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RESEARCH ARTICLE

On the Construction of Admissible Orders for Tuples and Its Application to Imprecise Risk Matrices

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Abstract

The imprecision inherent in human opinions is not properly modeled by crisp numbers. Other more complex structures like intervals or tuples capture better the imprecision of human assessments. This makes them very useful in decision problems. However, they cannot be easily compared. Despite they grasp better decision-makers inaccuracy, the lack of a natural total order for such structures makes the determination of the best alternative a difficult task. In this contribution, we explore how to obtain new total orders for (ordered) tuples paying special attention to admissible orders (total orders that extend the lattice order). The resulting orders are applied to four-dimensional ordered tuples that represent risk assessments in an imprecise environment. In addition, two case studies involving risk matrices in educational transport and the construction of a metro station are also provided.

Keywords Admissible order · Order between intervals · Risk matrix · Box · Order between boxes · Election methods

1 Introduction

Historically, reliability was defined as the probability for a system, machine, or device to perform its intended function adequately under some conditions along a specified period of time. Survival analysis is often used to study this type of behaviour (see [1]). Nowadays, this concept is known as mechanical reliability, when it is necessary to differentiate it from the concept of human reliability.

Human reliability has a central role in the study of the causes of errors in many fields. In this way, human failure has been pointed out as the most common cause of error by some researches (see [2–4]). Since IEEE published a report in this regard in 1972, many contributions have appeared on the topic (some examples can be found in [5–10]).

One of the central tools to study human reliability is the risk matrix (see, for instance, [11]). This matrix allows to assess the risk according to two properties, the severity of the consequences, and the likelihood of appearance. More specifically, both severity and likelihood are usually classified into 3–5 levels. Labels between "negligible" and "critically severe" are usually considered to assess severity and labels that go from "remote" to "nearly certain" are used to assess likelihood. These labels are identified with numbers, and given a value for the severity and another for the likelihood, the risk matrix returns a real number that is associated with the importance of the risk. This allows us to order the risks and to prioritise the prevention of the most important ones.

A decision-maker (usually an expert in the subject) assesses a level of severity and likelihood for each event, and combining that information, the risk matrix helps her to provide the risk values associated to each error. Let us observe that the decision-maker is forced to pick a precise level, even though this decision is subjective and, in general, a difficult task [12]. Here, a more flexible procedure would be desirable. Related to this, some alternatives concerning risk attitudes of the decision-makers have been studied in [13]. The authors also provide a detailed discussion on the existence of uncer-

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tainty at the three major phases of the risk-matrix-based assessment, mainly based on the paper of Ruan et al. [14].

The presence of uncertainty or indecision is a very well-known scenario in Decision-Making problems, even when the decision-maker that has to provide the assessment is an expert. Different approaches have been proposed to deal with this uncertainty as in [15]. When the evaluation is based on picking a value in a scale, as in the case of assessing severity and likelihood in risk matrices, intervals are a very common tool to model uncertainty [16, 17]. For the particular case of risk matrices, where two criteria (severity and likelihood) are considered to quantify a certain risk, it can be the case that two intervals have to be merged to determine the risk. The result is called a box in this contribution. The flexibility that this method provides to the decision-maker is very desirable, since it allows to model human perception in a more accurate way. As a counterpart, a method for ordering these resulting intervals and boxes is necessary, which is not an immediate task.

Other attempts to introduce uncertainly in risk matrices have been considered in the literature. For instance, in [18], the authors consider a continuous range for the evaluation of the severity and likelihood rather than a discrete set of labels. In [19] and [20], uncertainly is introduced to the labels using fuzzy linguistic labels. It must be noted that in these papers, the possible choices for the expert are precise. In the first case, the number of possible choices is increased to a continuous interval, while in the second one, the number of possible choices, regardless of the use of fuzzy linguistic labels, remains equal as in the classical case. In our approach, uncertainly appears by allowing the expert to hesitate between different labels.

Our contribution is twofold, theoretical, and applied. We start by providing general results on the construction of orders for tuples, and, in particular, for intervals and boxes. In a second stage, we apply those results to handle risk matrices with the objective of providing a solution in two different human reliability problems. The main contributions can be disclosed as follows:

- Total orders in $\mathbb{R}^n$ based on the type 1 lexicographic order are studied.
- The characterization of admissible orders constructed by a linear transformation and the type 1 lexicographic order is provided.
- The obtained orders are applied to obtain a method to order alternatives when using imprecise risk matrices.
- The benefits of the presented method are shown in two case studies regarding risks in educational transportation and the construction of metro stations.

We summarize the benefits and drawback of the presented method in Table 1.

This paper is organised as follows. In Sect. 2, some preliminaries regarding risk matrices and orders between intervals are presented. Section 3 contains a general theoretical study on ordering tuples. The concept of imprecise risk matrix is presented in Sect. 4, focusing on how to construct the risk intervals and boxes. In Sect. 5, two case studies regarding risks in an educational transportation example and the construction of a metro station are shown. The Borda Count and Condorcet Rankings are used to fuse the different ordinations obtained from applying different orders and to provide a final ranking. Section 6 contains some closing remarks and conclusions.

2 Preliminaries

In this section, we introduce some definitions and results regarding orders between intervals, focusing our study on admissible orders. We also recall the classical concept of risk matrix.

This section is essential to fix the notation used in the contribution and understand the proposed method, since orders between intervals and risk matrices are the main tools of our proposal.

2.1 Ordering Tuples

Recall that an order $\preceq$ in a universe X is a reflexive, antisymmetric, and transitive binary relation on X. A total or linear order is a partial order that is total, i.e., that for every pair of elements $a, b \in X$, at least $a \preceq b$ or $b \preceq a$. Otherwise, the order is said to be partial.

The set of real numbers $\mathbb{R}$ is endowed with the classical total order between numbers. This is not the case for $\mathbb{R}^2$ and, in general, for $\mathbb{R}^n$ with $n \geq 2$. There does not exist a "classical" total order in these sets.

The most commonly accepted way to compare intervals in $\mathbb{R}^n$ is the lattice order.

Definition 1 The lattice order on $\mathbb{R}^n$ is denoted as $\preceq_{Lo}$ and defined as $(a_1, \dots, a_n) \preceq_{Lo} (\bar{a}_1, \dots, \bar{a}_n)$ if and only if $a_i \leq \bar{a}_i$ for all $i \in \{1, \dots, n\}$.

It is an order for any $n \geq 1$, and therefore, it has been widely used to compare tuples. Let us remark that for $n = 1$, it becomes the classical total order in $\mathbb{R}$ and, therefore, it is total. Unfortunately, this is the only case. The lattice order is not total for $n \geq 2$. For example, the elements $(1, 4)$ and $(2, 3)$ in $\mathbb{R}^2$ are incomparable according to the lattice order. The lack of totalness is a handicap for many practical applications, specially in Decision-Making. Admissible orders try to overcome this problem. They were originally introduced to order elements in $\mathcal{L}([0, 1])$, but the definition can be easily translated to $\mathbb{R}^n$.

Table 1 Benefits and drawbacks of the presented method

Benefits	Drawbacks
1. The expert can hesitate between levels of likelihood and severity	1. It is not possible to consider unbalanced hesitation, i.e., the expert hesitates between A and B, but is more inclined to A
2. The semantic of the different choices of the expert are very intuitive	2. There is not a unique natural total order to apply to the resulting risks
3. The risk assignments are totally ordered	3. The ranking methods that can be considered to solve the problem of the choice of the total order may not provide a solution

Definition 2 [21] An admissible order $\preceq$ on $\mathbb{R}^n$ is a total order that refines the lattice order, i.e., is a total order, such that if $(a_1,\ldots,a_n) \preceq_{Lo} (\bar{a}_1,\ldots,\bar{a}_n)$, then also $(a_1,\ldots,a_n) \preceq (\bar{a}_1,\ldots,\bar{a}_n)$.

Two classical examples of admissible orders in $\mathbb{R}^2$ are the first and second lexicographic orders.

Definition 3 Given two elements (a_1,a_2) and $(\bar{a}_1,\bar{a}_2)$ in $\mathbb{R}^2$.

