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JORDAN BIMODULES OVER THE SUPERALGEBRAS $P(n)$ AND $Q(n)$

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Abstract. We extend the Jacobson's Coordinatization theorem to Jordan superalgebras. Using it we classify Jordan bimodules over superalgebras of types $Q(n)$ and $JP(n)$, $n \geq 3$. Then we use the Tits-Kantor-Koecher construction and representation theory of Lie superalgebras to treat the remaining case $Q(2)$.

Introduction

Throughout the paper all algebras are considered over a ground field F of characteristic $\neq 2$.

Let $G = \langle 1, e_i, i \geq 1 | e_i e_j + e_j e_i = 0 \rangle$ denote the Grassmann (or exterior) algebra. Then $G = G_{\bar{0}} + G_{\bar{1}}$ is a $\mathcal{Z}/2\mathcal{Z}$-graded algebra, where $G_{\bar{0}}$, $G_{\bar{1}}$ are linear spans of all tensors of even and odd length, respectively.

Let $\mathcal{V}$ be a variety of algebras defined by homogeneous identities (see [1], [20]). A superalgebra $A = A_{\bar{0}} + A_{\bar{1}}$ is said to be a $\mathcal{V}$- superalgebra if its *Grassmann envelope* $G(A) = A_{\bar{0}} \otimes G_{\bar{0}} + A_{\bar{1}} \otimes G_{\bar{1}}$ lies in $\mathcal{V}$.

C.T.C. Wall [19] proved that every associative simple finite-dimensional superalgebra over an algebraically closed field F is isomorphic to one of the superalgebras:

I) $A = M_{m+n}(F)$, $A_{\bar{0}} = \left\{ \begin{pmatrix} \star & 0 \\ 0 & \star \end{pmatrix}, A_{\bar{1}} = \begin{pmatrix} 0 & \star \\ \star & 0 \end{pmatrix} \right\}$ and

II) $A = Q(n) = \left\{ \begin{pmatrix} a & b \\ b & a \end{pmatrix} \mid a, b \in M_n(F) \right\}$

are associative superalgebras.

Given a homogeneous element $a \in A_{\bar{0}} \cup A_{\bar{1}}$, let $|a|$ denote its parity (0 or 1).

From the definition above it follows that a Jordan superalgebra is a $\mathcal{Z}/2\mathcal{Z}$-graded algebra $J = J_{\bar{0}} + J_{\bar{1}}$ satisfying the graded identities

$$xy = (-1)^{|x||y|} yx$$

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and

$$((xy)z)t + (-1)^{|y||z|+|y||t|+|z||t|}((xt)z)y + (-1)^{|x||y|+|x||z|+|x||t|+|z||t|}((yt)z)x$$
$$= (xy)(zt) + (-1)^{|y||z|}(xz)(yt) + (-1)^{|t|(|y|+|z|)}(xt)(yz).$$

If A is an associative (super)algebra, then the new operation $a \cdot b = \frac{1}{2}(ab + (-1)^{|a||b|}ba)$ defines a structure of a Jordan (super)algebra on A. We will denote this Jordan (super)algebra as $A^{(+)}$.

Similarly, the operation $[a, b] = ab - (-1)^{|a||b|}ba$ defines a Lie superalgebra $A^{(-)}$.

A graded linear map $\star : A \to A$ of an associative superalgebra is called a superinvolution if $(a^\star)^\star = a$, $(ab)^\star = (-1)^{|a||b|}b^\star a^\star$. Then the set of symmetric elements $H(A, \star)$ is a (Jordan) subsuperalgebra of $A^{(+)}$. Similarly the set of skewsymmetric elements $Skew(A, \star)$ is a Lie subsuperalgebra of $A^{(-)}$.

Let I_n, I_m be the identity matrices, t the transposition and $U = -U^t = -U^{-1} = \begin{pmatrix} 0 & -I_m \\ I_m & 0 \end{pmatrix}$. Then the mapping $\star : M_{n+2m}(F) \to M_{n+2m}(F)$ defined as

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}^\star = \begin{pmatrix} I_n & 0 \\ 0 & U \end{pmatrix} \begin{pmatrix} a^t & -c^t \\ b^t & d^t \end{pmatrix} \begin{pmatrix} I_n & 0 \\ 0 & U^{-1} \end{pmatrix}$$

is a superinvolution.

The Jordan (resp. Lie) superalgebra of symmetric (resp. skewsymmetric) elements is called the Jordan (resp. Lie) orthosymplectic superalgebra and denoted $Josp_{n,2m}(F) = H(M_{n+2m}(F), \star)$ (resp. $OSP_{n,2m}(F) = Skew(M_{n+2m}(F), \star)$).

The associative superalgebra $M_{n+n}(F)$ has another superinvolution:

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}^\sigma = \begin{pmatrix} d^t & -b^t \\ c^t & a^t \end{pmatrix}.$$

The Jordan (resp. Lie) superalgebra of symmetric (resp. skewsymmetric) elements is denoted by $JP_n(F)$ (resp. $P_n(F)$).

V. Kac [3] (see also I. Kantor [4]) classified simple finite dimensional Jordan superalgebras over an algebraically closed field F of zero characteristic. Simple finite dimensional Jordan superalgebras over fields of positive characteristics $\neq 2$ were classified in [15] and [9].

If J is a Jordan (super)algebra, a Jordan bimodule V over J is a $\mathcal{Z}/2\mathcal{Z}$-graded vector space with operations $V \times J \to V$, $J \times V \to V$ such that the split null extension $V + J$ is a Jordan (super)algebra (see [1]). Recall that the split null extension is the direct sum of vector spaces $V + J$ with the operation that extends the multiplication of J and the action of J on V while the product of two arbitrary elements in V is zero.

Given an arbitrary set X, there is a unique free J-bimodule $V(X)$ over the set of free generators X. If V' is a J-bimodule, then an arbitrary map $X \to V'$ uniquely extends to a homomorphism of bimodules $V(X) \to V'$.

Let X be a set consisting of one element. For an element $a \in J$ let $R_{V(X)}(a)$ denote the multiplication operator $R_{V(X)}(a) : V(X) \to V(X)$, $v \to va$.

The subalgebra $U(J)$ of the algebra of all linear transformations of $V(X)$ generated by the operators $R_{V(X)}(a), a \in J$, is called the multiplicative enveloping algebra of J.

Every Jordan bimodule over J is a right module over $U(J)$ and vice versa.

In [1], N. Jacobson developed the representation theory of semisimple finite dimensional Jordan algebras. He proved that:

i) if J is a finite dimensional Jordan algebra, then $\dim_F U(J) < \infty$,

ii) if J is a finite dimensional semisimple Jordan algebra, then $U(J)$ is semisimple as well. In particular, all bimodules over J are completely reducible.

iii) Moreover, he determined all irreducible bimodules over simple finite dimensional Jordan algebras.

The representation theory for various types of simple Jordan superalgebras was developed in [8], [17], [18], [10], [11], [12] and [13]. For the current status of the project, see the survey [13].

In this paper we classify unital bimodules over Jordan superalgebras of the remaining type $Q(n)^{(+)}$, $n \geq 2$ and extend the results of [12] for $JP(n), n \geq 3$ to arbitrary characteristics $\neq 2$.

