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On the Axial Symmetry of 2D Star-Shaped Sets

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Abstract

An essential aspect of the study of shapes is the symmetry because of its importance from a theoretical point of view and its applicability in multiple real-life problems. In this manuscript, the axial symmetry of 2D star-shaped sets is analyzed. For such a purpose, different measures of axial symmetry of a star-shaped set are proposed and the concept of a best symmetry axis is also introduced. By means of them, families of symmetry measures for star-shaped sets quantifying the degree of symmetry of a set of that class are introduced. All of them are discussed in detail, providing their main properties and the existence of at least a best axis of symmetry, which could be not unique, for any star-shaped set. Some examples illustrate the concepts and results of the manuscript.

Keywords Kernel of a star-shaped set · Measure of axial symmetry · Radial function · Star-shaped set · Steiner point · Symmetry axis

1 Introduction

Star-shaped sets come up naturally in many fields, particularly in geometry (see, for instance, [19] or [30]), as well as in functional analysis (see, for instance, [3]), computational geometry (see, for instance, [16] or [28]), approximation theory (see, for instance, [21]), optimization (see, for instance, [17] or [20]), etc. The notion of star-shapedness is a generalization of that of convexity. The idea is to consider sets where the visibility of every point is guaranteed at least from one point in the set, not requiring the visibility from every point. The study of the geometry of star-shaped sets started at the beginning of the 20th century and is still actively developing (see, for instance, [4, 5] or [15]) because of its numerous applications (see, for instance, [6, 8] or [11]).

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For a detailed and rigorous review of the fundamental mathematical notion of star-shapedness and a board spectrum of applications, the reader is referred, for instance, to [15].

One of the questions of interest related to star-shaped sets is symmetry. Symmetry of convex sets, mainly central symmetry, has been study in depth (see, for instance, [33, 34] or [32]). A wide variety of geometric and analytic constructions that quantify various concepts of symmetry for convex sets has been developed due to the application in a huge number of interrelated problems on symmetry. The work presented in [32] contains a detailed and rigorous analysis of those procedures to quantify symmetry of convex bodies. An affine measure of symmetry is defined as an affine invariant continuous function on the space of convex bodies provided with a suitable metric topology. That mapping tries to quantify the degree of symmetry of a convex set and reaches its smallest (or largest) value at symmetric convex sets. Some examples of applicability of axial symmetry of convex sets to image analysis are, for instance, [22] and [23].

Our aim is the study of measures of axial symmetry of 2D star-shaped sets, that is, relaxing the condition of convexity and considering only the star-shapedness requirement.

The study of measures of axial symmetry of a 2D star-shaped set that is developed in this manuscript is strongly related to the notions of *kernel*, *Steiner point*, *support function* and *radial function*. The kernel of a star-shaped set is made up of all points from which every point of the star-shaped set is visible. Different characterizations of the kernel of a star-shaped set have been developed like, for instance, those in [4, 13, 31] or [14]. The Steiner point of the kernel will play a relevant role in order to relate star-shaped sets to radial functions. The Steiner point of a set can be viewed as the mass center of such a set. The Steiner point was introduced by R. Steiner for convex polytopes of $\mathbb{R}^2$ in [29], and it was extended to convex polytopes of $\mathbb{R}^n$ by B. Grünbaum in [9] (see also [10]). It has been generalized to convex bodies of finite-dimensional vector spaces by G.C. Shephard in [26]. The Steiner point is defined in terms of the so-called support function. The radial function quantifies the distance from the origin to the frontier of a star-shaped set when the origin belongs to its kernel, in the direction given by a point of the unit sphere.

In the manuscript, we introduce measures of the degree of symmetry of a star-shaped set about a line and the concept of best symmetry axis for a star-shaped set. By means of them, measures of axial symmetry on the class of star-shaped sets are also defined.

We will introduce measures of axial symmetry about a line, measures of axial symmetry and the concept of best symmetry axis. All of them will be discussed in detail, providing the main properties and characteristics, and the existence of at least a best axis of symmetry for any star-shaped set.

The structure of the paper is the following. In Sect. 2, we collect the concepts and basic results that we need for the development of the manuscript. Section 3 is devoted to the introduction of measures of the degree of symmetry of a star-shaped set about a line and the concept of best symmetry axis for a star-shaped set. These concepts will lead to the definition of measures of axial symmetry on the family of star-shaped sets. Different results which justify the correctness of those definitions are developed in this section of the manuscript. In Sect. 4, relevant properties of the above-mentioned measures are developed. Examples illustrating the concepts and results of the paper are included in Sect. 5. Finally, Sect. 6 summarizes the main contributions of this manuscript.

2 Preliminaries

The concepts needed for the development of the manuscript are included in this section.

Let $\alpha \in \mathbb{R}$, $y, z \in \mathbb{R}^2$ and $A, B \subset \mathbb{R}^2$. The Minkowski addition of A and B is the set $A + B = \{a + b \in \mathbb{R}^2 \mid a \in A, b \in B\}$, $A + z$ will denote the set $A + \{z\}$, and αA will be the set $\{\alpha a \in \mathbb{R}^2 \mid a \in A\}$. If P is a real square matrix of order 2, PA will be the set $\{Pa \in \mathbb{R}^2 \mid a \in A\}$. Moreover, $[y, z]$ will stand for the set $\{\lambda y + (1-\lambda)z \mid \lambda \in [0, 1]\}$. Such a set and the corresponding sets $(y, z]$ (when $\lambda \in [0, 1)$), $[y, z)$ (when $\lambda \in (0, 1]$) and (y, z) (when $\lambda \in (0, 1)$) are referred to as intervals.

The open ball of radius $\epsilon > 0$ centered at $a \in \mathbb{R}^2$, when we consider the usual Euclidean metric on $\mathbb{R}^2$, will be denoted by $B_\epsilon(a)$. The interior of the set A in the usual topology of $\mathbb{R}^2$ will be denoted by A°.

The unit sphere in $\mathbb{R}^2$ will be represented by S^1, that is, $S^1 = \{u \in \mathbb{R}^2 \mid \|u\| = 1\}$, where $\|\cdot\|$ stands for the usual Euclidean norm on $\mathbb{R}^2$.

The scalar product in $\mathbb{R}^2$ will be denoted by $\langle \cdot, \cdot \rangle$.

We will denote by $O(2, \mathbb{R})$ the set of real orthogonal matrices of order 2, that is, the set of 2×2 matrices with elements in $\mathbb{R}$ which are orthogonal matrices.

The concept of star-shaped set is essential in the manuscript.

Definition 1 A set $A \subset \mathbb{R}^2$ is said to be a star-shaped set if there exists $a \in A$ such that $l \cap A$ is an interval for any line l with $a \in l$.

Basically, a set A is star-shaped if there exists a point in A from which it is possible to "see" all points of A. That is equivalent to the existence of a point $a \in A$ such that $[a, x] \subset A$ for all $x \in A$.

A relevant role will be played by the kernel of a star-shaped set.

Definition 2 Let $A \subset \mathbb{R}^2$ be a star-shaped set. The kernel of A, denoted by $Ker\, A$, is the set $Ker\, A = \{a \in A \mid l \cap A$ is an interval for any line l with $a \in l\}$.

The kernel of a star-shaped set A is made up of the points of the set from which all the points of A are visible.

The symbol $\mathcal{S}$ will stand for the class of compact star-shaped sets A of $\mathbb{R}^2$ such that $(Ker\, A)^\circ \neq \emptyset$.

It is known that if $A \in \mathcal{S}$, $Ker\, A$ is a convex subset of $\mathbb{R}^2$ (see [31]).

Let $\mathcal{B}$ be the class of non-empty bounded convex subsets of $\mathbb{R}^2$.

Definition 3 The support function of a set $B \in \mathcal{B}$ is the mapping $\gamma(\cdot, B) : S^1 \to \mathbb{R}$, with $\gamma(u, B) = \sup_{b \in B} \langle u, b \rangle$ for all $u \in S^1$.

Recall that for any $x \in \mathbb{R}^2$ and any $u \in S^1$, there exist $a_x \in \mathbb{R}$ and $x' \in \mathbb{R}^2$ such that $x = a_x u + x'$ and $\langle u, x' \rangle = 0$. Thus, $a_x = \langle u, x \rangle$. The number a_x is said to be the scalar orthogonal projection of x onto the line passing through the point $(0, 0) \in \mathbb{R}^2$ with direction u (see Fig. 1, left).

Given $B \in \mathcal{B}$ and $u \in S^1$, $\gamma(u, B) = \sup_{b \in B} \langle u, b \rangle = \sup_{b \in B} a_b$ (see Fig. 1, right).

The Steiner point of a non-empty bounded convex set will be key in the manuscript.

Definition 4 The Steiner point mapping is $s : \mathcal{B} \longrightarrow \mathbb{R}^2$, with

$$s(B) = 2 \int_{S^1} u\gamma(u, B)\; d\mu$$

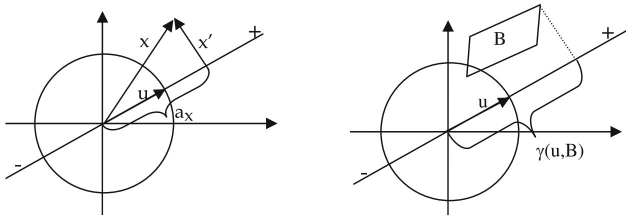


Fig. 1 Scalar orthogonal projection of the point x onto the line determined by $(0, 0)$ and u (left), support function of the set B at $u \in S^1$ (right)

for all $B \in \mathcal{B}$, where μ is the measure on S^1 proportional to the Lebesgue measure, satisfying that $\mu(S^1) = 1$.

The Steiner point of $B \in \mathcal{B}$, $s(B)$, can be viewed as the mass center of B.

The Steiner point mapping has been widely studied (see, for instance, [1, 2, 24] or [27] for the most relevant properties of the Steiner point mapping). It holds that,

1) for all $K \in \mathcal{B}$, $s(K) \in K$,

2) for all $K, L \in \mathcal{B}$ and for all $\lambda, \mu \in \mathbb{R}$, $s(\lambda K + \mu L) = \lambda s(K) + \mu s(L)$,

3) the Steiner point mapping commutes with affine isometries, i.e., for all $K \in \mathcal{B}$ and for all affine isometry $f : \mathbb{R}^2 \to \mathbb{R}^2$, it holds that $s(f(K)) = f(s(K))$.

