B$ and $g(\omega) = 0$ otherwise. This is what allows us to make intersection $\mathcal{D} \cap \mathcal{L}(B)$, or consider the inclusion between a subset of $\mathcal{L}(B)$ and $\mathcal{D}$.

$$\begin{aligned}\mathcal{D} &:= \mathcal{E}_{(\mathcal{D}_{\mathcal{B}} \cup \bigcup_B \mathcal{D}(B))} \\ &= \{f \geq g + h : g \in \mathcal{D}_{\mathcal{B}} \cup \{0\}, (\forall B)\ Bh \in \mathcal{D}(B) \cup \{0\}, \text{supp}_{\mathcal{B}} h \text{ finite } \} \setminus \{0\},\end{aligned} \tag{10}$$

where $\text{supp}_{\mathcal{B}}(h) := \{B \in \mathcal{B} : Bh \neq 0\}$ denotes the support of the gamble h with respect to the partition $\mathcal{B}$.

On the other hand, the smallest coherent set of desirable gambles satisfying (9) and conglomerability is given by [13, Prop. 29]:

$$\tilde{\mathcal{D}} := \{f \geq g + h : g \in \mathcal{D}_{\mathcal{B}} \cup \{0\}, (\forall B)\ Bh \in \mathcal{D}(B) \cup \{0\}\} \setminus \{0\}. \tag{11}$$

It is not difficult to establish that the two sets above satisfy negative conglomerability.

Proposition 4 *The sets $\mathcal{D}, \tilde{\mathcal{D}}$ defined by Eqs. (10) and (11) satisfy Co6.*

Proof Let us establish the result for $\mathcal{D}$, the proof for $\tilde{\mathcal{D}}$ being analogous.

Consider a gamble $f \in \mathcal{D}$. Then there are gambles g, h such that $f \geq g + h$, where $g \in \mathcal{D}_{\mathcal{B}} \cup \{0\}$ and $Bh \in \mathcal{D}(B) \cup \{0\}$ for every B. Assume that $Bf \notin \mathcal{D}(B)$ for any $B \in \mathcal{B}$; since $f \neq 0$, this implies that at least one of g, h must be different from zero. In fact, it must always be $g \neq 0$, since otherwise it would be $f \geq h$ and as a consequence $Bf \in \mathcal{D}(B)$ whenever $Bh \neq 0$.

Now, if $0 \neq Bf \notin \mathcal{D}$ then given that $Bh \in \mathcal{D}(B) \cup \{0\}$ then it must be $g(B) < 0$, or we would arrive at a contradiction; but since $g \in \mathcal{D}_{\mathcal{B}}$ there must also be some B for which $g(B) > 0$, and in that case it should be $Bf = 0 \Rightarrow Bh < 0$, also a contradiction. □

More generally, if we consider a coherent set of desirable gambles $\mathcal{D}$ and for each $B \in \mathcal{B}$ we let

$$\mathcal{D}(B) := \mathcal{D} \cap \mathcal{L}(B), \tag{12}$$

then it can easily be seen that

$$\mathcal{D} \text{ is } \mathcal{B}\text{-conglomerable } \Leftrightarrow \bigoplus_B \mathcal{D}(B) \subseteq \mathcal{D} \tag{13}$$

while

$$\begin{aligned}\mathcal{D} \text{ is negatively } \mathcal{B}\text{-conglomerable} &\Leftrightarrow \bigoplus_B \mathcal{D}(B)^c \subseteq \mathcal{D}^c \\ &\Leftrightarrow \mathcal{D} \subseteq \left(\bigoplus_B \mathcal{D}(B)^c\right)^c,\end{aligned} \tag{14}$$

where for any set of gambles $\mathcal{K}(B) \subseteq \mathcal{L}(B)$ we denote

$$\bigoplus_B \mathcal{K}(B) := \{f : (\forall B)\ Bf \in \mathcal{K}(B) \cup \{0\}\} \setminus \{0\}. \tag{15}$$

From Eqs. (13), (14) we deduce that $\mathcal{D}$ is positively and negatively conglomerable if and only if

$$\bigoplus_B \mathcal{D}(B) \subseteq \mathcal{D} \subseteq \left(\bigoplus_B \mathcal{D}(B)^c\right)^c.$$

Note that the inclusion $\bigoplus_B \mathcal{D}(B) \subseteq \left(\bigoplus_B \mathcal{D}(B)^c\right)^c$ always holds, since the sets $\bigoplus_B \mathcal{D}(B)$ and $\bigoplus_B \mathcal{D}(B)^c$ are disjoint. From these considerations we can establish the following result:

Proposition 5 *Consider a partition $\mathcal{B}$ of Ω and let $\mathcal{D}(B)$ be a coherent set of desirable gambles with respect to $\mathcal{L}(B)$.*

(a) *Any coherent set of gambles $\mathcal{D}$ such that $\bigoplus_B \mathcal{D}(B) \subseteq \mathcal{D} \subseteq \left(\bigoplus_B \mathcal{D}(B)^c\right)^c$ determines the sets of gambles $\mathcal{D}(B)$, $B \in \mathcal{B}$ by means of Eq. (12), and it is therefore conglomerable and negatively conglomerable.*
(b) *The smallest such set is $\bigoplus_B \mathcal{D}(B)$.*

Proof (a) Let $B \in \mathcal{B}$ and denote $\mathcal{D}'(B) := \{f \in \mathcal{D} : f = Bf\}$. Since $\mathcal{D}(B) \subseteq \bigoplus_B \mathcal{D}(B) \subseteq \mathcal{D}$ and by definition any $f \in \mathcal{D}(B)$ satisfies $f = Bf$, it follows that $\mathcal{D}(B) \subseteq \mathcal{D}'(B)$. On the other hand, if $f \in \mathcal{D}'(B) \setminus \mathcal{D}(B)$ then $f \in \mathcal{D}(B)^c \subseteq \bigoplus_B \mathcal{D}(B)^c$, a contradiction with $f \in \mathcal{D}$.
(b) The coherence of $\bigoplus_B \mathcal{D}(B)$ is a consequence of the coherence of the sets $\mathcal{D}(B)$ for every $B \in \mathcal{B}$. □

If we want to have a set of desirable gambles that is more informative than $\bigoplus_B \mathcal{D}(B)$ and is at the same time conglomerable and negatively conglomerable, we may consider the conglomerable natural extension of $\mathcal{D}_{\mathcal{B}} \cup \bigcup_B \mathcal{D}(B)$, defined by Eq. (11).

On the other hand, the set $\left(\bigoplus_B \mathcal{D}(B)^c\right)^c$ is not coherent in general and as a consequence there is not a unique largest set of gambles that induces $\mathcal{D}(B)$ and is at the same time conglomerable and negatively conglomerable. This is illustrated by the following example:

Example 7 Consider $\Omega := \{1, 2, 3, 4\}$, $B := \{1, 2\}$, $\mathcal{B} := \{B, B^c\}$ and the coherent sets of desirable gambles $\mathcal{D}(B) := \mathcal{L}^+(B), \mathcal{D}(B^c) := \mathcal{L}^+(B^c)$. Given $f := (1, 1, -2, -2)$ and $g := (-2 - 2, 1, 1)$ it holds that $f, g \in (\bigoplus_B \mathcal{D}(B)^c)^c$, because $Bf \in \mathcal{D}(B)$ and $B^c g \in \mathcal{D}(B^c)$. However, $f + g = (-1, -1, -1, -1) \in \bigoplus_B \mathcal{D}(B)^c$, or, equivalently, $f + g$ does not belong to $(\bigoplus_B \mathcal{D}(B)^c)^c$. Thus, this set is not coherent. Consider now

$$\mathcal{D}_1 := \{f : Bf \in \mathcal{L}^+\} \cup \{f : Bf = 0, B^c f \in \mathcal{L}^+\}.$$

- $\mathcal{D}_1$ is coherent: $\mathcal{L}^+ \subseteq \mathcal{D}_1$ and $0 \notin \mathcal{D}_1$ by construction; $f \in \mathcal{D}_1, \lambda > 0 \Rightarrow \lambda f \in \mathcal{D}_1$, by considering the two sets of gambles that build $\mathcal{D}_1$ separately; and $f, g \in \mathcal{D}_1 \Rightarrow f + g \in \mathcal{D}_1$, again dealing with the two sections of $\mathcal{D}_1$ separately.
- By construction, $\bigoplus_B \mathcal{D}(B) \subseteq \mathcal{D}_1 \subseteq \left(\bigoplus_B \mathcal{D}(B)^c\right)^c$.

