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ORIGINAL PAPER

Semi-Riemannian structures for symmetric Galilean spacetimes

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Abstract
We introduce semi-Riemannian structures well-adapted to certain fields of observers in a Galilean spacetime. The Levi-Civita connection of such a semi-Riemannian metric will allow us to obtain variational characterizations of the Galilean geodesics as well as global results on the topological and differentiable structure of the spacetime. Moreover, these new semi-Riemannian metrics provide a new way to compare the studied Newton–Cartan models with their relativistic counterparts.

Keywords Galilean connection · Semi-Riemannian metric · Koszul formula · Leibnizian vector field

Mathematics Subject Classification 53Z05 · 53C80 · 53B50

1 Introduction

The geometrization of Newton's theory of gravity began with Cartan in the early twentieth century [16, 17]. Nevertheless, it was not until the second half of the century when Newton–Cartan theory experienced its major boost. Indeed, a non exhaustive list of some advances made during these years include the study of the Galilean and Lorentzian groups in [26], which enabled the author to show that Newtonian gravity constitutes a limit of General Relativity and how free particles are represented by geodesic curves of certain connections in these two theories. Moreover, the existence of a symmetric connection in both theories, which appears due to the local character of the physical laws and mathematically codifies the inertia principle was pointed out in [40]. We should also highlight the introduction of new Newtonian models satisfying the cosmological principle [33], obtaining the Galilean

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analogue of the relativistic Robertson–Walker spacetimes. Moreover, in [19] the Newtonian theory is shown to be a limit of the Einstein–Klein–Gordon theory. In fact, this vision of the theory as a limit of General Relativity continued to be developed during these years [30], also concluding that the results obtained from Newton–Cartan cosmology were similar on not too large scales to the ones derived from Einstein's theory [37, 38].

In recent years, the complexity of General Relativity from a computational level compared to Newton–Cartan theory has provided new applications of this non-relativistic theory in cosmology [12], AdS/CFT correspondence [31], condensed matter systems [11], hydrodynamics [24], quantum collapse [35], quantum Hall effect [25] and other related phenomena. This variety of applications has renewed the interest in Newton–Cartan theory, leading to a revision of the geometric structures associated with a generalized Newton–Cartan theory [9] and new extensions such as non-relativistic strings [1, 10] and Newton–Cartan Supergravity [2]. Furthermore, new Newtonian models have been obtained such as the Newtonian Gödel spacetime [18], Galilean Generalized Robertson–Walker spacetimes [21], standard stationary Galilean spacetimes [22], as well as the embedding of these geometries in relativistic manifolds [4].

As it is well-known, the geometry in Newton–Cartan theory is more subtle than in Einstein's theory, i.e., Lorentzian geometry. This is due, among other things, to the different behaviour of the connections featuring in both models. Whereas in the Lorentzian setting there is a canonically associated Levi-Civita connection, in the Newtonian theory the requirements of the associated Galilean connection are to be compatible with the absolute clock and the space metric. As a consequence, we cannot expect to easily obtain a variational characterization of geodesics in Newton–Cartan theory, in contrast to what happens in a Lorentzian spacetime. Since only the restriction of this Galilean connection to the spacelike leaves is the Levi-Civita connection of the associated spatial metric, global results for the Galilean spacetime's geodesics are not so immediate. Therefore, the study of these connections has been a key point to better understand Newtonian gravity. Indeed, it was Cartan's starting point towards the geometrization of the theory [16, 17]. Since then, many authors have continued their characterization (see [5, 9, 40], for instance).

In this work, we study several relevant classes of Galilean spacetimes and relate their Galilean connections to certain Levi-Civita connections defined on the same manifold, which is adapted to a suitable field of observers of the Newton–Cartan model. Thus, by means of these semi-Riemannian metrics that are partly compatible with the Galilean connection we are able to prove new results on the geodesic connectedness and completeness of the spacetime as well as obtain variational characterizations of the spacetime's geodesics in certain non relativistic models. Note that the existence of a geodesic curve joining two given points of a semi-Riemannian manifold or, in a more general setting, of a smooth manifold endowed with an affine connection, is a well known and interesting problem [7]. From a physical point of view, our results on geodesic connectedness provide relevant information about whether two non simultaneous events lie in the trajectory of some free falling observer. Furthermore, obtaining completeness results for timelike trajectories is crucial in order to have a consistent physical theory, i.e., one where observers do not disappear nor suddenly appear.

In addition, these semi-Riemannian metrics associated to a Galilean spacetime provide new global results concerning the topological and differential structure of the Galilean spacetime as well as allow us to compare the different families under study with their relativistic counterparts. On the other hand, characteristic elements of the classical gravitational field theory, such as the potential (measured by the observer), are also related to the introduced metric tensor.

This article is organized as follows. In Sect. 2 we will describe the elements in Newton–Cartan theory that will be used throughout the article to obtain our main results. Section 3 is devoted to relate the connection of a Galilean spacetime with the Levi-Civita connection of a product semi-Riemannian manifold. As a consequence of Lemma 5 we are able to obtain a variational characterization of the Galilean connection's geodesics in Corollary 8, as well as determine the topological structure of the manifold defining the spacetime in Theorem 10 and the geodesic connectedness and completeness of the Galilean connection in Corollary 11. In Sect. 4 we extend our results to a more general family of stationary semi-Riemannian metrics and, thanks to the main result of this section (Lemma 12), we study the geodesic completeness of the Galilean connection in Theorems 20 and 23 and characterize the spacetime's structure (Theorems 17 and 22). Finally, in Sect. 5 we obtain similar results for spatially conformally Leibnizian spacetimes, relating their Galilean connection to the Levi-Civita connection of a semi-Riemannian warped product (Lemma 28) and analyzing the geodesic completeness in Theorem 30 and structure of the Galilean spacetime in Theorem 34. We also particularize our results for the family of Galilean Generalized Robertson-Walker spacetimes.

2 Preliminaries

A *Leibnizian* spacetime is defined as the triad (M, Ω, g), where M^{n+1} is a smooth connected manifold of arbitrary dimension $n + 1 \geq 2$ endowed with a Leibnizian structure (Ω, g) determined by a nowhere null differential 1-form $\Omega \in \Lambda^1(M)$ and a positive definite metric g on the kernel of Ω. Namely, if $\mathrm{An}(\Omega) = \{v \in TM\ ,\ \Omega(v) = 0\}$ is the smooth n-distribution induced on M by Ω and $\Gamma(TM)$ is the set of smooth vector fields on M, we can construct the subset $\Gamma(\mathrm{An}(\Omega)) = \{V \in \Gamma(TM)\ /\ V_q \in \mathrm{An}(\Omega),\ \forall p \in M\}$, where the map

$$g : \Gamma(\mathrm{An}(\Omega)) \times \Gamma(\mathrm{An}(\Omega)) \longrightarrow C^\infty(M),\ \ (V, W) \mapsto g(V, W),$$

is smooth, bilinear, symmetric and positive definite (see [8, 9] for further details).

In this ambient spacetime, a point $p \in M$ is called an *event*. The Euclidean vector space $(\mathrm{An}(\Omega_p), g_p)$ is called the *absolute space* at $p \in M$ and the linear form Ω_p is the *absolute clock* at p. A tangent vector $v \in T_pM$ is *spacelike* if $\Omega_p(v) = 0$ and *timelike* otherwise. In addition, if $\Omega_p(v) > 0$ (resp., $\Omega_p(v) < 0$), v is said to be *future* pointing (resp., *past* pointing).

An *observer* in a Leibnizian spacetime is a smooth curve $\gamma : I \subseteq \mathbb{R} \longrightarrow M$ whose velocity γ' is a unitary future pointing timelike vector field (i.e., $\Omega(\gamma'(s)) = 1$ for all $s \in I$). The parameter s is known as the *proper time* of the observer γ. A vector field $Z \in \Gamma(TM)$ with $\Omega(Z) = 1$ is called a *field of observers*, since its integral curves are observers in the Leibnizian spacetime.

When the smooth distribution $\mathrm{An}(\Omega)$ is integrable (equivalently, if the absolute clock Ω satisfies $\Omega \wedge d\Omega = 0$), the Leibnizian spacetime (M, Ω, g) is said to be *locally synchronizable*. In this case, Frobenius Theorem (see [41]) guarantees the existence of a foliation of the spacetime by a family of hypersurfaces tangent to the absolute space. In this case, it is well-known that at each $p \in M$ there is a neighborhood where $\Omega = \beta\, dT$, for certain smooth functions $\beta > 0$, T, and the hypersurfaces $\{T = \text{constant}\}$ locally coincide with a leaf of the foliation. Therefore, any observer can rescale its proper time to be synchronized with the "compromise time" T. If $d\Omega = 0$, the Leibnizian spacetime is called *proper time locally synchronizable* and, locally, $\Omega = dT$. In this case, observers are synchronized directly by its proper time (up to a constant). When $\Omega = dT$ for some function $T \in C^\infty(M)$, any observer may be assumed to be parametrized by the *absolute time function* T. We should highlight that the notion of (local and local proper time) synchronizability is intrinsic to the Leibnizian structure, applicable for every observer, in contrast to the relativistic setting, where the analogous concepts only have meaning for fields of observers.