Note that for any $A \in \mathcal{S}$, it holds that $Ker\, A$ belongs to $\mathcal{B}$, and so, it is possible to consider the Steiner point of that set.

Given $A \in \mathcal{S}$, let $\widetilde{A} = A - s(Ker\, A) = \{a - s(Ker\, A) \in \mathbb{R}^2 \mid a \in A\}$. Since rigid-body transformations do not change the shape of the body, we have that $\widetilde{A} \in \mathcal{S}$.

Let us consider the set $\mathcal{S}_1 = \{A - s(Ker\, A) \mid A \in \mathcal{S}\}$.

Let us prove that if $\mathcal{T} = \{A \in \mathcal{S} \mid s(Ker\, A) = (0, 0)\}$, then $\mathcal{S}_1 = \mathcal{T}$. Trivially, $\mathcal{T} \subset \mathcal{S}_1$; conversely, the definition of the kernel of a star-shaped set ensures that for all $A \in \mathcal{S}$, $Ker\,(A - s(Ker\, A)) = Ker\, A - s(Ker\, A)$, and property 2) of the Steiner point mapping provides that $s(Ker\,(A - s(Ker\, A))) = s(Ker\, A) - s(Ker\, A) = (0, 0)$. As a consequence, $\mathcal{S}_1 \subset \mathcal{T}$, and so, both sets are equal.

Now, let us consider the so-called radial function of a star-shaped set in $\mathcal{T}$.

Definition 5 Given $A \in \mathcal{T}$, the radial function of A is the mapping $\rho_A : S^1 \longrightarrow \mathbb{R}$, with $\rho_A(u) = \sup\{\alpha \geq 0 \mid \alpha u \in A\}$ for all $u \in S^1$.

A set in the class $\mathcal{T}$ is uniquely determined by its radial function, in the sense that given $A, B \in \mathcal{T}$, then $A = B$ if and only if $\rho_A = \rho_B$.

Thus, we can consider on $\mathcal{T}$ the metric $d_{\mathcal{T}} : \mathcal{T} \times \mathcal{T} \to \mathbb{R}$, given by

$$d_{\mathcal{T}}(A, B) = \sup_{u \in S^1} |\rho_A(u) - \rho_B(u)| = \|\rho_A - \rho_B\|_\infty,$$

for all $A, B \in \mathcal{T}$.

On the unit sphere S^1, we will consider the metric $d : S^1 \times S^1 \to \mathbb{R}$, with $d(u_1, u_2)$ equal to the angle determined by u_1 and u_2, that is, the angle given by the vectors $\overrightarrow{Ou_1}$ and $\overrightarrow{Ou_2}$ for all $u_1, u_2 \in S^1$, where $O = (0, 0)$ (see [31]).

It is known that for any $A \in \mathcal{T}$, the mapping $\rho_A : S^1 \longrightarrow \mathbb{R}$ is Lipschitz with respect to the metric d on S^1. Moreover, if $A \in \mathcal{T}$ and $r, R > 0$ satisfy that $rB_1(O) \subset A \subset RB_1(O)$, it holds that

$$|\rho_A(u_1) - \rho_A(u_2)| \leq R\left(\left(\frac{R}{r}\right)^2 - 1\right)^{\frac{1}{2}} d(u_1, u_2) \qquad (1)$$

for all $u_1, u_2 \in S^1$ (see [31]).

Thus, the mapping ρ_A is continuous with respect to the metric d on S^1 for any $A \in \mathcal{T}$. Moreover, it holds that $\rho_A(u) > 0$ for all $u \in S^1$; in fact, there exists $m_A > 0$ such that $\rho_A(u) \geq m_A$ for all $u \in S^1$.

We will denote by $C(S^1)$ the class of continuous mappings $f : S^1 \to \mathbb{R}$ with respect to the metric d on S^1. Note that for any $A \in \mathcal{T}$, it holds that $\rho_A \in C(S^1)$.

On the space $C(S^1)$, we will consider the L^p-norms with respect to the measure μ given by

$$\|f\|_p = \left(\int_{S^1} |f(u)|^p \, d\mu\right)^{1/p} \quad \text{if } p \in [1, \infty), \quad \text{and}$$

$$\|f\|_\infty = \sup_{u \in S^1} |f(u)|.$$

The corresponding d_p-metrics are given by $d_p(f, g) = \|f - g\|_p$, that is,

$$d_p(f, g) = \left(\int_{S^1} |f(u) - g(u)|^p \, d\mu\right)^{1/p} \quad \text{if } p \in [1, \infty),$$

and

$$d_\infty(f, g) = \sup_{u \in S^1} |f(u) - g(u)|.$$

One key element of our approach to axial symmetry will be the following mapping.

Consider $\theta \in [0, \pi]$. Let $f_\theta : \mathbb{R}^2 \to \mathbb{R}^2$ be the mapping which produces a symmetry about the line passing through the origin and with slope equal to $\tan(\theta)$, that is, $y = (\tan\theta)x$, with the agreement that when $\theta = \pi/2$, the line is $x = 0$. This mapping is given by $f_\theta(a) = R_\theta a$ for all $a \in \mathbb{R}^2$, with R_θ the matrix

$$R_\theta = \begin{pmatrix} \cos 2\theta & \sin 2\theta \\ \sin 2\theta & -\cos 2\theta \end{pmatrix} \in O(2, \mathbb{R}).$$

3 Measures of Axial Symmetry and the Best Axis of Symmetry

In this section, measures of the degree of symmetry of a star-shaped set about a line and the concept of best symmetry axis for a star-shaped set are introduced. On the basis of those elements, measures of axial symmetry on $\mathcal{S}$ are also defined. Different results which justify the correctness of those definitions are developed in this section of the manuscript.

Next, we justify the particular translation of a star-shaped set in $\mathcal{S}$ that we have considered to define the class $\mathcal{T}$. Basically, if a star-shaped set A has a symmetry axis l, we will see that l is also a symmetry axis of the kernel of A, and so, the Steiner point of that kernel is in l.

Proposition 1 *Let $A \in \mathcal{S}$ be symmetric about a line l. Then, l is a symmetry axis of $Ker\ A$.*

Proof Let $f^l : \mathbb{R}^2 \to \mathbb{R}^2$ be the mapping with $f^l(x)$ the symmetric point of x about the line l, for all $x \in \mathbb{R}^2$. Recall that f^l is an affine transformation.

If the result is false, there exists $x \in Ker\ A$ such that $f^l(x) \notin Ker\ A$. Note that $f^l(x) \in A$ since A is symmetric about l. Thus, there exists $y \in A$ and $\lambda \in [0, 1]$ with $\lambda f^l(x) + (1-\lambda)y \notin A$. Since A is symmetric about l, $f^l(\lambda f^l(x) + (1-\lambda)y) \notin A$. Because f^l is an affine transformation, $f^l(\lambda f^l(x) + (1-\lambda)y) = \lambda x + (1-\lambda) f^l(y)$. Observe that $f^l(y) \in A$ since $y \in A$ and A is symmetric about l, which implies that $x \notin Ker\ A$, and so a contradiction. □

Corollary 2 *Let $A \in \mathcal{S}$ be symmetric about a line l. Then, $s(Ker\ A) \in l$ and $A - s(Ker\ A)$ has a symmetry axis which contains the point $(0, 0)$.*

Proof It is well-known that if $B \subset \mathbb{R}^2$ is a symmetric region about l, then its Steiner point is located in l, and so, Proposition 1 proves the first part. For the second part, it is sufficient to take the axis $l - s(Ker\ A)$. □

Note that if A is symmetric about a line, then $A - s(Ker\ A)$ is symmetric about a line which contains the point $O = (0, 0)$. That is the reason to consider $\mathcal{T} = \{A - s(Ker\ A) \mid A \in \mathcal{S}\} = \{A \in \mathcal{S} \mid s(Ker\ A) = (0, 0)\}$ to introduce in a first step measures of axial symmetry. Observe that for those sets, the search of axis of symmetry is reduced to lines passing through the point $(0, 0)$. Note that for elements in $\mathcal{T}$, the radial function is well-defined. Moreover, in a similar way, if $A - s(Ker(A))$ is symmetric about a line l, then A is symmetric about the line $l + s(Ker(A))$.

Proposition 3 *Let $A \in \mathcal{T}$ and $\theta \in [0, \pi]$. It holds that $f_\theta(A) \in \mathcal{T}$.*

Proof Since f_θ is an homeomorphism and A is compact, $f_\theta(A)$ is compact.

Note that multiplying R_θ by A implies the application of an orthogonal transformation given by the matrix R_θ to A. Since rigid-body transformations do not change the shape of the body, we have that $f_\theta(A)$ is a star-shaped set.

Let us see that $Ker\ f_\theta(A) = f_\theta(Ker\ A)$.

Let $f_\theta(a)$ be an element in $Ker\ f_\theta(A)$. Consider l a line passing through a. Then $f_\theta(l)$ is a line such that $f_\theta(a) \in f_\theta(l)$, and so, $f_\theta(l) \cap f_\theta(A)$ is an interval. Applying the orthogonal transformation f_θ^{-1}, we have that $l \cap A$ is an interval. Then, a is an element in $Ker\ A$ and so, $Ker\ f_\theta(A) \subset f_\theta(Ker\ A)$. On the other hand, consider a an element in $Ker\ A$. Let l be a line passing through $f_\theta(a)$. Since $f_\theta^{-1}(l)$ is a line passing through a, $f_\theta^{-1}(l) \cap A$ is an interval. Thus, $l \cap f_\theta(A)$ is an interval and so, $f_\theta(a)$ is an element in $Ker\ f_\theta(A)$. Therefore, $Ker\ f_\theta(A) = f_\theta(Ker\ A)$.

