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Projection depth and L^r-type depths for fuzzy random variables

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ABSTRACT

Statistical depth functions are a standard tool in nonparametric statistics to extend order-based univariate methods to the multivariate setting. Since there is no universally accepted total order for fuzzy data (even in the univariate case) and there is a lack of parametric models, a fuzzy extension of depth-based methods is very interesting. In this paper, we adapt the multivariate depths projection depth and L^r-type depth functions to the fuzzy setting, proposing different generalizations for the L^r-type depths. We prove that the proposed fuzzy depth functions have very good properties, obtaining that the fuzzy projection depth is the second example in the literature to satisfy simultaneously the notion of semilinear and of geometric depth. This implies that the fuzzy projection depth is extremely well behave, to order fuzzy sets with respect to fuzzy random variables. Furthermore, we illustrate the good empirical behavior of the proposed fuzzy depth functions with a real data example of trapezoidal fuzzy sets and the used of fuzzy depths in depth-based classification procedures. Finally, as trapezoidal fuzzy sets can be represented by elements of $\mathbb{R}^4$, we justify our proposals by also showing empirically the superiority of the fuzzy depths over the multivariate projection depth applied to fuzzy sets.

1. Introduction

It has repeatedly been observed (see, e.g., [1,16]) that statistical analysis of fuzzy data faces several difficulties:

(a) The algebraic structure of fuzzy sets, which is not a linear space and lacks a subtraction operation.
(b) Fuzzy sets lack of a natural total order (even in $\mathbb{R}$) and many competing approaches to rank fuzzy numbers exist.
(c) There is a substantial lack of parametric models and no practically useful analog of the normal distribution.

In this situation, nonparametric methods taylored to the specific structure of fuzzy set spaces that incorporate a well-founded way to order a fuzzy data sample would be very interesting. That is exactly what *statistical depth for fuzzy data* [13] tries to achieve.

By definition, the medians are the points with respect to which at least half of the sample is smaller or equal and at least half of the sample is greater or equal. A seemingly innocuous rewording replaces ordering by geometry: the medians are the points that split the real line into two half-lines each of which contains at least half of the sample. The 10th percentile is more outlying because the two half-lines it defines divide the sample very unevenly. With this idea, Tukey [31] realized that, in order to extend the notion of *position* of a point in a sample to the multivariate setting, it suffices to replace half-lines by half-spaces. To each $x \in \mathbb{R}^p$,

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Tukey associated a depth value $D(x)$, calculated as the greatest lower bound of the proportion of the sample points contained in any half-space whose boundary passes through x. Like in the real line, if $D(x)$ is very small there exists a hyperplane through x splitting the sample very unevenly. That is, x is quite outlying. And $D(x)$ will be largest if the sample is split (by the worst-case hyperplane through x) as evenly as it is possible. Thus data themselves define a way to rank points according to their centrality or outlyingness, without requiring a total ordering in $\mathbb{R}^p$. Tukey's data-driven center-outward ordering is not unique. In time, more ways to assess statistical depth were discovered and eventually Zuo and Serfling [35] proposed a list of desirable properties for a statistical depth function. Depth functions in the literature often fail to satisfy all those properties perfectly. The dominant view is that this does not automatically disqualify a candidate depth function but it surely points out a weakness that should be taken into account in a practical context (see Remark 4.11 in this regard). Therefore, understanding the theoretical properties of each depth function is an important step to make an informed choice between them.

In [13], we proposed a defining list of desirable properties for statistical depth in the fuzzy case. Unlike with multivariate data, many different distances between fuzzy sets are available. Thus we suggested a definition of depth which only depends on the algebraic operations between fuzzy sets (semilinear depth functions) as well as a metric dependent definition (geometric depth functions) and studied the relationships between them. While there are approaches to depth in abstract metric spaces [4,7,24], our definition (see Properties P1–P4b below) was conceived with the specificities of fuzzy data in mind, and in particular it would make sense for (crisp) set-valued data as well. In connection to this, statistical depth functions for either set-valued or fuzzy data were also independently proposed by Cascos et al. [3] and Sinova [27]. This paper is part of an ongoing program to develop depth-based methods specifically designed for fuzzy data. As it happens that, although in theory, multivariate depths could be applied to many fuzzy settings (for instance, trapezoidal fuzzy sets can be seen as elements of $\mathbb{R}^4$), it happens in practice that they do not provide a meaningful order (see Section 5). In [13], besides proposing an abstract list of desirable properties, we studied a generalization of the Tukey depth to the fuzzy setting and showed that it fulfills all the proposed properties. Additionally, in [14] we studied several ways to adapt Liu's simplicial depth [19] and also their properties. Next it becomes necessary to establish whether some other popular and relevant statistical depth functions also admit adaptations and whether their properties are preserved in this more general setting. Once a library of depth functions becomes available, comparing their performance for specific purposes will be possible. In this paper, the projection and L^r-depth functions, initially defined in $\mathbb{R}^p$, are generalized to the fuzzy setting. We have selected them because they do not vanish outside the convex hull of the sample, as it happens with the previously studied Tukey and simplicial depths. Note that such vanishment is problematic for certain applications, such as clustering.

The projection depth [35, Example 2.4] of a point $x \in \mathbb{R}^p$ with respect to the distribution of a random vector X considers the projections of x in every direction and compares them with the univariate median of the corresponding projection of the distribution. In that sense, it measures the worst case of outlyingness of x with respect to the median of the distribution in any direction. It is formally defined as

$$PD(x;X) := \left(1 + O(x;X)\right)^{-1},$$

with

$$O(x;X) := \sup_{u \in \mathbb{S}^{p-1}} \frac{\left|\langle x, u\rangle - \operatorname{med}\left(\langle x, X\rangle\right)\right|}{\operatorname{MAD}\left(\langle u, X\rangle\right)}. \tag{1}$$

In (1), $\langle \cdot, \cdot \rangle$ denotes the usual inner product in $\mathbb{R}^p$, and $\mathbb{S}^{p-1} := \{x \in \mathbb{R}^p : \|x\| \leq 1\}$ the unit sphere, with $\|.\|$ the Euclidean norm on $\mathbb{R}^p$. Moreover, $\operatorname{med}(Y)$ and $\operatorname{MAD}(Y)$ denote the median and the median absolute deviation of a real random variable Y. Notice the set of all medians will be denoted by $\operatorname{Med}(Y)$ and the usual convention of defining $\operatorname{med}(Y)$ to be the midpoint of $\operatorname{Med}(Y)$ applies. The function O, which measures the outlyingness of a point with respect to the median, is widely considered in the literature. For instance, in the univariate case it appears in [23] and its multivariate version was introduced in [9]. In this paper, after providing the necessary notation and basic results on fuzzy sets, fuzzy random variables and statistical depth in Section 2, we generalize the concept of the projection depth to the fuzzy setting in Section 3. We do so by substituting the inner products in (1) by the support function for every direction $u \in \mathbb{S}^{p-1}$ and every level $\alpha \in [0,1]$. There, we also demonstrate that our proposal has extremely good properties, proving that the fuzzy projection depth is the second existing instance to satisfy simultaneously the semilinear and geometric depth notions [13]. Besides that, we prove that the fuzzy projection depth is actually a generalization of the, multivariate, projection depth; as they coincide when applied to the indicator function of a crisp vector and, respectively, the vector itself.