- We say that (a_1,a_2) is smaller than or equal to $(\bar{a}_1,\bar{a}_2)$ according to the type 1 or first lexicographic order, and we denote it as $(a_1,a_2) \preceq_{Lex1} (\bar{a}_1,\bar{a}_2)$, if and only if

$$a_1 < \bar{a}_1 \quad \text{or} \quad a_1 = \bar{a}_1 \text{ and } a_2 \le \bar{a}_2.$$

- We say that (a_1,a_2) is smaller than or equal to $(\bar{a}_1,\bar{a}_2)$ according to the type 2 or second lexicographic order, and we denote it as $(a_1,a_2) \preceq_{Lex2} (\bar{a}_1,\bar{a}_2)$, if and only if

$$a_2 < \bar{a}_2 \quad \text{or} \quad a_2 = \bar{a}_2 \text{ and } a_1 \le \bar{a}_1.$$

These two definitions can be easily generalized to $\mathbb{R}^n$.

Definition 4 Given two elements $(a_1,\ldots,a_n)$ and $(\bar{a}_1,\ldots,\bar{a}_n)$ in $\mathbb{R}^n$, we say that they are ordered according to the type 1 lexicographic order, and we denote it $(a_1,\ldots,a_n) \preceq_{Lex1} (\bar{a}_1,\ldots,\bar{a}_n)$, if and only if they are the same tuple or (assuming $a_0 = 0 = \bar{a}_0$), there exists some $k \in \{1,\ldots,n\}$, such that $a_i = \bar{a}_i$ for all $0 \le i < k$ and $a_k < \bar{a}_k$.

Definition 5 Given two elements $(a_1,\ldots,a_n)$ and $(\bar{a}_1,\ldots,\bar{a}_n)$ in $\mathbb{R}^n$, we say that they are ordered according to the type 2 lexicographic order, and we denote it $(a_1,\ldots,a_n) \preceq_{Lex2} (\bar{a}_1,\ldots,\bar{a}_n)$, if and only if they are the same tuple or (assuming $a_{n+1} = 0 = \bar{a}_{n+1}$), there exists some $k \in \{1,\ldots,n\}$, such that $a_i = \bar{a}_i$ for all $n+1 \ge i > k$ and $a_k < \bar{a}_k$.

Lemma 1 *Given two tuples, $(a_1,\ldots,a_n)$ and $(\bar{a}_1,\ldots,\bar{a}_n)$, we have that $(a_1,\ldots,a_n) \preceq_{Lex1} (\bar{a}_1,\ldots,\bar{a}_n)$ if and only if for every $k \in \{1,\ldots,n\}$, it holds that either (i) $\exists i < k$, such that $a_i < \bar{a}_i$ or (ii) $a_i = \bar{a}_i$ for all $i < k$ and $a_k \le \bar{a}_k$, where we assume $a_0 = \bar{a}_0$.*

Both relations, the type 1 and type 2 lexicographic orders, are admissible orders in $\mathbb{R}^n$, as we can see at the following result.

Lemma 2 *The type 1 and type 2 lexicographic orders are admissible orders in $\mathbb{R}^n$ for every $n \in \mathbb{N}$.*

Proof We provide the proof of the type 1 lexicographic order, the other one being analogous. For the case $n = 1$, it becomes the classical (order) in $\mathbb{R}$. Consider $n \ge 2$.

1. Reflexivity follows from the definition itself.
2. Antisymmetry: Assume $(a_1,\ldots,a_n) \preceq_{Lex1} (\bar{a}_1,\ldots,\bar{a}_n)$ and $(\bar{a}_1,\ldots,\bar{a}_n) \preceq_{Lex1} (a_1,\ldots,a_n)$. Assume also $(a_1,\ldots,a_n) \ne (\bar{a}_1,\ldots,\bar{a}_n)$, we get to a contradiction. Since $(a_1,\ldots,a_n) \preceq_{Lex1} (\bar{a}_1,\ldots,\bar{a}_n)$, then there exists $k \in \{1,\ldots,n\}$, such that $a_i = \bar{a}_i$ for all $i < k$ and $a_k < \bar{a}_k$. At the same time, since $(\bar{a}_1,\ldots,\bar{a}_n) \preceq_{Lex1} (a_1,\ldots,a_n)$, there exists $j \in \{1,\ldots,n\}$, such that $a_i = \bar{a}_i$ for all $i < j$ and $a_j < \bar{a}_j$. A contradiction.
3. Transitivity: Assume $(a_1,\ldots,a_n) \preceq_{Lex1} (\bar{a}_1,\ldots,\bar{a}_n)$ and $(\bar{a}_1,\ldots,\bar{a}_n) \preceq_{Lex1} (\hat{a}_1,\ldots,\hat{a}_n)$. Assume also that $(a_1,\ldots,a_n) \ne (\bar{a}_1,\ldots,\bar{a}_n)$ and $(\bar{a}_1,\ldots,\bar{a}_n) \ne (\hat{a}_1,\ldots,\hat{a}_n)$. Otherwise, the proof is trivial.
   Then, there exist $k, j \in 1,\ldots,n$, such that $a_i = \bar{a}_i$ for all $i < k$ and $a_k < \bar{a}_k$ and $\bar{a}_i = \hat{a}_i$ for all $i < j$ and $\bar{a}_j < \hat{a}_j$.
   If $k < j$, then $a_i = \bar{a}_i = \hat{a}_i$ for all $i < k$ and $a_k < \bar{a}_k = \hat{a}_k$, whereas $(a_1,\ldots,a_n) \preceq_{Lex1} (\hat{a}_1,\ldots,\hat{a}_n)$.
   If $k = j$, then $a_i = \bar{a}_i = \hat{a}_i$ for all $i < k$ and $a_k < \bar{a}_k < \hat{a}_k$, whereas $(a_1,\ldots,a_n) \preceq_{Lex1} (\hat{a}_1,\ldots,\hat{a}_n)$.
   If $k > j$, then $a_i = \bar{a}_i = \hat{a}_i$ for all $i < j$ and $a_j = \bar{a}_j < \hat{a}_j$, whereas $(a_1,\ldots,a_n) \preceq_{Lex1} (\hat{a}_1,\ldots,\hat{a}_n)$.
4. Totalness: Consider two different tuples, $(a_1,\ldots,a_n)$ and $(\bar{a}_1,\ldots,\bar{a}_n)$. Compare the first components. If $a_1 < \bar{a}_1$, then $(a_1,\ldots,a_n) \preceq_{Lex1} (\bar{a}_1,\ldots,\bar{a}_n)$. If $\bar{a}_1 < a_1$, then $(\bar{a}_1,\ldots,\bar{a}_n) \preceq_{Lex1} (a_1,\ldots,a_n)$. Finally, if $a_1 = \bar{a}_1$, compare the second components, a_2 and $\bar{a}_2$. If neither $(a_1,\ldots,a_n) \preceq_{Lex1} (\bar{a}_1,\ldots,\bar{a}_n)$ nor $(a_1,\ldots,a_n) \preceq_{Lex1} (\bar{a}_1,\ldots,\bar{a}_n)$ can be proven yet. The comparison continues with the third and subsequent components until either for some position, we have an inequality and, therefore, we can order the tuples

according to the type 1 lexicographic order or all the components are equal. In this case, they are the same interval, and therefore, $(a_1, \ldots, a_n) \preceq_{Lex1} (\bar{a}_1, \ldots, \bar{a}_n)$ and $(\bar{a}_1, \ldots, \bar{a}_n) \preceq_{Lex1} (a_1, \ldots, a_n)$.

5. Admissible: Let us consider $(a_1, \ldots, a_n) \preceq_{Lo} (\bar{a}_1, \ldots, \bar{a}_n)$, we have to prove that $(a_1, \ldots, a_n) \preceq_{Lex1} (\bar{a}_1, \ldots, \bar{a}_n)$. Observe that $(a_1, \ldots, a_n) \preceq (\bar{a}_1, \ldots, \bar{a}_n)$ means that $a_i \leq \bar{a}_i$ for every i. If $a_i = \bar{a}_i$ for all i, they are the same tuple and $(a_1, \ldots, a_n) \preceq_{Lex1} (\bar{a}_1, \ldots, \bar{a}_n)$. If not, let $k \in \{1, \ldots, n\}$ be the first component, such that $a_k \neq \bar{a}_k$. This implies, since $a_i \leq \bar{a}_i$ for all i, that $a_i = \bar{a}_i$ for all $i < k$ and $a_k < \bar{a}_k$ which implies $(a_1, \ldots, a_n) \preceq_{Lex1} (\bar{a}_1, \ldots, \bar{a}_n)$.