Since an orthogonal transformation preserves norms, we have that $(f_\theta(K))^\circ = f_\theta((K)^\circ)$ for any $K \subset \mathbb{R}^2$. Hence,

$$(Ker\ f_\theta(A))^\circ = (f_\theta(Ker\ A))^\circ = f_\theta((Ker\ A)^\circ),$$

which implies that $f_\theta(A) \in \mathcal{S}$ since $(Ker\ f_\theta(A))^\circ \neq \emptyset$.

Note that $A \in \mathcal{T}$, and so, $(0, 0) = s(Ker\ A)$. Since f_θ is linear, it holds that $(0, 0) = f_\theta((0, 0)) = f_\theta(s(Ker\ A))$. Moreover, f_θ is an orthogonal transformation, and so, property 3) of the Steiner point mapping ensures that $(0, 0) = f_\theta((0, 0)) = f_\theta(s(Ker\ A)) = s(f_\theta(Ker\ A)) = s(Ker\ f_\theta(A))$. Thus, $f_\theta(A) \in \mathcal{T}$. □

The above result permits to consider the radial function of the set $f_\theta(A)$ for any $A \in \mathcal{T}$ and any $\theta \in [0, \pi]$.

The following result characterizes the radial function of the set $f_\theta(A)$ in terms of the radial function of A.

Proposition 4 *Let $A \in \mathcal{T}$ and $\theta \in [0, \pi]$. It holds that $\rho_{f_\theta(A)}(u) = \rho_A(f_\theta(u))$ for all $u \in S^1$.*

Proof Note that $R_\theta \in O(2, \mathbb{R})$, thus $f_\theta(u) \in S^1$ for all $u \in S^1$. Since R_θ satisfies that $R_\theta^{-1} = R_\theta$, f_θ is a linear orthogonal transformation with $f_\theta = f_\theta^{-1}$. Now, for all $u \in S^1$, $\rho_{f_\theta(A)}(u) = \sup\{\alpha \in \mathbb{R} \mid \alpha u \in f_\theta(A)\} = \sup\{\alpha \in \mathbb{R} \mid \alpha f_\theta(u) \in A\} = \rho_A(f_\theta(u))$, which proves the result. □

Observe that an element $A \in \mathcal{T}$ is symmetric about a line if it is invariant under a reflection about such a line. Thus, $A \in \mathcal{T}$ is symmetric about the line $y = (\tan\theta)x$ if the radial function takes "the same values on both sides" of the line (see Fig. 2). This fact motivates the following definition.

Definition 6 Let $A \in \mathcal{T}$ and let $p \in [1, \infty]$. Define the mapping $\nabla_A^p : [0, \pi] \to \mathbb{R}$, with

$$\nabla_A^p(\theta) = \frac{1}{\|\rho_A\|_p} d_p(\rho_A, \rho_{f_\theta(A)})$$

for all $\theta \in [0, \pi]$. The mapping ∇_A^p is said to be the measure of axial symmetry of order p of the set A.

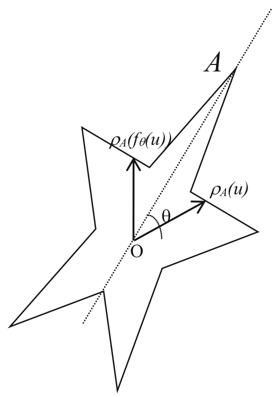


Fig. 2 Symmetry axis (dotted line) of star-shaped set $A \in \mathcal{T}$. Note that $\rho_A(f_\theta(u)) = \rho_A(u)$ for all $u \in S^1$

Observe that the line determined by the angle θ is a symmetry axis of $A \in \mathcal{T}$ if and only if $\nabla_A^p(\theta) = 0$, regardless of the value of p.

The quantity $\nabla_A^p(\theta)$ can be interpreted as a measure of the axial symmetry of A with respect to the line determined by the angle θ and passing through the point $(0, 0)$. The lower $\nabla_A^p(\theta)$, the greater the symmetry of A about $y = (\tan\theta)x$.

The value of p can be viewed as a permissiveness index of the lack of symmetry. The greater p, the lower the permissiveness. If we take $p = \infty$, and so $\nabla_A^\infty(\theta)$ is $d_\infty(\rho_A, \rho_{f_\theta(A)}) = \sup_{u\in S^1} |\rho_A(u) - \rho_{f_\theta(A)}(u)|$ except for a normalization factor, the measure of lack of axial symmetry of A is taken as the highest lack of symmetry between the points $\rho_A(u)$ and $\rho_{f_\theta(A)}(u)$ when u ranges all the sphere S^1.

On the contrary, if we take $p = 1$, the measure of lack of symmetry is

$$\int_{S^1} |\rho_A(u) - \rho_{f_\theta(A)}(u)|\, d\mu$$

except for a normalization factor, and so the measure is the average of the lack of axial symmetry between $\rho_A(u)$ and $\rho_{f_\theta(A)}(u)$ when u ranges all the sphere S^1.

Clearly, the above measures of axial symmetry about a line can be extended to elements of $\mathcal{S}$ by defining for all $A \in \mathcal{S}$,

$$\nabla_A^p(\theta) = \nabla_{A-s(Ker(A))}^p(\theta), \qquad (2)$$

where $\nabla_A^p(\theta)$ quantifies the degree of symmetry of A with respect to the line with slope θ passing through the point $s(Ker(A))$.

Note that ∇_A^p can be defined for any angle $z \in \mathbb{R}$ by taking $\nabla_A^p(z) = \nabla_A^p(\alpha)$ with $z = \alpha + [\frac{z}{\pi}]\pi$ where $\alpha \in [0, \pi]$ and $[\cdot]$ stands for the integer part function. Of course, working on a compact set instead of the real line offers clear advantages from mathematical and computational points of view.

An appealing matter in the axial symmetry analysis is the search of possible axes of symmetry, or the search of the best axes of symmetry. Roughly speaking, for a set $A \in \mathcal{T}$ and $p \in [1, \infty]$, the line determined by the angle θ is a better axis of symmetry than the line determined by the angle θ' if $\nabla_A^p(\theta) \leq \nabla_A^p(\theta')$.

As a consequence of this, we introduce the following definition.

Definition 7 Let $A \in \mathcal{T}$ and $p \in [1, \infty]$. The line determined by the angle $\widehat{\theta}$ passing through the point $(0, 0)$ is said to be a best axis of symmetry of order p of the set A, if $\nabla_A^p(\widehat{\theta}) \leq \nabla_A^p(\theta)$ for all $\theta \in [0, \pi]$.

By using formula (2), the concept of best axis of symmetry can be extended to star-shaped sets in $\mathcal{S}$. For those elements, the best axis is given by the angle $\widehat{\theta}$ and the point $s(Ker(A))$.

For simplicity, results will be developed for axial symmetry measures on $\mathcal{T}$, the counterpart on $\mathcal{S}$ being direct by means of formula (2).

The following results prove the existence of, at least, a best symmetry axis for any $A \in \mathcal{T}$.

Proposition 5 *Let $u \in S^1$ and $\theta, \theta' \in [0, \pi]$. It holds that*

$$d(f_\theta(u), f_{\theta'}(u)) = 2|\theta - \theta'|.$$

Proof Since $u \in S^1$, there exists $\alpha \in [0, 2\pi)$ with $u = (\cos\alpha, \sin\alpha)^t$. Thus,

$$f_\theta(u) = R_\theta u = \begin{pmatrix} \cos 2\theta & \sin 2\theta \\ \sin 2\theta & -\cos 2\theta \end{pmatrix} \begin{pmatrix} \cos\alpha \\ \sin\alpha \end{pmatrix} = (\cos(2\theta - \alpha),\ sin(2\theta - \alpha))^t.$$

In a similar way,

$$f_{\theta'}(u) = (cos(2\theta' - \alpha),\ \sin(2\theta' - \alpha))^t.$$

Since R_θ is orthogonal, $\|f_\theta(u)\| = \|u\| = \|f_{\theta'}(u)\| = 1$. Therefore, the cosine of the angle determined by $f_\theta(u)$ and $f_{\theta'}(u)$ is

$$\begin{aligned} &\langle f_\theta(u), f_{\theta'}(u)\rangle \\ &= \cos(2\theta - \alpha)\cos(2\theta' - \alpha) + \sin(2\theta - \alpha)\sin(2\theta' - \alpha) \\ &= \cos(2\theta - 2\theta'), \end{aligned}$$

which derives the result. □

Proposition 6 *Let $A \in \mathcal{T}$ and $p \in [1, \infty]$. The mapping $\nabla_A^p : [0, \pi] \to \mathbb{R}^2$ is continuous.*

Proof Let $\theta, \theta' \in [0, \pi]$. Consider the case $p \in [1, \infty)$.

By Proposition 4,

$$\nabla_A^p(\theta) = \frac{1}{\|\rho_A\|_p} d_p(\rho_A, \rho_A \circ f_\theta),$$

and so,

$$\begin{aligned}&|\nabla_A^p(\theta) - \nabla_A^p(\theta')|\\&= \frac{1}{\|\rho_A\|_p}|d_p(\rho_A, \rho_A \circ f_\theta) - d_p(\rho_A, \rho_A \circ f_{\theta'})|\\&\leq \frac{1}{\|\rho_A\|_p} d_p(\rho_A \circ f_\theta, \rho_A \circ f_{\theta'})\\&= \frac{1}{\|\rho_A\|_p}(\int_{S^1} |\rho_A(f_\theta(u)) - \rho_A(f_{\theta'}(u))|^p \, d\mu)^{1/p}.\end{aligned}$$

Since the radial function of A is Lipschitz with respect to the metric d on S^1, there exists a constant $M_A > 0$ such that

$$|\rho_A(f_\theta(u)) - \rho_A(f_{\theta'}(u))| \leq M_A d(f_\theta(u), f_{\theta'}(u))$$

for any $u \in S^1$. Note that formula (1) ensures that M_A can be taken as $R((\frac{R}{r})^2 - 1)^{\frac{1}{2}}$, where $R, r > 0$ are any values satisfying that $rB_1(O) \subset A \subset RB_1(O)$.