First, we adapt the arguments from [1] to obtain a Coordinatization theorem for Jordan superalgebras of capacity ≥ 3. The latter condition is satisfied for the superalgebras $JP(n)$, $Q(n)^{(+)}$, $n \geq 3$. Then we determine irreducible involutive alternative bimodules over the coordinate superalgebras of $JP(n)$, $Q(n)^{(+)}$, $n \geq 3$. This yields the classification of unital irreducible bimodules over $JP(n)$, $Q(n)^{(+)}$, $n \geq 3$. Recall that in [12] it was shown that the multiplicative enveloping algebra $U = U(J)$, $J = JP(n)$, $Q(n)^{(+)}$, $n \geq 3$, is a finite dimensional semisimple algebra; hence all Jordan bimodules over J are completely reducible. The classification of irreducible finite dimensional Jordan bimodules over $JP(n)$ (including the case $n = 2$) is obtained in [12] by different methods, though only over fields of characteristic zero.

In order to tackle the case $J = Q(2)^{(+)}$ we had to change the point of view and to resort to the study of root-graded modules over Lie superalgebras (as in [12]). This imposes stronger assumptions on the characteristic of the ground field: $\mathrm{char} F > 3$ or $= 0$.

We prove that $U(Q(n)^{(+)})$ is finite dimensional for all $n \geq 2$. If $\mathrm{char} F > 3$ or $= 0$, then the only unital irreducible Jordan bimodules over $Q(2)^{(+)}$ are the 4 nonisomorphic matrix bimodules over the same involutive alternative bimodules as in the case $n \geq 3$. The algebra $U(Q(2)^{(+)})$ is semisimple; that is, all unital Jordan bimodules over $Q(2)^{(+)}$ are completely reducible.

1. The Coordinatization theorem

Let J be a Jordan (super)algebra with an identity element 1. Let $e_1, \dots, e_n \in J_{\bar{0}}$ be pairwise orthogonal idempotents such that $\sum_{i=1}^n e_i = 1$. Then

$$J = \sum_{i \leq j} J_{ij},$$

where $J_{ii} = \{x \in J | xe_i = x\}$, $J_{ij} = \{x \in J | xe_i = xe_j = \frac{1}{2}x\}$.

It is easy to see [1] that $J_{ii}^2 \subseteq J_{ii}$, $J_{ij}J_{ii} \subseteq J_{ij}$, $J_{ij}^2 \subseteq J_{ii} + J_{jj}$, $J_{ij}J_{jk} \subseteq J_{ik}$, $J_{ii}J_{jj} = J_{ij}J_{kk} = (0)$ for distinct i, j, k.

The idempotents e_i, e_j, $i \neq j$ are said to be *strongly connected* if there exists an element $a_{ij} \in J_{ij}$ such that $a_{ij}^2 = e_i + e_j$. In this case denote $U_{(ij)} = U(a_{ij} + \sum_{k \neq i,j} e_k)$.

The following theorem is one of the cornerstones in the structure theory of Jordan algebras.

Theorem 1.1 ([1]). *Let J be a Jordan algebra with* 1, *which is a sum of $n \geq 3$ strongly connected orthogonal idempotents,* $1 = \sum_{i=1}^n e_i$, $a_{ij} \in J_{ij}$, $a_{ij}^2 = e_i + e_j$, $1 \leq i \neq j \leq n$.

(1) Consider the Peirce space $D = J_{12}$ *with the multiplication* $x \star y = 2xU_{(23)}.yU_{(13)}$. *Then D is an alternative algebra with the identity element a_{12} and the involution* $x \to \bar{x} = xU_{(12)}$. *If $n \geq 4$, then D is associative. The symmetric elements* $\{x \in D | x = \bar{x}\}$ *lie in the associative center of D.*

(2) J is isomorphic to the Jordan matrix algebra $H_n(D)$.

Our aim is to extend this theorem to Jordan superalgebras. Let $J = J_{\bar{0}} + J_{\bar{1}}$ be a unital Jordan superalgebra, $1 = \sum_{i=1}^n e_i$, $n \geq 3$, the idempotents $e_1, \ldots, e_n$ are pairwise orthogonal and strongly connected in $J_{\bar{0}}$; $a_{ij} \in (J_{\bar{0}})_{ij}$, $a_{ij}^2 = e_i + e_j$, $1 \leq i \neq j \leq n$. As above, consider the automorphisms $U_{(ij)} = U(a_{ij} + \sum_{k \neq i,j} e_k)$ of the superalgebra J. On the Peirce space J_{12} define the multiplication

$$x \star y = 2xU_{(23)}.yU_{(13)}.$$

It is easy to see that the Grassmann envelope of the superalgebra $D = (J_{12}, \star)$ is isomorphic to the Peirce subspace $G(J)_{12}$ with the operation $\star$. Part (1) of Jacobson's theorem above implies that D is an alternative superalgebra, where $x \to \bar{x} = xU_{(12)}, x \in D$ is a superinvolution. The symmetric elements lie in the associative center of D; if $n \geq 4$, then D is associative.

In order to prove that $J \simeq H_n(D)$, let's recall the isomorphism from part (2) of the Coordinatization theorem. Suppose that J is a Jordan algebra. Following [1] we will define 1-1 linear maps φ_{ij} from the alternative algebra D to all Peirce spaces J_{ij}, $1 \leq i \leq j \leq n$. Let $1 \leq i < j \leq n$. If $i = 1, j = 2$, then $\varphi_{12} = Id_D$. If $i = 1, j > 2$, then $\varphi_{1j} = U_{(2j)}$. If $i = 2$, then $\varphi_{2j} = U_{(1j)}$. Let $\varphi_{11} = 2R(a_{12})R(e_1)$, $\varphi_{ii} = \varphi_{11}U_{(1i)}$ for $i > 1$.

Define the linear mapping $\varphi : H_n(D) \to J$ via $(x_{ij})_{n \times m} \to \sum_{i=1}^n \varphi_{ii}(x_{ii}) + \sum_{i<j} \varphi_{ij}(x_{ij})$. In [1] it is proved that φ is an algebra isomorphism.

Now let's come back to the Jordan superalgebra J and define the linear mapping $\varphi : H_n(D) \to J$ as above. Applying Jacobson's theorem to the Grassmann envelopes we see that $\varphi \otimes Id : H_n(G(D)) \to G(J)$ is an algebra isomorphism. This implies that φ is an isomorphism as well.

A superinvolution $\sigma : A \to A$ in an alternative superalgebra is said to be *nuclear* if all symmetric elements lie in the associative center of A.

Let V be a bimodule over A. A linear mapping $\tau : V \to V$ is a superinvolution of the bimodule V if $\sigma + \tau$ is a superinvolution of the split extension $A + V$.

Let A be an alternative superalgebra with a nuclear superinvolution (if $n = 3$) or an associative superalgebra with a superinvolution (if $n \geq 4$). Then the superalgebra of Hermitian matrices $H_n(A)$ is a Jordan superalgebra with n strongly connected orthogonal idempotents.

Just as was done in [1], the Coordinatization theorem implies that the category of unital Jordan bimodules over $H_n(A)$ is equivalent to the category of alternative A-bimodules with a nuclear involution (if $n = 3$) or to the category of involutive associative bimodules (if $n \geq 4$).

2. Alternative bimodules

Let $A = (Fe + Fu) \oplus (Ff + Fv)$; $e^2 = e$, $eu = ue = u, u^2 = e$; $f^2 = f$, $fv = vf = v$, $v^2 = -f$. The algebra A is $\mathcal{Z}/2\mathcal{Z}$-graded: $A_{\bar{0}} = Fe + Ff$, $A_{\bar{1}} = Fu + Fv$, and thus is an associative superalgebra. The graded mapping $\sigma(e) = f$, $\sigma(f) = e$, $\sigma(u) = v, \sigma(v) = u$ is a superinvolution. It is easy to see that $H_n(A) \simeq Q(n)^{(+)}$.

Let $B = M_{1+1}(F)$ with the superinvolution

$$\begin{pmatrix} \alpha & \beta \\ \gamma & \xi \end{pmatrix} \to \begin{pmatrix} \xi & -\beta \\ \gamma & \alpha \end{pmatrix}.$$

Then $H_n(B) \simeq JP(n)$.