A vector field K in a Leibnizian spacetime is called *Leibnizian* [9, Def. 3] if the stages Φ_s of its local flows preserve the absolute clock and space, i.e.,

$$\Phi_s^*\Omega = \Omega, \qquad \text{and} \qquad \Phi_s^* g = g.$$

Both conditions are equivalent to the following ones.

(i) $\Omega([K, X]) = K(\Omega(X)), \quad \forall X \in \Gamma(TM)$.
(ii) $K(g(V, W)) = g([K, V], W) + g(V, [K, W]), \quad \forall V, W \in \Gamma(\mathrm{An}(\Omega))$.

Notice that (ii) is always well defined since $[K, V],\ [K, W] \in \Gamma\big(\mathrm{An}(\Omega)\big)$ by (i).

On the other hand, a connection on the spacetime is required in order to codify the inertia principle. Nevertheless, a Leibnizian structure does not have an associated canonical affine connection, so we need to introduce a compatible connection with the absolute clock Ω and the space metric g, i.e., a connection ∇ such that

(a) $\nabla\Omega = 0$ (equivalently, $\Omega(\nabla_X Y) = X(\Omega(Y))$ for any $X, Y \in \Gamma(TM)$).
(b) $\nabla g = 0$, i.e., $Z(g(V, W)) = g(\nabla_Z V, W) + g(\nabla_Z W, V)$ for any $Z \in \Gamma(TM)$ and V, W spacelike vector fields. (Note that (a) implies that $\nabla_Z V$ and $\nabla_Z W$ are spacelike vector fields).

Such a connection is called *Galilean* and its restriction to the spacelike leaves of the foliation coincides with the Levi-Civita connection g. The tetrad (M, Ω, g, ∇) formed by a Leibnizian spacetime endowed with a Galilean connection ∇ is called a *Galilean spacetime*. As usual, ∇ is said to be *symmetric* if its torsion vanishes identically ($\mathrm{Tor}_\nabla(X, Y) = \nabla_X Y - \nabla_Y X - [X, Y] \equiv 0$). From [9, Lemma 13] or [29, Thm. 7], the existence of a symmetric Galilean connection for a Leibnizian structure implies the proper time local synchronizability of the latter. Furthermore, using Poincaré's lemma, it is clear that if the spacetime is simply connected then there exists an absolute time function. A symmetric connection is also desirable from a physical perspective since it is completely determined by its geodesic trajectories, that is, by the free falling observers in M.

Given a field of observers Z in a Galilean spacetime (M, Ω, g, ∇), the *gravitational field* induced by ∇ in Z is given by the spacelike vector field $\mathcal{G}^Z = \nabla_Z Z$ and the *vorticity* or *Coriolis field* of Z is the 2-form $\omega_Z = \frac{1}{2}\mathrm{Rot}(Z)$, defined as

$$\omega_Z(V, W) = \frac{1}{2}\Big(g(\nabla_V Z, W) - g(\nabla_W Z, V)\Big), \qquad \forall V, W \in \Gamma(\mathrm{An}(\Omega)).$$

If the gravitational field a field of observers Z vanishes, i.e., $\mathcal{G}^Z \equiv 0$, it is said that Z is a *free falling* field of observers. On the other hand, the vorticity ω_Z allows us to introduce the notion of inertial field of observers in the next definition, following [9, Def. 4.24].

Definition 1 A field of observers Z in a Galilean spacetime is said to be *inertial* if it is Leibnizian and irrotational (i.e., $\omega_Z \equiv 0$).

Following [4], we can also define the *gravitational fieldstrength* measured by the field of observers Z as the 2-form on M, F^Z, given by

$$F^Z(X, Y) = g(\nabla_X Z, P^Z Y) - g(\nabla_Y Z, P^Z X)), \qquad \forall X, Y \in \Gamma(TM),$$

where $P^Z X = X - \Omega(X)Z$ is the natural spacelike projection for Z. This tensor field encodes all the information contained in the gravitational field $\mathcal{G}^Z$ and the Coriolis 2-form ω_Z via the relations

$$F^Z(Z, V) = g(\mathcal{G}^Z, V), \qquad F^Z(V, W) = \omega_Z(V, W), \qquad \forall V, W \in \Gamma(\mathrm{An}(\Omega)).$$

The importance of the gravitational field and the vorticity of a field of observers comes from the fact that they determine a unique symmetric Galilean geometry for proper time locally synchronizable spacetimes [9, Cor. 28]. Furthermore, that symmetric Galilean connection admits a formula 'à la Koszul' for the field of observers Z whose expression is

$$\nabla_X Y = P^Z(\nabla_X Y) + X(\Omega(Y))Z, \qquad \forall X, Y \in \Gamma(TM), \tag{1}$$

for each $V \in \Gamma(\mathrm{An}(\Omega))$,

$$\begin{aligned} 2g(P^Z(\nabla_X Y), V) = {} & X(g(P^Z Y, V)) + Y(g(P^Z X, V)) - V(g(P^Z X, P^Z Y) \\ & +2\,\Omega(X)\Omega(Y)g(\mathcal{G}^Z, V) \\ & +2\,\Omega(X)\omega_Z(P^Z Y, V) + 2\,\Omega(Y)\omega_Z(P^Z X, V) \\ & +\Omega(X)\Big(g([Z, P^Z Y], V) - g([Z, V], P^Z Y)\Big) \\ & -\Omega(Y)\Big(g([Z, P^Z X], V) + g([Z, V], P^Z X)\Big) \\ & +g([P^Z X, P^Z Y], V) - g([P^Z Y, V], P^Z X) \\ & -g([P^Z X, V], P^Z Y). \end{aligned} \tag{2}$$

Let us now recall the notion of affine transformation. Given a smooth manifold M endowed with a connection ∇, the automorphism $f : M \longrightarrow M$ is an affine transformation if $f_* \nabla_X Y = \nabla_{f_* X} f_* Y$, for $X, Y \in \mathfrak{X}(M)$. It is well-known that an automorphism is an affine transformation iff it maps geodesics onto geodesics and preserves the torsion tensor. Consequently, if the connection is symmetric, it suffices for a transformation to map geodesics onto geodesics to be affine. Moreover, a vector field $X \in \mathfrak{X}(M)$ is called affine iff the stages of its (local) flows are affine transformations. Namely, $X \in \mathfrak{X}(M)$ is called affine iff $L_X \nabla = 0$, where L denotes the Lie derivative. This notion also plays an important role in Galilean spacetimes, where we will use the following definition.

Definition 2 A Leibnizian vector field K in a Galilean spacetime (M, Ω, g, ∇) is called *Galilean* if it is affine for ∇, that is, $L_K \nabla = 0$.

The condition $L_K \nabla = 0$ can be characterized as follows:

$$[K, \nabla_Y X] = \nabla_{[K,Y]} X + \nabla_Y [K, X], \qquad \forall X, Y \in \Gamma(TM). \tag{3}$$

Equivalently, a Leibnizian vector field is Galilean if and only if its flow preserves the Galilean connection (see [28, Chap. 6]).

To conclude this section, let us show some basic examples of Galilean spacetimes.

Example 3 Consider $\mathbb{R}^{n+1}$ and let $\{t = x_0, x_1, \ldots, x_n\}$ be the standard Cartesian coordinates and $g = \sum_{i=1}^n dx_i^2$ be the Euclidean metric restricted to each leaf $\{t = \text{constant}\}$. The triad $(\mathbb{R}^{n+1}, dt, g)$ constitutes a Leibnizian structure. Endowing it with several compatible connections we obtain the next well known Galilean spacetimes:

- *Galilei–Newton spacetime*. It is an Aristotle spacetime endowed with the usual flat affine connection in $\mathbb{R}^{n+1}$, i.e., the Christoffel symbols identically vanish, $\Gamma^\lambda_{\mu\nu} = 0$, $\mu, \nu, \lambda \in \{0, 1, \cdots, n\}$.

- *Newton–Hooke spacetime*. It is an Aristotle spacetime endowed with a connection Γ whose only nonvanishing components are $\Gamma^i_{00} = -\frac{k}{\tau^2}x_i$, $i \in \{1, \ldots, n\}$, $\tau > 0$, and $k \in \mathbb{R}$. The constant k can take the values $k = +1$ (expanding Newton–Hooke spacetime), $k = -1$ (oscillating Newton–Hooke spacetime) or $k = 0$ (Galilei-Newton spacetime). A physical interpretation of these nonrelativistic cosmological models can be found in [20]. For this family of Galilean spacetimes the Lie algebra of Galilean vector fields is of maximal dimension $\frac{(n+1)(n+2)}{2}$.

3 Semi-Riemannian product metrics associated to Leibnizian vector fields of observers

In this section we study the relation between the connection of a Galilean spacetime (M, Ω, g, ∇) and the Levi-Civita connection of a semi-Riemannian manifold $(M, \overline{g}_\epsilon)$, where M is the same smooth manifold that describes the Galilean spacetime and $\overline{g}_\epsilon$ is a semi-Riemannian metric. Indeed, let Z be a field of observers in a Leibnizian spacetime (M, Ω, g). We define the following semi-Riemannian metrics in M associated to Z,

$$\overline{g}_\epsilon = \epsilon\, \Omega \otimes \Omega + g(P^Z \cdot, P^Z \cdot), \quad \epsilon = \pm 1, \tag{4}$$

i.e., $\overline{g}_\epsilon(X, Y) = \epsilon\, \Omega(X)\Omega(Y) + g(P^Z X, P^Z Y)$ for any $X, Y \in \Gamma(TM)$. Notice that $\overline{g}_\epsilon$ is clearly non-degenerate and the spacelike vectors are orthogonal to Z, $\mathrm{An}(\Omega) = Z^\perp$. Moreover, $\overline{g}_\epsilon$ is Riemannian if $\epsilon = 1$ and Lorentzian if $\epsilon = -1$.