Thus, we have that

$$\begin{aligned}&|\nabla_A^p(\theta) - \nabla_A^p(\theta')|\\&\leq \frac{1}{\|\rho_A\|_p} M_A(\int_{S^1} d(f_\theta(u), f_{\theta'}(u))^p \, d\mu)^{1/p}.\end{aligned}$$

By Proposition 5,

$$d(f_\theta(u), f_{\theta'}(u)) = 2|\theta - \theta'|,$$

which implies that

$$|\nabla_A^p(\theta) - \nabla_A^p(\theta')| \leq \frac{1}{\|\rho_A\|_p} M_A 2|\theta - \theta'|,$$

and so the result is proved when $p \in [1, \infty)$.

Consider $p = \infty$. Reasoning in a similar way, we obtain that

$$\begin{aligned}|\nabla_A^\infty(\theta) - \nabla_A^\infty(\theta')| &\leq \frac{1}{\|\rho_A\|_\infty} d_\infty(\rho_A \circ f_\theta, \rho_A \circ f_{\theta'})\\&= \frac{1}{\|\rho_A\|_\infty} \sup_{u \in S^1} |\rho_A(f_\theta(u)) - \rho_A(f_{\theta'}(u))|\\&\leq \frac{1}{\|\rho_A\|_\infty} M_A \sup_{u \in S^1} d(f_\theta(u), f_{\theta'}(u))\\&\leq \frac{1}{\|\rho_A\|_\infty} M_A 2|\theta - \theta'|,\end{aligned}$$

which derives the result in this case, and so the proof. □

As a consequence, we prove the existence of best symmetry axes.

Proposition 7 *Let $A \in \mathcal{T}$ and let $p \in [1, \infty]$. There exists at least one value $\widehat{\theta} \in [0, \pi]$ such that $\nabla_A^p(\widehat{\theta}) \leq \nabla_A^p(\theta)$ for all $\theta \in [0, \pi]$.*

Proof Proposition 6 assures that ∇_A^p is a continuous function on a compact set, which guarantees the existence of a minimum of the mapping. □

From now on, given $A \in \mathcal{T}$ and $p \in [1, \infty]$, we are denoting by $\widehat{\theta}_{A,p}$ a value in $[0, \pi]$ such that $\nabla_A^p(\widehat{\theta}_{A,p}) \leq \nabla_A^p(\theta)$ for all $\theta \in [0, \pi]$.

Next, we define the measure of axial symmetry of order p on $\mathcal{T}$.

Definition 8 Let $p \in [1, \infty]$. Define the mapping $\Delta^p : \mathcal{T} \to \mathbb{R}$ with $\Delta^p(A) = \nabla_A^p(\widehat{\theta}_{A,p})$ for all $A \in \mathcal{T}$. The mapping Δ^p is said to be the measure of axial symmetry of order p on $\mathcal{T}$.

The mapping Δ^p quantifies for each element of $\mathcal{T}$ its axial symmetry degree. For each element A in $\mathcal{T}$, the lower the value of $\Delta^p(A)$, the larger its axial symmetry. In fact, $A \in \mathcal{T}$ is symmetric about a line if and only if $\Delta^p(A) = 0$ for any $p \in [1, \infty]$. Clearly, Δ^p can be defined on $\mathcal{S}$ by taking for each $A \in \mathcal{S}$ the set $A - s(Ker(A))$, which is in $\mathcal{T}$.

4 On the Measures of Axial Symmetry

Relevant properties of the measures of axial symmetry of a set $A \in \mathcal{T}$, that is, ∇_A^p, and of the measures of axial symmetry on $\mathcal{T}$, that is, Δ^p, are developed in this part of the manuscript. Extensions to the class $\mathcal{S}$ are clear by means of the above comments.

We remind the concept of equicontinuity of a family of functions.

Definition 9 A family of functions $\{f_\alpha\}_{\alpha \in C}$ defined on a set $I \subset \mathbb{R}^m$ and with values in $\mathbb{R}^n$ is said to be equicontinuous, when for all $\varepsilon > 0$ there exists $\delta > 0$ such that for all $x, y \in I$ with $\|x - y\| < \delta$, and for all $\alpha \in C$, we have that $\|f_\alpha(x) - f_\alpha(y)\| < \varepsilon$.

The following result says that when a sequence $\{A_m\}_m \subset \mathcal{T}$ is convergent in the $d_{\mathcal{T}}$-metric, the family of functions $\{\nabla_{A_m}^p\}_m$ is equicontinuous.

Proposition 8 *Let $A_m, A \in \mathcal{T}$, $m \in \mathbb{N}$, such that $\lim_{m\to\infty} d_{\mathcal{T}}(A_m, A) = 0$. Then, the sequence of mappings $\{\nabla_{A_m}^p\}_m$ is equicontinuous for all $p \in [1, \infty]$.*

Proof According to the proof of Proposition 6, for all elements $A_m \in \{A_m\}_m$, $\theta, \theta' \in [0, \pi]$ and $p \in [1, \infty]$, it holds that

$$\begin{aligned}&|\nabla_{A_m}^p(\theta) - \nabla_{A_m}^p(\theta')|\\&\leq \frac{1}{\|\rho_{A_m}\|_p} R_m \left(\left(\frac{R_m}{r_m}\right)^2 - 1\right)^{\frac{1}{2}} 2|\theta - \theta'|,\end{aligned}$$

where r_m and R_m are any positive real numbers satisfying that $r_m B_1(O) \subset A_m \subset R_m B_1(O)$.

For each $m \in \mathbb{N}$, take r_m and R_m as follows,

$$r_m = \sup\{r > 0 \mid r B_1(O) \subset A_m\} \text{ and}$$
$$R_m = \inf\{R > 0 \mid A_m \subset R B_1(O)\}.$$

Because of the compactness of A_m, such supremum and infimum are a maximum and a minimum, respectively. Define in a similar way r_A and R_A for the set A.

Note that for all $u \in S^1$,

$$\begin{aligned}\rho_A(u) &= \rho_A(u) - \rho_{A_m}(u) + \rho_{A_m}(u)\\ &\leq \|\rho_A - \rho_{A_m}\|_\infty + \rho_{A_m}(u),\end{aligned}$$

and so,

$$\sup_{u\in S^1} \rho_A(u) - \sup_{u\in S^1} \rho_{A_m}(u) \leq \|\rho_A - \rho_{A_m}\|_\infty.$$

Similarly,

$$\sup_{u\in S^1} \rho_{A_m}(u) - \sup_{u\in S^1} \rho_A(u) \leq \|\rho_A - \rho_{A_m}\|_\infty.$$

Since $\lim\limits_{m\to\infty} d_{\mathcal{T}}(A_m, A) = \lim\limits_{m\to\infty} \|\rho_{A_m} - \rho_A\|_\infty = 0$, we obtain that $\lim\limits_{m\to\infty} R_m = R_A$.

Taking infima instead of suprema in the above development leads to $\lim\limits_{m\to\infty} r_m = r_A$.

Thus, there are $r_0 = \inf\{r_m\}_m > 0$ and $R_0 = \sup\{R_m\}_m > 0$ real numbers, such that for all $A_m, \theta, \theta' \in [0, \pi]$, and $p \in [1, \infty]$, we have that

$$\begin{aligned}&|\nabla^p_{A_m}(\theta) - \nabla^p_{A_m}(\theta')|\\ &\leq \frac{1}{\|\rho_{A_m}\|_p} R_0 \left(\left(\frac{R_0}{r_0}\right)^2 - 1\right)^{\frac{1}{2}} 2|\theta - \theta'|.\end{aligned}$$

On the other hand, the condition $\lim\limits_{m\to\infty} \|\rho_{A_m} - \rho_A\|_\infty = 0$ implies that $\lim\limits_{m\to\infty} \|\rho_{A_m} - \rho_A\|_p = 0$ for all $p \in [1, \infty]$ since $\|\rho_{A_m} - \rho_A\|_p \leq \|\rho_{A_m} - \rho_A\|_\infty$. Therefore, $\lim\limits_{m\to\infty} \|\rho_{A_m}\|_p = \|\rho_A\|_p$ for any $p \in [1, \infty]$. Thus, there exists $\rho_{0,p} = \inf\{\|\rho_{A_m}\|_p\}_m > 0$. This implies that

$$|\nabla^p_{A_m}(\theta) - \nabla^p_{A_m}(\theta')| \leq \frac{1}{\rho_{0,p}} R_0((\frac{R_0}{r_0})^2 - 1)^{\frac{1}{2}} 2|\theta - \theta'|$$

for all $m \in \mathbb{N}$, $\theta, \theta' \in [0, \pi]$ and $p \in [1, \infty]$. That is, the family $\{\nabla^p_{A_m}\}_m$ is equicontinuous whatever value of p. □

Under the above conditions, we will see that the sequence $\{\nabla^p_{A_m}\}_m$ tends to ∇^p_A in the pointwise convergence for any value $p \in [1, \infty]$.

Proposition 9 *Let $A_m, A \in \mathcal{T}$, $m \in \mathbb{N}$, such that $\lim_{m\to\infty} d_{\mathcal{T}}(A_m, A) = 0$. It holds that $\lim\limits_{m\to\infty} \nabla^p_{A_m}(\theta) = \nabla^p_A(\theta)$ for all $\theta \in [0, \pi]$ and $p \in [1, \infty]$.*

Proof It has been seen in the proof of Proposition 8, that $\lim\limits_{m\to\infty} \|\rho_{A_m}\|_p = \|\rho_A\|_p$ for all $p \in [1, \infty]$. On the other hand,

$$\begin{aligned}&|d_p(\rho_{A_m}, \rho_{f_\theta(A_m)}) - d_p(\rho_A, \rho_{f_\theta(A)})|\\ &\quad\leq d_p(\rho_{A_m}, \rho_A) + d_p(\rho_{f_\theta(A_m)}, \rho_{f_\theta(A)}),\end{aligned}$$

and by Proposition 4,

$$d_p(\rho_{A_m}, \rho_A) = d_p(\rho_{f_\theta(A_m)}, \rho_{f_\theta(A)}).$$

Thus,

$$\begin{aligned}&|d_p(\rho_{A_m}, \rho_{f_\theta(A_m)}) - d_p(\rho_A, \rho_{f_\theta(A)})|\\ &\quad\leq 2d_p(\rho_{A_m}, \rho_A) = 2\|\rho_{A_m} - \rho_A\|_p\\ &\quad\leq 2\|\rho_{A_m} - \rho_A\|_\infty = 2d_{\mathcal{T}}(\rho_{A_m}, \rho_A).\end{aligned}$$

Therefore,

$$\lim_{m\to\infty} |d_p(\rho_{A_m}, \rho_{f_\theta(A_m)}) - d_p(\rho_A, \rho_{f_\theta(A)})| = 0.$$

The above conclusions imply that

$$\begin{aligned}\lim_{m\to\infty} \nabla^p_{A_m}(\theta) &= \lim_{m\to\infty} \frac{1}{\|\rho_{A_m}\|_p} d_p(\rho_{A_m}, \rho_{f_\theta(A_m)})\\ &= \frac{1}{\|\rho_A\|_p} d_p(\rho_A, \rho_{f_\theta(A)}) = \nabla^p_A(\theta)\end{aligned}$$

for all $\theta \in [0, \pi]$ and $p \in [1, \infty]$. □

Now, we prove that the measures of axial symmetry of order p on $\mathcal{T}$ are continuous.