If V is a supermodule over a superalgebra A with a superinvolution $\star$, a bijective linear map (that we will denote also $\star$), $\star : V \to V$ is a *superinvolution* of V if the natural extension $\star$ to $A+V$ is a superinvolution of the split null extension $A+V$.

Notice that if $\star$ is a superinvolution of the supermodule V, then $-\star$ is a superinvolution as well.

Let V be an alternative bimodule over an alternative superalgebra C with a superinvolution $\star : C \to C$. Consider another copy of the vector space V, the 1-1 linear map $ex : V \to V^{ex}$ and define the multiplication $av^{ex} = (-1)^{|a||v|}(va^\star)^{ex}$, $v^{ex}a = (-1)^{|a||v|}(a^\star v)^{ex}$; $a \in C$, $v \in V$.

Then V^{ex} is an alternative bimodule over C, and $V \oplus V^{ex} \to V \oplus V^{ex}$, $v + w^{ex} \to w + v^{ex}$ is a superinvolution in the bimodule $V \oplus V^{ex}$.

Lemma 2.1. *(1) An irreducible involutive bimodule over an alternative superalgebra with a superinvolution is either an irreducible bimodule or isomorphic to $V \oplus V^{ex}$, where V is an irreducible bimodule.*

(2) $V \oplus V^{ex}$ is an irreducible involutive bimodule if and only if V is an irreducible bimodule, which does not have a superinvolution that is, $V \not\simeq V^{ex}$.

Proof. Part (1) is standard. Let us prove (2).

Suppose that $\sigma : V \to V$ is a superinvolution in the bimodule V. Then $\tau : V \to V^{ex}$, $v \to (v^\sigma)^{ex}$ is an isomorphism of bimodules. In this case, $\{v + v^\tau, v \in V\}$ is a proper involutive subbimodule of $V \oplus V^{ex}$.

On the other hand, let V be an irreducible bimodule and let W be a proper involutive subbimodule of $V \oplus V^{ex}$. Then $W \cap V = W \cap V^{ex} = (0)$.

Let $0 \neq v_1 + v_2^{ex} \in W$; $v_1, v_2 \in V$. For an arbitrary multiplication operator P (by elements from the superalgebra), $v_1 P = 0$ implies $v_2^{ex} P = 0$; otherwise $0 \neq (v_1 + v_2^{ex})P \in W \cap V^{ex}$. Hence $v_1 P \to v_2 P$ is an isomorphism of the bimodules $V \to V^{ex}$. The lemma is proved. □

Let $V = V_{\bar 0} + V_{\bar 1}$ be a bimodule over a superalgebra A. Consider the bimodule $V^{op} = V_{\bar 1}^{op} + V_{\bar 0}^{op}$, where the parity of the subspace $V_{\bar i}^{op}$ is different from $\bar i$ and the action of A is defined via

$$av^{op} = (-1)^{|a|}(av)^{op}, \;\; v^{op}a = (va)^{op}$$

for arbitrary $a \in A, \; v \in V$. The bimodule V^{op} is called the *opposite* of the bimodule V.

Let us proceed with the classification of alternative involutive unital bimodules with nuclear superinvolution over $M_{1+1}(F)$ with

$$\begin{pmatrix} \alpha & \beta \\ \gamma & \xi \end{pmatrix}^\star = \begin{pmatrix} \xi & -\beta \\ \gamma & \alpha \end{pmatrix}.$$

N. A. Pisarenko [14] proved that every alternative unital bimodule over $M_{1+1}(F)$ is associative and completely reducible and the only irreducible $M_{1+1}(F)$-bimodules are the regular bimodule $\mathrm{Reg}(M_{1+1}(F))$ and its opposite.

It is not difficult to check that the regular bimodule $\mathrm{Reg}(M_{1+1}(F))$ has two (up to isomorphism) superinvolutions, $\star$ and $-\star$. By Lemma 2.1 the only irreducible involutive bimodules over $M_{1+1}(F)$ are $\mathrm{Reg}(M_{1+1}(F))$ with the involution $\star$, $\mathrm{Reg}(M_{1+1}(F))$ with the involution $-\star$ and their opposites. This implies the following.

Theorem 2.2. *(1) Unital Jordan bimodules over $JP(n)$, $n \geq 3$ are completely reducible.*

(2) The only unital irreducible Jordan bimodules over $JP(n)$, $n \geq 3$ are:

(i) the regular bimodule,

(ii) the matrix bimodule over Reg($M_{1+1}(F)$) with the superinvolution $-\star$, which is isomorphic to the bimodule of skewsymmetric matrices in $M_{n+n}(F)$ with respect to the superinvolution σ (see page 2),

(iii) the opposites of (i) and (ii).

In [12] this theorem was proved over fields of zero characteristic.

Now let us consider alternative bimodules over the involutive algebra $A = (Fe + Fu) \oplus (Ff + Fv)$.

Lemma 2.3. *If V is an alternative unital A-bimodule with a nuclear involution, then V is an associative bimodule.*

Proof. Let $V \neq (0)$ be an alternative unital A-bimodule. Let us show that the identity map cannot be a superinvolution in V.

Suppose that Id_V is a superinvolution; that is, $ax = (-1)^{|a||x|}xa^\sigma$ for arbitrary elements $x \in V$, $a \in A$. Then $eVe = (0)$. Indeed, for $x \in eVe$ we have $x = ex = xf = 0$. Similarly, $fVf = (0)$.

Consider the operator $P : eVf \to eVf$, $x \to uxv$. Recall that, since the symmetric element $u + v$ lies in the associative center of $A + V$ it follows that $(ux)v = u(xv)$. We have $xP^2 = u(uxv)v = -exf = -x$. On the other hand $(ux)v = (-1)^{|x|}(xv)v = -(-1)^{|x|}x$ and therefore $xP^2 = x$. Hence $eVf = (0)$ and similarly $fVe = (0)$.

Now let $\star : V \to V$ be a nuclear superinvolution in V. Consider the subbimodule V' of V generated by all symmetric elements $x + x^\star$, $x \in V$. Then -$Id_{V/V'}$ is a superinvolution; hence $V/V' = (0)$.

Hence the bimodule V is generated by symmetric elements $x + x^\star$, $x \in V$, which lie in the associative center of $A+V$. This implies that V is an associative bimodule. The lemma is proved. □

It is well known that associative bimodules over a separable finite dimensional associative superalgebra are completely reducible.

Let us first determine irreducible unital associative bimodules V over the superalgebra $Fe + Fu$. Consider the operator $P : V \to V$, $x \to uxu$; $P^2 = Id_V$. Hence $V = V(1) \oplus V(-1)$, $V(i) = \{x \in V | P(x) = ix\}$. Since the decomposition above is again a direct sum of subbimodules it follows that $V = V(i)$, $i = \pm 1$. If $0 \neq x \in V_{\bar{0}}$, then x, ux is a base of V with a clearly defined action of A. We will denote these two nonisomorphic 2-dimensional bimodules as $V(i)$, $i = \pm 1$. Clearly $V(-1)$ is isomorphic to $V(1)^{op}$.

Now we will proceed with the classification of irreducible involutive unital associative A-bimodules.

Let $V = V_{\bar{0}} + V_{\bar{1}}$ be such a bimodule. Since $V = eVe + eVf + fVe + fVf$ is a direct sum of A-subbimodules and $(eVe)^\star = fVf$, $(fVf)^\star = eVe$, $(eVf)^\star = eVf$, $(fVe)^\star = fVe$ it follows that $V = eVe + fVf$ or $V = eVf$ or $V = fVe$.