- To prove that $\mathcal{D}_1$ does not have a coherent superset included in $\left(\bigoplus_B \mathcal{D}(B)^c\right)^c$, we are going to show that for any gamble $f \in \left(\bigoplus_B \mathcal{D}(B)^c\right)^c \setminus \mathcal{D}_1$, the set $\mathcal{D}_1 \cup \{f\}$ incurs a sure loss, i.e., that $\mathcal{E}_{\mathcal{D}_1 \cup \{f\}}$ is not coherent.
  Consider indeed such a gamble f. Since $f \in \left(\bigoplus_B \mathcal{D}(B)^c\right)^c$, it means that it does not belong to $\bigoplus_B \mathcal{D}(B)^c$; as a consequence, either Bf does not belong to $\mathcal{D}(B)^c$ or $B^c f$ does not belong to $\mathcal{D}(B^c)^c$; equivalently, this means that either $Bf \in \mathcal{D}(B) = \mathcal{L}^+(B)$ or $B^c f \in \mathcal{D}(B^c) = \mathcal{L}^+(B^c)$. The first case cannot be, because by assumption $f \notin \mathcal{D}_1$, and any gamble satisfying $Bf \in \mathcal{L}^+(B)$ belongs to $\mathcal{D}_1$. Also, if it were $Bf = 0$ and $B^c f \in \mathcal{L}^+(B^c)$, then also we would deduce that $f \in \mathcal{D}_1$, which has been ruled out by assumption. We conclude then that there must be some $\omega \in B$ for which $f(\omega) < 0$ and also that $B^c f \in \mathcal{L}^+(B^c)$.
  Let $\mu := \min_B Bf < 0$ and $\mu' := \max f > 0$; assume w.l.o.g that it is $\mu = f(1)$. Then the natural extension $\mathcal{E}_{\mathcal{D}_1 \cup \{f\}}$ should include the gamble $g := f + (-0.5\mu, -0.5\mu, -2\mu', -2\mu')$ given that the gamble

$$(-0.5\mu, -0.5\mu, -2\mu', -2\mu')$$

  is strictly positive on B and therefore belongs to $\mathcal{D}_1$; but g does not belong to $\left(\bigoplus_B \mathcal{D}(B)^c\right)^c$, because $Bg \notin \mathcal{L}^+(B)$ (since $g(1) = 0.5\mu < 0$) and $B^c g \notin \mathcal{L}^+(B^c)$ (since $g(3), g(4) \leq -\mu' < 0$).

Thus, $\mathcal{D}_1$ is a set of gambles that satisfies conglomerability and negative conglomerability, induces $\mathcal{D}(B), \mathcal{D}(B^c)$ and is undominated with respect to set inclusion. But it is not the only one: with a completely similar reasoning we can establish that so is

$$\mathcal{D}_2 = \{f : B^c f \in \mathcal{L}^+\} \cup \{f : B^c f = 0, Bf \in \mathcal{L}^+\}.$$

As a consequence, there is not a unique set of gambles that is undominated in the partial order determined by set inclusion. ♦

4.1 A Negatively Conglomerable Extension?

If we endorsed negative conglomerability as a structural assessment, then we would want to be able to transform any model into one that satisfies this condition with a minimal change. This would be an analogous process to that of natural extension mentioned in Sect. 2.

Thus, assume that $\mathcal{D}$ is a coherent set of desirable gambles that does not satisfy negative conglomerability. If we consider a superset $\mathcal{D}'$, it then follows that $\mathcal{D}(B) \subseteq \mathcal{D}'(B)$ for all B, whence $\left(\bigoplus_B \mathcal{D}(B)^c\right)^c \subseteq \left(\bigoplus_B \mathcal{D}'(B)^c\right)^c$. Recalling Eq. (14), we conclude that making $\mathcal{D}$ more precise may allow us to satisfy negative conglomerability, but this need not be the case; and a similar comment can be made if we make $\mathcal{D}$ more imprecise. In this section, we explore these two avenues in more detail. We begin with that of making the assessments more precise, which corresponds to adding gambles to our initial set $\mathcal{D}$.

Example 8 Consider the set of desirable gambles $\mathcal{K}$ from Example 2 and its natural extension $\mathcal{D}$. Then $\mathcal{D}$ does not satisfy negative conglomerability, because $I_{2n} - I_{2n-1} \notin$

$\mathcal{D}$ for any $n \in \mathbb{N}$ but $I_{\{2n:n\in\mathbb{N}\}} - I_{\{2n-1:n\in\mathbb{N}\}}$ does. $\mathcal{D}$ has subsets that satisfy negative conglomerability: one such set is the natural extension of $\mathcal{K}\setminus\{I_{\{2n:n\in\mathbb{N}\}} - I_{\{2n-1:n\in\mathbb{N}\}}\}$, which is equal to

$$\mathcal{E} := \{f : P(f) > 0\} \cup \text{posi}(\{I_{2n-1} - I_{2n} : n \in \mathbb{N}\}).$$

To see that it satisfies negative conglomerability, consider a gamble f such that $B_n f \notin \mathcal{E}$ for every n. If $B_n f \neq 0$, then either $P(B_n f) < 0$ or $B_n f = \lambda_n(I_{2n} - I_{2n-1})$ for some $\lambda_n > 0$ and $P(B_n f) = 0$; if there is some n with $P(B_n f) < 0$ we deduce that $P(f) < 0$, whence $f \notin \mathcal{E}$. If on the contrary $B_n f = \lambda_n(I_{2n} - I_{2n-1})$ for some $\lambda_n > 0$ whenever $B_n f \notin \mathcal{E}$, we also deduce that $f \notin \text{posi}(\{I_{2n-1} - I_{2n} : n \in \mathbb{N}\})$, and therefore again $f \notin \mathcal{E}$.