Let us state the following result, which is a direct consequence of the fact that the flow of a Leibnizian field of observers Z in a Galilean spacetime with symmetric connection preserves both Ω and g.

Proposition 4 *Let (M, Ω, g, ∇) be a Galilean spacetime with symmetric connection ∇. Then, every Leibnizian field of observers Z is a Killing vector field of the metric $\overline{g}_\epsilon$.*

These previous results allow us to obtain the relation between a symmetric connection of a Galilean spacetime and the connection the metrics $\overline{g}_\epsilon$ defined in (4). This result was already observed in [6, Prop. A17].

Lemma 5 *Let (M, Ω, g, ∇) be a Galilean spacetime with symmetric connection and let Z be a free falling inertial field of observers. Then, the Galilean connection ∇ is the Levi-Civita connection of the metrics $\overline{g}_\epsilon$.*

Proof Denoting by $\overline{\nabla}$ the Levi-Civita connection of the metric (4), our aim in this proof is to obtain that

$$T(A, B) = \overline{\nabla}_A B - \nabla_A B = 0,$$

for any vector fields $A, B \in \Gamma M$. Let $U, V, W \in \Gamma(\mathrm{An}(\Omega))$ denote spacelike vector fields throughout this proof. Due to the tensorial character of this expression, its symmetry because both connections are torsionfree and the fact that the flow of Z preserves the kernel of Ω, we only need to check that $T(U, V) = T(U, Z) = T(Z, Z) = 0$, with $[U, V] = [U, Z] = [V, Z] = 0$. Note that these assumptions on U, V, Z imply that $Z(g(U, V)) = 0$.

Firstly, in order to obtain that $T(U, V) = 0$ we trivially see that $\overline{g}_\epsilon(\overline{\nabla}_U V, W) = g(P^Z(\nabla_U V), W)$. Thus, $\overline{g}_\epsilon(T(U, V), W) = 0$. Moreover, using the Koszul formula for a semi-Riemannian metric and the Leibnizian character of Z we have

$$\overline{g}_\epsilon(\overline{\nabla}_U V, Z) = 0.$$

On the other hand,

$$\overline{g}_\epsilon(\nabla_U V, Z) = \epsilon\, \Omega(\nabla_U V) = \epsilon\, U(\Omega(V)) = 0.$$

Hence, we deduce that $T(U, V) = 0$ for U, V spacelike vector fields.

Secondly, let us see that $T(U, Z) = 0$. Since $\overline{g}_\epsilon(\overline{\nabla}_U Z, V) = -2\overline{g}_\epsilon(\overline{\nabla}_U V, Z) = 0$ and $2\overline{g}_\epsilon(\nabla_U Z, V) = 0$ due to (2) and the Leibnizian character of Z, we have $\overline{g}_\epsilon(T(U, Z), V) = 0$. Combining this with the fact that $\overline{g}_\epsilon(\overline{\nabla}_U Z, Z) = 0$ since $\overline{g}_\epsilon(Z, Z) = \epsilon$ and taking into account that $\overline{g}_\epsilon(\nabla_U Z, Z) = \epsilon\, \Omega(\nabla_U Z) = 0$, we obtain that $T(U, Z) = 0$.

Finally, $\overline{g}_\epsilon(\overline{\nabla}_Z Z, U) = Z(\overline{g}_\epsilon(Z, U)) - \overline{g}_\epsilon(\overline{\nabla}_Z U, Z) = 0$ since $\Omega([Z, U]) = 0$. Moreover, $\overline{g}_\epsilon(\overline{\nabla}_Z Z, Z) = 0$ because $\overline{g}_\epsilon(Z, Z) = \epsilon$. Consequently, $T(Z, Z) = 0$ since both $\overline{\nabla}_Z Z$ and $\nabla_Z Z$ identically vanish (the second one because Z is free falling). □

Remark 6 Note that if Z is a field of observers, the identity

$$2\, g(\nabla_X Z, Y) = L_Z g(X, Y) + \omega_Z(X, Y), \qquad \forall X, Y \in \Gamma(\mathrm{An}(\Omega)), \tag{5}$$

implies that if Z is a free falling inertial field of observers, then Z is ∇-parallel.[1]

Remark 7 As a consequence of Lemma 5 we obtain by means of a new approach the well known fact that a free falling inertial field of observers uniquely determines a symmetric Galilean connection [5, 9].

Using Lemma 5 we can obtain the geodesics of the Galilean connection as the critical points of the following functional.

Corollary 8 *Let (M, g, Ω, ∇) be a Galilean spacetime admitting a free falling inertial field of observers Z. Then, the geodesics of ∇ are locally critical points of the functional $\mathcal{L} : \mathcal{C}_p^q \longrightarrow \mathbb{R}$,*

$$\mathcal{L}[\gamma] = \frac{1}{2}\int_a^b \big[\epsilon\, \Omega(\gamma'(s))^2 + g(P^Z\gamma'(s), P^Z\gamma'(s))\big] ds, \tag{6}$$

where $\mathcal{C}_p^q = \{\gamma : [a, b] \longrightarrow M \text{ piecewise smooth observers}, \ \gamma(a) = p, \ \gamma(b) = q\}$.

Remark 9 This variational characterization of the Galilean geodesics may be seen as a particular case of the result given in [5] obtained using a completely different approach. In [5] the authors suppose the existence of an irrotational field of observers Z, with $\mathcal{G}^Z = \mathrm{grad}^g \phi$, $\phi \in \mathcal{C}^\infty(M)$, being grad^g the g-gradient along each leaf of $\mathrm{An}(\Omega)$. In that setting, the field of observers Z is not assumed to be Leibnizian, and the resulting Lagrangian is given by

$$\mathcal{L}^\phi[\gamma] = \int_a^b g^\phi(\gamma'(s), \gamma'(s))\, ds, \qquad g^\phi = \phi\, \Omega \otimes \Omega + g(P^Z\cdot, P^Z\cdot).$$

Note that g^ϕ can be a degenerate metric, in contrast to what happens to the ones defined in (4). However, since the functional $\mathcal{L}^\phi$ is invariant (up to a total derivative) under a change of field of observers, $Z \to Z + \mathrm{grad}^g f$, $f \in \mathcal{C}^\infty(M)$, an analogous argument to the one used in [5, Prop. 3.26] leads to another irrotational and free falling field of observers Z' with $\phi' = \epsilon$. In this way, $g^{\phi'} = \overline{g}_\epsilon$ and $\mathcal{L}^{\phi'}$ is, up to a constant, the Lagrangian (6).

[1] Identity (5) is derived using the symmetric character of ∇. In the more general context of an Aristotelian spacetime (see for instance [32, Def. 2.17]), i.e., a Galilean spacetime (M, Ω, g, ∇) endowed with a Carrollian structure (Z, h) – in our case, Z is a field of observers and $h = g(P^Z\cdot, P^Z\cdot)$ –, being ∇ a compatible (possibly torsional) connection; the closedness of Ω and the null expansion of g with respect to Z, $L_Z g = 0$, are the necessary and sufficient conditions for the existence of a Galilean connection where Z is a free falling inertial field of observers.

In addition, we can characterize the associated semi-Riemannian space $(M, \overline{g}_\epsilon)$.

Theorem 10 *Let (M, Ω, g, ∇) be a simply connected Galilean spacetime with symmetric connection and let Z be a free falling inertial field of observers. Suppose that Z is complete. Then, $(M, \overline{g}_\epsilon)$ is isometric to a Riemannian ($\epsilon = 1$) or Lorentzian ($\epsilon = -1$) product $(\mathbb{R} \times \mathcal{F}, \tilde{g}_\epsilon)$, where $\mathcal{F}$ is an integral maximal manifold of the distribution $\mathrm{An}(\Omega)$ and $\tilde{g}_\epsilon$ is the product metric*

$$\tilde{g}_\epsilon = \epsilon dt^2 + g_{\mathcal{F}},$$

where $g_{\mathcal{F}}$ denotes the restriction to $\mathcal{F}$ of the Riemannian metric g defined on $An(\Omega)$.

Proof From Proposition 4, Z is a Killing vector field for the metric $\overline{g}_\epsilon$. Consequently, the flow of Z sends leaves of $\mathrm{An}(\Omega)$ to leaves of $\mathrm{An}(\Omega)$. Taking this into account, the semi-Riemannian manifold $(M, \overline{g}_\epsilon)$ is endowed with two complementary foliations whose leaves intersect perpendicularly. Moreover, the leaves of both foliations are totally geodesic. Now, it is enough to apply de Rham decomposition theorem (see [36, Cor. 2]) to obtain the desired decomposition. □

Theorem 10 provides a way to ensure the geodesic connectedness of the Galilean spacetime as we can see in the following corollary.