The *L^r-depth* [35, Example 2.3] of $x \in \mathbb{R}^p$ with respect to the distribution of a random vector X is

$$L^r D(x;X) := \left(1 + \mathrm{E}[\|x - X\|_r]\right)^{-1}, \tag{2}$$

where $\mathrm{E}[\cdot]$ denotes the expected value and $\|\cdot\|_r$ is the r-norm in $\mathbb{R}^p$ (the same notation will be used for the L^r-norm in function spaces). The structure is similar to that of the projection depth, but now the function $\mathrm{E}[\|\cdot - X\|_r]$ measures the distance from a point to the distribution. In Section 4, we generalize the concept of L^r-type depth to the fuzzy setting. We do so in four different manners, two of them use the ρ_r metrics to generalize the multivariate $\|\cdot\|_r$ norms. For the other two, we define a family of metrics in the fuzzy space using the (mid, spr) decomposition of the support function. There, we prove that under certain scenarios the proposed fuzzy depth functions fulfill the semilinear and geometric notions.

An example of real fuzzy data is analyzed in Section 5. There, we show the good empirical behavior of all the proposed fuzzy depths. As the analyzed real dataset consists of trapezoidal fuzzy sets, which are identifiable with elements of $\mathbb{R}^4$, we also compare the proposed fuzzy depths with a multivariate depth. We show the empirical superiority on fuzzy data of fuzzy depths by making use of the projection depth. The selection leans on the extremely good theoretical properties obtained by the proposed fuzzy projection depth and L^r-type depths. Furthermore, we make use of our depth proposals in two depth based classification procedures, analyzing another real-world fuzzy dataset consisting of trapezoidal fuzzy sets.

Some final remarks close the paper in Section 6. All proofs are deferred to Appendix A.

2. Notation and preliminaries

2.1. Fuzzy sets

A function $A : \mathbb{R}^p \to [0,1]$ is called a *fuzzy set* on $\mathbb{R}^p$. Let $\alpha \in (0,1]$, the α-*level* of a fuzzy set A is defined to be $A_\alpha := \{x \in \mathbb{R}^p : A(x) \geq \alpha\}$ and $A_0 = \mathrm{clo}(\{x \in \mathbb{R}^p : A(x) > 0\})$, where $\mathrm{clo}(\cdot)$ denotes the closure of a set. By $\mathcal{F}_c(\mathbb{R}^p)$ we denote the set of all fuzzy sets A on $\mathbb{R}^p$ whose α-level is a non-empty compact and convex set for each $\alpha \in [0,1]$ For simplicity, we will just refer to the elements of $\mathcal{F}_c(\mathbb{R}^p)$ as fuzzy sets, although a general fuzzy set may not be in $\mathcal{F}_c(\mathbb{R}^p)$.

Let $\mathcal{K}_c(\mathbb{R}^p)$ denote the class of all non-empty compact and convex subsets of $\mathbb{R}^p$. Any set $K \in \mathcal{K}_c(\mathbb{R}^p)$ can be identified with a fuzzy set via its indicator function $\mathrm{I}_K : \mathbb{R}^p \to [0,1]$, where $\mathrm{I}_K(x) = 1$ if $x \in K$ and $\mathrm{I}_K(x) = 0$ otherwise. For any $K \in \mathcal{K}_c(\mathbb{R}^p)$, define $\|K\| = \max_{x\in K} \|x\|$.

The *support function* of a fuzzy set A is the mapping $s_A : \mathbb{S}^{p-1} \times [0,1] \to \mathbb{R}$ defined by $s_A(u,\alpha) := \sup_{v\in A_\alpha} \langle u, v\rangle$, for every $u \in \mathbb{S}^{p-1}$ and $\alpha \in [0,1]$.

In $\mathcal{F}_c(\mathbb{R})$ it is common to use *trapezoidal fuzzy numbers* (e.g., [17, Section 10.7]). For any real numbers $a \leq b \leq c \leq d$, the fuzzy set given by

$$\mathrm{Tra}(a,b,c,d)(x) := \begin{cases} \dfrac{x-a}{b-a} & \text{if } x \in [a,b), \\ 1 & \text{if } x \in [b,c], \\ \dfrac{x-c}{d-c} & \text{if } x \in (c,d], \\ 0 & \text{otherwise} \end{cases} \tag{3}$$

is called a trapezoidal fuzzy number.

2.2. Arithmetics and Zadeh's extension principle

Let $A, B \in \mathcal{F}_c(\mathbb{R}^p)$ and $\gamma \in \mathbb{R}$. According to [32], the operations *sum* and *product by a scalar* are defined by

$$(A+B)(t) := \sup_{x,y\in\mathbb{R}^p :\ x+y=t} \min\{A(x), B(y)\}, \text{ with } t \in \mathbb{R}^p,$$

$$(\gamma \cdot A)(t) := \sup_{x\in\mathbb{R}^p :\ t=\gamma\cdot x} A(x) = \begin{cases} A\left(\frac{t}{\gamma}\right), & \text{if } \gamma \neq 0 \\ I_{\{0\}}(t) & \text{if } \gamma = 0 \end{cases}, \text{ with } t \in \mathbb{R}^p.$$

Given $A, B \in \mathcal{F}_c(\mathbb{R}^p)$, $\gamma \in [0,\infty)$, $u \in \mathbb{S}^{p-1}$ and $\alpha \in [0,1]$, a useful relationship between the support function and these operations is the formula

$$s_{A+\gamma\cdot B}(u,\alpha) = s_A(u,\alpha) + \gamma \cdot s_B(u,\alpha). \tag{4}$$

The (mid / spr)-decomposition is a commonly used tool to deal with support functions of fuzzy sets. Given $A \in \mathcal{F}_c(\mathbb{R}^p)$ and s_A the support function of A, it can be expressed as

$$s_A(u,\alpha) = \mathrm{mid}(s_A)(u,\alpha) + \mathrm{spr}(s_A)(u,\alpha), \tag{5}$$

where, for all $u \in \mathbb{S}^{p-1}$ and $\alpha \in [0,1]$,

$$\begin{aligned} \mathrm{mid}(s_A)(u,\alpha) &:= (s_A(u,\alpha) - s_A(-u,\alpha))/2, \\ \mathrm{spr}(s_A)(u,\alpha) &:= (s_A(u,\alpha) + s_A(-u,\alpha))/2. \end{aligned} \tag{6}$$

A function $f : \mathcal{F}_c(\mathbb{R}^p) \to \mathbb{R}$ is *convex* if

$$f(\lambda \cdot A + (1-\lambda)\cdot B) \leq \lambda \cdot f(A) + (1-\lambda)\cdot f(B)$$

for all $\lambda \in [0,1]$ and $A, B \in \mathcal{F}_c(\mathbb{R}^p)$.