Proposition 10 *The mapping $\Delta^p : \mathcal{T} \to \mathbb{R}$ is continuous with respect to the metric $d_{\mathcal{T}}$ on $\mathcal{T}$ for all $p \in [1, \infty]$.*

Proof It is well-known that a convergent sequence of equicontinuous mappings on a compact set converges uniformly (see, for instance, [7]). Therefore, by Proposition 8 and 9, $\{\nabla^p_{A_m}\}_m$ converges uniformly to ∇^p_A. As a consequence,

$$\lim_{m\to\infty} \min_{\theta\in[0,\pi]} \nabla^p_{A_m}(\theta) = \min_{\theta\in[0,\pi]} \nabla^p_A(\theta),$$

equivalently,

$$\lim_{m\to\infty} \Delta^p(A_m) = \Delta^p(A).$$

□

It is important to remark that conditions to guarantee the continuity of Δ^p can be strongly weakened as we will see in Proposition 13. In such a result, we will analyze conditions for that continuity permitting rotations, symmetries and homotheties of star-shaped sets. Before that, we will see that those transformations do not entail changes of the measures of symmetry Δ^p with $p \in [1, +\infty]$.

The first result proves that the measures of axial symmetry are invariant under the product by a nonzero scalar.

Proposition 11 *Let $A \in \mathcal{T}$ and $\lambda \in \mathbb{R} \setminus \{0\}$, it holds that*

i) $\lambda A \in \mathcal{T}$,
ii) $\nabla^p_{\lambda A}(\theta) = \nabla^p_A(\theta)$ *for all* $\theta \in [0, \pi]$ *and* $p \in [1, \infty]$,
iii) $\Delta^p(\lambda A) = \Delta^p(A)$ *for all* $p \in [1, \infty]$.

The proof of Proposition 11 is included in Appendix A.

Below we relate the product of an orthogonal matrix by a star-shaped set and the measures of axial symmetry. We will see that axial symmetry measures on $\mathcal{T}$ are invariant under the product by orthogonal matrices. Moreover, the axial symmetry measures ∇^p_A and ∇^p_{PA}, with $A \in \mathcal{T}$ and $P \in O(2, \mathbb{R})$, will be related.

Let $P \in O(2, \mathbb{R})$. If $det(P) = -1$, where det stands for the determinant, there exists an angle α such that

$$P = \begin{pmatrix} \cos\alpha & \sin\alpha \\ \sin\alpha & -\cos\alpha \end{pmatrix}.$$

On the other hand, if $det(P) = 1$, there exists an angle α such that

$$P = \begin{pmatrix} \cos\alpha & -\sin\alpha \\ \sin\alpha & \cos\alpha \end{pmatrix}.$$

Proposition 12 *Let $A \in \mathcal{T}$ and $P \in O(2, \mathbb{R})$. Then,*

i) $PA \in \mathcal{T}$,
ii) $\nabla^p_{PA}(\theta) = \begin{cases} \nabla^p_A(\alpha - \theta) & \text{if } P = \begin{pmatrix} \cos\alpha & \sin\alpha \\ \sin\alpha & -\cos\alpha \end{pmatrix}, \\ \nabla^p_A(\theta - \alpha) & \text{if } P = \begin{pmatrix} \cos\alpha & -\sin\alpha \\ \sin\alpha & \cos\alpha \end{pmatrix}, \end{cases}$ *for all* $\theta \in [0, \pi]$ *and* $p \in [1, \infty]$, *with* $\nabla^p_A(\alpha - \theta) = \nabla^p_A(\mu)$ *where* $\mu \in [0, \pi]$ *and* $\alpha - \theta = \mu + [\frac{\alpha - \theta}{\pi}]\pi$, *and similarly,* $\nabla^p_A(\theta - \alpha)$,
iii) $\Delta^p(PA) = \Delta^p(A)$ *for all* $p \in [1, \infty]$.

The proof of Proposition 12 is included in Appendix A.

As we have mentioned before, Propositions 11 and 12 permit to prove the continuity of $\Delta^p : \mathcal{T} \to \mathbb{R}$ in Proposition 10 under milder conditions.

Proposition 13 *Let $A_m, A \in \mathcal{T}$, $m \in \mathbb{N}$, such that*

$$\lim_{m\to\infty} \left(\inf_{\lambda\in\mathbb{R},\, P\in O(2,\mathbb{R})} d_{\mathcal{T}}(\lambda P A_m, A) \right) = 0.$$

Then, $\lim_{m\to\infty} \Delta^p(A_m) = \Delta^p(A)$ *for all* $p \in [1, \infty]$.

Proof Note that for all $\varepsilon > 0$ there exists m_0 such that for all $m \geq m_0$,

$$\inf_{\lambda\in\mathbb{R},\, P\in O(2,\mathbb{R})} d_{\mathcal{T}}(\lambda P A_m, A) < \varepsilon,$$

and so, there are $\lambda_m \in \mathbb{R}$ and $P_m \in O(2, \mathbb{R})$ with $d_{\mathcal{T}}(\lambda_m P_m A_m, A) < \varepsilon$.

Define $B_m = \lambda_m P_m A_m$ for any $m \in \mathbb{N}$. In accordance with Proposition 11 and Proposition 12, $B_m \in \mathcal{T}$ and $\Delta^p(B_m) = \Delta^p(A_m)$ for all $m \in \mathbb{N}$.

By Proposition 10, $\lim_{m\to\infty} \Delta^p(B_m) = \Delta^p(A)$ for all $p \in [1, \infty]$, which proves the result. □

5 Examples

Some examples illustrating the methods and techniques proposed in this manuscript are developed in this section. The first example is about a star-shaped set with a unique axis of symmetry. We will see that the considered measure of axial symmetry takes on zero at one point of the interval $[0, \pi]$, namely, at the angle of the corresponding symmetry axis. In the second instance, a star-shaped set with no axes of symmetry will be considered. The corresponding measure of axial symmetry is strictly positive at any point. We will obtain its best axis of symmetry. Regarding the third example, it contains a star-shaped set with several axes of symmetry. Thus, the measure of axial symmetry will take on zero at different points which correspond to the angles which determine the axes of symmetry. The last example aims to visualize Proposition 9. Basically, when a sequence of star-shaped sets converges to another star-shaped set, so does the corresponding sequence of measures of axial symmetry.

It is well-known that the kernel of a star-shaped polygon in $\mathcal{S}$ can be obtained as follows. Each edge of the polygon determines a half-plane whose boundary lies on the line with the edge and that contains points of the interior of the polygon in a neighborhood of any point of the edge. The kernel of the polygon is given by the intersection of all those half-planes.

Algorithms for computing the kernel of a star-shaped polygon have been proposed in scientific literature. To the best of our knowledge, the first algorithm for that purpose was given by Michael I. Shamos and Dan Hoey in [25]. It is based on obtaining the intersection of the above-mentioned half-planes. A kernel can be found in $O(n \log n)$ where n is the number of vertices. A faster algorithm was given by

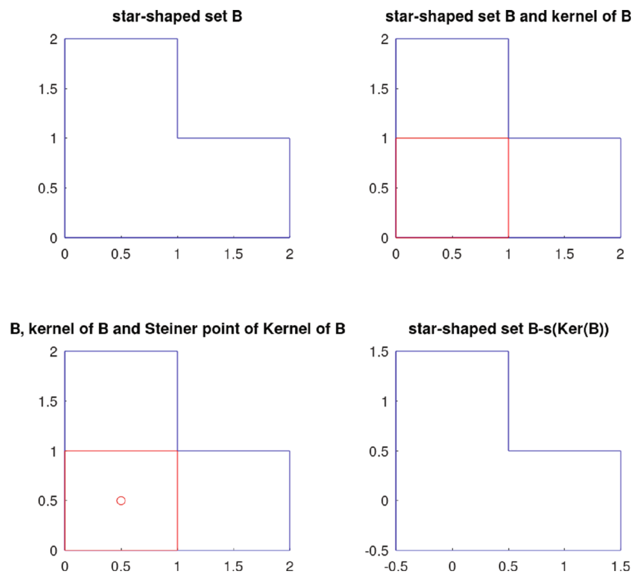


Fig. 3 Graphical representation of star-shaped B (top-left), the kernel of B (top-right), the Steiner point of the kernel of B (bottom-left) and the element of $\mathcal{T}$, $B - s(Ker\ B)$ (bottom-right), in Example 1

Franco P. Preparata an Der T. Lee in [18], with execution time $O(n)$.

We should indicate that all the computational aspects of the forthcoming examples were approached with the interactive software *Octave*, version 6.4.0, using the package *matgeom*.

Example 1 Consider the star-shaped polygon given by the vertices $(0, 0)$, $(2, 0)$, $(2, 1)$, $(1, 1)$, $(1, 2)$ and $(0, 2)$. Let B stand for such a set. The graphical representation of its frontier appears in Fig. 3, top-left. Clearly, $B \in \mathcal{S}$.