On the other hand, no superset of $\mathcal{D}$ satisfies negative conglomerability, because coherence prevents us from adding $I_{2n} - I_{2n-1}$ to $\mathcal{D}$ for any n and so we get a violation of negative conglomerability with $f := I_{\{2n:n\in\mathbb{N}\}} - I_{\{2n-1:n\in\mathbb{N}\}}$. ♦

On the other hand, even if a set of desirable gambles has some superset satisfying negative conglomerability, it may not have a smallest such superset. This is because the family of coherent and negatively conglomerable sets of gambles is not closed under arbitrary intersections (that is, it does not form a belief structure in the sense of [3]), as our next example shows:

Example 9 Consider the same setting as in the previous example and let now $\mathcal{D}'$ be the natural extension of

$$\mathcal{K}' := \{f : P(f) > 0\} \cup \{I_{\{2n:n\in\mathbb{N}\}} - I_{\{2n-1:n\in\mathbb{N}\}}\}.$$

Then any gamble g in $\mathcal{D}'$ will satisfy either $P(g) > 0$ or $g = \lambda(I_{\{2n:n\in\mathbb{N}\}} - I_{\{2n-1:n\in\mathbb{N}\}})$ for some $\lambda > 0$. As a consequence, given $f := I_{\{2n:n\in\mathbb{N}\}} - I_{\{2n-1:n\in\mathbb{N}\}}$, it holds that $B_n f = I_{2n} - I_{2n-1}$ does not belong to $\mathcal{D}'$ for any $n \in \mathbb{N}$, given that $P(B_n f) = 0$. Since on the other hand $f \in \mathcal{D}'$ by construction, $\mathcal{D}'$ does not satisfy negative conglomerability. On the other hand, if we let $\mathcal{D}'_n$ be the natural extension of $\mathcal{K}' \cup \{I_{2n} - I_{2n-1}\}$, it can be checked that $\mathcal{D}'_n$ is a negatively conglomerable superset of $\mathcal{D}'$. However, it holds that $\mathcal{D}' = \cap_n \mathcal{D}'_n$, from which we deduce that $\mathcal{D}'$ has no minimal negatively conglomerable superset. ♦

If a set of desirable gambles is not negatively conglomerable, one strategy to find a negatively conglomerable superset would be to add some gamble Bf whenever f entails a violation of negative conglomerability. The crudest approximation would be then to consider the set

$$\mathcal{D}' := \{f \in \mathcal{D} : (\forall B \in \mathcal{B})\ Bf \notin \mathcal{D}\} \tag{16}$$

and take the natural extension $\mathcal{E}$ of

$$\mathcal{D} \cup \{Bf : f \in \mathcal{D}', -Bf \notin \mathcal{D}\}.$$

However, this procedure may not determine a coherent set of desirable gambles, even if such a set exists:

Example 10 Let $\Omega := \{1, 2, 3, 4, 5, 6\}$, $B_1 := \{1, 2\}$, $B_2 = \{3, 4\}$, $B_3 := \{5, 6\}$ and $\mathcal{B} := \{B_1, B_2, B_3\}$. Let P be the uniform distribution and let $\mathcal{D}$ be the natural extension of $\{f : P(f) > 0\} \cup \{g_1, g_2, g_3, g_4\}$, where $g_1 := (1, -2, 1, 0, 0, 0)$, $g_2 := (1, 0, 1, -2, 0, 0)$, $g_3 := (0, 0, -2, 1, 0, 1)$, $g_4 := (0, 0, 0, 1, -2, 1)$. It is not difficult to establish that $\mathcal{D}$ is coherent: indeed, for any gamble $g \in \mathcal{D}$ such that $P(g) = 0$, there are $\lambda_i \geq 0$, $i = 1, \ldots, 4$, not all of them equal to 0, such that $g = \sum_{i=1}^{4} \lambda_i g_i$; but if $\lambda_1 > 0$ or $\lambda_2 > 0$ we obtain $g(1) = \sum_{i=1}^{4} \lambda_i g_i(1) > 0$, and if $\lambda_3 > 0$ or $\lambda_4 > 0$ we obtain $g(6) = \sum_{i=1}^{4} \lambda_i g_i(6) > 0$. As a consequence, 0 cannot belong to $\mathcal{D}$ and this implies that this set is coherent.

It follows from coherence that both $f_1 := (1, -1, 1, -1, 0, 0) = 0.5 \cdot (1, -2, 1, 0, 0, 0) + 0.5 \cdot (1, 0, 1, -2, 0, 0)$ and $f_2 := (0, 0, -1, 1, -1, 1) = 0.5 \cdot (0, 0, -2, 1, 0, 1) + 0.5 \cdot (0, 0, 0, 1, -2, 1)$ belong to $\mathcal{D}$. Moreover, Bf_1 and Bf_2 do not belong to $\mathcal{D}$ for any $B \in \mathcal{B}$:

- In the case of f_1, it follows from our comments above in the case of B_2, B_3; on the other hand, $B_1 f_1 = (1, -1, 0, 0, 0, 0)$ cannot be expressed as a linear combination of g_1, g_2, whence it does not belong to $\mathcal{D}$.
- In the case of f_2, it follows from our comments above in the case of B_1, B_2; on the other hand, $B_3 f_2 = (0, 0, 0, 0, -1, 1)$ cannot be expressed as a linear combination of g_3, g_4, whence it does not belong to $\mathcal{D}$.

As a consequence, $f_1, f_2 \in \mathcal{D}'$. Thus, if we follow the above procedure $\mathcal{E}$ should include both $(0, 0, 1, -1, 0, 0)$ and $(0, 0, -1, 1, 0, 0)$, leading to an incoherence.

Nevertheless $\mathcal{D}$ has some negatively conglomerable superset: it suffices to consider the natural extension $\mathcal{E}^*$ of $\mathcal{D} \cup \{(1, -1, 0, 0, 0, 0), (0, 0, 0, 0, -1, 1)\}$:

- $0 \notin \mathcal{E}^*$ because, reasoning as before, we can prove that any $f \in \mathcal{E}^*$ such that $P(f) = 0$ will satisfy $f(1) > 0$ or $f(6) > 0$. As a consequence, $\mathcal{E}^*$ is coherent.
- To see that $\mathcal{E}^*$ is negatively conglomerable, consider f such that $Bf \notin \mathcal{E}^*$ for all $B \in \mathcal{B}$ but $f \in \mathcal{E}^*$. Then it should be $P(Bf) \leq 0$ for all B and $P(f) \geq 0$, from which it follows that $P(f) = 0$, $P(Bf) = 0$ for all B and that either $f(1) > 0$ or $f(6) > 0$. In the first case, $B_1 f = (\lambda, -\lambda, 0, 0, 0, 0)$ for $\lambda > 0$, whence $B_1 f \in \mathcal{E}^*$; in the second, $B_3 f = (0, 0, 0, 0, -\mu, \mu)$ for $\mu > 0$, whence $B_3 f \in \mathcal{E}^*$. ♦

Remark 2 The above example suggests a procedure for establishing a negatively conglomerable extension: for any gamble $f \in \mathcal{D}'$ we make a selection $H(f) \in \mathcal{B}$ satisfying $H(f)f, -H(f)f \notin \mathcal{D}$, and then take $\mathcal{E}_{\mathcal{D} \cup \mathcal{D}_1}$, for

$$\mathcal{D}_1 := \{H(f)f : f \in \mathcal{D}'\}.$$

It can be checked that if $\mathcal{E}_{\mathcal{D} \cup \mathcal{D}_1}$ is coherent, then it is negatively conglomerable: any element g of $\mathcal{E}_{\mathcal{D} \cup \mathcal{D}_1}$ will be of the form $g = \lambda_1 g_1 + \sum_{i=2}^{n} \lambda_i g_i$, for $g_1 \in \mathcal{D}$, $g_2, \ldots, g_n \in \mathcal{D}_1$, $\lambda_1, \ldots, \lambda_n \geq 0$. If g entails a violation of negative conglomerability then for any B with $Bg \neq 0$ it should be $Bg = B(\lambda_1 g_1 + \sum_{i=2}^{n} \lambda_i g_i)$; noting that $Bg_i \in \mathcal{E}_{\mathcal{D} \cup \mathcal{D}_1} \cup \{0\}$

for $i = 2, \ldots, n$, we deduce that $Bg_1 \notin \mathcal{E}_{\mathcal{D}\cup\mathcal{D}_1}$ for any $B \in \mathcal{B}$. But then this implies in particular that $Bg_1 \notin \mathcal{D}$ for every $B \in \mathcal{B}$, whence $g_1 \in \mathcal{D}'$; by construction we should have $Bg_1 \in \mathcal{E}_{\mathcal{D}\cup\mathcal{D}_1}$ for $B = H(g)$, so we obtain a contradiction.