Zadeh's extension principle [33] allows to apply a crisp function $f : \mathbb{R}^p \to \mathbb{R}^p$ to a fuzzy set $A \in \mathcal{F}_c(\mathbb{R}^p)$, obtaining a new fuzzy set $f(A) \in \mathcal{F}_c(\mathbb{R}^p)$ with

$$f(A)(t) := \sup\{A(y) : y \in \mathbb{R}^p, f(y) = t\}$$

for all $t \in \mathbb{R}^p$.

Let $M \in \mathcal{M}_{p\times p}(\mathbb{R})$ be a regular matrix, $A \in \mathcal{F}_c(\mathbb{R}^p)$ a fuzzy set and let $f : \mathbb{R}^p \to \mathbb{R}^p$ be the function given by $f(x) = M \cdot x$. The application of Zadeh's extension principle results in the fuzzy set $f(A) = M \cdot A$ defined as

$$(M \cdot A)(t) = \sup\{A(y) : y \in \mathbb{R}^p, M \cdot y = t\}$$

From [13, Proposition 7.2],

$$s_{M\cdot A}(u,\alpha) = \left\| M^T \cdot u \right\| \cdot s_A\left(\frac{1}{\left\| M^T \cdot u \right\|} \cdot M^T \cdot u, \alpha \right) \tag{7}$$

for any $A \in \mathcal{F}_c(\mathbb{R}^p)$, $M \in \mathcal{M}_{p\times p}(\mathbb{R})$ a regular matrix, $u \in \mathbb{S}^{p-1}$ and $\alpha \in [0,1]$.

2.3. Metrics between fuzzy sets

Given fuzzy sets $A, B \in \mathcal{F}_c(\mathbb{R}^p)$, define

$$d_r(A,B) := \begin{cases} \left(\int_{[0,1]} \left(d_H(A_\alpha, B_\alpha) \right)^r \mathrm{d}\nu(\alpha) \right)^{1/r}, & r \in [1,\infty) \\ \sup_{\alpha\in[0,1]} d_H(A_\alpha, B_\alpha), & r = \infty, \end{cases} \tag{8}$$

where

$$d_H(S,T) := \max\left\{ \sup_{s\in S} \inf_{t\in T} \parallel s - t \parallel, \sup_{t\in T} \inf_{s\in S} \parallel s - t \parallel \right\}$$

is the *Hausdorff metric* between elements of $\mathcal{K}_c(\mathbb{R}^p)$ and ν denotes the Lebesgue measure over $[0,1]$. The metric space $(\mathcal{F}_c(\mathbb{R}^p), d_r)$ is separable and non-complete for any $r \in (1,\infty)$, while the metric space $(\mathcal{F}_c(\mathbb{R}^p), d_\infty)$ is non-separable and complete [8].

L^r-type metrics can be considered using the support function [8]. Given $A, B \in \mathcal{F}_c(\mathbb{R}^p)$ and $r \geq 1$,

$$\rho_r(A,B) := \left(\int_{[0,1]} \int_{\mathbb{S}^{p-1}} |s_A(u,\alpha) - s_B(u,\alpha)|^r \, \mathrm{d}\mathcal{V}_p(u) \, \mathrm{d}\nu(\alpha) \right)^{1/r}, \tag{9}$$

where $\mathcal{V}_p$ denotes the normalized Haar measure in $\mathbb{S}^{p-1}$.

2.4. Fuzzy random variables

Let $(\Omega, \mathcal{A})$ be a measurable space. A function $\Gamma : \Omega \to \mathcal{K}_c(\mathbb{R}^p)$ is a *random compact set* [22] if $\{\omega \in \Omega : \Gamma(\omega) \cap K \neq \emptyset\} \in \mathcal{A}$ for all $K \in \mathcal{K}_c(\mathbb{R}^p)$, or equivalently if Γ is Borel measurable with respect to the Hausdorff metric. According to [25], a function $\mathcal{X} : \Omega \to \mathcal{F}_c(\mathbb{R}^p)$ is called a *fuzzy random variable* if the α-level $\mathcal{X}_\alpha(\omega)$ is a random compact set for all $\alpha \in [0,1]$ where $\mathcal{X}_\alpha : \Omega \to \mathcal{K}_c(\mathbb{R}^p)$ is defined as $\mathcal{X}_\alpha(\omega) := \{x \in \mathbb{R}^p : \mathcal{X}(\omega)(x) \geq \alpha\}$ for any $\omega \in \Omega$.

Let us denote by $L^0[\mathcal{F}_c(\mathbb{R}^p)]$ the class of all fuzzy random variables on $(\Omega, \mathcal{A})$. For any $r \in [1,\infty)$, we denote by $L^r[\mathcal{F}_c(\mathbb{R}^p)]$ the subset of fuzzy random variables in $L^0[\mathcal{F}_c(\mathbb{R}^p)]$ such that $E[\|\mathcal{X}_0\|^r] < \infty$. Fuzzy random variables in $L^1[\mathcal{F}_c(\mathbb{R}^p)]$ are called *integrably bounded*.

The support function of a fuzzy random variable $\mathcal{X}$ is the function $s_{\mathcal{X}} : \mathbb{S}^{p-1} \times [0,1] \times \Omega \to \mathbb{R}$ with $s_{\mathcal{X}}(u,\alpha,\omega) := s_{\mathcal{X}(\omega)}(u,\alpha)$ for all $u \in \mathbb{S}^{p-1}, \alpha \in [0,1]$ and $\omega \in \Omega$. Throughout the paper, the probability space associated with a fuzzy random variable is denoted by $(\Omega, \mathcal{A}, \mathbb{P})$.

2.5. Symmetry and depth: semilinear and geometric notions

In [13], we proposed two notions of symmetry in the fuzzy setting, the F-symmetry notion, based in the support function, and the $(\mathrm{mid}, \mathrm{spr})$-notion, based on the $(\mathrm{mid}, \mathrm{spr})$-decomposition. Given a fuzzy random variable $\mathcal{X} : \Omega \to \mathcal{F}_c(\mathbb{R}^p)$ and a fuzzy set $A \in \mathcal{F}_c(\mathbb{R}^p)$,

- $\mathcal{X}$ is *F-symmetric* with respect to A if

  $$s_A(u,\alpha) - s_{\mathcal{X}}(u,\alpha) =^d s_{\mathcal{X}}(u,\alpha) - s_A(u,\alpha),$$

  for all $(u,\alpha) \in \mathbb{S}^{p-1} \times [0,1]$, where $=^d$ represents being equal in distribution.

- $\mathcal{X}$ is said to be (mid, spr)-*symmetric* with respect to A if

$$\mathrm{mid}(s_A(u,\alpha)) - \mathrm{mid}(s_{\mathcal{X}}(u,\alpha)) =^d \mathrm{mid}(s_{\mathcal{X}}(u,\alpha)) - \mathrm{mid}(s_A(u,\alpha)) \text{ and}$$
$$\mathrm{spr}(s_A(u,\alpha)) - \mathrm{spr}(s_{\mathcal{X}}(u,\alpha)) =^d \mathrm{spr}(s_{\mathcal{X}}(u,\alpha)) - \mathrm{spr}(s_A(u,\alpha)),$$

  for all $(u,\alpha) \in \mathbb{S}^{p-1} \times [0,1]$.