Until now, we have studied orders in $\mathbb{R}^n$. However, in this paper, we will be specially interested in a particular case of tuples, the ordered ones. Thus, if we denote as $\mathcal{C}^n(\mathbb{R})$ the set of ordered tuples in $\mathbb{R}^n$: $\mathcal{C}^n(\mathbb{R}) = \{(a_1, \ldots, a_n) \mid a_i \in \mathbb{R},$ for $1 \leq i \leq n$ and $a_i \leq a_{i+1}$ for $i \leq n-1\}$, we can restrict the previous orders to this set. It is immediate to prove that the lattice order, the type 1 and type 2 lexicographic orders in $\mathcal{C}^n(\mathbb{R})$ keep being a partial order and two admissible orders, respectively. From now on, we will use the notation $[a_1, \ldots, a_n]$ to refer to the members of $\mathcal{C}^n(\mathbb{R})$ to emphasize that the elements of the tuple are ordered.

2.2 Risk Matrices

Theoretical studies about ordering ordered tuples will be applied to Human Reliability, where risk matrices are an essential tool. Here, the main ideas related to these matrices are recalled.

Risk assessment matrices are a tool that given some subjective considerations, allows to quantify the relevance of a risk. They were put forward to classify the possible risks. The main objective is to identify the most important ones, and to devote appropriate resources to prevent them. The subjective considerations are two, the level of severity, that is related to how big the impact or the consequences of the error could be, and the likelihood, that is related to how often we expect this situation to happen. In particular, this type of matrices is defined as follows.

Definition 6 ([11, 19]) If severity is classified into m levels and likelihood is classified into n levels, a $n \times m$ matrix R is said to be a risk matrix if its values are associated with a certain combination of the levels of severity (identified with the columns of the matrix) and the levels of likelihood (identified with the rows)

$$
\begin{array}{c}
\\ L_n \\ \vdots \\ L_i \\ \vdots \\ L_1
\end{array}
\begin{array}{c}
\begin{array}{ccccc} S_1 & \ldots & S_j & \ldots & S_m \end{array} \\
\begin{pmatrix}
r_{n1} & \ldots & r_{nj} & \ldots & r_{nm} \\
\vdots & \vdots & \vdots & \vdots & \vdots \\
r_{i1} & \ldots & r_{ij} & \ldots & r_{im} \\
\vdots & \vdots & \vdots & \vdots & \vdots \\
r_{11} & \ldots & r_{1j} & \ldots & r_{1m}
\end{pmatrix}
\end{array}
$$

Let us make some additional remarks about the above definition:

- The number of levels of severity and likelihood can vary depending on the particular problem, but they are typically between 3 and 5.
- The values of a risk matrix are known as risk values and determined only by the severity and likelihood levels.
- The risk values' computation can be summarized as a logic implication: if the likelihood has level a and the severity has level b, then the risk value is r_{ab}.
- Both the levels of severity and likelihood are ordered. The levels of severity increase from left to right and the levels of likelihood from bottom to top. Accordingly, the entries of the matrix, the risk values, are increasing $r_{ij} \geq r_{kl}$ for $i \leq k$ and $j \geq l$. Higher risk scores are assigned to higher levels of severity and/or likelihood.

The dimensions as well as the values of the risk matrix are very dependent on the field of application. In this paper, we consider the most typical ones (see [11, 22]) which assign 5 levels for both severity and likelihood. In particular, the likelihood axis is divided into the levels: Remote (R), Unlikely (UL), Likely (L), Highly Likely (HL), and Nearly Certain (NC). Similarly, the severity is divided into the following categories: Negligible (N), Minor (MI), Moderate (MO), Serious (S), and Critical (C).

Then, the risk values are obtained by multiplying the number associated with the levels of severity and likelihood. There are different criteria to do that and so there is not a unique risk matrix, although the general ideas are common to all the usual considered matrices. In our case, these values are numbers between 1 (Negligible and Remote) and 25 (Critical and Nearly Certain). The output risk values are classified into zones, depending on how important the risk is. In particular, the risk values from 1 to 3 are considered low and are coloured in green; risk values between 4 and 6 are moderate, with a yellow colour; the values between 8 and 12 are considered highly important and coloured in orange and finally the rest of the values are considered severe and are represented in red. The final form of the aforementioned risk matrix is shown in Table 2.

Table 2 Original risk assessment matrix

Likel. \ Sev.	Negligible	Minor	Moderate	Serious	Critical
Nearly Certain	5	10	15	20	25
Highly Likely	4	8	12	16	20
Likely	3	6	9	12	15
Unlikely	2	4	6	8	10
Remote	1	2	3	4	5

The levels of severity and likelihood are assessed by a decision maker, usually a person, that can doubt between two levels. Our proposal to capture this hesitancy is to allow the decision-maker to provide intervals of values instead of precise scores when assessing severity and/or likelihood as we will explain in detail in Sect. 4. When both (severity and likelihood of a risk) are assigned an interval, the final score of the risk cannot be a number, but an interval where the bounds are delimited by intervals, which we call a box. If we want to know the most important risks, we need an order to compare these new elements. In the following section, we study orders for tuples in general, which will be later applied in the context of Risk Analysis.

3 On New Orders in $\mathbb{R}^n$

In this section, we provide theoretical results that allow to build new orders from existing ones. In particular, we will show that the most well-known admissible orders are in fact transformations of the type 1 lexicographic order. We also provide a characterization that shows how the transformation has to be to obtain an admissible order.

Proposition 1 *Let F be a injection in a set X endowed with a total order $\preceq_T$. Then, the binary relation $\mathcal{R}$ defined over X as*

$$x\mathcal{R}y \text{ if and only if } F(x) \preceq_T F(y)$$

for every $x, y \in X$ is a total order.

Proof We prove that $\mathcal{R}$ is reflexive, antisymmetric, transitive, and total.

Reflexive: since $\preceq_T$ is reflexive, $F(x) \preceq_T F(x)$ for all $x \in X$.

Antisymmetric: Assume both $x\mathcal{R}y$ and $y\mathcal{R}x$, this is equivalent to $F(x) \preceq_T F(y)$ and $F(y) \preceq_T F(x)$. Since $\preceq_T$ is antisymmetric, then $F(x) = F(y)$, and since F is injective, this implies $x = y$.

Transitive: Assume $x\mathcal{R}y$ and $y\mathcal{R}z$, then $F(x) \preceq_T F(y)$ and $F(y) \preceq_T F(z)$. Since $\preceq_T$ is transitive, $F(x) \preceq_T F(z)$, and therefore, $x\mathcal{R}z$.

Total: Since $\preceq_T$ is total, for every pair of elements $x, y \in X$, it holds that either $F(x) \preceq_T F(y)$ or $F(y) \preceq_T F(x)$. Therefore, at least $x\mathcal{R}y$ or $y\mathcal{R}x$.

A particular case that will be very relevant in our practical application is that one in which the injection is obtained by multiplying the tuple by a regular matrix. In particular:

Corollary 1 *Let A be a $n \times n$ matrix. The binary relation $\mathcal{R}$ defined over $\mathbb{R}^n \times \mathbb{R}^n$ as*

$$(a_1, \ldots, a_n)\mathcal{R}(\bar{a}_1, \ldots, \bar{a}_n) \\ \text{if and only if } A(a_1, \ldots, a_n)' \preceq_{Lex1} A(\bar{a}_1, \ldots, \bar{a}_n)'$$

is a total order if and only if A is full rank.

Proof If A is full rank, the function $F : \mathbb{R}^n \to \mathbb{R}^n$ defined as $F((a_1, \ldots, a_n)) = A(a_1, \ldots, a_n)'$ is a bijection. On the other hand, $\preceq_{Lex1}$ is a total order over $\mathbb{R}^n$. It follows from Proposition 1 that $\mathcal{R}$ is a total order.

On the other hand, if A is not full rank, then there is a tuple $(a_1, \ldots, a_n) \neq (0, \ldots, 0)$, such that $A(a_1, \ldots, a_n)' = (0, \ldots, 0) = A(0, \ldots, 0)$. Then, $\mathcal{R}$ is not antisymmetric, since $(a_1, \ldots, a_n)\mathcal{R}(0, \ldots, 0)$ and $(0, \ldots, 0)\mathcal{R}(a_1, \ldots, a_n)$ but $(a_1, \ldots, a_n) \neq (0, \ldots, 0)$.

As mentioned above, the lattice order is assumed to be the most natural one. This is why, admissible orders are specially interesting. They keep the essence of the lattice order, but in addition, they are total. In the next result, we characterize the matrices that lead to admissible orders. It can be considered as a generalization of Proposition 2 in [23], where an analogous result was proven for the interval [0, 1].