The frontier of the kernel of B appears in red color in Fig. 3, top-right, namely the kernel of B is $[0, 1]^2$.

The Steiner point of the kernel of B is $(1/2, 1/2)$, which appears in red color in Fig. 3, bottom-left.

Finally, $B - s(Ker\ B)$ is included in Fig. 3, bottom-right. Note that $B - s(Ker\ B) \in \mathcal{T}$ and any possible symmetry axis of $B - s(Ker\ B)$ should contain the point $(0, 0)$, as Corollary 2 reads.

Let $A = B - s(Ker\ B)$. The mapping $\nabla_A^\infty : [0, \pi] \to \mathbb{R}$, with

$$\nabla_A^\infty(\theta) = \frac{1}{\|\rho_A\|_\infty} d_\infty(\rho_A, \rho_{f_\theta(A)}) \text{ for all } \theta \in [0, \pi],$$

was calculated in a mesh of 501 points of the interval $[0, \pi]$, namely, from 0 to π with a step of $\pi/500$. Note that the values of the mapping ∇_A^∞ at $\theta = 0$ and at $\theta = \pi$ are equal since the line $y = (\tan\theta)x$ is the same for both angles.

Regarding the radial functions in $\nabla_A^\infty(\theta)$, that is, $\rho_A, \rho_{f_\theta(A)} : S^1 \to \mathbb{R}$, the unit sphere S^1 was divided by means of a mesh with 1000 equidistant points. The points of S^1 were parametrized in terms of the corresponding angle of the interval $[0, 2\pi]$. Thus, this interval was divided with a mesh containing points from 0 to 2π with a step of $2\pi/1000$.

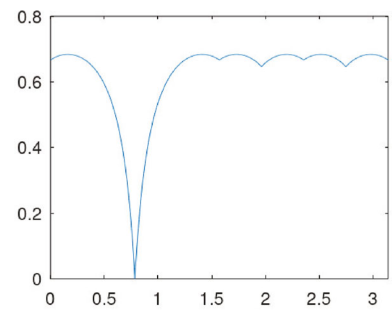


Fig. 4 Graphical representation of the mapping ∇_A^∞ with $A = B - s(Ker\ B)$ in Example 1. Horizontal axis for angles θ in $[0, \pi]$, vertical axis for values of $\nabla_A^\infty(\theta)$

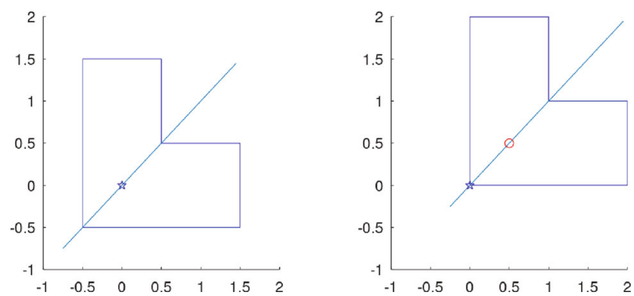


Fig. 5 Representation of star-shaped set A with its symmetry axis (left), and of star-shaped set B with its symmetry axis (right), in Example 1

To calculate the corresponding radial functions at those points of the unit sphere, the intersection of A with edges determined by the points $(0, 0)$ and the corresponding $u \in S^1$ were calculated.

We have taken $p = \infty$ in the measure of axial symmetry of the set A, that is, ∇_A^∞, to detect possible lacks of symmetry "rapidly", but other values of p, not so sensitive to those lacks, can be considered at this step.

The graphical representation of ∇_A^∞ was depicted by means of linear interpolation at the points of the mesh in $[0, \pi]$. It appears in Fig. 4.

The minimum value of the mapping is 0, which appears only once. That indicates the existence of a unique symmetry axis of A. The value in which the minimum is reached, is the angle which determines the symmetry axis of A which contains the point $(0, 0)$. That value corresponds to the point $\pi/4$, that is, there exists a unique best axis of symmetry determined by the angle $\widehat{\theta}_{A,\infty} = \pi/4$. Therefore, the line $y = (\tan(\pi/4))x$, that is, $y = x$, is a symmetry axis of A, and in this particular case, it is also a symmetry axis of B, see Fig. 5.

Clearly enough, $\Delta^\infty(A) = \nabla_A^\infty(\widehat{\theta}_{A,\infty}) = 0$.

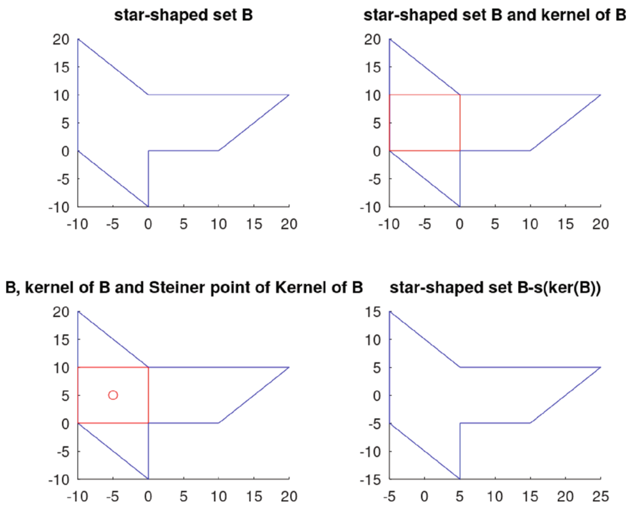


Fig. 6 Graphical representation of star-shaped set B (top-left), the kernel of B (top-right), the Steiner point of the kernel of B (bottom-left) and the element of $\mathcal{T}$, $B - s(Ker\ B)$ (bottom-right), in Example 2

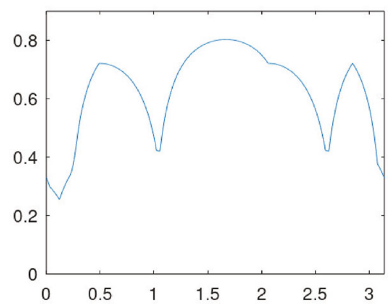


Fig. 7 Graphical representation of the mapping ∇_A^∞ with $A = B - s(Ker\ B)$ in Example 2. Horizontal axis for angles θ in $[0, \pi]$, vertical axis for values of $\nabla_A^\infty(\theta)$

Example 2 The second example includes a star-shaped set with no axial symmetries. That is given by the vertices $(-10, 20)$, $(-10, 0)$, $(0, -10)$, $(0, 0)$, $(10, 0)$, $(20, 10)$ and $(0, 10)$. Let us denote it by B.

The graphical representations of B, the kernel of B, the Steiner point of that kernel, which is $(-5, 5)$, and $A = B - s(Ker\ B)$, appear in Fig. 6.

All the computational aspects of this instance regarding the calculations of radial functions, best axes of symmetry and the measure of axial symmetry are as in Example 1.

The graphical representation of ∇_A^∞ is included in Fig. 7.

The minimum value of the mapping is 0.2558. That means that there are not symmetry axes of A, and so, neither has B.

The best axis of symmetry corresponds to the angle in which the above minimum is reached. In this case, $\widehat{\theta}_{A,\infty} = 0.1194$.

Therefore, the line $y = (\tan(0.1194))x$ is the best symmetry axis of A, and the best symmetry axis of B has the same slope but passing through the point $(-5, 5)$. The graphical representations of A and B jointly with their best symmetry axes are included in Fig. 8.

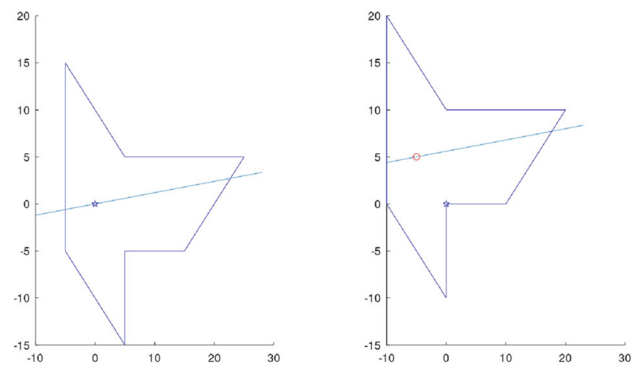


Fig. 8 Representation of star-shaped set A with its best symmetry axis (left), and of star-shaped set B with its best symmetry axis (right), in Example 2

In this case, $\Delta^\infty(A) = \nabla_A^\infty(\widehat{\theta}_{A,\infty}) = 0.2558$.

Example 3 The following instance includes a star-shaped set with multiple axes of symmetry. Consider B the "star" given by the vertices $(-2.5, 5)$, $(-1.25, 1.25)$, $(-5, 2.5)$, $(-2.5, 0)$, $(-5, -2.5)$, $(-1.25, -1.25)$, $(-2.5, -5)$, $(0, -2.5)$, $(2.5, -5)$, $(1.25, -1.25)$, $(5, -2.5)$, $(2.5, 0)$, $(5, 2.5)$, $(1.25, 1.25)$, $(2.5, 5)$ and $(0, 2.5)$. The graphical representation of that set appears in Fig. 9, top-left.

The kernel of B is the polygon with vertices $(0, 0.8333)$, $(-0.625, 0.625)$, $(-0.8333, 0)$, $(-0.625, -0.625)$, $(0, -0.8333)$, $(0.625, -0.625)$, $(0.8333, 0)$ and $(0.625, 0.625)$. It appears in Fig. 9, top-right.

The Steiner centroid of the kernel is $(0, 0)$, see Fig. 9, bottom-left, and so $A = B - s(Ker\ B) = B$, see Fig. 9, bottom-right.

All the computational aspects of this example are like those in Example 1.

Figure 10 contains the graphical representation of ∇_A^∞. That mapping takes on the value 0 at five points, namely, $0, \pi/4, \pi/2, 3\pi/4$ and π.

That means that there are four best axes of symmetry (those of 0 and π are the same). The graphical representation of those axes, jointly with B, appears in Fig. 11. Note that $\widehat{\theta}_{A,\infty} = 0, \pi/4, \pi/2, 3\pi/4, \pi$.

Obviously, $\Delta^\infty(A) = \nabla_A^\infty(\widehat{\theta}_{A,\infty}) = 0$.