There it is also proved that, for all $u \in \mathbb{S}^{p-1}$ and $\alpha \in [0,1]$,

$$s_A(u,\alpha) \in \mathrm{Med}(s_{\mathcal{X}}(u,\alpha)) \text{ if } \mathcal{X} \text{ is } F\text{-symmetric with respect to } A \tag{10}$$

and

$$\mathrm{mid}(s_A)(u,\alpha) \in \mathrm{Med}(\mathrm{mid}(s_{\mathcal{X}})(u,\alpha)) \text{ and } \mathrm{spr}(s_A)(u,\alpha) \in \mathrm{Med}(\mathrm{spr}(s_{\mathcal{X}})(u,\alpha)) \tag{11}$$

if $\mathcal{X}$ is (mid, spr)-symmetric with respect to A.

In [13], we introduced the following two abstract definitions of a statistical depth function for fuzzy data. Let us consider $\mathcal{H} \subseteq L^0[\mathcal{F}_c(\mathbb{R}^p)]$ and $\mathcal{J} \subseteq \mathcal{F}_c(\mathbb{R}^p)$, with $\mathcal{H}$ and $\mathcal{J}$ being non-empty, and a mapping $D(\cdot;\cdot) : \mathcal{J} \times \mathcal{H} \to [0,\infty)$. Let $A \in \mathcal{J}$ be such that $D(A;\mathcal{X}) = \sup\{D(B;\mathcal{X}) : B \in \mathcal{J}\}$ and let $d : \mathcal{F}_c(\mathbb{R}^p) \times \mathcal{F}_c(\mathbb{R}^p) \to [0,\infty)$ be a metric. Consider the following properties, which are required to hold for any such A.

P1. $D(M \cdot U + V; M \cdot \mathcal{X} + V) = D(U;\mathcal{X})$ for any regular matrix $M \in \mathcal{M}_{p\times p}(\mathbb{R})$, any $U, V \in \mathcal{J}$ and any $\mathcal{X} \in \mathcal{H}$.
P2. For any symmetric fuzzy random variable $\mathcal{X} \in \mathcal{H}$ (for some notion of symmetry), $D(U;\mathcal{X}) = \sup_{B\in\mathcal{F}_c(\mathbb{R}^p)} D(B;\mathcal{X})$, where $U \in \mathcal{J}$ is a center of symmetry of $\mathcal{X}$.
P3a. $D(A;\mathcal{X}) \geq D((1-\lambda)\cdot A + \lambda \cdot U;\mathcal{X}) \geq D(U;\mathcal{X})$ for all $\lambda \in [0,1]$ and all $U \in \mathcal{F}_c(\mathbb{R}^p)$.
P3b. $D(A;\mathcal{X}) \geq D(U;\mathcal{X}) \geq D(V;\mathcal{X})$ for all $B, C \in \mathcal{J}$ satisfying $d(A,V) = d(A,U) + d(U,V)$.
P4a. $\lim_{\lambda\to\infty} D(A + \lambda \cdot U;\mathcal{X}) = 0$ for all $U \in \mathcal{J} \setminus \{\mathrm{I}_{\{0\}}\}$.
P4b. $\lim_{n\to\infty} D(A_n;\mathcal{X}) = 0$ for every sequence $\{A_n\}_n$, with $A_n \in \mathcal{J}$ for all $n \in \mathbb{N}$, such that $d(A_n, A) \to \infty$.

These properties adapt to the specificities of fuzzy data the defining properties of a statistical depth function in multivariate analysis [35]. As defined in [13], D is a *semilinear depth function* if it satisfies P1, P2, P3a and P4a. It is a *geometric depth function* with respect to a metric d if it satisfies P1, P2, P3b and P4b for that metric.

2.6. Banach spaces

A *Banach space* is a real normed space $(\mathbb{E}, \|\cdot\|)$ whose induced metric is complete.

Definition 2.1 *([11])*. Let $(\mathbb{E}, \|\cdot\|)$ be a Banach space. It is said to be *strictly convex* if $x = y$ whenever $\|(1/2)\cdot(x+y)\| = \|x\| = \|y\|$ for every $x, y \in \mathbb{E}$.

The property of strict convexity in Banach spaces plays a crucial role in the relation between properties P3a and P3b of the semilinear and geometric depth notions, which is due to [13, Theorem 5.4]. In this work, we make use of strict convexity in proving, in Sections 3 and 4, property P3b for the projection and the L^r-type fuzzy depths.

The Cartesian product $\mathbb{E} \times \mathbb{F}$ of two Banach spaces $(\mathbb{E}, \|\cdot\|_{\mathbb{E}})$ and $(\mathbb{F}, \|\cdot\|_{\mathbb{F}})$ can be endowed with an r-norm

$$\|(x,y)\|_r = (\|x\|_{\mathbb{E}}^r + \|y\|_{\mathbb{F}}^r)^{1/r}.$$

The resulting Banach space is denoted by $\mathbb{E} \oplus_r \mathbb{F}$.

3. Projection depth and its properties

3.1. Definition

In this section, we introduce a statistical depth function inspired by the multivariate projection depth. We extend the notion of projection depth by replacing in (1) the product functionals $\langle u, \cdot\rangle$ by the support functionals $s_{\cdot}(u,\alpha)$. A rationale for this adaptation is given in [13, Section 6]. We introduce the concept of projection depth into the fuzzy framework for three different reasons. Firstly, our adaptation aims at creating a library of depth functions, specifically tailored for fuzzy sets and for subsequent applications to fuzzy real data. The need of this will be seen later in Section 5, where we show that multivariate depth functions do not have a good performance when applied to fuzzy sets. Secondly, in both [13] and [14], we adapt the halfspace and the simplicial depth to the fuzzy setting. Both depth measures face the issue of vanishing outside the convex hull of the sample and we aim in this paper at depth functions that behave well in this matter. The projection depth does not vanish at any point of the space, an important property for the purpose of ordering elements according to their depth value. Finally, the projection depth has a robust construction, a property expected from depth functions.