Corollary 11 *Let (M, Ω, g, ∇) be a simply connected Galilean spacetime with symmetric connection and assume that there exists a geodesic complete maximal integral submanifold of the distribution $\mathrm{An}(\Omega)$. If the spacetime admits a complete free falling inertial field of observers then its Galilean connection is geodesically complete. Moreover, given two arbitrary points $p, q \in M$ there exists a geodesic of the Galilean connection from p to q.*

Proof From Theorem 10, the semi-Riemannian manifold $(M, \overline{g}_\epsilon)$ is complete. Now, it is enough to call Lemma 5 and Hopf-Rinow's theorem. □

4 Stationary Galilean spacetimes via certain stationary semi-Riemannian metrics

In this section we extend the previous results for more general semi-Riemannian metrics. Namely, we will relate the symmetric connection of a Galilean spacetime with the Levi-civita connection of certain 'stationary' metrics. Namely, the next result extends Lemma 5 to this setting.

Lemma 12 *Let Z be a Leibnizian field of observers in a Galilean spacetime (M, Ω, g, ∇) with symmetric connection. Suppose that:*

(i) The gravitational field of Z, $\mathcal{G}^Z$, is the g-gradient (along a leaf of $\mathrm{An}(\Omega)$) of a Z-invariant positive function α, i.e,

$$\mathcal{G}^Z = -\frac{\epsilon}{2}\mathrm{grad}^g \alpha, \quad \text{with} \quad Z(\alpha) = 0. \tag{7}$$

(ii) There exists a Z-invariant 1-form μ on M verifying $\mu(Z) = 0$, such that the vorticity of Z satisfies

$$2\,\omega_Z(V, W) = d\mu(V, W) \quad \text{for all} \quad V, W \in \Gamma(\mathrm{An}(\Omega)). \tag{8}$$

Let us denote by $\overline{\nabla}$ *the Levi-Civita connection of the 'stationary' metric given by*

$$\overline{g}_\epsilon^{\alpha,\mu} = \epsilon\alpha\,\Omega\otimes\Omega + \mu\otimes\Omega + \Omega\otimes\mu + g(P^Z\cdot, P^Z\cdot), \quad \text{with} \quad \epsilon = \pm 1.$$

Then, the following identity holds for $X, Y, F \in \Gamma(TM)$

$$\overline{g}_\epsilon^{\alpha,\mu}(\overline{\nabla}_X Y, F) = \overline{g}_\epsilon^{\alpha,\mu}(\nabla_X Y, F) + \frac{\epsilon}{2}\,\Omega\odot d\alpha\,(X,Y)\Omega(F) + \frac{1}{2}\left((\nabla_X\mu)(Y) + (\nabla_Y\mu)(X)\right)\Omega(F), \tag{9}$$

where $\Omega\odot d\alpha = \Omega\otimes d\alpha + d\alpha\otimes\Omega$. *Additionally, the field of observers* Z *is a Killing vector field of* $\overline{g}_\epsilon^{\alpha,\mu}$.

Proof As in the proof of Lemma 5, we want to obtain for any vector fields $A, B, C \in \Gamma(TM)$ that
$\overline{g}_\epsilon^{\alpha,\mu}(T(A,B),C) = \frac{\epsilon}{2}\,\Omega\odot d\alpha\,(A,B)\Omega(C) + \frac{1}{2}\left((\nabla_A\mu)(B) + (\nabla_B\mu)(A)\right)\Omega(C)$,
being $T(A,B) = \overline{\nabla}_A B - \nabla_A B$ and $\Omega\odot d\alpha = \Omega\otimes d\alpha + d\alpha\otimes\Omega$. Throughout this proof, U, V, W will denote spacelike vector fields satisfying $[U,V] = [U,Z] = [V,Z] = 0$.

As a first step, focusing on $T(U,V)$, we can easily see that for spacelike vector fields $\overline{g}_\epsilon^{\alpha,\mu}(T(U,V),W) = 0$. Moreover, using the classical Koszul formula for a semi-Riemannian metric and the Leibnizian character of Z we have

$$2\overline{g}_\epsilon^{\alpha,\mu}(\overline{\nabla}_U V, Z) = U(\mu(V)) + V(\mu(U)) + \mu([U,V]) = U(\mu(V)) + V(\mu(U)). \tag{10}$$

Moreover, recalling that $\Omega(\nabla_U V) = U(\Omega(V)) = 0$ for spacelike vectors we have

$$\overline{g}_\epsilon^{\alpha,\mu}(\nabla_U V, Z) = \mu(\nabla_U V).$$

Combining these expresions we obtain

$$2\overline{g}_\epsilon^{\alpha,\mu}(T(U,V),Z) = (\nabla_U\mu)(V) + (\nabla_V\mu)(U).$$

Secondly, let us compute $T(U,Z)$ (the case $T(Z,U)$ follows by symmetry). Indeed, from (10) we get

$$2\overline{g}_\epsilon^{\alpha,\mu}(\overline{\nabla}_U Z, V) = 2U(\overline{g}_\epsilon^{\alpha,\mu}(Z,V)) - 2\overline{g}_\epsilon^{\alpha,\mu}(Z,\overline{\nabla}_U V) = U(\mu(V)).$$

Using now (2) as well as recalling that $\Omega(\nabla_U Z) = U(\Omega(Z)) = 0$ yields

$$\begin{aligned} 2\overline{g}_\epsilon^{\alpha,\mu}(\nabla_U Z, V) &= Z(g(U,V)) + 2\omega_Z(U,V) - g([Z,U],V) - g([Z,V],U) \\ &= 2\omega_Z(U,V). \end{aligned}$$

Combining both expressions and using the Leibnizian character of Z again we deduce

$$2\overline{g}_\epsilon^{\alpha,\mu}(T(U,Z),V) = d\mu(U,V) - 2\omega_Z(U,V) = 0,$$

where the last equality follows from (8). Furthermore, we also have

$$2\overline{g}_\epsilon^{\alpha,\mu}(\overline{\nabla}_U Z, Z) = U(\overline{g}_\epsilon^{\alpha,\mu}(Z,Z)) = \epsilon\, U(\alpha).$$

Since $\overline{g}_\epsilon^{\alpha,\mu}(\nabla_U Z, Z) = \mu(\nabla_U Z)$, we get from the previous equation

$$2\overline{g}_\epsilon^{\alpha,\mu}(T(U,Z),Z) = \epsilon\, d\alpha(U) - 2\mu(\nabla_U Z) = \epsilon\, d\alpha(U) + (\nabla_U\mu)(Z) + (\nabla_Z\mu)(U),$$

where the last equality comes from the fact that $L_Z\mu = 0$.

Our third and final step is to compute $T(Z,Z)$. From the classical Koszul formula we get

$$2\overline{g}_\epsilon^{\alpha,\mu}(\overline{\nabla}_Z Z, U) = 2Z(\mu(U)) - \epsilon\, U(\alpha) + 2\mu([U,Z]) = 2Z(\mu(U)) - \epsilon\, U(\alpha).$$

Since $\overline{g}_{\epsilon}^{\alpha,\mu}(\nabla_Z Z, U) = g(\mathcal{G}^Z, U)$, using again $L_Z\mu = 0$ and (7) we obtain

$$2\overline{g}_{\epsilon}^{\alpha,\mu}(T(Z,Z),U) = -\epsilon\, U(\alpha) - 2g(\mathcal{G}^Z, U) = 0.$$

We finish the third step of the proof combining

$$2\overline{g}_{\epsilon}^{\alpha,\mu}(\overline{\nabla}_Z Z, Z) = Z(\overline{g}_{\epsilon}^{\alpha,\mu}(Z,Z)) = \epsilon\, Z(\alpha) = 0,$$

and $\overline{g}_{\epsilon}^{\alpha,\mu}(\nabla_Z Z, Z) = \mu(\nabla_Z Z)$ to conclude

$$\overline{g}_{\epsilon}^{\alpha,\mu}(T(Z,Z),Z) = (\nabla_Z\mu)(Z).$$

The last assertion is a straightforward consequence of the Leibnizian character of Z. □

Remark 13 Notice that from a physical point of view the metrics $\overline{g}_{\epsilon}^{\alpha,\mu}$ are well adapted to the observers in Z. Indeed, it is clear from (9) that the gravitational field measured by an observer in Z satisfies $\mathcal{G}^Z = \overline{\nabla}_Z Z$. On the other hand, the skew-symmetric (0, 2)-tensor field $\overline{\mathrm{Rot}_{\epsilon}}^{\alpha,\mu}(Z)$ is given by

$$\overline{\mathrm{Rot}_{\epsilon}}^{\alpha,\mu}(Z) = \epsilon d\alpha \wedge \Omega + d\mu.$$

Consequently,

$$\overline{\mathrm{Rot}_{\epsilon}}^{\alpha,\mu}(Z)(V,W) = d\mu(V,W) = 2\omega_Z(V,W),$$

for all $V, W \in \Gamma(\mathrm{An}(\Omega))$. Moreover, it follows from (2) that the tensors $\mathcal{G}^Z$ and ω_Z determine the Galilean connection of the spacetime. Therefore, the metrics $\overline{g}_{\epsilon}^{\alpha,\mu}$ contain all the physical and mathematical information of the Galilean spacetime.