Nevertheless, a word of caution should be given: the procedure above may not always be practical, in that the evaluation of all the possible selections H for the gambles f where negative conglomerability is violated may not be computable depending on the size of the partition $\mathcal{B}$ and of the class $\mathcal{D}'$. ♦

It is somewhat instructive to compare the situation with the condition of conglomerability. Although the conglomerable natural extension may not exist [13, Remark 1], when a coherent set of desirable gambles has a coherent and conglomerable superset then it has a smallest such superset, in contradistinction with what we have seen for negative conglomerability.

Since the procedure of considering a negatively conglomerable superset may not always be applicable, next we go in the opposite direction: we look for a subset of $\mathcal{D}$ that satisfies negative conglomerability. Note that, unlike what happened before, such a set always exists: we can for instance take the set $\mathcal{L}^+$ of non-negative gambles. Our goal is then to look for the greatest such subset, which would correspond to the idea of making the minimal correction from our set $\mathcal{D}$ that achieves negative conglomerability. In this sense, it is not difficult to establish an upper bound of this set:

Proposition 6 *Let $\mathcal{D}$ be a set of gambles that is coherent and conglomerable, and let $\mathcal{D}'$ be given by Eq. (16).*

(a) *For every $f \in \mathcal{D}'$ there is some $B \in \mathcal{B}$ such that $B(-f) \notin \mathcal{D}$.*
(b) *Let $\mathcal{E}$ be the natural extension of $\mathcal{D} \setminus \mathcal{D}'$. If it is negatively conglomerable, then it is the largest subset of $\mathcal{D}$ to be so.*

Proof We begin with the first statement. Ex-absurdo, if for a given $f \in \mathcal{D}'$ it held that $B(-f) \in \mathcal{D}$ for every $B \in \mathcal{B}$ then by conglomerability it would be $-f \in \mathcal{D}$, and this, together with coherence, contradicts that $f \in \mathcal{D}' \subseteq \mathcal{D}$.

For the second statement, note that the coherence of $\mathcal{D}$ implies the coherence of $\mathcal{E}$. On the other hand, for any $\mathcal{D}^* \subseteq \mathcal{D}$ that is negatively conglomerable it must be $\mathcal{D}^* \cap \mathcal{D}' = \emptyset$, since otherwise there would be some gamble $f \in \mathcal{D}^*$ such that $Bf \notin \mathcal{D}^*$ for every B, entailing a violation of negative conglomerability. Therefore, $\mathcal{D}^* \subseteq \mathcal{E}$. Thus, if $\mathcal{E}$ is negatively conglomerable then it is the largest such subset of $\mathcal{D}$. □

Nevertheless, the set $\mathcal{E}$ in the above construction need not satisfy negative conglomerability:

Example 11 Let $\Omega := \{1, 2, 3, 4\}$, $B := \{1, 2\}$, $\mathcal{B} := \{B, B^c\}$ and P the linear prevision associated with the uniform distribution. Let $\mathcal{D}$ be the natural extension of $\{f : P(f) > 0\} \cup \{(1, -2, 1, 0), (1, 0, 1, -2)\}$. Then $\mathcal{D}$ is coherent, observing that we cannot obtain the zero gamble as a positive linear combination of the gambles $(1, -2, 1, 0), (1, 0, 1, -2)$. To illustrate that it violates negative conglomerability, let $f := (1, -1, 1, -1)$. Then

$$f = 0.5 \cdot (1, -2, 1, 0) + 0.5 \cdot (1, 0, 1, -2) \Rightarrow f \in \mathcal{D};$$

however, $Bf = (1, -1, 0, 0) \notin \mathcal{D}$ because $P(Bf) = 0$ and the only gambles in $\mathcal{D}$ with zero prevision are the linear combinations of $(1, -2, 1, 0)$, $(1, 0, 1, -2)$, which are always strictly positive in 1 and on 3. Similarly, $B^c f = (0, 0, 1, -1) \notin \mathcal{D}$ and as a consequence negative conglomerability is violated.

Let us establish that

$$\mathcal{D}' = \{(\lambda_1 + \lambda_2, -2\lambda_1, \lambda_1 + \lambda_2, -2\lambda_2) : \lambda_1, \lambda_2 > 0\}.$$

First of all, $\mathcal{D}' \subseteq \{f \in \mathcal{D} : P(f) = 0\}$, since $P(f) > 0$ implies that either $P(Bf) > 0$ or $P(B^c f) > 0$ and in any of the two cases we conclude that Bf or $B^c f$ belongs to $\mathcal{D}$. Secondly,

$$\{f \in \mathcal{D} : P(f) = 0\} = \{(\lambda_1 + \lambda_2, -2\lambda_1, \lambda_1 + \lambda_2, -2\lambda_2) : \lambda_1, \lambda_2 \geq 0, \max\{\lambda_1, \lambda_2\} > 0\}.$$

However, when $\lambda_1 = 0$ in the equation above we obtain that $Bf \in \mathcal{D}$ and when $\lambda_2 = 0$ we get $B^c f \in \mathcal{D}$. Thus, it must be $\lambda_1, \lambda_2 > 0$ for f to belong to $\mathcal{D}'$.

From this we deduce that the natural extension of $\mathcal{D} \backslash \mathcal{D}'$ shall still include the natural extension of $\{f : P(f) > 0\} \cup \{(1, -2, 1, 0), (1, 0, 1, -2)\}$, and as a consequence we obtain that $\mathcal{E} = \mathcal{D}$.

This also tells us that there may not be a largest subset of $\mathcal{D}$ that is negatively conglomerable: for instance, the natural extension $\mathcal{D}_1$ of $\{f : P(f) > 0\} \cup \{(1, -2, 1, 0)\}$ is negatively conglomerable, because the only gambles it includes with zero prevision are of the type $(\lambda, -2\lambda, \lambda, 0)$ for some $\lambda > 0$, and these cannot be used to violate negative conglomerability; similarly, the natural extension $\mathcal{D}_2$ of $\{f : P(f) > 0\} \cup \{(1, 0, 1, -2)\}$ also satisfies negative conglomerability. But the only coherent superset of $\mathcal{D}_1 \cup \mathcal{D}_2$ that is included in $\mathcal{D}$ is $\mathcal{D}$ itself, which does not satisfy negative conglomerability. ♦

5 The Precise Case

In the previous section we have studied the basic properties of negative conglomerability in contraposition with those related to the conglomerability of a coherent set of desirable gambles. Next we shall analyse in more detail the case where the set of desirable gambles is associated with a linear prevision.

We begin by investigating if in that case we can establish additional implications to those already present in Proposition 2. In this respect, it is quite immediate to prove the following:

Proposition 7 *If $\mathcal{D}$ induces a linear prevision P and a linear conditional prevision $P(\cdot|\mathcal{B})$, then:*

(a) *Co7 $\Rightarrow$ Co8;*
(b) *Co4 $\Leftrightarrow$ Co8;*
(c) *Co3 $\Leftrightarrow$ Co7.*

Proof (a) To prove that Co7 $\Rightarrow$ Co8, simply observe that for any gamble f, it holds that $0 \geq P(G(f|\mathcal{B})) = P(f - P(f|\mathcal{B})) = P(f) - P(P(f|\mathcal{B}))$, whence $P \leq P(P(\cdot|\mathcal{B}))$.

(b) For Co4 $\Leftrightarrow$ Co8, note that $P(P(\cdot|\mathcal{B})) \leq P \Leftrightarrow P(P(\cdot|\mathcal{B})) = P \Leftrightarrow P(P(\cdot|\mathcal{B})) \geq P$, since if there was some gamble f such that $P(P(f|\mathcal{B})) < P(f)$ then by conjugacy it would be $P(P(-f|\mathcal{B})) > P(-f)$.