Case 1. $V = eVe + fVf$.

It is easy to see that in this case eVe is an irreducible unital bimodule over $Fe + Fu$. Hence $eVe \simeq V(1)$ or $eVe \simeq V(-1)$ and $V \simeq V(1) \oplus V(1)^{ex}$ or $V \simeq V(-1) \oplus V(-1)^{ex}$. These two bimodules are the opposites.

Case 2. $V = eVf$.

Let us show that $V_{\bar{0}}$ has a nonzero symmetric element. Indeed, otherwise $x^\star = -x$ for an arbitrary $x \in V_{\bar{0}}$. Then $(uxv)^\star = -v^\star x^\star u^\star = uxv = 0$. Since $u^2 = e$, $v^2 = -f$, this implies that $x = 0$, a contradiction. So, choose $0 \neq x \in V_{\bar{0}}$, $x = x^\star$. As we have seen above, $(uxv)^\star = -uxv$ in this case; hence the elements x, uxv are linearly independent. Multiplying both elements by the invertible element u on the left, we conclude that the odd elements ux, xv are also linearly independent. We have $(ux)^\star = xv$. Hence x, uxv, ux, xv span an involutive A-bimodule. Hence $V = Fx + Fuxv + Fux + Fxv$.

Case 3. $V = fVe$.

As in the previous case we can choose $0 \neq x \in V_{\bar{0}}$, $x = x^\star$. Hence $V = Fx + Fvxu + Fxu + Fvx$.

Theorem 2.4. *(1) Unital Jordan bimodules over $Q(n)^{(+)}$, $n \geq 3$ are completely reducible.*

(2) The only unital irreducible Jordan bimodules over $Q(n)^{(+)}$, $n \geq 3$ are the bimodules of Hermitian $n \times n$ matrices over the four irreducible involutive A-bimodules above. The bimodules of the cases 2, 3 are isomorphic to their opposite bimodules.

Remark. The four irreducible unital Jordan $Q(n)^{(+)}$-bimodules above have a different presentation. The first two of them come from the associative $Q(n)$-bimodules $M_n(V(\pm 1))$. If $\sqrt{-1} \in F$, then the second two Jordan bimodules are the same matrix modules $M_n(V(\pm 1))$ but with a "twisted" action. The mapping $\star : A \to A$, $(\alpha e + \beta u)^\star = \alpha e + \sqrt{-1}\beta u$ is a pseudoinvolution (see [12]). It extends to a pseudoinvolution $\star : Q(n) \to Q(n)$, $(a_{ij}) \to (a_{ji}^\star)$. Define the action of $Q(n)^{(+)}$ on $M_n(V(\pm 1))$ via $a \cdot x = \frac{1}{2}(ax + (-1)^{|a||x|} x a^\star)$, $a \in Q(n)$, $x \in M_n(V(\pm 1))$.

3. Multiplicative enveloping algebra of $Q(2)^{(+)}$

In [12] it was shown that the multiplicative enveloping algebra $U(J)$ of a finite dimensional simple Jordan superalgebra, containing 3 orthogonal idempotents in its even part, is finite dimensional. The latter assumption is essential as $U(D_t)$ and $U(JP(2))$, for example, are infinite dimensional (see [10]). In this chapter we prove, however, that $U(Q(2)^{(+)})$ is finite dimensional.

Theorem 3.1. $\dim U(Q(2)^{(+)}) < \infty$.

Proof. As in the introduction, we consider the one-generator free unital module V over $J = Q(2)^{(+)}$ and denote $R(a) = R_V(a)$, the right multiplication operator. The multiplicative enveloping algebra U is generated by the subspace $R(J)$. The algebra U acts on any bimodule over J, including J itself.

Denote $D(x,y) = R(x)R(y) - (-1)^{|x||y|}R(y)R(x)$.

We will need the following well-known identities (see [1], [20]).

(1) $R(x)R(y)R(z) + (-1)^{|y||z|+|x||y|+|x||z|}R(z)R(y)R(x) + (-1)^{|y||z|}R((xz)y) = R(xy)R(z) + (-1)^{|y||z|}R(xz)R(y) + (-1)^{|x||y|+|x||z|}R(yz)R(x)$,

(2) $D(x,y)$ acts on J as a superderivation,

(3)$D(xy,z) = D(x,yz) + (-1)^{|x||y|}D(y,xz)$,

(4) $R(x)R(y)R(z) = \frac{1}{2}(-(-1)^{|y||z|}R((xz)y) + R(xy)R(z)$
$+(-1)^{|z||y|}R(xz)R(y) + (-1)^{|x|(|y|+|z|)}R(yz)R(x) + R(x)D(y,z)$
$+(-1)^{|z||y|}D(x,z)R(y) + (-1)^{|z|(|x|+y|)}R(z)D(x,y))$.

We say that an operator $R(a_1)\cdots R(a_k)$, $a_i \in J_{\bar 0}\cup J_{\bar 1}$ is irreducible if it does not lie in $\sum_{i=1}^{k-1}\underbrace{R(J)\cdots R(J)}_{i}$.

Step 1 (N. Jacobson, [1]). If $a_i \in J_{\bar 0}$, $1\le i\le k$ and $R(a_1)\cdots R(a_k)$ is irreducible, then $k\le 8$. Indeed, by the identity (1), the element

$$R(a_1)\cdots R(a_k) + \sum_{i=1}^{k-1}\underbrace{R(J)\cdots R(J)}_{i} \in \sum_{i=1}^{k}\underbrace{R(J)\cdots R(J)}_{i} / \sum_{i=1}^{k-1}\underbrace{R(J)\cdots R(J)}_{i}$$

is skew-symmetric in $a_1, a_3, a_5, \ldots$. This implies the claim.

Step 2. Suppose that $a_i \in J_{\bar 0}\cup J_{\bar 1}$, and the operator $R(a_1)\cdots R(a_k)$ is irreducible. Then $|\{i \mid 1\le i\le k,\ a_i\in J_{\bar 0}\}|\le 12$.

If $a_i, a_{i+1} \in J_{\bar 0}$, then "push" them to the left via the Jordan identity (4). If $a_i, a_{i+1}\in J_{\bar 1}$ then "push" them to the right via the Jordan identity.

We will get

$$R(a_1)\cdots R(a_k)$$
$$\in \sum R(b_1)\cdots R(b_r)(\prod_{i=1}^{t}R(x_i)R(c_i))R(z_1)\cdots R(z_s) + \sum_{i=1}^{k-1}\underbrace{R(J)\cdots R(J)}_{i}$$

and for each summand $r+2t+s=k$; $b_1,\ldots,b_r,c_1,\ldots,c_t\in J_{\bar 0}$; $x_1,\ldots,x_t,z_1,\ldots,z_s\in J_{\bar 1}$ and $b_1,\ldots,b_r,x_1,\ldots,x_t,c_1,\ldots,c_t,z_1,\ldots,z_s$ is a permutation of $a_1,\ldots,a_k$.

The expression $\prod_{i=1}^{t}R(x_i)R(c_i)$ is skew-symmetric in $c_1,\ldots,c_t$ modulo $\sum_{j=1}^{2t-1}\underbrace{R(J)\cdots R(J)}_{j}$. Hence $t\le 4$. By Step 1, $r\le 8$. This implies the assertion.

We will denote an even element $\begin{pmatrix} a & 0\\ 0 & a\end{pmatrix}\in J_{\bar 0}$ as a and an odd element $\begin{pmatrix} 0 & b\\ b & 0\end{pmatrix}\in J_{\bar 1}$ as $\bar b$, where $a,b\in M_2(F)$.