Proposition 2 *Let A be a $n \times n$ matrix. The binary relation $\mathcal{R}$ defined over $\mathbb{R}^n$ as*

$$(a_1, \ldots, a_n)\mathcal{R}(\bar{a}_1, \ldots, \bar{a}_n) \\ \text{if and only if } A(a_1, \ldots, a_n)' \preceq_{Lex1} A(\bar{a}_1, \ldots, \bar{a}_n)'$$

is an admissible order in $\mathbb{R}^n$ if and only if A is full rank and for every column of A, the first non-null element is positive.

Proof Let us start proving that it is a sufficient condition. Since A is full rank, then it follows from Proposition 1 that the binary relation $\mathcal{R}$ is a total order. Thus, we only have to prove that it refines the lattice order.

Assume $(a_1, \ldots, a_n) \preceq_{Lo} (\bar{a}_1, \ldots, \bar{a}_n)$, this is equivalent to

$$a_j \leq \bar{a}_j, \quad \forall j. \tag{1}$$

On the other hand, by hypothesis, it holds that $A_{1j} \geq 0$ for all j, then

$$\begin{aligned}\sum_{j=1}^{n} A_{1j}a_j &= \sum_{A_{1j}=0} A_{1j}a_j + \sum_{A_{1j}>0} A_{1j}a_j = \sum_{A_{1j}>0} A_{1j}a_j \\ &\leq \sum_{A_{1j}>0} A_{1j}\bar{a}_j = \sum_{A_{1j}=0} A_{1j}\bar{a}_j + \sum_{A_{1j}>0} A_{1j}\bar{a}_j = \sum_{j=1}^{n} A_{1j}\bar{a}_j.\end{aligned}$$

Therefore

$$\sum_{j=1}^{n} A_{1j}a_j \leq \sum_{j=1}^{n} A_{1j}\bar{a}_j. \tag{2}$$

We have to prove that

$$\left(\sum_{j=1}^{n} A_{1j}a_j, \ldots, \sum_{j=1}^{n} A_{nj}a_j\right) \preceq_{Lex1} \left(\sum_{j=1}^{n} A_{1j}\bar{a}_j, \ldots, \sum_{j=1}^{n} A_{nj}\bar{a}_j\right).$$

Using Lemma 1, we will prove that for all $k \in \{1, \ldots, n\}$, either (i) $\exists i < k$, such that $\sum_{j=1}^{n} A_{ij}a_j < \sum_{j=1}^{n} A_{ij}\bar{a}_j$ or (ii) $\sum_{j=1}^{n} A_{ij}a_j = \sum_{j=1}^{n} A_{ij}\bar{a}_i$ for all $i < k$ and $\sum_{j=1}^{n} A_{kj}a_j \leq \sum_{j=1}^{n} A_{kj}\bar{a}_j$, where we assume $\sum_{j=1}^{n} A_{0j}a_j = \sum_{j=1}^{n} A_{0j}\bar{a}_i$.

The case $k = 1$ follows from Eq. 2. We will prove that for all $k \in \{2, \ldots, n\}$:

- either $\exists i < k$ such that $\sum_{j=1}^{n} A_{ij}a_j < \sum_{j=1}^{n} A_{ij}\bar{a}_j$
- or $\sum_{j=1}^{n} A_{ij}a_j = \sum_{j=1}^{n} A_{ij}\bar{a}_j$ for all $i \leq k-1$ and in this case, $a_j = \bar{a}_j$ for all j such that $A_{ij} \neq 0$ for some $i \leq k-1$ and $\sum_{j=1}^{n} A_{kj}a_j \leq \sum_{j=1}^{n} A_{kj}\bar{a}_j$.

Case $k = 2$. If $\sum_{j=1}^{n} A_{1j}a_j < \sum_{j=1}^{n} A_{1j}\bar{a}_j$, it holds for $k = 2$.

Otherwise, by Eq. 2, $\sum_{j=1}^{n} A_{1j}a_j = \sum_{j=1}^{n} A_{1j}\bar{a}_j$. This implies that

$$a_j = \bar{a}_j \quad \forall j \mid A_{1j} > 0. \tag{3}$$

Moreover

$$\begin{aligned}\sum_{j=1}^{n} A_{2j}a_j &= \sum_{A_{2j}<0} A_{2j}a_j + \sum_{A_{2j}=0} A_{2j}a_j + \sum_{A_{2j}>0} A_{2j}a_j \\ &= \sum_{A_{2j}<0} A_{2j}a_j + \sum_{A_{2j}>0} A_{2j}a_j.\end{aligned}$$

If $A_{2j} < 0$, by hypothesis on A, it must hold that $A_{1j} > 0$, and therefore, by Eq. 3, also $a_j = \bar{a}_j$. It then follows that $\sum_{A_{2j}<0} A_{2j}a_j = \sum_{A_{2j}<0} A_{2j}\bar{a}_j$. Therefore

$$\begin{aligned}\sum_{j=1}^{n} A_{2j}a_j &= \sum_{A_{2j}<0} A_{2j}a_j + \sum_{A_{2j}>0} A_{2j}a_j \\ &\leq \sum_{A_{2j}<0} A_{2j}\bar{a}_j + \sum_{A_{2j}>0} A_{2j}\bar{a}_j = \sum_{j=1}^{n} A_{2j}\bar{a}_j.\end{aligned}$$

Assume the result holds for $k-1$, this is, assume that

- either $\exists i < k-1$ such that $\sum_{j=1}^{n} A_{ij}a_j < \sum_{j=1}^{n} A_{ij}\bar{a}_j$
- or $\sum_{j=1}^{n} A_{ij}a_j = \sum_{j=1}^{n} A_{ij}\bar{a}_j$ for all $i < k-1$, and in this case, $a_j = \bar{a}_j$ for all j, such that $A_{ij} \neq 0$ for some $i < k-1$ and $\sum_{j=1}^{n} A_{k-1j}a_j \leq \sum_{j=1}^{n} A_{k-1j}\bar{a}_j$.

Let us prove it for k. If $\sum_{j=1}^{n} A_{ij}a_j < \sum_{j=1}^{n} A_{ij}\bar{a}_j$ for some $i < k$, the proof follows. Otherwise, $\sum_{j=1}^{n} A_{ij}a_j = \sum_{j=1}^{n} A_{ij}\bar{a}_j$ for all $i < k$, in particular for $i = k-1$. In this case

$$\begin{aligned}&\sum_{A_{k-1j}<0} A_{k-1j}a_j + \sum_{A_{k-1j}>0} A_{k-1j}a_j \\ &\quad = \sum_{A_{k-1j}<0} A_{k-1j}\bar{a}_j + \sum_{A_{k-1j}>0} A_{k-1j}\bar{a}_j.\end{aligned} \tag{4}$$

If $A_{k-1j} < 0$, by the hypothesis on A, necessarily $A_{ij} > 0$ for some $i < k-1$. It follows from the hypothesis of induction that then $a_j = \bar{a}_j$. Therefore, $\sum_{A_{k-1j}<0} A_{k-1j}a_j = \sum_{A_{k-1j}<0} A_{k-1j}\bar{a}_j$, whereas

$$\sum_{A_{k-1j}>0} A_{k-1j}a_j = \sum_{A_{k-1j}>0} A_{k-1j}\bar{a}_j.$$

Therefore, $a_j = \bar{a}_j$ for all j, such that $A_{k-1j} \neq 0$.