Remark 14 The existence hypothesis of such a function α and a 1-form μ in Lemma 12 are known in some of the existing literature as the conditions that a torsionfree Galilean spacetime should verify to be a 'Newtonian' manifold (see [4, 5]). In the terminology of [5], this assumption can be expressed by means of the existence of a global 1-form in M, $A^Z = -\frac{\epsilon}{2}\alpha\Omega + \mu$, such that the gravitational fieldstrength is given by $F^Z = d A^Z$. Therefore, the stationary metric $\overline{g}^{\alpha,\mu}$ corresponds with the Lagrangian metric $g(P^Z\cdot, P^Z\cdot) + 2\,\Omega \odot A^Z$ given in [5, Table II, Sect. D]. The positivity of α is imposed to ensure that the latter is non-degenerate. However, we should highlight that the concept of Newtonian manifold is not unanimous in the existing literature. For instance, in [4, 5], a torsionfree Galilean spacetime is called Newtonian if there exists a field of observers N such that the gravitational fieldstrength F^N is closed, i.e., locally $F^N = d A^N$, where the 1-form A^N is called the gravitational potential. However, in [9, 21, 22], a (torsionfree) Newtonian spacetime is defined as a Galilean spacetime where there exists an inertial free falling field of observers and the leaves of the spacelike foliation are flat.

Remark 15 The exactness assumption on the vorticity is quite natural. For instance, in the Lorentzian framework it is even assumed in the definition. In addition, the Z-invariance of μ implies that $L_Z\omega_Z = 0$, which means that the observers in Z are in a 'stationary' rotation.

Remark 16 Another form for writing (9) is

$$\overline{\nabla}_X Y = \nabla_X Y + \frac{1}{2}\left(\epsilon\,\Omega \odot d\alpha\,(X,Y) + (\nabla_X\mu)(Y) + (\nabla_Y\mu)(X)\right)\Omega^*, \qquad (11)$$

where Ω^* is the vector field $\overline{g}_{\epsilon}^{\alpha,\mu}$-metrically equivalent to Ω. This may be expressed as,

$$\Omega^* = \frac{1}{\epsilon\alpha - \|\mu\|^2}(Z - \mu^{\star}),$$

being $\mu^\star$ the unique spacelike vector field satisfying $g(\mu^\star, V) = \mu(V)$ for any spacelike vector V, and $\|\mu\|^2 = g(\mu^\star, \mu^\star)$.

Moreover, we can characterize the Lorentzian structure associated to the Galilean one in Lemma 12.

Theorem 17 *Under the assumptions of Lemma 12, if M is simply connected and the vector field of observers Z is complete, then the Lorentzian manifold $(M, \overline{g}^{\alpha,\mu}_{\epsilon=-1})$ is isometric to a relativistic standard stationary spacetime.*

Proof Observe that any leaf of the foliation $\mathcal{F}$ defined by $\mathrm{An}(\Omega)$ is spacelike for the metric $\overline{g}^{\alpha,\mu}_{\epsilon=-1}$. Furthermore, the existence of a global time function and the structure of the Galilean spacetime ensure that any integral curve of Z crosses exactly once any fixed leaf $\mathcal{F}_0$ of the foliation $\mathcal{F}$. Therefore, it is enough to apply [27, Lemma 3.3] to conclude the proof. □

A Galilean spacetime (M, Ω, g, ∇) is said to be *stationary* if it admits a future timelike Galilean vector field (see [22, Def. 1]). Therefore, if the 1-form μ given in Lemma 12 is also ∇-parallel, we obtain in the next proposition a characterization of stationary Galilean spacetimes.

Proposition 18 *A vector field Z satisfying the assumptions of Lemma 12 for a ∇-parallel 1-form μ is affine for the Galilean connection. Consequently, the Galilean spacetime is stationary in the sense of [22, Def. 1].*

Proof It is enough to show that the flow of Z is an affine transformation, which is a direct consequence of (11) and the Killing character of Z. □

Remark 19 Recall that a relativistic spacetime is called stationary if it admits a globally defined timelike Killing vector field. A standard stationary spacetime $(M, \overline{g})$ is given by a product smooth manifold $\mathbb{R} \times S$, where (S, g) is a Riemannian manifold, endowed with the Lorentzian metric

$$\overline{g} = -\alpha\, dt \otimes dt + \mu \otimes dt + dt \otimes \mu + g, \tag{12}$$

where α is a positive function and μ a 1-form, both defined (defined by the existence of a complete timelike Killing vector field) obeys the causality condition of being distinguishing, then it is isometric to a standard stationary spacetime (see [27] for details).

If we compare the stationary Galilean case defined in Lemma 12 with a relativistic stationary spacetime as described above, the first thing we can find is that the distribution given by the rest-spaces at each point of the Killing field of observers is not integrable in the relativistic case. Nevertheless, in the Galilean one the integrability of the spacelike distribution is clear, constituting the absolute space (being independent of the observer).

Moreover, despite the fact that the tensors $\mathrm{Rot}_{\overline{g}}$ (relativistic case) and $\overline{\mathrm{Rot}}^{\alpha,\mu}_{\epsilon=-1}$ (Galilean case) are formally analogous, whereas in the Galilean case the rotational tensor relative to Z is given by the 2-form $d\mu$ (acting on vectors of the absolute space), this does not happen in the relativistic case for the rest-spaces given at each point by $Z^\perp$. From a physical point of view, this makes a difference in the way an observer in Z measures 'infinitesimally close' events in each model. Indeed, if $\gamma(t)$ is an observer in Z, the spacelike n-plane $\{\gamma(t)\}^\perp$ is the instantaneous space for the observer in the event $\gamma(t)$. Since Z is Killing, the correspondence $t \longrightarrow \{\gamma(t)\}^\perp$ yields, via the parallel translation the Fermi-Walker connection in the relativistic case and the parallel translation the Galilean connection in the other one, a one parameter group of rotations on an associated n-Euclidean plane. Namely, we can say that the ‘infinitesimally close’ observers to γ are spatially rotating around it and the observer γ measures a different rate of rotation for these ‘infinitesimally close’ observers in each model (see [13, 14]).

Once the stationary character of the Galilean spacetime is ensured by Proposition 18 we can study its geodesic completeness in the next result.

Theorem 20 *Let Z be a complete Leibnizian field of observers in a simply connected Galilean spacetime (M, Ω, g, ∇) with symmetric connection. Suppose that:*

(i) *The gravitational field of Z, $\mathcal{G}^Z$, is the g-gradient (along a leaf of* $\mathrm{An}(\Omega)$*) of a Z-invariant positive function α, i.e, (7) holds.*
(ii) *There exists a Z-invariant and ∇-parallel 1-form μ on M, verifying $\mu(Z) = 0$, such that the vorticity of Z satisfies*

$$2\,\omega_Z(V, W) = d\mu(V, W) \quad \text{for all} \quad V, W \in \Gamma(\mathrm{An}(\Omega)). \tag{13}$$

If some maximal integral submanifold of the distribution $\mathrm{An}(\Omega)$ *is complete with its Riemannian metric g and the norm of the gravitational field $||\mathcal{G}^Z||_g$ is bounded on this integral submanifold, then each inextensible geodesic of the Galilean connection is complete.*

Proof Poincaré’s lemma ensures the existence of a function $\mathcal{T}$ such that $\Omega = d\mathcal{T}$. Consider the Riemannian metric on M given by $g_R = d\mathcal{T} \otimes d\mathcal{T} + g(P^Z\cdot, P^Z\cdot)$. Let φ be the global flow of the vector field Z and let $\mathcal{F}_0$ be a leaf of the foliation induced by $\mathrm{An}(\Omega)$. Define the map

$$\Phi : \mathbb{R} \times \mathcal{F}_0 \longrightarrow M, \quad \Phi(t, p) = \varphi(t, p).$$

Thanks to the completeness of Z (and the existence of $\mathcal{T}$) we have that Φ is a diffeomorphism. Indeed, it is enough to see that the orbits of Z meet the integral submanifolds of $\mathrm{An}(\Omega)$ only once. Reasoning by contradiction, let us suppose that an integral curve of Z, γ, cuts twice the same leaf $\mathcal{F}_b$, $b \in \mathbb{R}$. In that case, there are two values $s_1, s_2 \in I$, $s_1 < s_2$, such that $(T \circ \gamma)(s_1) = (T \circ \gamma)(s_2)$. Since the real function $T \circ \gamma : I \longrightarrow \mathbb{R}$ is smooth, Rolle’s theorem ensures the existence of $s^* \in (s_1, s_2)$ such that

$$\frac{d}{ds}(T \circ \gamma)(s^*) = 0 \qquad \Longleftrightarrow \qquad dT\big(Z\big(\gamma(s^*)\big)\big) = 0,$$

which is a contradiction, since Z is a field of observers.

Using the pull back $\Phi^* g$ we obtain an isometry between (M, g_R) and $(\mathbb{R} \times \mathcal{F}_0, \Phi^* g)$. As a consequence, the Riemannian manifold (M, g_R) is complete. Moreover, since Proposition 18 ensures that Z is a Galilean vector field, we can call [22, Thm. 17] to finish the proof. □

4.1 Static Galilean spacetimes

Lemma 12 can be particularized for the case of an associated static semi-Riemannian metric, obtaining the following result.