Step 3. $D(\bar e_{12},\bar e_{12}) = 2D(\bar e_{11}\cdot e_{12},\bar e_{12}) = 2D(\bar e_{11}, e_{12}\cdot\bar e_{12}) + 2D(e_{12},\bar e_{11}\cdot\bar e_{12}) = 0$.

Similarly, $D(\bar e_{21},\bar e_{21}) = 0$.

Furthermore, $D(\bar e_{11},\bar e_{12}) = 2D(\bar e_{11},\bar e_{12}\cdot e_{22}) = D(e_{12},e_{22})\in D(J_{\bar 0},J_{\bar 0})$.

Similarly, $D(\bar e_{ii},\bar e_{jk})\in D(J_{\bar 0},J_{\bar 0})$, where $1\le j\ne k\le 2$, $1\le i\le 2$.

Finally, $D(\bar e_{12},\bar e_{21}) = 2D(\bar e_{11}\cdot e_{12},\bar e_{21}) = 2D(\bar e_{11},e_{12}\cdot\bar e_{21}) + 2D(e_{12},\bar e_{11}\cdot\bar e_{21}) = D(\bar e_{11},\bar e_{11}+\bar e_{22}) - D(e_{12},e_{21}) = D(\bar e_{11},\bar e_{11}) - D(e_{12},e_{21})$.

Similarly, $D(\bar e_{12},\bar e_{21}) = D(\bar e_{22},\bar e_{22}) + D(e_{12},e_{21})$.

We have proved that

$$D(J_{\bar 1},J_{\bar 1})\subseteq FD(\bar e_{11},\bar e_{11})+D(J_{\bar 0},J_{\bar 0})$$
$$=FD(\bar e_{22},\bar e_{22})+D(J_{\bar 0},J_{\bar 0})=FD(\bar e_{12},\bar e_{21})+D(J_{\bar 0},J_{\bar 0}).$$

Step 4. In view of the identities (1), (2) and (3) it is sufficient to bound the length of irreducible operators of the type

$$U=R(a_1)\cdots R(a_r)(\prod_{i=1}^{t}R(x_i)R(b_i))R(y_1)\cdots R(y_\nu)(\prod_{i=1}^{\mu}D(z_i,u_i)),$$

where $a_1,\dots,a_r,b_1,\dots,b_t\in J_{\bar 0};\ x_1,\dots,x_t,y_1,\dots,y_\nu,z_1,\dots,z_\mu,u_1,\dots,u_\mu\in J_{\bar 1}$, $r\le 8$, $t\le 4$ and $\nu\le 2$.

Step 5. For even elements a,b of $J_{\bar 0}$ we denote $U(a)=2R(a)^2-R(a^2)$, $U(a,b)=R(a)R(b)+R(b)R(a)-R(ab)$. Since V is a unital module it follows that $Id_V=U(e_{11}+e_{22})=U(e_{11})+U(e_{22})+U(e_{11},e_{22})$.

We claim that $U(e_{11})U(J)\subseteq U(e_{11})\sum_{i=0}^{18}\underbrace{R(J)\cdots R(J)}_{i}$.

Indeed, in the multiplication operator above $D(z_1,u_1)$ can be moved to the left modulo shorter operators. By step 3, $U(e_{11})D(z_1,u_1)\in U(e_{11})(FD(\bar e_{22},\bar e_{22})+D(J_{\bar 0},J_{\bar 0}))\subseteq U(e_{11})D(J_{\bar 0},J_{\bar 0})$.

In this way we can get rid of all the derivations $D(z_i,u_i)$, $1\le i\le\mu$.

Similarly, $U(e_{22})U(J)\subseteq U(e_{22})\sum_{i=0}^{18}\underbrace{R(J)\cdots R(J)}_{i}$.

Finally, $U(e_{11},e_{22})D(z_1,u_1)\in U(e_{11},e_{22})(FD(\bar e_{12},\bar e_{21})+D(J_{\bar 0},J_{\bar 0}))$, $U(e_{11},e_{22})$, $D(\bar e_{12},\bar e_{21})=U(e_{11},e_{22})D(\bar e_{12},\bar e_{21})(U(e_{11})+U(e_{22}))$.

Hence

$$U(e_{11},e_{22})U(J)\subseteq U(e_{11},e_{22})\sum_{i=0}^{18}\underbrace{R(J)\cdots R(J)}_{i}$$
$$+U(e_{11},e_{22})D(\bar e_{12},\bar e_{21})U(e_{11})\sum_{i=0}^{18}\underbrace{R(J)\cdots R(J)}_{i}$$
$$+U(e_{11},e_{22})D(\bar e_{12},\bar e_{21})U(e_{22})\sum_{i=0}^{18}\underbrace{R(J)\cdots R(J)}_{i}.$$

We have that $\dim\sum_{i=0}^{18}\underbrace{R(J)\cdots R(J)}_{i}<1+8+\cdots+8^{18}<8^{19}$. Hence $\dim U(J)<5.8^{19}$. The theorem is proved. $\square$

4. General facts

Let us recall some constructions relating Lie and Jordan algebras.

Definition 4.1 ([7])**.** A Jordan (super)pair $P=(P^-,P^+)$ is a pair of vector (super)spaces with a pair of trilinear operations

$$\{\ ,\ ,\ \}:P^-\times P^+\times P^-\to P^-,\{\ ,\ ,\ \}:P^+\times P^-\times P^+\to P^+$$

that satisfies the following identities:

(P.1) $\{x^\sigma,y^{-\sigma},\{x^\sigma,z^{-\sigma},x^\sigma\}\}=\{x^\sigma,\{y^{-\sigma},x^\sigma,z^{-\sigma}\},x^\sigma\}$,

(P.2) $\{\{x^\sigma,y^{-\sigma},x^\sigma\},y^{-\sigma},u^\sigma\}=\{x^\sigma,\{y^{-\sigma},x^\sigma,y^{-\sigma}\},u^\sigma\}$,

(P.3) $\{\{x^\sigma,y^{-\sigma},x^\sigma\},z^{-\sigma},\{x^\sigma,y^{-\sigma},x^\sigma\}\}=\{x^\sigma,\{y^{-\sigma},\{x^\sigma,z^{-\sigma},x^\sigma\},y^{-\sigma}\},x^\sigma\}$,

for every $x^\sigma,u^\sigma\in P^\sigma$, $y^{-\sigma},z^{-\sigma}\in P^{-\sigma}$, $\sigma=\pm$.

Let $L = L_{-1} + L_0 + L_1$ be a $\mathcal{Z}$-graded Lie (super)algebra. Then (L_{-1}, L_1) is a Jordan (super)pair.

For an arbitrary Jordan (super)pair $P = (P^-, P^+)$, there exists a unique Z-graded Lie (super)algebra $K = K_{-1} + K_0 + K_1$ such that $(K_{-1}, K_1) \simeq P$, $K_0 = [K_{-1}, K_1]$ and for every 3-graded Lie (super)algebra $L = L_{-1} + L_0 + L_1$, an arbitrary homomorphism of the Jordan pairs $P \to (L_{-1}, L_1)$ uniquely extends to a homomorphism of Lie (super)algebras $K \to L$.

We will refer to $K = K(P)$ as the Tits-Kantor-Koecher (in short **TKK) construction** of the pair P.

If J is a Jordan superalgebra, then (J^-, J^+) is a Jordan superpair. The Lie superalgebra $K = K(J^-, J^+)$ is called the TKK-construction of J.

Let $J = J_{\bar{0}} + J_{\bar{1}}$ be a simple finite dimensional Jordan superalgebra. Let us consider $L = K(J)$ its TKK-construction.

If V is a Jordan bimodule over J, then the null extension $V + J$ is a Jordan superalgebra, so we can consider its TKK Lie superalgebra $K(V + J) = (V^- + J^-) + [V^- + J^-, V^+ + J^+] + (V^+ + J^+)$.