We still need to prove that $\sum_{j=1}^{n} A_{kj}a_j \leq \sum_{j=1}^{n} A_{kj}\bar{a}_j$ to close the proof by induction. Observe that if $A_{kj} < 0$, necessarily $A_{ij} > 0$ for some $i < k$ and $a_j = \bar{a}_j$. Therefore

$$\sum_{j=1}^{n} A_{kj}a_j = \sum_{A_{kj}<0} A_{kj}a_j + \sum_{A_{kj}>0} A_{kj}a_j$$

$$= \sum_{A_{kj}<0} A_{kj}\bar{a}_j + \sum_{A_{kj}>0} A_{kj}a_j$$
$$\leq \sum_{A_{kj}<0} A_{kj}\bar{a}_j + \sum_{A_{kj}>0} A_{kj}\bar{a}_j = \sum_{j=1}^{n} A_{kj}\bar{a}_j.$$

On the other hand, the necessary condition is also fulfilled. Assume that there exists a column j, such that $A_{kj} < 0$ for some $k \in \{1, \ldots, n\}$ and $A_{ij} = 0$ for all $i < k$. Take $a_l = \bar{a}_l = 0$ for all $l < j$, $a_j = 1$, $\bar{a}_j = 2$ and $a_l = \bar{a}_l = 2$ for all $l > j$. This is

$$(a_1, \ldots, a_n) = (0, \ldots, 0, 1, 2, \ldots, 2)$$
$$(\bar{a}_1, \ldots, \bar{a}_n) = (0, \ldots, 0, 2, 2, \ldots, 2).$$

It is clear that $(a_1, \ldots, a_n) \preceq_{Lo} (\bar{a}_1, \ldots, \bar{a}_n)$. However, $A(a_1, \ldots, a_n)' \not\preceq_{Lex1} A(\bar{a}_1, \ldots, \bar{a}_n)'$. Observe that for $i < k$, since $A_{ij} = 0$, it holds that

$$\textstyle\sum_{l=1}^{n} A_{il}a_l = A_{ij} + 2\sum_{l=j+1}^{n} A_{il} = 2\sum_{l=j+1}^{n} A_{il}$$
$$= 2\sum_{l=j}^{n} A_{il} = \sum_{l=1}^{n} A_{il}\bar{a}_l.$$

And for k

$$\sum_{l=1}^{n} A_{kl}a_l = A_{kj} + 2\sum_{l=j+1}^{n} A_{kl} > 2A_{kj} + 2\sum_{l=j+1}^{n} A_{kl}$$
$$= \sum_{l=1}^{n} A_{kl}\bar{a}_l,$$

where the inequality comes from $A_{kj} < 0$ and, therefore, $A_{kj} > 2A_{kj}$. With this, we prove that $A(a_1, \ldots, a_n)' \not\preceq_{Lex1} A(\bar{a}_1, \ldots, \bar{a}_n)'$.

As a direct consequence, we can build new lexicographic orders just by changing the order of comparison of the components of the tuples.

Definition 7 Let $(i_1, \ldots, i_n)$ be the permutation of $\{1, \ldots, n\}$ that takes element i_k to position k, then the ordering in $\mathbb{R}^n$ generated from permutation $(i_1, \ldots, i_n)$ is denoted $\preceq_{i_1 \ldots i_n}$ and defined as

$$(a_1, \ldots, a_n) \preceq_{i_1 \ldots i_n} (\bar{a}_1, \ldots, \bar{a}_n)$$

if and only if they are the same tuple or (assuming $a_{i_0} = 0 = \bar{a}_{i_0}$), there exists some $k \in \{1, \ldots, n\}$, such that $a_{i_j} = \bar{a}_{i_j}$ for all $j < k$ and $a_{i_k} < \bar{a}_{i_k}$.

Example 1 In $\mathbb{R}^4$, the permutation $(2, 1, 4, 3)$ produces the order $\preceq_{2143}$ that is based on first comparing the second components; if there is a tie, then comparing the first components; if there is again a tie, it compares the forth components and if they are still equal, finally comparing the third components.

Observe that $\preceq_{1,\ldots,n}$ is the type 1 lexicographic order.

As mentioned above, Proposition 2 allows to prove easily that the ordering of Definition 7 is an admissible order for any permutation $(i_1, \ldots, i_n)$.

Corollary 2 *For any permutation $(i_1, \ldots, i_n)$, the lexicographic order $\preceq_{i_1 \ldots i_n}$ is a total order.*

Proof First, notice that the order $\preceq_{i_1 \ldots i_n}$ can be obtained from permuting the components of the tuples according to permutation $(i_1, \ldots, i_n)$ and then applying the type 1 lexicographic order:

$$(a_1, \ldots, a_n) \preceq_{i_1 \ldots i_n} (\bar{a}_1, \ldots, \bar{a}_n) \Leftrightarrow (a_{i_1}, \ldots, a_{i_n})$$
$$\preceq_{Lex1} (\bar{a}_{i_1}, \ldots, \bar{a}_{i_n}).$$

With this in mind, call $I_{i_1 \ldots i_n}$ the matrix obtained from applying permutation $(i_1, \ldots, i_n)$ to the rows of the identity matrix of dimension n. It is easy to see that $(a_{i_1}, \ldots, a_{i_n}) = I_{i_1 \ldots i_n}(a_1, \ldots, a_n)'$. The result follows from Proposition 2. Notice that $I_{i_1 \ldots i_n}$ is a full rank matrix, such that the first (and only) non-null element of every column is positive.

The characterization of Proposition 2 still holds if we restrict to $\mathcal{C}^n(\mathbb{R})$, this is, to ordered tuples:

Corollary 3 *Let A be a full rank $n \times n$ matrix. The binary relation $\mathcal{R}$ defined over $\mathcal{C}^n(\mathbb{R})$ as*

$$[a_1, \ldots, a_n]\mathcal{R}[\bar{a}_1, \ldots, \bar{a}_n] \text{ if and only if } A(a_1, \ldots, a_n)'$$
$$\preceq_{Lex1} A(a_1, \ldots, a_n)'$$

is an admissible order in $\mathcal{C}^n(\mathbb{R})$ if and only if the first non-null element of all the columns of A are positive.

Proof It follows from Proposition 2. Just observe that the elements used in the proof of necessity, $(a_1, \ldots, a_n) = (0, \ldots, 0, 1, 2, \ldots, 2)$ and $(\bar{a}_1, \ldots, \bar{a}_n) = (0, \ldots, 0, 2, 2, \ldots, 2)$ are in fact elements of $\mathcal{C}^n(\mathbb{R})$.

As an immediate corollary, we can in particular construct more linear orders over $\mathcal{C}^4(\mathbb{R})$ just by considering the 24 possible permutations of the rows of I_4. All these matrices satisfy that they have a non-null element (equal to 1) per column.

Corollary 4 *Let (i_1, i_2, i_3, i_4) be a permutation of the set $\{1, 2, 3, 4\}$ and let $I_{i_1 \ldots i_4}$ be the matrix obtained from applying the permutation (i_1, i_2, i_3, i_4) to the rows of the identity matrix I_4. The relation $\preceq_{i_1 \ldots i_4}$ defined as*

$$[a_1, a_2, a_3, a_4] \preceq_{i_1 \ldots i_4} [\bar{a}_1, \bar{a}_2, \bar{a}_3, \bar{a}_4]$$
$$\Updownarrow$$
$$I_{i_1 \ldots i_4}(a_1, a_2, a_3, a_4)' \preceq_{1234} I_{i_1 \ldots i_4}(\bar{a}_1, \bar{a}_2, \bar{a}_3, \bar{a}_4)'$$

for any $[a_1, a_2, a_3, a_4], [\bar{a}_1, \bar{a}_2, \bar{a}_3, \bar{a}_4] \in \mathcal{C}^4(\mathbb{R})$, *is an admissible order in* $\mathcal{C}^4(\mathbb{R})$.

The orders constructed in the previous corollary are the lexicographic orders in $\mathcal{C}^4(\mathbb{R})$ and they will be essential in this study.

Finally, we will relate the set of ordered tuples to the set of real intervals. Let us recall that given $a, b \in \mathbb{R}$ with $a \leq b$, the closed interval $[a, b]$ is the set $[a, b] = \{r \in \mathbb{R} \mid a \leq r \leq b\}$. If we denote the set of all intervals in the real line as $\mathcal{L}(\mathbb{R}) = \{[a, b] \mid a, b \in \mathbb{R} \text{ and } a \leq b\}$ and the set of subintervals of $[0, 1]$ as $\mathcal{L}([0, 1])$, that is, $\mathcal{L}([0, 1]) = \{[a, b] \mid a, b \in \mathbb{R} \text{ and } 0 \leq a \leq b \leq 1\}$, it is immediate that the sets $\mathcal{L}(\mathbb{R})$ and $\mathcal{L}([0, 1])$ are isomorphic to the particular cases of ordered tuples $\mathcal{C}^2(\mathbb{R})$ and $\mathcal{C}^2([0, 1])$, respectively.

Thus, if we restrict us to $\mathcal{L}(\mathbb{R})$, the only possible lexicographic orders are $Lex1$ and $Lex2$. If we restrict the order to real numbers, we recover the usual order in $\mathbb{R}$.