(c) With a similar reasoning we obtain that Co3 $\Leftrightarrow$ Co7: we only need to establish that $P(G(\cdot|\mathcal{B})) \leq 0 \Leftrightarrow P(G(\cdot|\mathcal{B})) = 0 \Leftrightarrow P(G(\cdot|\mathcal{B})) \geq 0$, using again that by conjugacy $P(G(f|\mathcal{B}) < 0$ implies that $P(G(-f|\mathcal{B})) > 0$.

□

It is also immediate to establish that conglomerability and negative conglomerability are equivalent in the particular case where $\mathcal{D}$ is a maximal set of desirable gambles:

Proposition 8 *Let $\mathcal{D}$ be a maximal set of desirable gambles.*

(a) *$\mathcal{D}$ is $\mathcal{B}$-conglomerable $\Leftrightarrow$ $\mathcal{D}$ is negatively $\mathcal{B}$-conglomerable.*

(b) *Co5 $\Rightarrow$ Co1.*

Proof (a)($\Rightarrow$) Consider a gamble $f \neq 0$ such that $Bf \notin \mathcal{D}$ for every $B \in \mathcal{B}$. By maximality, it follows that $B(-f) = -Bf \in \mathcal{D} \cup \{0\}$ for every B. Applying conglomerability, it follows that $-f \in \mathcal{D}$, since $-f \neq 0$, and by coherence $f \notin \mathcal{D}$.

($\Leftarrow$) Conversely, consider a gamble $f \neq 0$ such that $Bf \in \mathcal{D} \cup \{0\}$ for every $B \in \mathcal{B}$. Then $-Bf \notin \mathcal{D}$ for every $B \in \mathcal{B}$, whence by negative conglomerability $-f \notin \mathcal{D}$ and by maximality of $\mathcal{D}$ we deduce that $f \in \mathcal{D}$.

(b) Consider a gamble f such that $P(f|\mathcal{B}) \in \mathcal{D}$. If it was $f \notin \mathcal{D}$ then, since it must be $f \neq 0$ (or by coherence it would be $P(f|\mathcal{B}) = 0 \notin \mathcal{D}$), maximality would imply that $-f \in \mathcal{D}$, whence by Co5 $P(-f|\mathcal{B}) \in \mathcal{D}$, a contradiction. □

Figure 3 summarises the implications between the conditions under maximality. Let us show that no additional implication holds:

Example 12 (*Co6 $\not\Rightarrow$ Co5; Co2 $\not\Rightarrow$ Co1; Co3 $\not\Rightarrow$ Co1*). Let $\Omega := \{1, 2, 3, 4\}$, $B := \{1, 2\}$, $\mathcal{B} := \{B, B^c\}$, P the uniform probability distribution and $\mathcal{D}$ the maximal set of gambles given by $\mathcal{D} = \{f : P(f) > 0\} \cup \mathcal{D}'$, where

$$\mathcal{D}' = \{f : P(f) = 0 \text{ and } (f(1) > 0 \text{ or } 0 = f(1) < f(2) \text{ or } 0 = f(1) = f(2) < f(3))\}.$$

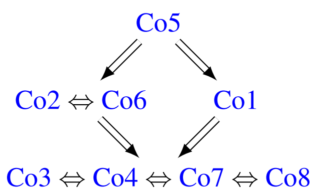


Fig. 3 Implications between conditions Co1–Co8 for maximal coherent sets of desirable gambles

This set is coherent: it holds that $\mathcal{L}^+ \subseteq \{f : P(f) > 0\} \subseteq \mathcal{D}$; $0 \notin \mathcal{D}$ by construction; and given $f_1, \ldots, f_n \in \mathcal{D}$ and $\lambda_1, \ldots, \lambda_n > 0$, if $P(f_i) > 0$ for some i we deduce that $P(\sum_{i=1}^n \lambda_i f_i) > 0$, whence $\sum_{i=1}^n \lambda_i f_i \in \mathcal{D}$; and if $P(f_i) = 0$ for every i then it must be $f_i \in \mathcal{D}'$ for all i, whence $\sum_{i=1}^n \lambda_i f_i \in \mathcal{D}' \subseteq \mathcal{D}$.

Now, given $g := (-1, 2, -1, 0) \notin \mathcal{D}$ we obtain that $P(g|\mathcal{B}) = (0.5, 0.5, -0.5, -0.5) \in \mathcal{D}$, whence Co1 does not hold. Since $\mathcal{D}$ is coherent and $\mathcal{B}$ is finite we deduce that it is conglomerable, and applying Proposition 8, also negatively conglomerable. The same proposition tells us that Co5 does not hold, given that we have established that Co1 does not. Finally, since from Proposition 1, Co2 implies Co3, we deduce that Co3 does not imply Co1 either. ♦

Example 13 (*Co1 $\not\Rightarrow$ Co2; Co4 $\not\Rightarrow$ Co2; Co1 $\not\Rightarrow$ Co5*). Let $\mathcal{D}$ be the coherent set of desirable gambles from Example 2; for any maximal superset $\mathcal{D}'$ it holds that the gamble $f := I_{\{2n-1:n\in\mathbb{N}\}} - I_{\{2n:n\in\mathbb{N}\}} \notin \mathcal{D}'$ because $-f \in \mathcal{D} \subseteq \mathcal{D}'$ but $Bf \in \mathcal{D}'$ for every $B \in \mathcal{B}$, whence Co2 is not satisfied.

Let us give a maximal superset of $\mathcal{D}$ satisfying Co1. For this, consider

$$\mathcal{D}_1 := \left\{f : P(P(f|\mathcal{B})) = 0, P(f|B_{n_f}) > 0, \text{ where } n_f := \min\{n : P(f|B_n) \neq 0\}\right\}$$

and $\mathcal{D}'$ a maximal superset of $\mathcal{D} \cup \mathcal{D}_1$. To ensure that such a set exists, we need to show that $\mathcal{D} \cup \mathcal{D}_1$ avoids sure loss, that is, that $\mathcal{E}_{\mathcal{D}\cup\mathcal{D}_1}$ is coherent. Looking at the definitions in Example 2, we deduce that the elements in $\mathcal{D} \cup \mathcal{D}_1$ satisfy either $P(f) > 0$, $P(f|B_{n_f}) > 0$ or are a positive linear combination of gambles in

$$\{I_{\{2n:n\in\mathbb{N}\}} - I_{\{2n-1:n\in\mathbb{N}\}}\} \cup \{I_{2n-1} - I_{2n} : n \in \mathbb{N}\}.$$

But then it is not possible to obtain 0 in $\text{posi}(\mathcal{D} \cup \mathcal{D}_1)$, from which we deduce that this set is coherent, and as a consequence it has some maximal superset.

Next, if $P(f|\mathcal{B}) \in \mathcal{D}'$ then either $P(P(f|\mathcal{B})) = P(f) > 0$ or $P(P(f|\mathcal{B})) = 0$ and the smallest non-zero value of $P(f|\mathcal{B})$ is positive, in which case also $f \in \mathcal{D}_1 \subseteq \mathcal{D}'$. Thus, Co1 holds.

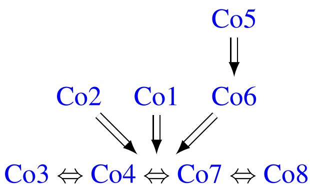
This shows that Co1 $\not\Rightarrow$ Co2 and therefore also Co4 $\not\Rightarrow$ Co2 and Co1 $\not\Rightarrow$ Co5. ♦

On the other hand, Figure 4 summarises the implications between the conditions in the precise case (when $\mathcal{D}$ is not necessarily maximal).