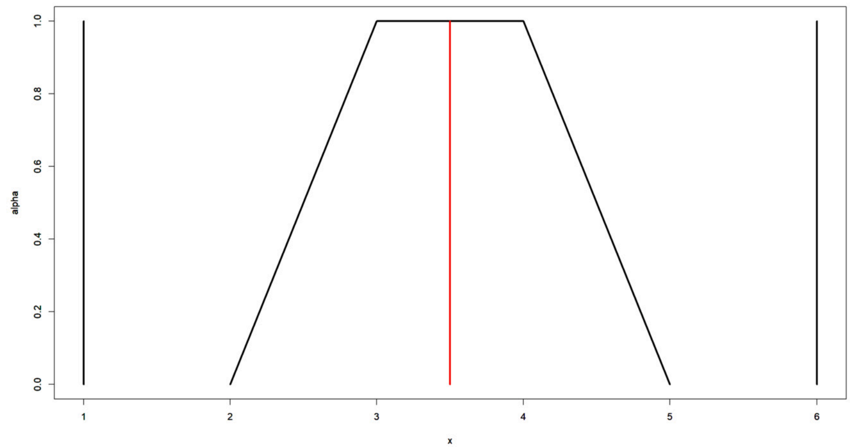

Fig. 1. Representation in black of the sets A (left), B (right) and C (middle) and in red the median of the fuzzy random variable $\mathcal{X}$. (For interpretation of the colors in the figure(s), the reader is referred to the web version of this article.)

Definition 3.1. The *projection depth* based on $\mathcal{J} \subseteq \mathcal{F}_c(\mathbb{R}^p)$ and $\mathcal{H} \subseteq L^0[\mathcal{F}_c(\mathbb{R}^p)]$, with $\mathcal{H}$ and $\mathcal{J}$ being non-empty and the subset $\mathcal{H}$ not containing degenerate fuzzy random variables, of a fuzzy set $A \in \mathcal{J}$ with respect to a fuzzy random variable $\mathcal{X} \in \mathcal{H}$ is

$$D_{FP}(A;\mathcal{X}) := \left(1 + O(A;\mathcal{X})\right)^{-1},$$

where

$$O(A;\mathcal{X}) := \sup_{u\in\mathbb{S}^{p-1},\alpha\in[0,1]} \frac{|s_A(u,\alpha) - \mathrm{Med}(s_{\mathcal{X}}(u,\alpha))|}{\mathrm{MAD}(s_{\mathcal{X}}(u,\alpha))}. \tag{12}$$

The usual convention of taking the mid-point of the interval of medians when the median is not unique is adopted, both in the numerator and the denominator.

The hypothesis that $\mathcal{H}$ does not contain degenerate fuzzy random variables is a technical hypothesis implemented to prevent null denominators in the definition of the O function. Going back to the multivariate function O, commented in Section 1, we have that it measures the worst case of outlyingness of a point with respect to the univariate median for every direction $u \in \mathbb{S}^{p-1}$. Here we follow the same idea, measuring the worst case of outlyingness over $\mathbb{S}^{p-1}$ and $[0,1]$ of the support function of a fuzzy set with respect the univariate median of the support function of the fuzzy random variable.

In the following examples we show the performance of the proposed fuzzy projection depth and in what sense it measures the outlyingness of the support function of fuzzy sets.

Example 3.2. Let $(\{\omega_1,\omega_2\},\mathcal{P}(\{\omega_1,\omega_2\}),\mathbb{P})$ be a probability space such that $\mathbb{P}(\{\omega_1\}) = \mathbb{P}(\{\omega_2\})$ and $\mathcal{X} : \{\omega_1,\omega_2\} \to \mathcal{F}_c(\mathbb{R})$ a fuzzy random variable such that $A := \mathcal{X}(\omega_1) = \mathrm{I}_{\{1\}}$ and $B := \mathcal{X}(\omega_2) = \mathrm{I}_{\{6\}}$. Let us consider the trapezoidal fuzzy set $C = \mathrm{Tra}(2,3,4,5)$. These three sets are represented in black in Fig. 1.

To compute the fuzzy projection depth of C with respect to $\mathcal{X}$ we have to compute the support functions of the above fuzzy sets. It is easy to see that, for every $u \in \mathbb{S}^0$ and $\alpha \in [0,1]$, their support functions are $s_A(u,\alpha) = u$, $s_B(u,\alpha) = 6u$ and

$$s_C(u,\alpha) = \begin{cases} 5-\alpha, & \text{if } u = 1 \\ -2-\alpha, & \text{if } u = -1. \end{cases}$$

The median and the median absolute deviation of the support function of the fuzzy random variable are respectively $\mathrm{Med}(s_{\mathcal{X}}(u,\alpha)) = 7u/2$ and $\mathrm{MAD}(s_{\mathcal{X}}(u,\alpha)) = 5/2$.

Let us compute the value $O(C;\mathcal{X})$. According to [28], the 1-median of the fuzzy random variable $\mathcal{X}$ is $\mathrm{I}_{\{7/2\}}$ (see the red line in Fig. 1). As the function O measures the worst case of outlyingness of the support function, we have that $O(C;\mathcal{X})$ takes the supremum at $\alpha = 0$, as these are the farthest values of the support function of C to the support function of the 1-median of the fuzzy random variable. The value between $u = 1$ and $u = -1$ is the same because the fuzzy set C is symmetric with respect to line $x = 7/2$.

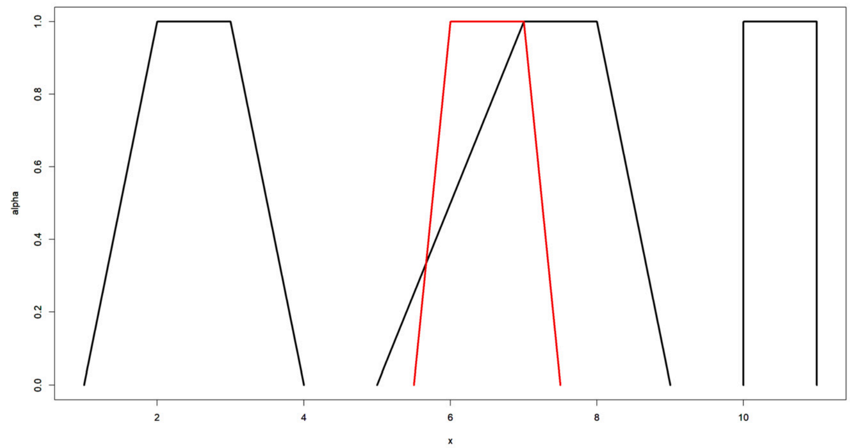

Fig. 2. Representation of Example in black of the sets A (left), B (right) and C (middle) and in red the median of the fuzzy random variable $\mathcal{X}$.

$$
\begin{aligned}
O(C;\mathcal{X}) &= \sup_{\alpha\in[0,1]} \max\left\{\frac{|5-\alpha-7/2|}{5/2}, \frac{|-2-\alpha+7/2|}{5/2}\right\}\\
&= \sup_{\alpha\in[0,1]} \max\left\{\frac{3-2\alpha}{5}, \frac{3-2\alpha}{5}\right\}\\
&= \sup_{\alpha\in[0,1]} \frac{3-2\alpha}{5} = \frac{3}{5}.
\end{aligned}
$$

Finally, the projection depth of C with respect to $\mathcal{X}$ is

$$
D_{FP}(C;\mathcal{X}) = (1+O(C;\mathcal{X}))^{-1} = (1+3/5)^{-1} = 5/8.
$$

In the example above, we have computed the projection depth of a symmetric fuzzy set with respect to a fuzzy random variable that takes values on crisp sets. In the following example, we compute the projection depth of a more complex scenario.