Corollary 21 *Let Z be an inertial field of observers in a Galilean spacetime (M, Ω, g, ∇) with symmetric connection. Suppose that the gravitational field of Z, $\mathcal{G}^Z$, is the g-gradient (along a leaf of* $\mathrm{An}(\Omega)$*) of a Z-invariant positive function α, i.e., (7) holds.*

Let us denote by $\overline{\nabla}$ the Levi-Civita connection of the ‘static’ metric

$$\overline{g}^{\alpha}_{\epsilon} = \epsilon\alpha\,\Omega \otimes \Omega + g(P^Z\cdot, P^Z\cdot), \quad \text{with} \quad \epsilon = \pm 1. \tag{14}$$

Then, the following identity holds

$$\overline{\nabla}_X Y = \nabla_X Y + \Theta(X, Y)\, Z, \tag{15}$$

where $\Theta = d\phi \otimes \Omega + \Omega \otimes d\phi$ *and* $\alpha = e^{2\phi}$. *In addition, the field of observers* Z *is a Killing vector field of* $\overline{g}^{\alpha}_{\epsilon}$.

Proof Taking $\mu = 0$ in Lemma 12 and computing the vector field $\overline{g}^{\alpha}_{\epsilon}$-metrically equivalent to Ω, Ω^*,

$$\Omega^* = \frac{\epsilon}{\alpha} Z,$$

formula (15) is directly obtained from (11). □

Notice that from Proposition 18 the Galilean spacetime in Corollary 21 is static in the sense of [22, Def. 1]. In this static case we can also determine the structure of the associated semi-Riemannian manifold.

Theorem 22 *Let* Z *be a complete inertial field of observers in a simply connected Galilean spacetime* (M, Ω, g, ∇) *with symmetric connection. Suppose that the gravitational field of* Z, $\mathcal{G}^Z$, *is the* g*-gradient (along a leaf of* $\mathrm{An}(\Omega)$*) of a* Z*-invariant positive function* α, *i.e, (7) holds.*

Then, the semi-Riemannian manifold $(M, \overline{g}^{\alpha}_{\epsilon})$ *is isometric to the warped product* $(\mathbb{R} \times S, \overline{g}_{\epsilon})$, *where*

$$\overline{g}_{\epsilon} = \epsilon\, \alpha\, dt^2 + g,$$

being (S, g) *a Riemannian manifold isometric to a maximal integral submanifold of the distribution* $\mathrm{An}(\Omega)$, *with its Riemannian metric.*

Proof Taking into account that the stages of the global flow of Z are isometries relative to the metric $\overline{g}^{\alpha}_{\epsilon}$, all the leaves of the foliation defined by $\mathrm{An}(\Omega)$ are isometric with respect to $\overline{g}^{\alpha}_{\epsilon}$ and totally geodesic. On the other hand, $\overline{\nabla}_Z Z = \nabla_Z Z$. Thus, $\overline{g}^{\alpha}_{\epsilon}(\overline{\nabla}_Z Z, Z) = 0$, i.e., the integral curves of Z are extrinsic circles, that is, 1-dimensional totally umbilic submanifolds with constant mean curvature. Consequently, since we have two complementary foliations whose leaves intersect perpendicularly we can call [36, Cor. 1] to obtain the desired isometry. □

To conclude this section, we will provide the following result concerning the geodesic completeness of static Galilean spacetimes.

Theorem 23 *Let* Z *be a complete inertial field of observers in a simply connected Galilean spacetime* (M, Ω, g, ∇) *with symmetric connection. Suppose that the gravitational field of* Z, $\mathcal{G}^Z$, *is the* g*-gradient (along a leaf of* $\mathrm{An}(\Omega)$*) of a* Z*-invariant positive function* α, *i.e, (7) holds.*

If some maximal integral submanifold of the distribution $\mathrm{An}(\Omega)$ *is complete with its Riemannian metric* g *and the norm of the gravitational field* $||\mathcal{G}^Z||_g$ *is bounded on this integral submanifold, then each inextensible geodesic of the Galilean connection is complete.*

Proof Making use of Proposition 18, Z is a Galilean vector field. On the other hand, the metric $\overline{g}_{\epsilon} = \epsilon\, \alpha\, dt^2 + g$ given in Theorem 22 is complete (see [34, Lemma 7.40]). We complete the proof using [22, Thm. 17]. □

5 Spatially conformally Leibnizian spacetimes and their associated semi-Riemannian metrics

In this section we will extend our results to the class of Galilean spacetimes known as spatially conformally Leibnizian spacetimes (see [21]). Let us recall some basic definitions in these ambient spacetimes.

Definition 24 Let (M, Ω, g) be a Leibnizian spacetime and K a vector field satisfying

$$\Omega([K, V]) = 0 \quad \text{for all} \quad V \in \Gamma(\mathrm{An}(\Omega)).$$

The vector field K is called *spatially conformally Leibnizian* vector field if the Lie derivative of the absolute space metric satisfies

$$L_K g = 2\rho\, g, \tag{16}$$

for some smooth function $\rho \in C^\infty(M)$.

Note that the first condition in the previous is equivalent to the preservation of $\mathrm{An}(\Omega)$ by the flow of K.

Remark 25 If M is additionally endowed with a Galilean structure, (M, Ω, g, ∇), where ∇ is a symmetric connection, we have

$$L_K\Omega(Y) = d\big(\Omega(K)\big)(Y) - K\big(\Omega(Y)\big).$$

Hence, if the function $\Omega(K)$ is spatially invariant, it follows

$$L_K\Omega(V) = 0, \qquad \forall V \in \Gamma\big(\mathrm{An}(\Omega)\big),$$

and then

$$\Omega([K, V]) = -L_K\Omega(V) = 0, \qquad \forall V \in \Gamma\big(\mathrm{An}(\Omega)\big).$$

Consequently, assumption (16) can be stated as

$$K(g(V, W)) = 2\rho\, g(V, W) + g([K, V], W) + g([K, W], V), \qquad \forall V, W \in \Gamma(\mathrm{An}(\Omega)). \tag{17}$$

Now, let us recall the notion of irrotational conformally Leibnizian spacetime introduced in [21].

Definition 26 Let (M, Ω, g, ∇) be a Galilean spacetime, whose absolute clock is closed $(d\Omega = 0)$. If M admits a timelike vector field $K \in \Gamma(TM)$ satisfying

$$\nabla_X K = \rho\, X, \qquad \forall X \in \Gamma(TM),\ \ where\ \rho \in C^\infty(M), \tag{18}$$

M is called *irrotational conformally Leibnizian spacetime* (ICL).

Remark 27 Notice that condition (18) directly implies that K is conformally Leibnizian and $\mathrm{Rot}(K)(V, W) = 0$, for all spacelike vector fields V, W.

We are now in a position to formulate the following result relating the symmetric Galilean connection of a spatially conformally Leibnizian spacetime and the Levi-Civita connection of a certain class of semi-Riemannian metrics.

Lemma 28 *Let (M, Ω, g, ∇) be a Galilean spacetime with symmetric connection endowed with an irrotational future timelike spatially conformally Leibnizian vector field K. Suppose that its conformal factor ρ is spatially invariant, i.e., $V(\rho) = 0$ for any V spacelike vector field and consider the following metric on M,*

$$\overline{g}_{\epsilon,\varphi} = \epsilon\, \Omega \otimes \Omega + e^{2\varphi}\, g(P^Z\cdot, P^Z\cdot), \quad \text{with} \quad \epsilon = \pm 1, \tag{19}$$

where Z is the field of observers K, $Z = K/\Omega(K)$, and φ is a spatially invariant smooth function on M. Denoting by $\overline{\nabla}$ the Levi-Civita connection $\overline{g}_{\epsilon,\varphi}$, the following relation holds for $X, Y \in \Gamma(M)$

$$\begin{aligned}\overline{\nabla}_X Y = \nabla_X Y + d\varphi(X) P^Z Y + d\varphi(Y) P^Z X - \Omega(X)\Omega(Y)\mathcal{G}^Z \\ -\epsilon \left(Z(\varphi) + \frac{\rho}{\Omega(K)} \right) \overline{g}_{\epsilon,\varphi}(P^Z X, P^Z Y) Z,\end{aligned} \tag{20}$$

where $\mathcal{G}^Z$ is the gravitational field of Z.

Proof We will use the same ideas as in the proofs of Lemmas 5 and 12 and compute $\overline{g}_{\epsilon,\varphi}(T(A, B), C)$ for certain vector fields $A, B, C \in \Gamma(TM)$, where $T(A, B) = \overline{\nabla}_A B - \nabla_A B$. Again, throughout this proof, U, V, W will denote spacelike vector fields satisfying $[U, V] = [U, Z] = [V, Z] = 0$.