In addition to the other lexicographic orders, Corollary 2 allows to connect another well-known admissible order with the type 1 lexicographic order: the Xu and Yager order. Recall that this order is defined over $\mathcal{L}([0, 1])$ and denoted $\preceq_{XY}$. It states that $[a_1, a_2] \preceq_{XY} [\bar{a}_1, \bar{a}_2]$ if and only if $a_1 + a_2 < \bar{a}_1 + \bar{a}_2$ or $(a_1 + a_2 = \bar{a}_1 + \bar{a}_2$ and $a_2 - a_1 \leq \bar{a}_2 - \bar{a}_1)$.

This definition can be extended to $\mathcal{C}^4(\mathbb{R})$ as follows.

Definition 8 The Xu-Yager order over $\mathcal{C}^4(\mathbb{R})$ is denoted as $\preceq_{XY}$ and defined as: $[a, b, c, d] \preceq_{XY} [\bar{a}, \bar{b}, \bar{c}, \bar{d}]$ if and only if one of the following conditions are fulfilled:

- $a + b + c + d < \bar{a} + \bar{b} + \bar{c} + \bar{d}$
- $a + b + c + d = \bar{a} + \bar{b} + \bar{c} + \bar{d}$ and $(c - a) + (d - b) < (\bar{c} - \bar{a}) + (\bar{d} - \bar{b})$
- $a + b + c + d = \bar{a} + \bar{b} + \bar{c} + \bar{d}$, $(c - a) + (d - b) = (\bar{c} - \bar{a}) + (\bar{d} - \bar{b})$ and $a < \bar{a}$
- $a + b + c + d = \bar{a} + \bar{b} + \bar{c} + \bar{d}$, $(c - a) + (d - b) = (\bar{c} - \bar{a}) + (\bar{d} - \bar{b})$, $a = \bar{a}$ and $c \leq \bar{c}$.

Corollary 5 *The binary relation* $\preceq_{XY}$ *is an admissible order in* $\mathcal{C}^4(\mathbb{R})$.

Proof The proof is immediate from Proposition 2, since we only have to consider the matrix

$$A = \begin{pmatrix} 1 & 1 & 1 & 1 \\ -1 & -1 & 1 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix},$$

which is a full rank matrix and the first element of each column is positive.

When restricted to intervals, the latter order behaves as the Xu–Yager classical order. Alternative definitions of extensions of the Xu–Yager order can be considered. In particular, we can change the third and fourth conditions for any pair that makes it an admissible order.

4 Risk Matrices Under Uncertainty

Our main proposal is to apply the previous studies about ordering tuples to manage risk matrices when uncertainty appears.

When a decision-maker uses the risk matrix in Table 2, a level of likelihood and severity must be chosen. However, there are some cases where the boundaries between, for example, a remote and an unlikely likelihood are not very clear. Thus, it may be quite natural for the expert to assign a likelihood level between remote and unlikely instead of choosing one of them. The same happens with severity, the expert may prefer to pick an intermediate level rather than to be forced to decide between two of them. The here-proposed method adds the possibility to the decision-maker to make this type of decisions. Thus, we have to introduce a definition of imprecise risk matrix. We also have to propose how to order the resulting risk assignments.

Continuing with the previous example, if the expert is allowed to assign an intermediate level of likelihood between remote and unlikely and severity is, for instance, Moderate, then the associated risk value must be in the interval $[3, 6]$, since, looking at Table 2, Remote-Moderate has a risk value of 3 and Unlikely-Moderate is assigned a risk value of 6. We can follow the same procedure for all the pairs that involve an intermediate level and a non-intermediate level, being the risk assignments intervals instead of real numbers. A first step on the construction of the imprecise risk matrix in this context is presented in Table 3.

Notice that now the rows and the columns include the intermediate levels. For example, the label NC-HL refers to a level of intermediate likelihood between Nearly Certain and Highly Likely and the label MI-MO refers to the severity intermediate level between Minor and Moderate. We also notice that we have filled in the cells in the intersection of an intermediate level and a non-intermediate level. The associated risk is represented with an interval that covers all the possible risk values.

However, there are still some empty cells in the imprecise risk matrix that we are constructing. They are those associated with the cases when both severity and likelihood are imprecise. To ease the representation, Table 4 only represents the submatrix containing the "Remote" and "Unlikely" levels of likelihood and the "Minor" and "Moderate" levels of severity.

An intuitive way to solve the cases of empty cells is to consider *an intersection of two intervals of intervals*, similarly to what we have done when only one of the labels was imprecise. Thus, in the example of Table 4, we must con-

Table 3 First step for the construction of an imprecise risk matrix

Likel	Sev. N	N-MI	MI	MI-MO	MO	MO-S	S	S-C	C
NC	5	[5, 10]	10	[10, 15]	15	[15, 20]	20	[20, 25]	25
NC-HL	[4, 5]		[8, 10]		[12, 15]		[16, 20]		[20, 25]
HL	4	[4, 8]	8	[8, 12]	12	[12, 16]	16	[16, 20]	20
HL-L	[3, 4]		[6, 8]		[9, 12]		[12, 16]		[15, 20]
L	3	[3, 6]	6	[6, 9]	9	[9, 12]	12	[12, 15]	15
L-UL	[2, 3]		[4, 6]		[6, 9]		[8, 12]		[10, 15]
UL	2	[2, 4]	4	[4, 6]	6	[6, 8]	8	[8, 10]	10
UL-R	[1, 2]		[2, 4]		[3, 6]		[4, 8]		[5, 10]
R	1	[1, 2]	2	[2, 3]	3	[3, 4]	4	[4, 5]	5

sider all the possible intervals between [2, 4] and [3, 6] that are also between [2, 3] and [4, 6]. This leads to the concept of box, formally defined as follows.

Definition 9 Let $a, b, c, d \in \mathbb{R}$ be four real numbers, such that $a \leq b \leq c \leq d$. The set $\{[x, y] \in \mathcal{L}(\mathbb{R}) : [a, c] \preceq_{Lo} [x, y] \preceq_{Lo} [b, d]$ and $[a, b] \preceq_{Lo} [x, y] \preceq_{Lo} [c, d]\}$ is said to be a box with extremes a, b, c and d.

A box is determined by the four (ordered) bounds a, b, c, and d, equivalently, by the ordered tuple $[a, b, c, d] \in \mathcal{C}^4(\mathbb{R})$. The converse also holds. Any ordered tuple in $\mathcal{C}^4(\mathbb{R})$ allows to build a box. Therefore, by abuse of notation, we use $\mathcal{C}^4(\mathbb{R})$ to refer to the set of all boxes.

We can obtain an equivalent but easier definition for a box.

Proposition 3 *Let $[a, b, c, d] \in \mathcal{C}^4(\mathbb{R})$. We have that*

$$[a, b, c, d] = \{[x, y] \in \mathcal{L}(\mathbb{R})) : a \leq x \leq b \text{ and } c \leq y \leq d\}.$$

Proof The definition of a box implies that $[a, c] \preceq_{Lo} [x, y] \preceq_{Lo} [b, d]$ and $[a, b] \preceq_{Lo} [x, y] \preceq_{Lo} [c, d]$, which is equivalent to $a \leq x \leq b$, $c \leq y \leq d$, $a \leq x \leq c$ and $b \leq y \leq d$. Since we have that $b \leq c$, the latter conditions can be reduced only to $a \leq x \leq b$ and $c \leq y \leq d$.

Observe that we can express any interval and any real number as a box. In particular, any interval $[a, b] \in \mathcal{L}(\mathbb{R})$ can be expressed as a box with the form $[a, a, b, b]$ and any real number $a \in \mathbb{R}$ as $[a, a, a, a]$, so this concept generalizes the concept of interval, and the concept of real number too.

From this equivalent definition, we can also deduce that a box is a particular type of bidimensional hyper rectangle or 2-orthotrope (see [24]). On the other hand, if we consider the identification of a box $[a, b, c, d]$ and the 4-tuple (a, b, c, d), the box concept coincides with the one of 4-dimensional interval proposed in [25].

Now, with the concept of box and the previous ideas about risk matrices, we can fill in the empty cells in Table 3. In general, an imprecise risk matrix has boxes as entries.