Example 3.3. Let $(\{\omega_1,\omega_2\}, \mathcal{P}(\{\omega_1,\omega_2\}), \mathbb{P})$ be a probability space such that $\mathbb{P}(\{\omega_1\}) = \mathbb{P}(\{\omega_2\})$ and $\mathcal{X} : \{\omega_1,\omega_2\} \to \mathcal{F}_c(\mathbb{R})$ a fuzzy random variable with $A := \mathcal{X}(\omega_1) = \mathrm{Tra}(1,2,3,4)$ and $B := \mathcal{X}(\omega_2) = \mathrm{I}_{[10,11]}$. Let us consider the trapezoidal fuzzy set $C = \mathrm{Tra}(5,7,8,9)$. We have displayed these fuzzy sets in black in Fig. 2.

To obtain the projection depth of C with respect to $\mathcal{X}$, we compute the support functions of the above fuzzy sets. It is easy to see that, for every $u \in \mathbb{S}^0$ and $\alpha \in [0,1]$, they are

$$
s_A(u,\alpha) = \begin{cases} 4-\alpha, & \text{if } u = 1\\ -1-\alpha, & \text{if } u = -1, \end{cases}
$$

$$
s_B(u,\alpha) = \begin{cases} 11, & \text{if } u = 1\\ -10, & \text{if } u = -1 \end{cases}
$$

and

$$
s_C(u,\alpha) = \begin{cases} 9-\alpha, & \text{if } u = 1\\ -5-2\alpha, & \text{if } u = -1. \end{cases}
$$

The median and median absolute deviation of the support function of the fuzzy random variable are, respectively,

$$
\mathrm{Med}(s_{\mathcal{X}}(u,\alpha)) = \begin{cases} \dfrac{15-\alpha}{2}, & \text{if } u = 1\\ -\dfrac{11+\alpha}{2}, & \text{if } u = -1 \end{cases}
$$

and

$$\mathrm{MAD}(s_{\mathcal{X}}(u,\alpha)) = \begin{cases} \dfrac{14+\alpha}{2}, & \text{if } u=1 \\ \dfrac{9-\alpha}{2}, & \text{if } u=-1. \end{cases}$$

Now, we compute the value $O(C;\mathcal{X})$. According to the computation of the support function of the median of the fuzzy random variable $\mathcal{X}$ and the definition of 1-median in [28], we have that the 1-median of $\mathcal{X}$ is the trapezoidal fuzzy set $\mathrm{Med}(\mathcal{X}) = \mathrm{Tra}(11/2, 6, 7, 15/2)$ (see the red trapezoid in Fig. 2). Thus, the case of $u=1$, the worst case of outlyingness between the fuzzy set C and the 1-median of $\mathcal{X}$ happens for $\alpha=0$. On the other hand, for the case of $u=-1$, the worst case happens for $\alpha=1$.

$$\begin{aligned} O(C;\mathcal{X}) &= \sup_{u\in\mathbb{S}^0,\alpha\in[0,1]} \frac{|s_C(u,\alpha)-\mathrm{Med}(s_{\mathcal{X}}(u,\alpha))|}{\mathrm{MAD}(s_{\mathcal{X}}(u,\alpha))} \\ &= \sup_{u\in\mathbb{S}^0,\alpha\in[0,1]} \max\left\{\frac{|(3-\alpha)/2|}{(14+\alpha)/2}, \frac{|(1-3\alpha)/2|}{(9-\alpha)/2}\right\} \\ &= \sup_{u\in\mathbb{S}^0,\alpha\in[0,1]} \max\left\{\frac{3-\alpha}{14+\alpha}, \frac{|1-3\alpha|}{9-\alpha}\right\} \\ &= \max\left\{\frac{3}{14}, \frac{1}{4}\right\} = \frac{1}{4} \end{aligned}$$

Finally, the projection depth of C with respect to $\mathcal{X}$ is

$$D_{FP}(C;\mathcal{X}) = (1+O(C;\mathcal{X}))^{-1} = (1+1/4)^{-1} = 4/5.$$

In what follows, we consider the particular case of the function D_{FP} based on

$$\mathcal{J} = \left\{ \mathrm{I}_{\{x\}} \in \mathcal{F}_c(\mathbb{R}^p) : x \in \mathbb{R}^p \right\},$$

showing D_{FP} generalizes the multivariate projection depth. This happens since the subset $\mathcal{J} \subset \mathcal{F}_c(\mathbb{R}^p)$, equipped with the fuzzy operations, exhibits a behavior analogous to $\mathbb{R}^p$. Particularly, the result states that the projection depth with respect to any fuzzy random variable with images in $\mathcal{J}$ coincides with the multivariate projection depth. Thus, in this context, our proposal is a generalization of the multivariate projection depth within the fuzzy framework.

Proposition 3.4. *Let $\mathcal{J} = \{I_{\{x\}} : x \in \mathbb{R}^p\}$. For any random vector X on $\mathbb{R}^p$ and any $x \in \mathcal{J}$,*

$$D_{FP}\left(I_{\{x\}}; I_{\{X\}}\right) = PD(x;X).$$

The proof follows directly from the fact that $s_A(u,\alpha) = \langle u, x\rangle$ for any $A = \mathrm{I}_{\{x\}}$, $u \in \mathbb{S}^{p-1}$ and $\alpha \in [0,1]$.

3.2. Properties

We will now show that projection depth, like Tukey depth [13], is both a semilinear depth function and a geometric depth function.

Theorem 3.5. *D_{FP} satisfies properties P1, P2 with F-symmetry, P3a and P4a. Moreover, it satisfies P3b for ρ_r if $r \in (1,\infty)$ and P4b for ρ_r if $r \in [1,\infty)$ and d_r if $r \in [1,\infty]$.*

Corollary 3.6. *When using the F-symmetry notion, D_{FP} is a semilinear depth function and a geometric depth function for the ρ_r distance for any $r \in (1,\infty)$.*

The next result shows that D_{FP} is not a geometric depth function for the d_r metrics. Using [13, Example 5.6], it is proved by counterexample that D_{FP} violates property P3b for some metrics.

Proposition 3.7. *D_{FP} is not a geometric depth function for the d_r-distance for any $r \in [1,\infty]$.*

4. L^r-type depths and their properties

4.1. Definitions

We present several approaches to statistical depth for fuzzy data inspired by multivariate L^r-depth. As it is apparent from (2), a distance between fuzzy data is required. A natural L^r-type distance is the ρ_r metric defined above. We introduce the concept of L^r-type depths into the fuzzy setting for the same first two reasons we explained for the projection depth, in previous section. Firstly, we are motivated to create a library of depth functions for fuzzy sets, as multivariate depths do not behave well in this setting. Secondly, L^r-type depth functions are based on metrics, thus any element of the space has null depth.