Indeed, the equivalence in Proposition 8(a) does not hold in this scenario, as shown by Example 4 (where $\mathcal{D}$ is conglomerable but not negatively conglomerable) and [13, Example 2] (where $\mathcal{D}$ is not conglomerable but it is negatively conglomerable).

In this case Co5 implies neither Co2 nor Co1:

Fig. 4 Implications between conditions Co1–Co8 when $\mathcal{D}$ induces precise P, $P(\cdot|\mathcal{B})$



Example 14 Consider $\Omega := \mathbb{N}$, $B_n := \{2n-1, 2n\}$, $\mathcal{B} := \{B_n : n \in \mathbb{N}\}$, and let $P_{\mathcal{B}}$ be a linear prevision on $\mathcal{L}_{\mathcal{B}}$ satisfying $P_{\mathcal{B}}(B_n) = 0$ for all n. Consider on the other hand $P(\cdot|B_n)$ the uniform linear prevision, and let $\mathcal{D}_{\mathcal{B}}$, $\mathcal{D}(B)$, $B \in \mathcal{B}$ be the associated sets of strictly desirable gambles. Let $\mathcal{D}$ be the natural extension of $\mathcal{D}_{\mathcal{B}} \cup \bigcup_n \mathcal{D}(B_n)$, given by Eq. (10). Then $\mathcal{D}$ induces the linear prevision $P := P_{\mathcal{B}}(P(\cdot|\mathcal{B}))$.

$\mathcal{D}$ does not satisfy Co2, since the gamble f given by $f(2n-1) = 1 + \frac{1}{n}$, $f(2n) = -1$ for every n does not belong to $\mathcal{D}$, but $B_n f \in \mathcal{D}(B_n) \subseteq \mathcal{D}$ for every n because $P(f|B_n) > 0$.

On the other hand, it satisfies Co5: if $f \in \mathcal{D}$ then it is $f \geq g + h$ for some $g \in \mathcal{D}_{\mathcal{B}}$, $h \in \mathcal{E}_{\cup_n \mathcal{D}(B_n)}$. From this it follows that $P(f|B_n) \geq P(g|B_n) + P(h|B_n) \geq P(g|B_n) = g(B_n)$ and as a consequence $P(f|\mathcal{B})$ belongs to $\mathcal{D}_{\mathcal{B}} \subseteq \mathcal{D}$. Thus,

$$f \in \mathcal{D} \Rightarrow P(f|\mathcal{B}) \in \mathcal{D},$$

which is an equivalent formulation of Co5.

Finally, to show that Co1 does not hold either, we are going to use that, in the precise case,

$$\text{Co5} + \text{Co1} \Rightarrow \text{Co2}.$$

Indeed, if $Bf \in \mathcal{D} \cup \{0\}$ for every B then it must be $P(f|B) > 0$ whenever $Bf \neq 0$ (or the gamble $g = Bf$ would entail a violation of Co5), whence $P(f|\mathcal{B}) \in \mathcal{L}^+ \subseteq \mathcal{D}$ and then by Co1 it follows that $f \in \mathcal{D}$.

Since we know that in this case $\mathcal{D}$ satisfies Co5 and not Co2, we deduce that it cannot satisfy Co1 either. ♦

5.1 Full Negative Conglomerability

Should we endorse negative conglomerability as a structural assessment, we may do it irrespective of the partition $\mathcal{B}$ on which the partial preferences are dependent on. This procedure leads to what we called in Definition 2 *full negative conglomerability*, in analogy to the notion of full conglomerability discussed by Walley in [24, Section 6.9] and by Schervisch, Seidenfeld and Kadane in [18]; full conglomerability is close, but not equivalent to, countable additivity [11]. Let us investigate next what would be the implications of full negative conglomerability, and in particular if it also has a connection with countable additivity.

In the case of full conglomerability, we proved in [13] that $\underline{P}$ is fully conglomerable if and only if we have the equality $\underline{P} = \sup_{\mathcal{B}} \underline{P}(\underline{P}(\cdot|\mathcal{B}))$. On the other hand, the set of desirable gambles $\mathcal{D}$ in Example 5 satisfies full conglomerability (trivially, because we have a finite referential space) and also full negative conglomerability, and the unconditional and conditional lower previsions it induces satisfies $\underline{P}(f) > \underline{P}(\underline{P}(f|\mathcal{B}))$ for certain gamble f and a certain partition $\mathcal{B}$. The example also shows that full negative conglomerability does not imply linearity of the associated lower prevision; a simpler example can be built by considering the vacuous coherent set of desirable gambles $\mathcal{L}^+$.

On the other hand, we can derive the following from Proposition 2:

Proposition 9 *Let $\mathcal{D}$ be a coherent set of desirable gambles and let $\underline{P}$, $\underline{P}(\cdot|\mathcal{B})$ be the unconditional and conditional lower previsions it induces by means of Eqs.* (2), (3).

(a) *If $\mathcal{D}$ is a set of strictly desirable gambles, then $\underline{P} \leq \inf_{\mathcal{B}} \underline{P}(\underline{P}(\cdot|\mathcal{B})) \Rightarrow \mathcal{D}$ fully negatively conglomerable, which in turn holds iff $\underline{P}(G(f|\mathcal{B})) \leq 0$ for every f and every $\mathcal{B}$.*
(b) *If $\mathcal{D}$ is a maximal set of gambles, then it is fully negatively conglomerable iff it is fully conglomerable.*

Proof (a) This follows from Co8⇒Co6⇔Co7 in Proposition 2(b).
(b) This is a consequence of Proposition 8(a). □

6 Negative Conglomerability and Dilation

There is an interesting connection between the notion of negative conglomerability and the phenomenon of *dilation* for sets of probabilities, discussed in [6, 7, 20, 22]. Dilation occurs when our beliefs become more imprecise no matter which is the conditioning event we observe, and it is something that can take place even when the exhaustive family of pairwise disjoint events is finite.

Definition 4 (*Dilation*) Let $\mathcal{D}$ be a coherent set of desirable gambles and let $\underline{P}$, $\underline{P}(\cdot|\mathcal{B})$ be the unconditional and conditional coherent lower prevision it induces by means of Eqs. (2), (3). Let $\overline{P}$, $\overline{P}(\cdot|\mathcal{B})$ be their conjugate *upper* previsions, given by $\overline{P}(f) = -\underline{P}(-f)$ and $\overline{P}(f|\mathcal{B}) = -\underline{P}(-f|\mathcal{B})$ for every $f \in \mathcal{L}$. We say that $\underline{P}$ *weakly dilates* at a gamble f when

$$(\forall B \in \mathcal{B})\ \underline{P}(f|B) \leq \underline{P}(f) \leq \overline{P}(f) \leq \overline{P}(f|B),$$

and that it *strictly dilates* at f when

$$(\forall B \in \mathcal{B})\ \underline{P}(f|B) < \underline{P}(f) \leq \overline{P}(f) < \overline{P}(f|B).$$

As we shall show next, when our model suffers from dilation at a gamble f, it will also violate the condition of negative conglomerability under some mild conditions:

Proposition 10 *Let $\mathcal{D}$ be a coherent set of desirable gambles and let $\underline{P}$, $\underline{P}(\cdot|\mathcal{B})$ be the unconditional and conditional coherent lower prevision it induces by means of Eqs.* (2), (3). *Let f be a gamble where $\underline{P}$ strictly dilates and such that*

$$\sup_B \underline{P}(f|B) < \underline{P}(f) \leq \overline{P}(f) < \inf_B \overline{P}(f|B), \tag{17}$$

Then there is some $\mu \in \mathbb{R}$ such that $\mathcal{D}$ violates negative conglomerability on $f - \mu$.