Denote $K(V) = V^- + [V^-, J^+] + [J^-, V^+] + V^+ \leq K(V + J)$. Then $K(V)$ is a Lie module over the subalgebra $J^- + [J^-, J^+] + J^+$ which is isomorphic to $K(J)$.

Let W be the maximal $K(J)$-submodule, which is contained in $K(V)_0 = [V^-, J^+] + [J^-, V^+]$. Let $\bar{K}(V) = K(V)/W$.

The following two lemmas were proved in [12].

Lemma 4.2 ([12])**.** *Let J be a unital Jordan (super)algebra and let V_1, V_2 be two unital Jordan J-bimodules. The following assertions are equivalent:*
(1) $V_1 \simeq V_2$,
(2) $K(V_1) \simeq K(V_2)$,
(3) $\bar{K}(V_1) \simeq \bar{K}(V_2)$.

Lemma 4.3 ([12])**.** *For a unital Jordan bimodule V over a unital Jordan (super)algebra J, the following assertions are equivalent:*
(1) V is an irreducible J-bimodule,
(2) $\bar{K}(V)$ is an irreducible $K(J)$-module.

The Tits-Kantor-Koecher Lie superalgebra of $J = Q(2)^{(+)}$ is the Lie superalgebra $L = \{\begin{pmatrix} a & b \\ b & a \end{pmatrix} | a, b \in M_4(F), \mathrm{tr}(b) = 0\} = [Q(4)^-, Q(4)^-]$.

We will denote the element $\begin{pmatrix} a & 0 \\ 0 & a \end{pmatrix}$ as a and the element $\begin{pmatrix} 0 & b \\ b & 0 \end{pmatrix}$ as $\bar{b}$.

Let $H = \{\mathrm{diag}(\alpha_1, \alpha_2, \alpha_3, \alpha_4) | \sum_{i=1}^4 \alpha_i = 0\}$ be a Cartan subalgebra of $[L_{\bar{0}}, L_{\bar{0}}]$.

Let $\Lambda = \bigoplus_{i=1}^4 \mathcal{Z} w_i / \mathcal{Z}(w_1 + \cdots + w_4)$ be the free abelian group of rank 3. The associative algebra $M_4(F)$ is Λ-graded with $\deg(e_{ij}) = w_j - w_i + \mathcal{Z}(w_1 + \cdots + w_4)$, $1 \leq i, j \leq 4$. This gradation induces a Λ-gradation of the Lie superalgebra L.

Clearly, $L_0 = \{a + \bar{b}$, where both a and b are diagonal and $\mathrm{tr}(b) = 0\}$.

An arbitrary element $\lambda = \sum_{i=1}^4 \lambda_i w_i + \mathcal{Z}(w_1 + \cdots + w_4)$ induces a functional on H. If $h = \mathrm{diag}(\alpha_1, \ldots, \alpha_4)$, $\sum_{i=1}^4 \alpha_i = 0$, we let $\langle \lambda, h \rangle = \sum_{i=1}^4 \lambda_i \alpha_i$. Thus, $[a, h] = \langle \lambda, h \rangle a$ for elements $a \in L_\lambda$, $h \in H$.

Let $\{V_i\}$ be the family of the four finite dimensional irreducible unital bimodules over $J_{\bar{0}} = M_2(F)^+$. Consider the modules $\{K(V_i)\}_i$ over $K(J_{\bar{0}}) = sl(3)$. From the description of the modules $K(V_i)$ (see [1], [12]) it follows that the Λ-gradation can be extended to those modules, $K(V_i) = \sum_{\lambda\in\Lambda} K(V_i)_\lambda$ and for arbitrary elements $a \in K(V_i)_\lambda,\ h \in H$ we have $ah = \langle a, h\rangle a$.

Let $\Delta = \{0 \neq \pm w_i \pm w_j + \mathcal{Z}(w_1 + \cdots + w_4), 1 \leq i, j \leq 4\}$.

In [12] it was shown that $K(V_i) = \sum_{\lambda\in\Delta\cup\{0\}} K(V_i)_\lambda$.

Lemma 4.4. *Let $\alpha, \beta \in \{0 \neq \pm w_i \pm w_j\}$.*

(1) If $\langle\alpha, h\rangle = \langle\beta, h\rangle$ for all $h \in H$, then $\alpha - \beta \in \mathcal{Z}(w_1 + \cdots + w_4)$.

(2) If $\langle w_i - w_j + \alpha + \beta, h\rangle = 0$ for all $h \in H$, then $w_i - w_j + \alpha + \beta + \mathcal{Z}(w_1 + \cdots + w_4) = 0$ in Λ.

Proof. The assertion (1) is obvious. Let's prove (2). We have $w_i - w_j + \alpha + \beta = \sum_{\mu=1}^4 k_\mu w_\mu$, $\sum_{\mu=1}^4 k_\mu$ is even, $\sum_{\mu=1}^4 |k_\mu| \leq 6$.

Suppose at first that at least one k_μ is equal to zero. Let $k_4 = 0$. Then $\sum_{\mu=1}^3 k_\mu\alpha_\mu = 0$ for all $\alpha_1, \alpha_2, \alpha_3 \in F$. Hence k_1, k_2, k_3 are divisible by $p = \mathrm{char} F$.

If $p \geq 7$, then $k_1 = k_2 = k_3 = 0$ since $\sum_{\mu=1}^3 |k_\mu| \leq 6$.

If $p = 5$, then at most one $k_\mu, 1 \leq \mu \leq 3$ is not equal to zero and equal to ± 5. This contradicts the fact that $\sum_{\mu=1}^4 k_\mu$ is even.

From now on we will assume that all k_μ are different from zero. Suppose that at least one of them is equal to ± 1. Without loss of generality we can assume that $k_4 = -1$. Then $\langle\sum_{\mu=1}^3 (k_\mu + 1)w_\mu, h\rangle = (0)$. Hence k_1+1, k_2+1, k_3+1 are divisible by p. If $p \geq 7$, then at most one of $k_\mu + 1$ is not equal to zero. In this case $p = 7$, $k_\mu = 6$, $k_\nu = -1$ for $\nu \neq \mu$, $1 \leq \nu \leq 3$.

Then, $\sum_{\mu=1}^4 k_\mu = 3$, an odd number.

Hence $k_1 = k_2 = k_3 = k_4 = -1$, which means that $\sum_{\mu=1}^4 k_\mu w_\mu + \mathcal{Z}(w_1 + \cdots + w_4) = 0$ in Λ.

Let $p = 5$. If $k_1 + 1 = \pm 5, k_2 + 1 = \pm 5$, then $|k_1| + |k_2| > 6$.

If $k_1 + 1 = \pm 5, k_2 = k_3 = k_4 = -1$, then again $\sum_{\mu=1}^4 |k_\mu| \geq 7$.

Hence $k_1 = k_2 = k_3 = k_4 = -1$ and again $\sum_{\mu=1}^4 k_\mu w_\mu + \mathcal{Z}(w_1 + \cdots + w_4) = 0$ in Λ.

Finally if $|k_\mu| \geq 2$ for all μ, then $\sum_{\mu=1}^4 |k_\mu| \geq 8$, a contradiction. The lemma is proved. □

Remark. If $p = 3$, then $\alpha = \beta = w_i - w_j$ and $\alpha = w_i - w_k$, $\beta = 2w_l$, where i, j, k, l are distinct, are counterexamples to the assertion (2).