Definition 10 An imprecise risk matrix based on m levels of severity and n levels of likelihood is a $n \times m$ matrix with the following entries:

$$
\begin{array}{c|ccccc|}
 & \dots & j & [j, j+1] & j+1 & \dots \\
\hline
\vdots & \ddots & \dots & & \dots & \\
i+1 & \dots & (i+1)j & [(i+1)j, (i+1)(j+1)] & (i+1)(j+1) & \dots \\
[i, i+1] & \dots & [ij, (i+1)j] & * & [i(j+1), (i+1)(j+1)] & \dots \\
i & \dots & ij & [ij, i(j+1)] & i(j+1) & \dots \\
\vdots & & \dots & & \dots & \ddots
\end{array}
$$

Table 4 Risk value for a level of likelihood between remote and unlikely and a level of severity between serious and critical

		Severity		
	UL	MI 4	MI-MO [4, 6]	MO 6
Likelihood	UL-R	[2, 4]	?	[3, 6]
	R	2	[2, 3]	3

for $1 \leq i \leq n-1$ and $1 \leq j \leq m-1$ where $*$ is the box

$$[ij,\ \min\{i(j+1),(i+1)j\},\\ \max\{i(j+1),(i+1)j\}),\ (i+1)(j+1)].$$

Observe that, when possible, we have simplified the notation and written some entries as numbers or intervals, but all the elements of the matrix are boxes actually.

Thus, we obtain the imprecise risk matrix, where the risk assignment to the intersection of two labels is a real number, the risk assignment to the intersection of a label and an intermediate label is an interval and the intersection of two intermediate labels is assigned a box. The complete imprecise risk matrix considered in our study can be found in Table 5.

Since the risk assessments of this matrix are boxes, an order that allows to determine the most important risks is necessary in $\mathcal{C}^4(\mathbb{R})$.

In the previous section, we have proven how to obtain new admissible orders from other ones and the final outcome obviously depends on the chosen order. For instance, if we consider the first lexicographical order, we can colour the imprecise risk matrix using the same legend as in the classical risk matrix, the green colour for the boxes with value lower or equal to 3; yellow if the value is greater than 3 but lower or equal to 7; orange for the boxes greater than 7 and lower or equal to 12 and red for the greatest values. The coloured imprecise risk matrix can be found in Table 6.

Additionally, we can build the Hasse diagram (see [26]) over the risk assignment of the imprecise risk matrix, as we can see in Fig. 1. Notice that a change on the choice of the admissible order for boxes may change the order of the different boxes at the imprecise risk matrix and, therefore, the Hasse diagram too. For instance, [4, 6, 6, 9], associated with (L-Ul)-(MI-MO), and 5 = [5, 5, 5, 5], associated with NC-N, fulfils that [4, 6, 6, 9] $\preceq_{1234}$ 5, but 5 $\preceq_{4321}$ [4, 6, 6, 9].

5 Illustrative Examples

In this section, we use the imprecise risk matrix in two different examples. In particular, we first present a toy example based on educational transportation and later a real example regarding the construction of a metro station.

5.1 An example regarding educational transportation

Consider the following seven different mistakes or errors a local bus driver can make given in [27]:

E1 Going through a yellow light,
E2 Going through a red light,
E3 Driving under the influence of drugs,
E4 Do not use the turn signal,
E5 Waving at another bus driver,
E6 Failing to give the correct change by more than 50 cents,
E7 Failing to give the correct change by 50 cents or less.

Some errors are likely but not very dangerous and others are extremely dangerous but highly unlikely. Since the available resources are limited, the objective is to compare the associated risks and identify the top ones (considering both likelihood and severity) so as to focus the effort on preventing them.

Based on previous studies, the likelihood and severity of the errors, as well as the risk assignment applying the imprecise risk matrix, can be determined as shown in Table 7.

As we have said before, the choice of the admissible order may change the order among the errors. In general, this choice is arbitrary. In the following, we will use ranking methods to fuse the information from some admissible orders to achieve a consensus order. Thus, we can combine the information given by the different orders with their different criteria.

In this example, we use the 24 lexicographic orders introduced in Corollary 4. Although we have used 24 different orders, only 3 different ordinations of the errors appear

- $E1 \geq E2 \geq E4 \geq E3 \geq E5 \geq E7 \geq E6$
- $E1 \geq E2 \geq E4 \geq E3 \geq E5 \geq E6 \geq E7$
- $E2 \geq E1 \geq E4 \geq E3 \geq E5 \geq E7 \geq E6$

all of them eight times.

The combination of this information can be done by means of the Borda Count Method ([28]). It is a family of positional voting rules that regards in the frequency of the possible ordinations. Then, a votation matrix is build, that is, a $n \times n$ matrix $\mathcal{O}$, for n elements, such that $\mathcal{O}_{i,j}$ is the number of times that the ith element is ordered after the j-th element.

Table 5 Complete imprecise risk matrix

Likel	Sev. N	N-MI	MI	MI-MO	MO	MO-S	S	S-C	C
NC	5	[5, 10]	10	[10, 15]	15	[15, 20]	20	[20, 25]	25
NC-HL	[4, 5]	[4, 5, 8, 10]	[8, 10]	[8, 10, 12, 15]	[12, 15]	[12, 15, 16, 20]	[16, 20]	[16, 20, 20, 25]	[20, 25]
HL	4	[4, 8]	8	[8, 12]	12	[12, 16]	16	[16, 20]	20
HL-L	[3, 4]	[3, 4, 6, 8]	[6, 8]	[6, 8, 9, 12]	[9, 12]	[9, 12, 12, 16]	[12, 16]	[12, 15, 16, 20]	[15, 20]
L	3	[3, 6]	6	[6, 9]	9	[9, 12]	12	[12, 15]	15
L-UL	[2, 3]	[2, 3, 4, 6]	[4, 6]	[4, 6, 6, 9]	[6, 9]	[6, 8, 9, 12]	[8, 12]	[8, 10, 12, 15]	[10, 15]
UL	2	[2, 4]	4	[4, 6]	6	[6, 8]	8	[8, 10]	10
UL-R	[1, 2]	[1, 2, 2, 4]	[2, 4]	[2, 3, 4, 6]	[3, 6]	[3, 4, 6, 8]	[4, 8]	[4, 5, 8, 10]	[5, 10]
R	1	[1, 2]	2	[2, 3]	3	[3, 4]	4	[4, 5]	5

Table 6 Imprecise risk matrix

Sev. / Likel.	N	N-MI	MI	MI-MO	MO	MO-S	S	S-C	C
NC	5	[5,10]	10	[10,15]	15	[15,20]	20	[20,25]	25
NC-HL	[4,5]	[4,5,8,10]	[8,10]	[8,10,12,15]	[12,15]	[12,15,16,20]	[16,20]	[16,20,20,25]	[20,25]
HL	4	[4,8]	8	[8,12]	12	[12,16]	16	[16,20]	20
HL-L	[3,4]	[3,4,6,8]	[6,8]	[6,8,9,12]	[9,12]	[9,12,12,16]	[12,16]	[12,15,16,20]	[15,20]
L	3	[3,6]	6	[6,9]	9	[9,12]	12	[12,15]	15
L-UL	[2,3]	[2,3,4,6]	[4,6]	[4,6,6,9]	[6,9]	[6,8,9,12]	[8,12]	[8,10,12,15]	[10,15]
UL	2	[2,4]	4	[4,6]	6	[6,8]	8	[8,10]	10
UL-R	[1,2]	[1,2,2,4]	[2,4]	[2,3,4,6]	[3,6]	[3,4,6,8]	[4,8]	[4,5,8,10]	[5,10]
R	1	[1,2]	2	[2,3]	3	[3,4]	4	[4,5]	5

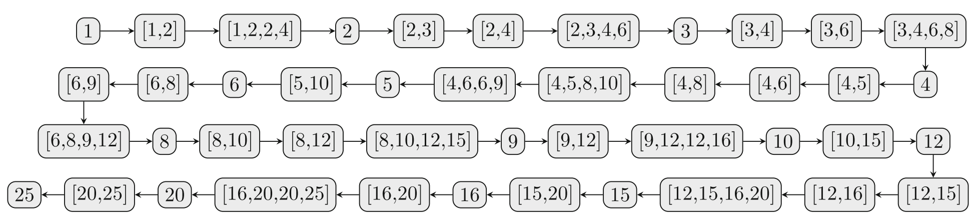

Fig. 1 Hasse diagram for all the risk assignments of the imprecise risk matrix using he admissible order $\preceq_{1234}$

The votation matrix associated to our example can be found in Table 8.