Example 4 The purpose of this instance is illustrating Proposition 9. Take B of Example 1 and B_k the star-shaped set given by the vertices $(0, 0)$, $(2+k, 0)$, $(2+k, 1)$, $(1, 1)$, $(1, 2)$ and $(0, 2)$ with $k > 0$. Roughly speaking, the basis of the L-shape of B is "enlarged" on the right-hand side k units in B_k.

Let $A = B - s(Ker\ B)$ and $A_k = B_k - s(Ker\ B_k)$. Note that $Ker(B) = Ker(B_k) = [0, 1] \times [0, 1]$.

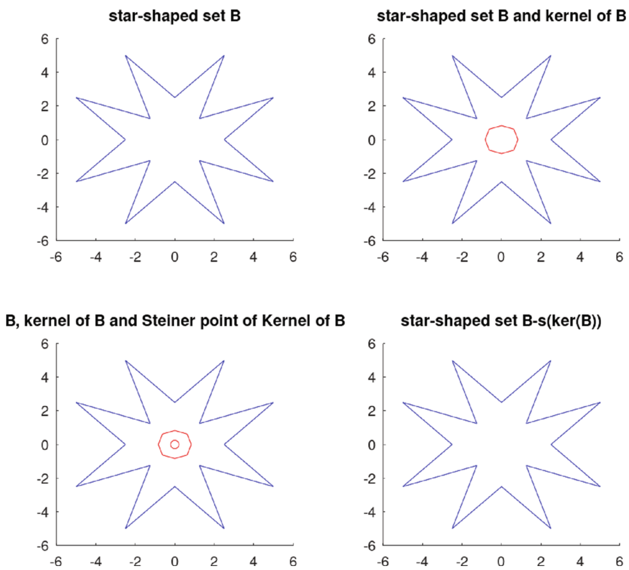


Fig. 9 Graphical representations of star-shaped B (top-left), the kernel of B (top-right), the Steiner point of the kernel of B (bottom-left) and the element of $\mathcal{T}$, $B - s(Ker\ B)$ (bottom-right), in Example 3

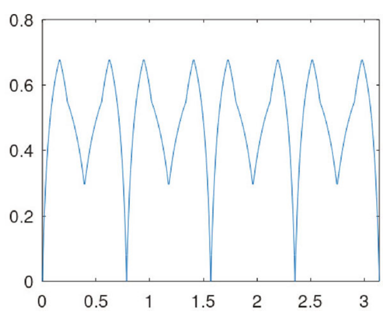


Fig. 10 Graphical representation of the mapping ∇_A^∞ with $A = B - s(Ker\ B)$ in Example 3. Horizontal axis for angles θ in $[0, \pi]$, vertical axis for values of $\nabla_A^\infty(\theta)$

It can be seen that $\lim_{k\to 0^+} d_{\mathcal{T}}(A_k, A) = 0$. By Proposition 9, it holds that $\lim_{k\to 0^+} \nabla_{A_k}^p(\theta) = \nabla_A^p(\theta)$ for any $\theta \in [0, \pi]$ and any $p \in [1, \infty]$. In fact, by the proof of Proposition 10, that convergence is uniform for any $p \in [1, \infty]$.

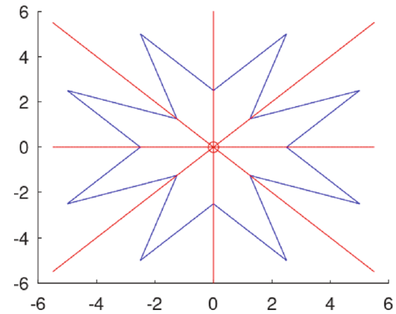


Fig. 11 Representation of star-shaped set B with its symmetry axes in Example 3

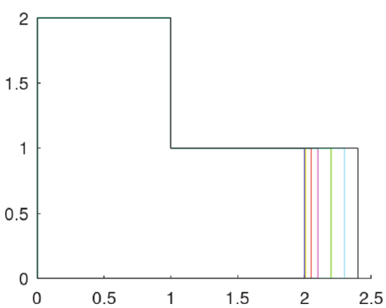
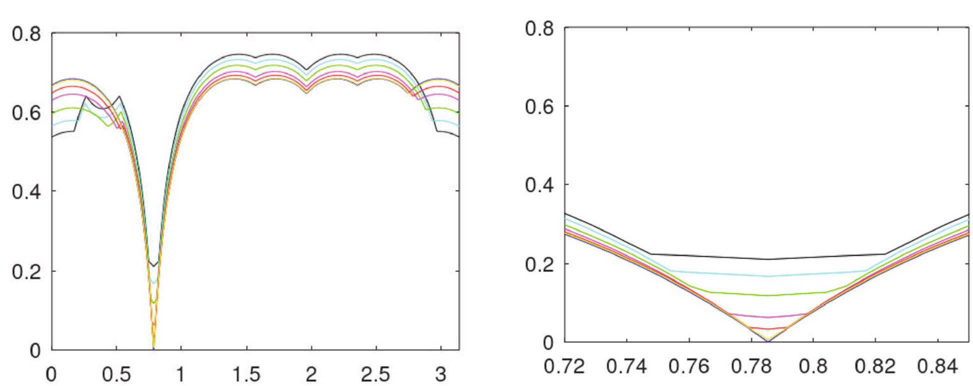


Fig. 12 Graphical representation of B and B_k with k equal to 0.4, 0.3, 0.2, 0.1, 0.05 and 0.01, in Example 4. B in blue color, $B_{0.01}$ in yellow, $B_{0.05}$ in red, $B_{0.1}$ in magenta, $B_{0.2}$ in green, $B_{0.3}$ in cyan and $B_{0.4}$ in black

We have taken k equal to 0.4, 0.3, 0.2, 0.1, 0.05 and 0.01 to visualize the result. The graphical representation of B and B_k for those values of k appears in Fig. 12.

Figure 13 contains the graphical representation of the mappings $\nabla_{A_k}^\infty$ and ∇_A^∞ for the above values of k. The latter is represented in blue color (see also Fig. 4). It takes on zero at the point $\pi/4$. The remaining measures of axial symmetry are strictly positive at any point of the interval $[0, \pi]$, note that A_k does not have any axis of symmetry. Observe that as k decreases, $\nabla_{A_k}^\infty$ tends to be nearer to ∇_A^∞ as Proposition 9 and 10 indicate. Note that the lower the value of k, the lower the minimum of $\nabla_{A_k}^\infty$.

Fig. 13 Graphical representation of $\nabla_{A_k}^\infty$ and ∇_A^∞ with k equal to 0.4, 0.3, 0.2, 0.1, 0.05 and 0.01 in Example 4. ∇_A^∞ in blue color, $\nabla_{A_{0.01}}^\infty$ in yellow, $\nabla_{A_{0.05}}^\infty$ in red, $\nabla_{A_{0.1}}^\infty$ in magenta, $\nabla_{A_{0.2}}^\infty$ in green, $\nabla_{A_{0.3}}^\infty$ in cyan and $\nabla_{A_{0.4}}^\infty$ in black. Horizontal axis for angles θ in $[0, \pi]$, vertical axis for values of $\nabla_A^\infty(\theta)$ (left) and in a neighborhood of $\pi/4$ (right)



6 Conclusions

Measures of symmetry for convex bodies have been deeply studied in mathematical literature. To the best of our knowledge, that is not the case when we consider the class of star-shaped sets. This manuscript tries to fill that gap for the case of 2D star-shaped sets.

For such a purpose, we propose a family of measures which quantify the degree of symmetry of a star-shaped set with respect to a line, that is, a set of axial symmetry measures. They are based on the comparison of the values that the radial function of a centered star-shaped set takes on both sides of the axis. A symmetry about an axis implies that the radial function should assume the same values on both sides. A great discrepancy of those values means that the symmetry about such a line is very poor; on the contrary, similar values lead to a large degree of symmetry.

On the basis of these measures, we introduce the concept of best axis of symmetry, as that determined by the angle which minimizes a measure of axial symmetry, that is, the angle whose corresponding line shows the largest degree of axial symmetry. These concepts permit to introduce families of symmetry measures for star-shaped sets, quantifying the degree of symmetry of a set of that family.

All the above concepts are studied in detail, proving that they satisfy adequate properties. Some examples illustrate the concepts and results of the manuscript.

Appendix A Proofs of Propositions 11 and 12

In this appendix, we include the proofs of Propositions 11 and 12. The first result proves that the measures of axial symmetry are invariant under the product by a nonzero scalar. The second shows that axial symmetry measures on $\mathcal{T}$ are invariant under the product by orthogonal matrices.

Proof of Proposition 11.

Proof $i)$ Consider $f_\lambda : \mathbb{R}^2 \to \mathbb{R}^2$, with $f_\lambda(x) = \lambda x$ for all $x \in \mathbb{R}^2$. That is an homomorphism and so, $f_\lambda(A) = \lambda A$ is compact.

If $a \in Ker\, A$, for all $\alpha \in [0, 1]$ and for all $b \in A$, $f_\lambda(\alpha a + (1-\alpha)b) = \alpha\lambda a + (1-\alpha)\lambda b \in f_\lambda(A) = \lambda A$. As a result, λA is a star-shaped set and $\lambda Ker\, A \subset Ker\, \lambda A$.

In a similar way, by means of f_λ^{-1}, we have that $Ker\, \lambda A \subset \lambda Ker\, A$, and so $Ker\, \lambda A = \lambda Ker\, A$.

Let us see that $(\lambda K)^\circ = \lambda K^\circ$ for any $\lambda \in \mathbb{R} \setminus \{0\}$ and any $K \subset \mathbb{R}^n$.

Consider $\lambda k \in (\lambda K)^\circ$, there exists $\epsilon > 0$ such that $B_\epsilon(\lambda k) \subset \lambda K$. Let $x \in B_{\frac{\epsilon}{|\lambda|}}(k)$. Then, $\lambda x \in B_\epsilon(\lambda k) \subset \lambda K$, and thus, $x \in K$. Hence, $B_{\frac{\epsilon}{|\lambda|}}(k) \subset K$ and then $k \in K^\circ$. As a consequence, $(\lambda K)^\circ \subset \lambda K^\circ$. On the other hand, f_λ is an open mapping, and so, $\lambda K^\circ \subset (\lambda K)^\circ$.