Let us begin by noticing that from Definition 24 we have $V(\Omega(K)) = K(\Omega(V)) - \Omega([K, V]) = 0$, for any $V \in \Gamma(\mathrm{An}(\Omega))$. Therefore, since $\Omega(K)$ is spatially invariant due to $\nabla\Omega = 0$, the vorticity of Z vanishes. In addition, a straightforward computation gives

$$Z\big(g(U, V)\big) = 2\, \frac{\rho}{\Omega(K)} g(U, V).$$

Our first step now will be to compute $T(U, V)$. For spacelike vector fields, using the spatial invariance of φ and recalling that for a conformal metric $\widetilde{g} = e^{2\psi} g$ the Levi-Civita connection satisfies $\widetilde{\nabla}_U V = \nabla_U V + U(\psi)V + V(\psi)U - g(U, V)\nabla\psi$, we obtain that $\overline{g}_{\epsilon,\varphi}(T(U, V), W) = 0$. In addition, the classical Koszul formula for a semi-Riemannian metric, (17) and the fact that $\overline{g}_{\epsilon,\varphi}(\nabla_U V, Z) = 0$ yield

$$\overline{g}_{\epsilon,\varphi}(T(U, V), Z) = -\left(Z(\varphi) + \frac{\rho}{\Omega(K)} \right) \overline{g}_{\epsilon,\varphi}(U, V).$$

As a second step, let us compute $T(U, Z)$ (the case $T(Z, U)$ easily follows from the symmetry of the connections). Using again Koszul formula, we have

$$\begin{aligned}2\overline{g}_{\epsilon,\varphi}(\overline{\nabla}_U Z, V) &= e^{2\varphi}\left\{2Z(\varphi)g(U, V) + Z(g(U, V)) - g([Z, U], V) - g([Z, V], U)\right\} \\ &= e^{2\varphi}\left\{2Z(\varphi)g(U, V) + Z(g(U, V))\right\}.\end{aligned}$$

From (2) and the fact that $\omega_Z = 0$ we also get

$$2\overline{g}_{\epsilon,\varphi}(\nabla_U Z, V) = e^{2\varphi}\left\{Z(g(U, V)) - g([Z, U], V) - g([Z, V], U)\right\} = e^{2\varphi}\, Z(g(U, V)).$$

Combining both expressions we obtain

$$\overline{g}_{\epsilon,\varphi}(T(U, Z), V) = Z(\varphi)\overline{g}_{\epsilon,\varphi}(U, V).$$

Taking now into account that $2\overline{g}_{\epsilon,\varphi}(\overline{\nabla}_U Z, Z) = U(\overline{g}_{\epsilon,\varphi}(Z, Z)) = 0$ and $\overline{g}_{\epsilon,\varphi}(\nabla_U Z, Z) = \epsilon\, \Omega(\nabla_U Z) = 0$, we obtain $\overline{g}_{\epsilon,\varphi}(T(U, Z), Z) = 0$.

Finally, to compute $T(Z, Z)$ we can use again Koszul formula to deduce $\overline{g}_{\epsilon,\varphi}(\overline{\nabla}_Z Z, U) = 0$, which combined with $2\overline{g}_{\epsilon,\varphi}(\overline{\nabla}_Z Z, Z) = Z(\overline{g}_{\epsilon,\varphi}(Z, Z)) = 0$ and the spatial character of $\mathcal{G}^Z = \nabla_Z Z$ directly yields $T(Z, Z) = -\mathcal{G}^Z$. □

The relation between these connections in Lemma 28 can be simplified in the particular case where the variation of the warping function φ along the integral curves of K is related to the conformal factor ρ, obtaining the following result.

Corollary 29 *Let (M, Ω, g, ∇) be a Galilean spacetime with symmetric connection endowed with an irrotational future timelike spatially conformally Leibnizian vector field K. Suppose that its conformal factor ρ is spatially invariant, i.e., $V(\rho) = 0$ for any V spacelike vector field. Let Z be the field of observers K, $Z = K/\Omega(K)$, and φ a spatially invariant smooth function on M satisfying*

$$K(\varphi) + \rho = 0. \tag{21}$$

Consider the following metric on M,

$$\overline{g}_{\epsilon,\varphi} = \epsilon\, \Omega \otimes \Omega + e^{2\varphi}\, g(P^Z\cdot, P^Z\cdot), \quad \text{with} \quad \epsilon = \pm 1. \tag{22}$$

Denoting by $\overline{\nabla}$ the Levi-Civita connection $\overline{g}_{\epsilon,\varphi}$, the following relation holds for $X, Y \in \Gamma(M)$

$$\overline{\nabla}_X Y = \nabla_X Y + d\varphi(X) P^Z Y + d\varphi(Y) P^Z X - \Omega(X)\Omega(Y)\mathcal{G}^Z, \tag{23}$$

where $\mathcal{G}^Z$ is the gravitational field of Z. Moreover, the field of observers Z is a Killing vector field of $\overline{g}_{\epsilon,\varphi}$.

We can also guarantee the completeness of the free falling observers' trajectories in these Galilean spacetimes under appropriate assumptions in the next theorem.

Theorem 30 *Let (M, Ω, g, ∇) be a symmetric Galilean spacetime endowed with a future timelike irrotational and spatially conformally Leibnizian vector field K. Suppose that its conformal factor ρ is spatially invariant and let us consider a smooth spatially invariant function, φ, satisfying (21). Suppose that:*

- (i) *For each free falling observer γ, the function $-\frac{\rho}{\Omega(K)}$ along γ is bounded from above by a constant $c_\gamma > 0$.*
- (ii) *There exist some constants $A, C > 0$ such that*

$$\|\mathcal{G}^Z_p\|_g \leq A\, \overline{d}(p, p_0) + C, \qquad \forall p \in M,$$

for a fixed point p_0, being $\mathcal{G}^Z_\gamma$ the gravitational field the field of observers $Z = K/\Omega(K)$ and $\overline{d}$ the Riemannian distance of $\overline{g}_{\epsilon,\varphi}$ with $\epsilon = +1$.

If $(M, \overline{g}_{\epsilon=+1,\varphi})$ is complete, then each inextensible free falling observer is complete towards the future.

Proof The proof of this result is quite similar to the one of [15, Thm. 3] and [23, Thm. 7]. Nevertheless, we include here the proof of our theorem for the sake of completeness. Let $\gamma : I = [0, b) \longrightarrow M$ be an inextensible free falling observer and, without loss of generality, choose $p_0 = \gamma(0)$. From (23), γ satisfies the following differential equation,

$$\frac{\overline{D}\gamma'}{dt} = 2\, d\varphi(\gamma')\, P^Z\gamma' - \mathcal{G}^Z_\gamma, \tag{24}$$

where $\mathcal{G}^Z_\gamma$ is the gravitational field Z along γ and $\frac{\overline{D}}{dt}$ is the covariant derivative of $\overline{g}_{\epsilon,\varphi}$ with $\epsilon = +1$, which from now on we will denote by $\overline{g}$. Denoting by

$$u(t) = \overline{g}(\gamma'(t), \gamma'(t)),$$

it is enough to prove that u is bounded by a positive constant in $[0, b)$. On the one hand, let us define the quadratic operator $F : \Gamma(TM) \longrightarrow \Gamma(TM)$ given by $F(X) = 2\, d\varphi(X)\, P^Z X$. Using the expression of the metric $\overline{g}$, we can easily check that

$$\overline{g}(\gamma', F(\gamma')) = 2\, e^{2\varphi}\, Z(\varphi)\, g(P^Z\gamma', P^Z\gamma') \leq 2\, Z(\varphi)\, \overline{g}(\gamma', \gamma') = -(\frac{\rho}{\Omega(K)} \circ \gamma)\, \overline{g}(\gamma', \gamma').$$

On the other hand, we note that $\|\mathcal{G}^Z_p\|_{\overline{g}} = e^{\varphi}\, \|\mathcal{G}^Z_p\|_g$. Since φ is spatially invariant we deduce that φ is bounded along each observer γ defined on a bounded real interval. This fact and *(ii)* imply that there exists a constant $r_\gamma > 0$ such that

$$\|\mathcal{G}^Z_{\gamma(t)}\|_g \leq r_\gamma(1 + |\gamma(t)|), \qquad \forall t \in I, \tag{25}$$

being $|\gamma(t)| = \overline{d}(\gamma(t), \gamma(0))$. Thus, from (24) and (25) we obtain,

$$u'(t) = 2\,\overline{g}(\gamma'(t), F(\gamma'(t))) + 2\,\overline{g}(\gamma'(t), \mathcal{G}^Z_{\gamma(t)}) \leq (2\,c_\gamma + 1)u(t) + 2\,r_\gamma^2|\gamma(t)|^2 + 2\,r_\gamma^2,$$

where we have used *(i)* and (25). Considering now $v(t) = \int_0^t u(s)\, ds$ and taking into account that $|\gamma(t)| \leq b\, \int_0^t u(s)\, ds$, we get

$$v''(t) \leq k_1 v'(t) + k_2 v(t) + k_3,$$

for some constants $k_1, k_2, k_3 > 0$. Finally, choosing the unique solution of the above equation such that $v(0) = 0$ and $v'(0) = u(0)$, we can use a classic argument for sub-solutions of ordinary differential equations (see [39, Lemma 1.1]) to obtain the desired bound of $u(t)$ in I and complete the proof. □

Remark 31 Note that if in Theorem 30 we replace *(i)* by the following assumption:

(i') For each free falling observer γ, the function $\frac{\rho}{\Omega(K)}$ along γ is bounded from above by a constant $c'_\gamma > 0$.