Proof By constant additivity, there exists some $\mu \in \mathbb{R}$ such that $\underline{P}(f + \mu|B) < 0 < \underline{P}(f + \mu)$ for every $B \in \mathcal{B}$. This means that $f + \mu$ belongs to the set of desirable gambles $\mathcal{D}$, while from Eq. (3) it follows that $B(f + \mu) \notin \mathcal{D}$ for any $B \in \mathcal{B}$. Thus, $f + \mu$ entails a violation of negative conglomerability. □

We should remark that condition (17) is equivalent to strict dilation when the partition $\mathcal{B}$ is finite, while it is slightly more restrictive in the case of infinite partitions $\mathcal{B}$.

On the other hand, it may be that a set of desirable gambles $\mathcal{D}$ does not satisfy negative conglomerability with a gamble f but there is no dilation on $f + \mu$ for any real number μ, as our next example illustrates:

Example 15 Consider $\Omega := \{1, 2, 3, 4\}$, $B := \{1, 2\}$, $\mathcal{B} := \{B, B^c\}$. Take the linear previsions associated with $P_1 := (0.1, 0.1, 0.3, 0.5)$, $P_2 := (0.1, 0.7, 0.1, 0.1)$, and let $\underline{P} := \min\{P_1, P_2\}$. Give the gamble $f := (-5, 5, -5, 5)$, we obtain:

- $P_1(Bf) = 0$, $P_1(B^c f) = 1 \Rightarrow P_1(f) = 1$;
- $P_2(Bf) = 3$, $P_2(B^c f) = 0 \Rightarrow P_2(f) = 3$.

Thus, $\underline{P}(f) = 1$, $\overline{P}(f) = 3$. On the other hand,

- $P_1(f|B) = 0$, $P_1(f|B^c) = 1.25$;
- $P_2(f|B) = 3.75$, $P_2(f|B^c) = 0$.

Thus, $[\underline{P}(f|B), \overline{P}(f|B)] = [0, 3.75]$ and $[\underline{P}(f|B^c), \overline{P}(f|B^c)] = [0, 1.25]$, meaning that the model does not dilate in f. It also follows that there is no dilation on $f + \mu$ for any real number μ either, given that the lengths of the intervals determined by the lower and upper previsions will not change, due to constant additivity.

On the other hand, since $\underline{P}(Bf) = 0 = \underline{P}(B^c f)$ it follows that Bf, $B^c f$ are not strictly desirable and therefore they do not belong to $\mathcal{D}$, while f does because $\underline{P}(f) > 0$. Thus, the set of strictly desirable gambles associated with $\underline{P}$ does not satisfy negative conglomerability. ♦

As an illustration of Proposition 10, next we analyse [24, Example 6.4.3] from the point of view of negative conglomerability:

Example 16 A fair coin is tossed twice, so that the second toss may be dependent on the first one but we ignore the degree of dependence. This means that the probability of Heads is known to be equal to 0.5 both for the first and for the second toss, but the second toss may always give the same outcome as the first, always the opposite, or something in between. We may model this scenario by considering the possibility space $\Omega := \{(H_1, H_2), (H_1, T_2), (T_1, H_2), (T_1, T_2)\}$ and by letting $\underline{P}$ be the lower envelope of the linear previsions associated with the mass functions $(0.5, 0, 0, 0.5)$, $(0, 0.5, 0.5, 0)$. If we consider the partition $\mathcal{B} := \{B, B^c\}$, where

$$B := \{(H_1, H_2), (H_1, T_2)\},$$

it is $\underline{P}(B) = \overline{P}(B) = 0.5$. Now let f be the indicator function of $\{(H_1, H_2), (T_1, H_2)\}$. It holds that $\underline{P}(f) = \overline{P}(f) = 0.5$ while

$$\underline{P}(f|B) = 0 < \overline{P}(f|B) = 1 \text{ and } \underline{P}(f|B^c) = 0 < \overline{P}(f|B^c) = 1.$$

In other words, before we observe the outcome of the first toss we have a precise probability of 0.5 for Heads in the second toss, while after having observed the outcome of the first toss we are completely ignorant, giving the [0, 1] interval, and this no matter which is the outcome we have observed. Thus, there is dilation.

We also get that $\underline{P}(Bf) = \underline{P}(B^c f) = 0$, meaning that the gamble $f - 0.4$ is strictly desirable while $B(f - 0.4)$ and $B^c(f - 0.4)$ are not. In other words, $f - 0.4$ entails a violation of negative conglomerability.

In this example, the marginal extension $\underline{P}(\underline{P}(\cdot|\mathcal{B}))$ produces

$$\underline{P}(f) = 0.5 \min\{f(H_1, H_2), f(H_1, T_2)\} + 0.5 \min\{f(T_1, H_2), f(T_1, T_2)\},$$

that gives $\underline{P}(H_2) = \underline{P}(T_2) = 0$; this means that with the marginal extension we obtain the vacuous model on the second toss. ♦

7 On the Rationality Status of Negative Conglomerability

Section 6 has laid the ground for a possible conflict between negative conglomerability and dilation. The latter is a relatively well-known concept, and one that is more and more regarded as a logical, if puzzling, consequence of imprecision in probability. Therefore if rationality would endorse negative conglomerability in a temporal setting, as it happens with conglomerability, one should conclude that dilation violates rationality in such a context, which would be quite a surprising outcome. It would also mean that negative conglomerability should have a very wide scope. In the following we proceed to clarify the matter.

Let us first recapitulate the state of affairs with regard to conglomerability. We can make things very simple by regarding $\mathcal{D}$ as our set of beliefs at present time and letting $\mathcal{D}^B$—for all B in a given partition $\mathcal{B}$—be our future beliefs, which we will hold in case B occurs. As we have argued in past work [25], our future beliefs are automatically conglomerable, in the sense that their implications go beyond their separate specifications, and must be represented by their so-called 'conglomerable natural extension':

$$\oplus_B(\mathcal{D}^B) := \left\{ f \in \mathcal{L}(\Omega) : f = \sum_{B \in \mathcal{B}} B g^B, g^B \in \mathcal{D}^B \cup \{0\} \right\} \setminus \{0\}.$$

This is a coherent and conglomerable set of desirable gambles obtained by piece-wise composing the gambles across the future sets $\mathcal{D}^B$, $B \in \mathcal{B}$.

To see how all this implies the conglomerability of set $\mathcal{D}$, we need three assumptions:[2]

- The first is that we are in a setting of 'perfect information', which means that all the information we have about Ω is available at present time; in turn this implies that our assessments of desirability may change in time only as a result of further reflection (namely, computation) that is allowed by the extra time.
- The second assumption is one of 'reliability', which requires that future beliefs are no less precise than present beliefs. If we let $\mathcal{D}|B := \{f \in \mathcal{D} : f = Bf\}$ be the set of our beliefs conditional on B, reliability corresponds to the following:

$$(\forall B \in \mathcal{B})\ \mathcal{D}|B \subseteq \mathcal{D}^B.$$

  This assumption appears to have been first made by Walley [24, Section 6.1.2]. Its rationale is that in defining present beliefs, we should never make stronger assessments than the evidence allows. In practice, this means that we make our present assessment the more imprecise the less information we have to define them. Note that reliability implies

$$\oplus_B \mathcal{D}|B \subseteq \oplus_B(\mathcal{D}^B),$$

  where $\oplus_B \mathcal{D}|B$ denotes the conglomerable natural extension of sets $\mathcal{D}|B$, $B \in \mathcal{B}$, and is defined via Eq. (15).
- The third and last assumption, which perhaps was somewhat implicit in Walley's reliability proposal, is one of belief 'refinement': namely, that our future beliefs $\mathcal{D}^B$ are better than present ones $\mathcal{D}|B$ (for all $B \in \mathcal{B}$). The idea is that the extra time to reflect on the problem allows us to improve on our original assessments by possibly assessing that some additional (cf. reliability) gambles are actually desirable too.