Now, $(Ker\, \lambda A)^\circ = (\lambda Ker\, A)^\circ = \lambda (Ker\, A)^\circ \neq \emptyset$. Thus, $\lambda A \in \mathcal{S}$.

We have that $s(Ker\, \lambda A) = s(\lambda Ker\, A)$. According to property 2) of the Steiner point, $s(Ker\, \lambda A) = \lambda s(Ker\, A) = (0, 0)$. Thus, $\lambda A \in \mathcal{T}$.

$ii)$ Observe that $\rho_{\lambda A}(u) = \lambda \rho_A(u)$ for all $u \in S^1$ and $\lambda \in (0, \infty)$. When $\lambda \in (-\infty, 0)$, $\rho_{\lambda A}(u) = -\lambda \rho_A(-u)$ for all $u \in S^1$.

As a consequence, $\|\rho_{\lambda A}\|_p = |\lambda|\, \|\rho_A\|_p$ for all $p \in [1, \infty]$ and $\lambda \in \mathbb{R} \setminus \{0\}$. Given $\theta \in [0, \pi]$, it holds that

$$\begin{aligned}
\nabla^p_{\lambda A}(\theta) &= \frac{1}{\|\rho_{\lambda A}\|_p} d_p(\rho_{\lambda A}, \rho_{f_\theta(\lambda A)}) \\
&= \frac{1}{\|\rho_{\lambda A}\|_p} d_p(\rho_{\lambda A}, \rho_{\lambda A} \circ f_\theta) \\
&= \frac{1}{|\lambda| \|\rho_A\|_p} |\lambda| d_p(\rho_A, \rho_A \circ f_\theta) \\
&= \frac{1}{\|\rho_A\|_p} d_p(\rho_A, \rho_{f_\theta(A)}) = \nabla^p_A(\theta),
\end{aligned}$$

where the second equality follows from Proposition 4.

Statement $iii)$ is a consequence of $ii)$. □

Proof of Proposition 12.

Proof $i)$ Multiplying P by A is an orthogonal transformation of A given by the matrix P. Since rigid-body transformations do not change the shape of the body, PA is a compact star-shaped set.

In a similar way to the proof of Proposition 3, we have that $Ker(PA) = P Ker(A)$ and $P(Ker\, A)^\circ = (P Ker\, A)^\circ = (Ker\, PA)^\circ$. Thus, $PA \in \mathcal{S}$. Applying property 3) of the Steiner point mapping, it holds that $s(Ker\, PA) = s(P Ker\, A) = P s(Ker\, A) = (0, 0)$. Then, $PA \in \mathcal{T}$.

$ii)$ Note that for every $u \in S^1$, $\rho_{PA}(u) = \sup\{\alpha \in \mathbb{R} \mid \alpha u \in PA\} = \sup\{\alpha \in \mathbb{R} \mid \alpha P^{-1} u \in A\} = \rho_A(P^{-1}u)$.

Let $p \in [1, \infty)$, then

$$\begin{aligned}
\nabla^p_{PA}(\theta) &= \frac{1}{\|\rho_{PA}\|_p} d_p(\rho_{PA}, \rho_{f_\theta(PA)}) \\
&= \frac{1}{\|\rho_{PA}\|_p} d_p(\rho_{PA}, \rho_{PA} \circ f_\theta) \\
&= \frac{1}{\|\rho_A \circ P^{-1}\|_p} d_p(\rho_A \circ P^{-1}, \rho_A \circ P^{-1} \circ f_\theta) \\
&= \frac{1}{\left(\int_{S^1} |\rho_A(P^{-1}u)|^p d\mu\right)^{\frac{1}{p}}} \\
&\quad \times \left(\int_{S^1} |\rho_A(P^{-1}u) - \rho_A(P^{-1} f_\theta(u))|^p d\mu\right)^{\frac{1}{p}}.
\end{aligned}$$

Consider the mapping $g : S^1 \longrightarrow S^1$, with $g(u) = Pu$ for all $u \in S^1$. Observe that $\mu(g^{-1}(S)) = \mu(S)$ for any measurable set $S \subset S^1$.

By a change of variable (see, for instance, [12]),

$$\nabla^p_{PA}(\theta) = \frac{1}{\left(\int_{S^1} |\rho_A(u)|^p d\mu\right)^{\frac{1}{p}}} \times \left(\int_{S^1} |\rho_A(u) - \rho_A(P^{-1} f_\theta(Pu))|^p d\mu\right)^{\frac{1}{p}}.$$

If $det(P) = -1$,

$$P = \begin{pmatrix} \cos\alpha & \sin\alpha \\ \sin\alpha & -\cos\alpha \end{pmatrix}$$

for some α and so,

$$P^{-1} R_\theta P = \begin{pmatrix} \cos 2(\alpha-\theta) & \sin 2(\alpha-\theta) \\ \sin 2(\alpha-\theta) & -\cos 2(\alpha-\theta) \end{pmatrix} = R_{\alpha-\theta}.$$

Then,

$$\begin{aligned} \nabla^p_{PA}(\theta) &= \frac{1}{\left(\int_{S^1} |\rho_A(u)|^p d\mu\right)^{\frac{1}{p}}} \times \left(\int_{S^1} |\rho_A(u) - \rho_A(f_{\alpha-\theta}(u))|^p d\mu\right)^{\frac{1}{p}} \\ &= \frac{1}{\|\rho_A\|_p} d_p(\rho_A, \rho_A \circ f_{\alpha-\theta}) \\ &= \frac{1}{\|\rho_A\|_p} d_p(\rho_A, \rho_{f_{\alpha-\theta}(A)}) \\ &= \nabla^p_A(\alpha-\theta). \end{aligned}$$

When $det(P) = 1$ with

$$P = \begin{pmatrix} \cos\alpha & -\sin\alpha \\ \sin\alpha & \cos\alpha \end{pmatrix}$$

for some α,

$$P^{-1} R_\theta P = \begin{pmatrix} \cos 2(\theta-\alpha) & \sin 2(\theta-\alpha) \\ \sin 2(\theta-\alpha) & -\cos 2(\theta-\alpha) \end{pmatrix} = R_{\theta-\alpha}.$$

Then,

$$\begin{aligned} \nabla^p_{PA}(\theta) &= \frac{1}{\left(\int_{S^1} |\rho_A(u)|^p d\mu\right)^{\frac{1}{p}}} \times \left(\int_{S^1} |\rho_A(u) - \rho_A(f_{\theta-\alpha}(u))|^p d\mu\right)^{\frac{1}{p}} \\ &= \frac{1}{\|\rho_A\|_p} d_p(\rho_A, \rho_A \circ f_{\theta-\alpha}) \\ &= \frac{1}{\|\rho_A\|_p} d_p(\rho_{PA}, \rho_{f_{\theta-\alpha}(A)}) = \nabla^p_A(\theta-\alpha). \end{aligned}$$

Regarding the case $p = \infty$,

$$\begin{aligned} \|\rho_{PA}\|_\infty &= \sup_{u\in S^1} |\rho_{PA}(u)| = \sup_{u\in S^1} |\rho_A(P^{-1}u)| \\ &= \sup_{u\in S^1} |\rho_A(u)| = \|\rho_A\|_\infty \end{aligned}$$

since $\{P^{-1}u \mid u \in S^1\} = S^1$. Thus,

$$\begin{aligned} d_\infty(\rho_{PA}, \rho_{f_\theta(PA)}) &= \sup_{u\in S^1} |\rho_{PA}(u) - \rho_{f_\theta(PA)}(u)| \\ &= \sup_{u\in S^1} |\rho_{PA}(u) - \rho_{PA}(f_\theta(u))| \\ &= \sup_{u\in S^1} |\rho_A(P^{-1}u) - \rho_A(P^{-1} f_\theta(u))| \\ &= \sup_{u\in S^1} |\rho_A(P^{-1}Pu) - \rho_A(P^{-1} R_\theta Pu)|. \end{aligned}$$

If $det(P) = -1$, then $P^{-1} R_\theta P = R_{\alpha-\theta}$ for some α and thus,

$$\begin{aligned} &\sup_{u\in S^1} |\rho_A(P^{-1}Pu) - \rho_A(P^{-1} R_\theta Pu)| \\ &\quad= \sup_{u\in S^1} |\rho_A(u) - \rho_A(R_{\alpha-\theta}u)| \\ &\quad= \sup_{u\in S^1} |\rho_A(u) - \rho_A(f_{\alpha-\theta}(u))| \\ &\quad= \sup_{u\in S^1} |\rho_A(u) - \rho_{f_{\alpha-\theta}(A)}(u)| = d_\infty(\rho_A, \rho_{f_{\alpha-\theta}(A)}). \end{aligned}$$

Hence, $\nabla^\infty_{PA}(\theta) = \nabla^\infty_A(\alpha-\theta)$.

Reasoning in a similar way when $det(P) = 1$, we obtain that $\nabla^\infty_{PA}(\theta) = \nabla^\infty_A(\theta-\alpha)$ for some α.

$iii)$ It follows from $ii)$. □

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Declarations

Conflict of interest The authors have no conflict of interest to declare that are relevant to the content of this article.

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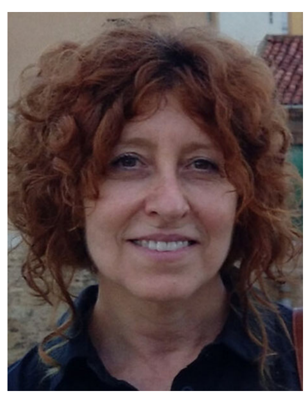

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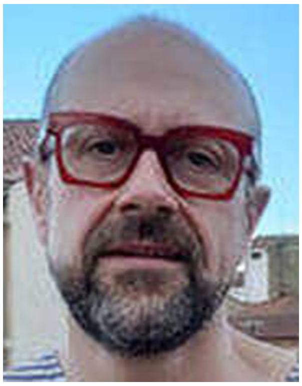

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