As can be seen in the votation matrix, its elements are multiples of 8, since all the possible ordinations have frequency 8. To obtain the Borda Count Ranking, we just need to add the values of the votation matrix row by row, obtaining the Borda Counts α_i with $i \in \{E1, \ldots, E7\}$. In our case, we have that $\alpha_{E1} = 136$, $\alpha_{E2} = 128$, $\alpha_{E3} = 72$, $\alpha_{E4} = 96$, $\alpha_{E5} = 48$, $\alpha_{E6} = 8$ and $\alpha_{E_7} = 16$. Then, ordering the Borda Counts using the usual order for real numbers, we obtain the final ranking

$$E1 \geq E2 \geq E4 \geq E3 \geq E5 \geq E7 \geq E6.$$

Table 7 Likelihood, severity, and risk assignment assigned to the mistakes

Error	Likelihood	Severity	Risk assignment
E1	HL-L	M-S	[12, 15, 16, 20]
E2	L	C	15
E3	UL-R	C	[5, 10]
E4	L-UL	S-C	[8, 10, 12, 15]
E5	NC	N	5
E6	UL-R	N-MI	[1, 2, 2, 4]
E7	L-UL	N	[2, 3]

Table 8 Votation matrix associated with the educational transport example when using the 24 lexicographical orders

$\mathcal{O}$	$E1$	$E2$	$E3$	$E4$	$E5$	$E6$	$E7$
$E1$	0	16	24	24	24	24	24
$E2$	8	0	24	24	24	24	24
$E3$	0	0	0	0	24	24	24
$E4$	0	0	24	0	24	24	24
$E5$	0	0	0	0	0	24	24
$E6$	0	0	0	0	0	0	8
$E7$	0	0	0	0	0	16	0

Therefore, the most important risks are related, respectively, with going through a yellow light and a red light, while the less important are related to failing to give the correct change.

The Condorcet Ranking (see, for instance, [29, 30]) is based on the most frequent dominance of an element over the others. Using this ranking method, we have that $A \geq B$ is the frequency of $A \geq B$ over the possible ordinations is greater or equal than the frequency of $B \geq A$. Focusing on our example, the error $E1$ is the greatest in 16 of the 24 orders, so it is the greatest error with respect to the Condorcet Ranking. Computing the rest of the cases, we obtain the following ranking:

$$E1 \geq E2 \geq E4 \geq E3 \geq E5 \geq E7 \geq E6,$$

which is, in this example, the same as with the Borda Count Ranking. Having the same ranking when using the Borda Count Ranking and the Condorcet Ranking is quite usual, although there are some examples in which the resulting raking is different (see [29]).

Thus, using ranking methods, we have been able to fuse the information given by the lexicographical orders and, therefore, the most critical error can be identify.

5.2 A Real Example Regarding the Construction of a Metro Station

In [18], the problem of risk assignment in the construction of the station C12 *Nowy Świat-Uniwersytet* in the metro system of Warsaw is studied. In particular, 19 different errors regarding political, economic, social, and technological factors were considered. For more detailed description of the problem and the considered sources of risk, we refer the reader to [18].

- E1 Obtaining administrative decisions and opinions from the owners of uninventoried infrastructure networks.
- E2 Financial consequences of acquiring opinions and positions from the owners of uninventoried infrastructure networks.
- E3 Long tendering procedures.
- E4 Gaining access to the land.
- E5 Acquiring administrative decisions in the pre-tender stage.
- E6 Adverse geotechnical conditions.
- E7 Actual technical condition of neighbouring buildings.
- E8 Damage to infrastructural networks not marked on maps.
- E9 Quality of the execution works.
- E10 Withdrawing the project from the list of indicative projects.
- E11 Protests against localization of the station.
- E12 A collapse.
- E13 Substantial increase in project costs as a result of overall increase in the cost of building materials and salaries.
- E14 Lack of coordination between parties of the investment process.
- E15 Unexplored ordnance and misfires.
- E16 Parameters of existing infrastructure in reference to the possibility of connecting new facilities during the construction.
- E17 High financial expectations of the land owners.
- E18 Quality of the project.
- E19 Changes in law.

Table 9 Assignation of likelihood and severity imprecise labels given a numeric value

Range of values	Likelihood label	Severity label
[1, 1.25)	R	N
[1.25, 1.75]	UL-R	N-MI
(1.75, 2.25)	UL	MI
[2.25, 2.75]	L-UL	MI-MO
(2.75, 3.25)	L	MO
[3.25, 3.75]	HL-L	MO-S
(3.75, 4.25)	HL	S
[4.25, 4.75]	HL-NC	S-C
(4.75, 5]	NC	C

The likelihood and severity of each of the errors were ranked with a number in the interval $[1, 5]$. To obtain our imprecise labels, we have performed the assignation provided in Table 9.

Admittedly, the definition of the boundaries of the values assigned to each imprecise label is quite arbitrary. Intuitively, considering that the values $1, 2, 3, 4, 5$ represent the labels R, UL, L, HL, and NC for the likelihood and N, MI, MO, S, and C for the severity, values that are closer to them should be assigned with non-imprecise labels, while values in the mid-

Table 10 Likelihood, severity, risk assignment, and position assigned to the different errors

Error	Likelihood	Severity	Risk assignment	Position
E1	5	5	25	1^{st}
E2	[4, 5]	[4, 5]	[16, 20, 20, 25]	3^{rd}
E3	4	5	20	2^{nd}
E4	[4, 5]	3	[12, 15]	5^{th}
E5	[4, 5]	[3, 4]	[12, 15, 16, 20]	4^{th}
E6	5	[1,2]	[5, 10]	10^{th}
E7	3	[2, 3]	[6, 9]	$7-8^{th}$
E8	4	2	8	6^{th}
E9	[3, 4]	2	[6, 8]	9^{th}
E10	[2, 3]	2	[4, 6]	13^{th}
E11	[2, 3]	3	[6, 9]	$7-8^{th}$
E12	[4, 5]	[1, 2]	[4, 5, 8, 10]	11^{th}
E13	4	1	4	$14-15^{th}$
E14	4	1	4	$14-15^{th}$
E15	[1, 2]	[4, 8]	[4, 8, 8, 16]	12^{th}
E16	[3, 4]	1	[3, 4]	17^{th}
E17	3	1	3	18^{th}
E18	[3, 4]	[1, 2]	[3, 4, 6, 8]	16^{th}
E19	[2, 3]	[1, 2]	[2, 3, 4, 6]	19^{th}

dle range between them should be associated with imprecise labels.

After the determination of the labels is done, each of the 19 errors has its likelihood, severity, and risk (see the first four columns of Table 10). Then, we can use the admissible order $\preceq_{1234}$ to order the different cases (see last column of Table 10). If different orders wanted to be considered, then a similar procedure as in the latter example, using ranking methods, can be applied.

Therefore, the main risks are the ones related to owners of uninventoried infrastructure networks and tendering procedures, due their both big likelihood and consequences. On the other hand, changes in law and high financial expectations of the land owners are seen as the less important risks.

6 Conclusions

In this contribution, we have studied how to order tuples of dimension greater than two and, in particular, ordered tuples. We have provided general results on how to obtain new admissible orders for tuples from known ones. For example, we have proven that the generalization of the Xu-Yager order for four-dimensional tuples can be obtained from the type 1 lexicographic order.

The previous theoretical study has been applied to the context of Human Reliability, in particular, to Risk Assessment. The concept of imprecise risk matrix has been introduced. In this matrix, the expert or decision-maker is allowed to choose intermediate levels of severity and likelihood for each evaluated error. The combination of those two assessments provides a level of risk. The use of our definition of imprecise risk matrix allows flexibility in the risk assignments. They are not necessarily real number any longer, but can be represented by more general mathematical concepts such as intervals or boxes. The concept of box has also been introduced in this paper as a generalization of intervals to capture risk assessments. The theoretical study developed in the first part of the manuscript allows us to order the new structures. Therefore, it allows to determine the most important errors.

A case study regarding educational transportation has been presented. In this example, different errors were compared using the imprecise risk matrix and the 24 lexicographical orders for boxes. To obtain a consensus ranking for all the orders, election methods such as the Borda Count Ranking and the Condorcet Ranking have been used.

The main limitation of the presented method is the huge amount of possible admissible orders that can be chosen to order the tuples in practical applications. Even with the use of the Borda Count Ranking or the Condorcet Ranking when considering more than one order, these ranking methods could fail to obtain a unique winner.

An open problem concerning the construction of linear orders is to determine what kind of admissible orders can be obtained as a non-linear transformation of the type 1 lexicographic order. Concerning the application, it would be interesting to be able to consider a third criteria, in addition to likelihood and severity, to evaluate errors. This would lead to work with three-dimensional matrices, and the imprecise risk assignment could be represented by three (or higher)-dimensional boxes. The use of more election methods, such as the Kemeny ranking, to get a consensus order on the risk assignments could also be explored.

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Declarations

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