Let V be a unital Jordan bimodule over $J = Q(2)^+$. Then V is a direct sum of irreducible bimodules over $J_{\bar{0}} = M_2(F)^+$. This defines the decomposition $K(V) = \sum_{\lambda\in\{0\}\cup\Delta} K(V)_\lambda$. By Lemma 4.4(1), each nonzero $K(V)_\lambda$ is an eigenspace with respect to the action of H.

From Lemma 4.4(2) it follows that $K(V)_\lambda L_\alpha \subseteq K(V)_{\lambda+\alpha}$ for any $\alpha \in \{w_i - w_j, 1 \leq i, j \leq 4\}$. Indeed, each nonzero vector from $K(V)_\lambda L_\alpha$ is an eigenvector with respect to the action of h, which belongs to the eigenfunctional $h \to \langle\lambda + \alpha, h\rangle$. Hence there exists $\beta \in \{0\} \cup \Delta$ such that $K(V)_\beta \neq (0)$ and $\langle\lambda + \alpha, h\rangle = \langle\beta, h\rangle$ for all $h \in H$. By Lemma 4.3(1), $\lambda + \alpha = \beta$.

We have proved 4.4.

Lemma 4.5. *The decomposition $K(V) = \sum_{\lambda\in\{0\}\cup\Delta} K(V)_\lambda$ makes $K(V)$ a Λ-graded L-module.*

Consider a functional $f : \bigoplus_{i=1}^4 \mathcal{Z}w_i \to \mathcal{Z}$ such that $f(w_1 + \cdots + w_4) = 0$ and all $\pm f(w_i)$ are distinct. For example, $f(w_1) = 4$, $f(w_2) = -3$, $f(w_3) = 1$, $f(w_4) = -2$.

Let $\Delta_+ = \{\gamma \in \Delta | f(\gamma) > 0\}$, $\Delta_- = \{\gamma \in \Delta | f(\gamma) < 0\}$, $L_+ = \sum_{\gamma\in\Delta_+} L_\gamma$, $L_- = \sum_{\gamma\in\Delta_-} L_\gamma$, $L = L_- + L_0 + L_+$.

Let M be an irreducible module over L_0. From $[(L_0)_{\bar 1}, (L_0)_{\bar 1}] = (L_0)_{\bar 0}$ it follows that $M_{\bar 0} \neq (0)$.

Lemma 4.6. *For an arbitrary $\lambda \in \Delta$ there exists at most one irreducible Λ-graded module V over the Lie superalgebra L, such that $V = V_0 + \sum_{\alpha\in\Delta} V_\alpha$, $V_\lambda \neq (0)$, $V_\lambda L_+ = (0)$ and the L_0-module V_λ is isomorphic to M.*

Proof. Choose a nonzero element $x \in M_{\bar 0}$ and consider the right ideal $I = \{a \in U(L_0) | xa = 0\}$ of $U(L_0)$, $M \simeq U(L_0)/I$.

The Λ-gradation on L extends to the Λ-gradation on $U(L)$.

Consider the free one-generated $U(L)$-module $W = wU(L)$. Assigning the degree λ to w we make W a Λ-graded module. Let W' be the submodule of W generated by wI, wL_+ and $\sum_{\alpha\notin\{0\}\cup\Delta} W_\alpha$. Let $\bar W = W/W'$. Since the L_0-module $\bar W_\lambda$ is a homomorphic image of M it follows that either $\bar W_\lambda = (0)$, in which case the module of the lemma does not exist, or $\bar W_\lambda \simeq M$. In the latter case, $\bar W$ has a unique proper submodule, which implies the lemma. □

We say that a Λ-graded L-module V is Δ-graded if $V = \sum_{\alpha\in\{0\}\cup\Delta} V_\alpha$ and V is generated by $V = \sum_{\alpha\in\Delta} V_\alpha$.

If $\lambda \in \Delta$, $V_\lambda \neq (0)$, $V_\lambda L_+ = (0)$ and V_λ generates V, then we say that λ is the highest weight of the Δ-graded module V.

Lemma 4.7. *Only $2w_1, w_1 - w_2, -2w_2$ can be highest weights of a Δ-graded L-module.*

Proof. Let V be a Δ-graded L-module. Suppose that $V_{2w_1} = V_{w_1-w_2} = V_{-2w_2} = (0)$. Since $VL^3_{w_i-w_j} = (0)$, $1 \le i \neq j \le 4$ and $\operatorname{char} F \ge 5$, it follows that the Weyl group acts on V permuting weight spaces. This implies that $V_{2w_i} = V_{w_i-w_j} = V_{-2w_i} = 0$ for all $1 \le i \neq j \le 4$.

We have $V_{w_1+w_2}\bar e_{12} \subseteq V_{2w_2} = (0)$, $V_{w_1+w_2}\bar e_{21} \subseteq V_{2w_1} = (0)$.

Hence $V_{w_1+w_2}[\bar e_{12}, \bar e_{21}] = V_{w_1+w_2}(e_{11} + e_{22}) = (0)$.

On the other hand $V_{w_1+w_2}(e_{11} - e_{22}) = \langle w_1 + w_2, e_{11} - e_{22}\rangle V_{w_1+w_2} = (0)$. Hence $V_{w_1+w_2}e_{11} = V_{w_1+w_2}e_{22} = (0)$.

We also have $V_{w_1+w_2}\bar e_{34} \subseteq V_{w_1+w_2+w_4-w_3} = V_{-2w_3} = (0)$, $V_{w_1+w_2}\bar e_{43} \subseteq V_{-2w_4} = (0)$; hence $V_{w_1+w_2}(e_{33} + e_{44}) = (0)$.

On the other hand, $V_{w_1+w_2}(e_{33} - e_{44}) = (0)$, which implies $V_{w_1+w_2}e_{ii} = 0$, $1 \le i \le 4$. However, for an arbitrary element $v \in V_{w_1+w_2}$ we have $v(e_{11} - e_{33}) = v$. Hence $V_{w_1+w_2} = V_{w_i+w_j} = (0)$, $1 \le i \neq j \le 4$.

Similarly, $V_{-w_i-w_j} = (0)$, $1 \le i \neq j \le 4$. This contradicts the assumption that V is generated by $\sum_{\alpha\in\Delta} V_\alpha$. The lemma is proved. □

Denote

$$z = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \in L_{\bar 0}$$

a central element. Clearly, $L_0 = H + Fz + \bar H$.

Let V be a Δ-graded L-module of the highest weight $2w_1$. Let $2 \le i \neq j \le 4$. Then $V_{2w_1}(e_{ii} - e_{jj}) = (0)$, $V_{2w_1}\bar e_{ij} = V_{2w_1}\bar e_{ji} = (0)$; hence $V_{2w_1}(e_{ii} + e_{jj}) = (0)$.

This implies $V_{2w_1}e_{ii} = (0)$. On the other hand, for an arbitrary element $v \in V_{2w_1}$ we have $v(e_{11} - e_{22}) = 2v$. Hence $ve_{ii} = 2\delta_{i1}v$, $1 \leq i \leq 4$.

The element z acts on V as the multiplication by 2. Again, if $2 \leq i \neq j \leq 4$, then $v_{2w_1}e_{ij} = V_{2w_1}\bar{e}_{ji} = (0)$; hence $V_{2w_1}(\overline{e_{ii} - e_{jj}}) = (0)$.

Denote $x = \overline{e_{11} - e_{22}}$. Then $x^2 = \frac{1}{2}(e_{11} + e_{22})$, $vx^2 = v$ for $v \in L_{2w_1}$. Thus, the even and the odd parts of V_{2w_1} can be identified, $V_{2w_1} = (V_{2w_1})_{\bar{0}} + (V_{2w_1})_{\bar{0}}x$.