The latter assumption implies that if a gamble f is known to belong to $\oplus_B(\mathcal{D}^B)$, then it should belong to $\mathcal{D}$ too. Taking into account reliability, this implies that

$$\oplus_B \mathcal{D}|B \subseteq \mathcal{D},$$

which is precisely the definition of conglomerability for sets of desirable gambles. Note that we would not be able to apply the same reasoning in a setting of imperfect information: there would be no reason to assume reliability, for instance, as our future beliefs might as well be more imprecise than present ones in the light of new information.

In summary, in a temporal setting, and under the three assumptions above, conglomerability turns out to be a rationality requirement. Since a setting of this type roughly, if implicitly, coincides with that of automatically 'updating' probabilistic beliefs in time, we deduce that conglomerability has a wide scope in practice.

Let us now turn our attention to negative conglomerability in the same setting as above. Reasoning as before, the reliability assumption can equivalently be formulated

[2] This argument essentially goes back to [26, Section 6.1].

as $(\mathcal{D}^B)^c \subseteq (\mathcal{D}|B)^c$ for any $B \in \mathcal{B}$. This leads us to conclude that for the 'negatively conglomerable natural extension' $\oplus_B(\mathcal{D}^B)^c := \{f \in \mathcal{L}(\Omega) : (\forall B \in \mathcal{B})\ Bf \notin \mathcal{D}^B\}$ it should hold:

$$\oplus_B(\mathcal{D}^B)^c \subseteq \oplus_B(\mathcal{D}|B)^c.$$

And finally, belief 'refinement' tells us that

$$\oplus_B(\mathcal{D}^B)^c \subseteq \mathcal{D}^c,$$

which we could rephrase as the requirement that a gamble that is not considered desirable in the future should not be considered desirable at present time.

However, the two previous inclusions do *not* imply that we should also have the inclusion

$$\oplus_B(\mathcal{D}|B)^c \subseteq \mathcal{D}^c,$$

as it may still happen that $f \in \oplus_B(\mathcal{D}|B)^c$ while $f \in \mathcal{D}$. As a consequence, negative conglomerability cannot be given a rationality status that allows us to require it, at least in a temporal setting. And in turn it follows that there is no reason we can devise as to why negative conglomerability should be widely applied, other than being an innate feature of models constructed via marginal extension.

Note that we could reach similar conclusions from a sensitivity analysis point of view, just because of the asymmetry between the notions of conglomerability and negative conglomerability: in fact, if a gamble f is desirable with respect to $\mathcal{D}$, it means that it is also desirable for any precise model that is compatible with $\mathcal{D}$; however, if a gamble is not desirable, it ensues that it is not desirable for at least one precise compatible model, but it might be desirable for others. As a consequence, we may reject negative conglomerability on the grounds that the non-desirability of Bf (for all $B \in \mathcal{B}$) for some compatible precise model does not imply the existence of a model for which f is not desirable.

In the same sensitivity analysis vein, one may be tempted to consider an alternative definition of negative conglomerability, such as:

$$(\forall B \in \mathcal{B})\ \overline{P}(Bf) \leq 0 \Rightarrow \overline{P}(f) \leq 0, \tag{18}$$

where we are equating the non-desirability of f with the fact that we are not disposed to pay any positive amount for it for any compatible precise model. Nevertheless, if we use conjugacy, we obtain that Eq. (18) is equivalent to

$$(\forall B \in \mathcal{B})\ \underline{P}(Bf) \geq 0 \Rightarrow \underline{P}(f) \geq 0,$$

or, equivalently,

$$(\forall B \in \mathcal{B})\ Bf \in \overline{\mathcal{D}} \Rightarrow f \in \overline{\mathcal{D}},$$

meaning that we are back to conglomerability (formulated using sets of gambles).

That negative conglomerability has no actual support as a rationality condition, in the light of the discussion above, somewhat solves the conflict between dilation and negative conglomerability, in that one cannot argue against dilation based on its incompatibility with negative conglomerability. It rather seems to be the case that dilation represents a broad range of problems for which negative conglomerability is just an inappropriate concept.

Let us make all this more concrete going back to the jogging example in the Introduction, by giving it a new twist: the weather forecast is accessible only through a friend, Joe, who unpredictably sometimes, just for fun, reports the reversed forecast (from good to bad and vice versa); we assume moreover that the actual forecast is good 50% of the times. In other words, this is the coin-tossing problem of Example 16 in disguise. We obtain that Alice, who desires jogging whether or not the reported forecast is good, desires going also when Joe fails to report the forecast. However nothing prevents Bob, who desires jogging neither when the report is good nor when it is bad, from desiring it when the report does not reach him.

In order to make this clearer, let us consider the events $B :=$ 'the forecast is of good weather' and $A :=$ 'Joe reports good weather'. By considering the assessments $P(B) = 0.5$ and $p := P(\text{'Joe reports the truth'}) \in [0, 1]$, we obtain that the set of possible probabilities is given by the following table.

Event	AB	A^cB	AB^c	A^cB^c
Probability	$\frac{p}{2}$	$0.5 - \frac{p}{2}$	$0.5 - \frac{p}{2}$	$\frac{p}{2}$

Now, assume for Bob that we consider a positive utility of, say, 2 if the report is positive and a negative utility of -1 is the report is negative. This produces a gamble f for which $f(AB) = f(AB^c) = 2$, $f(A^cB) = f(A^cB^c) = -1$. If we compute its associated lower prevision, we obtain that $\underline{P}(Bf) = -0.5 = \underline{P}(B^cf)$, so the gamble is desirable neither if the report is good nor if it is bad. However, $\underline{P}(f) = 1$, meaning that the gamble is desirable before Joe provides his report. In other words, in this context a violation of negative conglomerability is reasonable.

Note that if for Alice we consider another gamble g satisfying that both Bg and B^cg are desirable, this means that $\underline{P}(Bg) > 0$, $\underline{P}(B^cg) > 0$, and from this and coherence we deduce that $\underline{P}(g) > 0$, whence g is desirable. Thus, conglomerability follows here from the rationality criteria of coherence, as it should in this finitary context.

8 Conclusions

At the beginning of this journey, we were uncertain as to the respective roles of conglomerability and its negative counterpart, in particular because intuition may mistakenly lead to regard them as similar, and with a similar scope. In hindsight we see that these two notions are very different. This may look surprising at first also because both conglomerability and negative conglomerability collapse to the same notion of so-called *disintegrability* in the case of precise probability [1]. Probably the most important difference between them is that conglomerability has a rationality status that negative conglomerability has not.

Despite this, negative conglomerability is still going to be a frequent feature of our models, given the relation that we described it holds with marginal extension: it will be the case, for instance, when we construct our assessments in a modular fashion through hierarchical representations such as that of trees. On the opposite side, we shall never meet such a feature whenever our model dilates, as discussed in this paper. In fact, a large part of this paper has been spent on investigating the relations of conglomerability and negative conglomerability with a number of other notions in the literature. We have eventually given quite a complete picture of these relations in the imprecise and precise case that may provide future work with a solid technical reference.

And indeed with respect to future work, we envisage a few possibilities: given that negative conglomerability is about not desiring gambles, it seems natural to try to work out its relation with the accept-reject desirability framework by [16], where indeed one can reject a gamble other than accept it. Another interesting avenue would be studying whether negative conglomerability may take a stronger role in the case of non-linear desirability theory [12]; in such a case desirable sets are more general than convex cones and one might imagine special types of sets that imply negative conglomerability. Finally, since there is a significant relation between dilation and independence, as detailed by [14, 22], it would be interesting to study how negative conglomerability is affected by independence considerations.

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