Definition 4.1. For any $r \in [1, \infty)$, the *r-natural depth* based on $\mathcal{J} \subseteq \mathcal{F}_c(\mathbb{R}^p)$ and $\mathcal{H} \subseteq L^1[\mathcal{F}_c(\mathbb{R}^p)]$, with $\mathcal{H}$ and $\mathcal{J}$ being non-empty, of a fuzzy set $A \in \mathcal{J}$ with respect to a fuzzy random variable $\mathcal{X} \in \mathcal{H}$ is

$$D_r(A; \mathcal{X}) := \left(1 + \mathrm{E}[\rho_r(A, \mathcal{X})]\right)^{-1}.$$

The reason to consider $\mathcal{H} \subseteq L^1[\mathcal{F}_c(\mathbb{R}^p)]$ is to avoid having an infinite expectation in the definition. While it is possible to define D_r as being identically zero in that case (see [13, Example 5.9]), a null depth function is not desirable in practice, e.g., in classification problems.

Definition 4.2. For any $r \in [1, \infty)$, the *r-natural raised depth* based on $\mathcal{J} \subseteq \mathcal{F}_c(\mathbb{R}^p)$ and $\mathcal{H} \subseteq L^r[\mathcal{F}_c(\mathbb{R}^p)]$, with $\mathcal{H}$ and $\mathcal{J}$ being non-empty, of a fuzzy set $A \in \mathcal{J}$ with respect to a random variable $\mathcal{X} \in \mathcal{H}$ is

$$RD_r(A; \mathcal{X}) := \left(1 + E[\rho_r(A, \mathcal{X})^r]\right)^{-1}.$$

In the context of the r-natural raised depth, we consider the subsets $\mathcal{H} \subseteq L^r[\mathcal{F}_c(\mathbb{R}^p)]$ to prevent the denominator from having an infinite expectation.

Another possibility is to define an L^r-type depth by using the mid and spr functions, through which the location and the shape of the fuzzy sets are described. With that aim, denoting by $\|\cdot\|_r$ the norm of the Banach space $L^r\left(\mathbb{S}^{p-1} \times [0,1], \mathcal{V}_p \otimes \nu\right)$, we define

$$d_{r,\theta}(A, B) := \left[\|\mathrm{mid}(s_A) - \mathrm{mid}(s_B)\|_r^r + \theta \cdot \|\mathrm{spr}(s_A) - \mathrm{spr}(s_B)\|_r^r\right]^{1/r} \tag{13}$$

for any $A, B \in \mathcal{F}_c(\mathbb{R}^p)$, $r \in [1, \infty)$ and $\theta \in [0, \infty)$. This is a straightforward generalization of the distance $d_{2,\theta}$ in [30]. For $\theta > 0$, $d_{r,\theta}$ is a metric, as it identifies isometrically each $A \in \mathcal{F}_c(\mathbb{R}^p)$ with the element $(\mathrm{mid}(s_A), \mathrm{spr}(s_A))$ of the Banach space

$$L^r\left(\mathbb{S}^{p-1} \times [0,1], \mathcal{V}_p \otimes \nu\right) \oplus_r L^r\left(\mathbb{S}^{p-1} \times [0,1], \theta^{1/r} \cdot (\mathcal{V}_p \otimes \nu)\right).$$

In the case $\theta = 0$ it depends only on mid and it is just a pseudometric. We will use this case for a counterexample (Proposition 4.24).

The definitions introduce a parameter θ in order to control the relative importance of the shape and location of the fuzzy sets. That resembles what happens in function spaces with the Sobolev distances. As before, we give two proposals: one based on $d_{r,\theta}$ and another on $d^r_{r,\theta}$.

Definition 4.3. For any $r \in [1, \infty)$ and $\theta \in [0, \infty)$, the (r, θ)-*location depth* based on $\mathcal{J} \subseteq \mathcal{F}_c(\mathbb{R}^p)$ and $\mathcal{H} \subseteq L^r[\mathcal{F}_c(\mathbb{R}^p)]$, with $\mathcal{H}$ and $\mathcal{J}$ being non-empty, of a fuzzy set $A \in \mathcal{J}$ with respect to a fuzzy random variable $\mathcal{X} \in \mathcal{H}$ is

$$D^\theta_r(A; \mathcal{X}) := \left(1 + \mathrm{E}[d_{r,\theta}(A, \mathcal{X})]\right)^{-1}.$$

Definition 4.4. For any $r \in [1, \infty)$ and $\theta \in [0, \infty)$, the (r, θ)-*location raised depth* based on $\mathcal{J} \subseteq \mathcal{F}_c(\mathbb{R}^p)$ and $\mathcal{H} \subseteq L^r[\mathcal{F}_c(\mathbb{R}^p)]$, with $\mathcal{H}$ and $\mathcal{J}$ being non-empty, of a fuzzy set $A \in \mathcal{J}$ with respect to a random variable $\mathcal{X} \in \mathcal{H}$ is

$$RD^\theta_r(A; \mathcal{X}) := \left(1 + \mathrm{E}[d_{r,\theta}(A, \mathcal{X})^r]\right)^{-1}.$$

In the context of D^θ_r and RD^θ_r, we consider the subsets $\mathcal{H} \subseteq L^r[\mathcal{F}_c(\mathbb{R}^p)]$ to avoid the denominator having an infinite expectation.

The particular case of D^θ_2 in the real line was discussed in [27, Section 6]. Yet another similar definition, but involving only the spread and not the mid, is used in [13, Example 5.7] to show that P3a does not imply P3b in general.

Remark 4.5. The general structure of the definitions above is

$$D(A; \mathcal{X}) = (1 + \phi(E[d(A, \mathcal{X})]))^{-1}$$

where d is a metric in $\mathcal{F}_c(\mathbb{R}^p)$ and ϕ is an appropriate increasing (and convex, for some arguments in the sequel) function with $\phi(0) = 0$. While this type of object makes sense in a general metric space, the next subsection will focus on whether it satisfies properties which are specific to the context of fuzzy sets.

Remark 4.6. Definitions 4.1 through 4.4 adapt the multivariate notion of L^r-depth to the fuzzy setting but are not generalizations of it. The reason is that the r-norm distance between two points of $\mathbb{R}^p$ does not equal the ρ_r- or $d_{r,\theta}$-distance between their indicator functions. Take, for instance, $x = (2,3)$ and $y = (3,7)$. We have $\|x-y\|_1 = 1+4 = 5$ whereas

$$\rho_1\left(I_{\{x\}}, I_{\{y\}}\right) = \int_0^{2\pi} \left|\cos\theta + 4\cdot\sin\theta\right| \mathrm{d}\nu(\theta) = 4\sqrt{17}.$$

Observe $d_{r,\theta}\left(\mathrm{I}_{\{x\}}, \mathrm{I}_{\{y\}}\right) = \rho_r\left(\mathrm{I}_{\{x\}}, \mathrm{I}_{\{y\}}\right)$ for all $\theta \in [0,\infty)$ since their spread is the null function.

The following result states that functions of the form of L^r-type depths satisfy property P3a under certain convexity assumptions.