If $\overline{\text{diag}(\alpha_1, \alpha_2, \alpha_3, \alpha_4)} \in (L_0)_{\bar{1}}$, $\sum_{i=1}^4 \alpha_i = 0$, then $\overline{\text{diag}(\alpha_1, \alpha_2, \alpha_3, \alpha_4)} = \alpha_1 x + \alpha_3\overline{e_{33} - e_{22}} + \alpha_4\overline{e_{44} - e_{22}}$.

If $v_1, \ldots, v_r$ is a base of $(V_{2w_1})_{\bar{0}}$, then the L_0-module V_{2w_1} is a direct sum of r isomorphic irreducible 2-dimensional L_0-modules, $V_{2w_1} = \bigoplus_{i=1}^r (Fv_i + Fv_i x)$.

Now suppose that V is a Δ-graded L-module such that $V_{2w_i} = (0)$, $1 \leq i \leq 4$, but $V_{w_1 - w_2} \neq (0)$.

Then for an arbitrary element $v \in V_{w_1 - w_2}$ we have $v(e_{11} - e_{22}) = 2v$. Arguing as above we get $v\bar{e_{12}} = v\bar{e_{21}} = 0$, which implies $v(e_{11} + e_{22}) = 0$. Hence $ve_{11} = v$, $ve_{22} = -v$.

For $3 \leq i \neq j \leq 4$ we have $v(e_{ii} - e_{jj}) = v(e_{ii} + e_{jj}) = 0$; hence $ve_{ii} = 0$.

In this case $Vz = (0)$.

From $V_{w_1 - w_2}[e_{34}, \bar{e}_{43}] = (0)$ we deduce that $V_{w_1 - w_2}\overline{e_{33} - e_{44}} = (0)$.

Denote $x = \overline{e_{11} - e_{33}}$, $y = \overline{e_{22} - e_{44}}$. Then, for an arbitrary element $v \in V_{w_1 - w_2}$ we have $vx^2 = \frac{1}{2}v$, $vy^2 = -\frac{1}{2}v$, $v(xy + yx) = 0$.

Consider the operator $\varphi : V_{w_1 - w_2} \to V_{w_1 - w_2}$, $\varphi(v) = (vx)y$. Then $\varphi^2(v) = \frac{1}{4}v$. The decomposition $V_{w_1 - w_2} = V_{w_1 - w_2}(\frac{1}{2}) \oplus V_{w_1 - w_2}(-\frac{1}{2})$, where $V_{w_1 - w_2}(i) = \{v \in V_{w_1 - w_2} | \varphi(v) = iv\}$ is a direct sum of L_0-modules.

Each summand $V_{w_1 - w_2}(i)$ is a direct sum of isomorphic copies of the irreducible 2-dimensional L_0-modules $Fv + Fvx$, the element v is even, $vy = ivx$, $(vx)y = iv$, $i = \pm\frac{1}{2}$.

If $V_{2w_i} = V_{w_i - w_j} = (0)$, $1 \leq i \neq j \leq 4$, then arguing as above we can show that z acts on V as multiplication by -2 and V_{-2w_2} is a direct sum of isomorphic copies of a uniquely determined irreducible 2-dimensional module over L_0.

Recall that for all highest weights γ the irreducible components of the bimodule V_γ are isomorphic to their opposites.

Now we are ready to classify irreducible unital Jordan bimodules over $J = Q(2)^+$.

Let V be such a bimodule. Then $\overline{K(V)}$ is an irreducible Δ-graded module over the Lie superalgebra L. Let $\lambda \in \Delta$ be the highest weight of $\overline{K(V)}$.

The L_0-module $\overline{K(V)}_\lambda$ is irreducible. If $\lambda = 2w_1$ or $-2w_2$, then the L_0-module $\overline{K(V)}_\lambda$ is uniquely determined. If $\lambda = w_1 - w_2$, then there are two possibilities for the L_0-module $\overline{K(V)}_\lambda$. By Lemma 4.7 there are at most 4 possibilities for the module $\overline{K(V)}$; hence, by Lemma 4.2, there are at most four nonisomorphic bimodules over J, all of them isomorphic to their opposites. The 4 Hermitian 2×2 matrices over the 4 nonisomorphic irreducible involutive associative bimodules over the algebra $A = (Fe + Fu) \oplus (Ff + Fv)$ provide these bimodules. We proved the following theorem:

Theorem 4.8. *Let* char$F > 3$. *Then an arbitrary irreducible unital bimodule over* $Q(2)^{(+)}$ *is isomorphic to the bimodule of Hermitian* 2×2 *matrices over one of the* 4 *irreducible involutive associative bimodules over the algebra* A.

Now our aim is to establish that all unital Jordan bimodules over $Q(2)^{(+)}$ are completely reducible.

Lemma 4.9. *(1) Every homomorphism of unital Jordan J-bimodules $V_1 \to V_2$ gives rise to a homomorphism of L-modules $\bar{K}(V_1) \to \bar{K}(V_2)$.*

(2) If $V_1 \to V_2$ is an embedding, then $\bar{K}(V_1) \to \bar{K}(V_2)$ is an embedding.

Proof. By the universal property of $K(V_1)$ a homomorphism $V_1 \to V_2$ gives rise to a homomorphism $\varphi : K(V_1) \to \bar{K}(V_2)$. Let W be the largest submodule of $K(V_1)$ lying in $[V_1^-, J^+] + [V_1^+, J^-]$. The image of W lies in $[V_2^-, J^+] + [V_2^+, J^-]$ and therefore is zero. This proves (1).

If $V_1 \to V_2$ is an embedding, then the kernel of $\bar{K}(V_1) \to \bar{K}(V_2)$ has zero intersection with V_1^- and with V_1^+; hence it is zero. The lemma is proved. □

Theorem 4.10. *Every unital Jordan J-bimodule is completely reducible.*

Proof. Let V_1, V_2 be irreducible unital Jordan J-bimodules and let $(0) \to V_1 \to V \to V_2 \to (0)$ be a short exact sequence. It gives rise to $(0) \to \bar{K}(V_1) \to \bar{K}(V) \to \bar{K}(V_2) \to (0)$.

We do not claim that this sequence is exact, but its restrictions $(0) \to V_1^{\pm} \to V^{\pm} \to V_2^{\pm} \to (0)$ are exact.

Suppose at first that the irreducible modules $\bar{K}(V_1)$, $\bar{K}(V_2)$ have different highest weights. Then $\bar{K}(V_1)(z-\alpha) = \bar{K}(V_2)(z-\beta) = 0$, $\alpha \neq \beta$. Hence $V^{\pm}(z-\alpha)(z-\beta) = (0)$. Now $V = \mathrm{Ker}(z-\alpha) \oplus \mathrm{Ker}(z-\beta)$ is a direct sum of Jordan bimodules.

Now let $\bar{K}(V_1)$, $\bar{K}(V_2)$ have the same highest weight γ (which does not imply that they are isomorphic if $\gamma = w_1 - w_2$). We have shown above that for each of the highest weights $\gamma = 2w_1, w_1 - w_2, -2w_2$, the action of L_0 on $\bar{K}(V)_\gamma$ is completely reducible.

Hence $\bar{K}(V)_\gamma = \bar{K}(V_1)_\gamma \oplus M$. Let W be the submodule of $\bar{K}(V)$ generated by M. It is easy to see that $W \cap \bar{K}(V)_\gamma = M$. Hence $W \cap \bar{K}(V_1) = (0)$.

Since every nonzero submodule of $\bar{K}(V)$ has a nonzero intersection with V^- it follows that $W \cap V^- \neq (0)$. Now $\{v \in V \mid v^- \in W\}$ is a nonzero J-subbimodule of V which has zero intersection with V_1. This proves that $V \simeq V_1 \oplus V_2$. The theorem is proved. □

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