Then, the completeness towards the past of γ is ensured.

We can simplify the assumptions in Theorem 30, obtaining the following corollary.

Corollary 32 *Let (M, Ω, g, ∇) be a simply connected Galilean spacetime with symmetric connection endowed with a future timelike irrotational and spatially conformally Leibnizian vector field K. Suppose that its conformal factor ρ is spatially invariant and let us consider a smooth spatially invariant function, φ, satisfying (21). Suppose that:*

(i) *The functions $\frac{\rho}{\Omega(K)}$ and $\|\mathcal{G}^Z\|_g$ are bounded on M.*
(ii) *The field of observers K and a leaf of the foliation defined by the distribution* $\mathrm{An}(\Omega)$ *are complete.*

Then, each inextensible free falling Galilean observer is complete both towards the future and the past.

5.1 Irrotational conformally Leibnizian Galilean spacetimes

For an ICL Galilean spacetime, Lemma 28 particularizes to the following corollary.

Corollary 33 *Let* (M, Ω, g, ∇) *be an ICL Galilean spacetime with symmetric connection, future timelike conformal vector field* K *and conformal factor* ρ. *Consider the field of observers* $Z = \frac{K}{\Omega(K)}$ *and the metric*

$$\overline{g}_{\epsilon,\varphi} = \epsilon\, \Omega \otimes \Omega + e^{2\varphi}\, g(P^Z\cdot, P^Z\cdot), \quad \text{with} \quad \epsilon = \pm 1,$$

where φ *is a spatially invariant smooth function on* M. *Denoting by* $\overline{\nabla}$ *the Levi-Civita connection* $\overline{g}_{\epsilon,\varphi}$, *the following relation holds for* $X, Y \in \Gamma(M)$.

$$\begin{aligned} \overline{\nabla}_X Y = \nabla_X Y + d\varphi(X) P^Z Y + d\varphi(Y) P^Z X \\ -\epsilon \left(Z(\varphi) + \frac{\rho}{\Omega(K)} \right) \overline{g}_{\epsilon,\varphi}(P^Z X, P^Z Y) Z. \end{aligned} \tag{26}$$

Moreover, K *is a spatially conformal vector field of* $\overline{g}_{\epsilon,\varphi}$ *with conformal factor* $\rho + K(\varphi)$. *In particular, if* φ *is constant,* K *is a conformal vector field of* $\overline{g}_{\epsilon,\varphi}$ *with conformal factor* ρ.

Proof From (18) we obtain $\rho = \frac{K(\Omega(K))}{\Omega(K)}$, which is spatially invariant and $\mathcal{G}^Z = \nabla_Z Z = 0$. Thus, we obtain (26) using (20). The last assertions follow from the next two computations.

$$\overline{g}_{\epsilon,\varphi}(\overline{\nabla}_V K, V) = (\rho + K(\varphi))\overline{g}_{\epsilon,\varphi}(V, V), \quad \forall V \in \mathrm{An}(\Omega).$$
$$\overline{g}_{\epsilon,\varphi}(\overline{\nabla}_V K, V) = \rho\, \overline{g}_{\epsilon,\varphi}(Z, Z).$$

□

Moreover, we can characterize the structure of the semi-Riemannian manifold related to an ICL Galilean spacetime by means of the following result, which also provides a better understanding of the meaning of assumption (21).

Theorem 34 *Let* (M, Ω, g, ∇) *be a simply connected ICL Galilean spacetime with symmetric connection, future timelike conformal vector field* K *and conformal factor* ρ. *If the field of observers* $Z = \frac{K}{\Omega(K)}$ *is complete then the semi-Riemannian manifold* $(M, \overline{g}_{\epsilon,\varphi})$ *with* φ *a spatially invariant smooth function on* M *is isometric to a warped product* $(\mathbb{R} \times S, \overline{g}_\epsilon)$, *where*

$$\overline{g}_\epsilon = \epsilon\, dt^2 + e^{2\varphi} g, \quad \text{with} \quad \epsilon = \pm 1,$$

being (S, g) *isometric to a leaf of the foliation defined by the distribution* $\mathrm{An}(\Omega)$, *with the metric* g. *In addition, if* φ *satisfies* (21), *then* $(M, \overline{g}_{\epsilon,\varphi})$ *is isometric to a product* $(\mathbb{R} \times S, \overline{g}_\epsilon)$, *where*

$$\overline{g}_\epsilon = \epsilon\, dt^2 + g, \quad \text{with} \quad \epsilon = \pm 1$$

and (S, g) *is isometric to a leaf of the foliation defined by the distribution* $\mathrm{An}(\Omega)$, *with the metric* g.

Proof Observe that $\overline{\nabla}_Z Z = \nabla_Z Z = 2\,\mathcal{G}^Z = 0$. On the other hand, using (20) we obtain

$$\overline{\nabla}_V Z = \nabla_V Z + d\varphi(Z)V = \frac{1}{\Omega(K)}(\rho + K(\varphi))V,$$

for all $V \in \Gamma(\mathrm{An}(\Omega))$. Therefore, the leaves of the foliation given by the distribution $\mathrm{An}(\Omega)$ are totally umbilic submanifolds of $(M, \overline{g}_\epsilon)$ with mean curvature H given by

$$H = \frac{\epsilon}{n}\mathrm{trace}(A) = -\frac{\epsilon}{\Omega(K)}(\rho + K(\varphi)),$$

where $AV = -\overline{\nabla}_V Z$ denotes the shape operator associated to Z. Thus, from [36, Thm. 1] we deduce that $(M, \overline{g}_{\epsilon,\varphi})$ is isometric to a warped product $(\mathbb{R} \times S, \overline{g}_\epsilon = \epsilon\, dt^2 + e^{2\varphi} g)$. Moreover, if (21) holds, then the leaves of the foliation given by $\mathrm{An}(\Omega)$ are totally geodesic and the second statement is obtained from [36, Cor. 2]. □

We conclude particularizing our results for an important class of ICL Galilean spacetimes.

Example 35 (GGRW spacetimes) recall that a Galilean spacetime (M, Ω, g, ∇) is called a *Galilean Generalized Robertson–Walker spacetime (GGRW)* [21, Def. 1] if $M = I \times F$, where $I \subseteq \mathbb{R}$, (F, h) is an n-dimensional connected Riemannian manifold, $\Omega = d\pi_I$ and g is the restriction to the bundle $\mathrm{An}(\Omega)$ of the (degenerate) metric on M given by

$$\overline{g} = (f \circ \pi_I)^2\, \pi_F^* h, \tag{27}$$

where π_I, π_F are, respectively, the canonical projections onto the open interval I and the fiber F; $f \in C^\infty(I)$ and ∇ is the only symmetric Galilean connection on M such that

$$\nabla_{\partial_t}\partial_t = 0, \qquad \text{and} \qquad \mathrm{Rot}\,\partial_t = 0, \tag{28}$$

being $\partial_t = \partial/\partial t$ the global coordinate vector field $t := \pi_I$.

This family of Galilean geometric models are the classical version of the relativistic Generalized Robertson-Walker spacetimes introduced in [3].

As it is discussed in [21], the vector field $K = (f \circ \pi_I)\, \partial_t$ is a future timelike irrotational spatially conformally Leibnizian vector field with conformal factor $\rho = f' \circ \pi_I$. In addition, M is an ICL spacetime. Hence, from Corollary 33, the relation between ∇ and the Levi-Civita connection of the metric

$$\overline{g}_{\epsilon,\varphi} = \epsilon\, \Omega \otimes \Omega + \pi_F^* h,$$

which is denoted by $\overline{\nabla}$, is

$$\overline{\nabla} = \nabla + d\varphi \otimes d\pi_I + d\pi_I \otimes d\varphi,$$

where $\varphi = -\log(f \circ \pi_I) + C,\ C \in \mathbb{R}$. In addition, the vector field ∂_t is a Killing vector field of $\overline{g}_{\epsilon,\varphi}$.

6 Conclusion

In this work, we study several relevant classes of Galilean spacetimes and relate their symmetric Galilean connections to certain Levi-Civita connections defined on the same manifold, adapted to a suitable field of observers of the Newton–Cartan model. Using these semi-Riemannian metrics that are partly compatible with the Galilean connection we are able to prove new results on the geodesic connectedness and completeness of the spacetime as well as obtain variational characterizations of the spacetime's geodesics. Moreover, we also provide new global results concerning the topological and differential structure of the Galilean spacetime.

By means of progressively weakening the assumptions on the field of observers in the Galilean spacetime we can relate the symmetric connection of a Galilean spacetime with the Levi-Civita connection of more general semi-Riemannian metrics. Namely, when the field of observers is free falling and inertial (Sect. 3), the associated semi-Riemannian metric is a product one. Relaxing our assumptions on the field of observers to certain hypotheses on their gravitational field and vorticity the associated semi-Riemannian metric is a stationary one (Sect. 4), which includes the static case when the observers are inertial (Sect. 4.1). Finally, the symmetric Galilean connection of a spatially conformally Leibnizian spacetime is related to the Levi-Civita connection of a semi-Riemannian metric that includes the family of Generalized Robertson-Walker Lorentzian metrics (Sect. 5).

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