Lemma 4.7. *If $C(\cdot,\mathcal{X})$ is a convex function then the function $(1+E[C(\cdot,\mathcal{X})])^{-1}$ satisfies P3a for every $\mathcal{X} \in L^0[\mathcal{F}_c(\mathbb{R}^p)]$ such that $E[C(I_{\{0\}},\mathcal{X})] < \infty$.*

This lemma and its proof are analogous to the multivariate result [35, Theorem 2.4], since $C(\cdot,\mathcal{X})$ and P3a maintain the structure of their multivariate analogues.

Proposition 4.8. *Let $r \in [1,\infty)$, $\theta \in [0,\infty)$ and $\mathcal{X} \in L^r[\mathcal{F}_c(\mathbb{R}^p)]$. The functions $\rho_r(\cdot;\mathcal{X})$, $\rho_r(\cdot;\mathcal{X})^r$, $d_{r,\theta}(\cdot;\mathcal{X})$ and $d_{r,\theta}(\cdot;\mathcal{X})^r$ are convex.*

4.2. Properties

4.2.1. Affine invariance

The next example shows that neither D_r, RD_r, D_r^θ nor RD_r^θ are affine invariant in the sense of property P1; the same happens in the multivariate case [35].

Example 4.9. Let $\{\{\omega_1,\omega_2\}, \mathcal{P}(\{\omega_1,\omega_2\}), \mathbb{P}\}$ be a probability space with $\mathbb{P}(\{\omega_1\}) = \mathbb{P}(\{\omega_2\}) = 1/2$.

(i) Let $\mathcal{X}(\omega_1) := \mathrm{I}_{[1,2]}$ and $\mathcal{X}(\omega_2) := \mathrm{I}_{[5,7]}$. Taking $A = \mathrm{I}_{[3,4]}$, after some algebra we have, for any $r \in [1,\infty)$,

$$\mathrm{E}(\rho_r(A,\mathcal{X})) = \frac{1}{2}\cdot\left[2 + \left(\frac{3^r+2^r}{2}\right)^{1/r}\right]$$

and

$$\mathrm{E}(\rho_r(A,\mathcal{X})^r) = \frac{1}{2}\cdot\left[2^r + \frac{3^r+2^r}{2}\right].$$

Thus,

$$D_r(A;\mathcal{X}) = \left(2 + \frac{1}{2}\cdot\left(\frac{3^r+2^r}{2}\right)^{1/r}\right)^{-1} > 0$$

and

$$RD_r(A;\mathcal{X}) = \left(1 + 3\cdot\left(2^{r-2} + \frac{3^{r-1}}{4}\right)\right)^{-1} > 0.$$

Considering the matrix $M := (5) \in \mathcal{M}_{1\times 1}(\mathbb{R})$,

$$M\cdot\mathcal{X}(\omega_1) = \mathrm{I}_{[5,10]},\ M\cdot\mathcal{X}(\omega_2) = \mathrm{I}_{[25,35]} \text{ and } M\cdot A = \mathrm{I}_{[15,20]}.$$

Therefore, for every $r \in [1,\infty)$,

$$E[\rho_r(M\cdot A; M\cdot\mathcal{X}) = 5E[\rho_r(A;\mathcal{X})]$$

whence $D_r(M\cdot A; M\cdot\mathcal{X}) \neq D_r(A;\mathcal{X})$ and $RD_r(M\cdot A; M\cdot\mathcal{X}) \neq RD_r(A;\mathcal{X})$.

(ii) Let $\mathcal{X}(\omega_1) := \mathrm{I}_{[0,2]}$ and $\mathcal{X}(\omega_2) := \mathrm{I}_{[2,3]}$. Taking $A = \mathrm{I}_{[1,2]}$, we obtain for any $r \in [1,\infty)$ and $\theta \in (0,\infty)$

$$\mathrm{E}[d_{r,\theta}(A,\mathcal{X})] = \frac{1}{2}\cdot\left(1 + \frac{(1+\theta)^{1/r}}{2}\right)$$

and

$$E[d_{r,\theta}(A,\mathcal{X})^r] = \frac{1}{2}\cdot\left(1+\frac{1+\theta}{2^r}\right).$$

Thus,

$$D_r^{\theta}(A;\mathcal{X}) = \left(1+\frac{1}{2}\cdot\left[1+\frac{(1+\theta)^{1/r}}{2}\right]\right)^{-1} > 0$$

and

$$RD_r^{\theta}(A;\mathcal{X}) = \left(1+\frac{1}{2}\cdot\left[1+\frac{1+\theta}{2^r}\right]\right)^{-1} > 0.$$

Now, for $M = (2) \in \mathcal{M}_{1\times 1}(\mathbb{R})$,

$$M\cdot\mathcal{X}(\omega_1) = \mathrm{I}_{[0,4]},\, M\cdot\mathcal{X}(\omega_2) = \mathrm{I}_{[4,6]} \text{ and } M\cdot A = \mathrm{I}_{[2,4]}.$$

Therefore,

$$E[d_{r,\theta}(M\cdot A, M\cdot\mathcal{X})] = 1+\frac{(1+\theta)^{1/r}}{2}$$

and

$$E[d_{r,\theta}(M\cdot A, M\cdot\mathcal{X})^r] = 2^{r-1}\cdot\left(1+\frac{1+\theta}{2^r}\right).$$

For every $r\in[1,\infty)$ and $\theta\in(0,\infty)$,

$$D_r^{\theta}(M\cdot A; M\cdot\mathcal{X}) \neq D_r^{\theta}(A;\mathcal{X}) \text{ and } RD_r^{\theta}(M\cdot A; M\cdot\mathcal{X}) \neq RD_r^{\theta}(A;\mathcal{X}).$$

Let us consider the following property, weaker than P1.

P1$*$. $D(M\cdot A+B; M\cdot\mathcal{X}+B) = D(A;\mathcal{X})$ for any orthogonal matrix $M\in\mathcal{M}_{p\times p}(\mathbb{R})$ and $A, B\in\mathcal{F}_c(\mathbb{R}^p)$.

This property (called *rigid-body invariance*) was shown to hold in the multivariate case in [35]. The following result states that D_r, RD_r, D_r^{θ} and RD_r^{θ} are invariant when the matrix $M\in\mathcal{M}_{p\times p}(\mathbb{R})$ is orthogonal. That is due to the fact that $\|M^T\cdot u\| = 1$ for all $u\in\mathbb{S}^{p-1}$ if M is orthogonal. Note that the M's in Example 4.9 are not orthogonal matrices, because their determinant is not ± 1.

Proposition 4.10. *Let $\mathcal{J}\subseteq\mathcal{F}_c(\mathbb{R}^p)$, $\mathcal{H}_1 = L^1[\mathcal{F}_c(\mathbb{R}^p)]$ and $\mathcal{H}_r\subseteq L^r[\mathcal{F}_c(\mathbb{R}^p)]$. Property P1$*$ is satisfied by D_r based on $\mathcal{J}$ and $\mathcal{H}_1$ and RD_r based on $\mathcal{J}$ and $\mathcal{H}_r$, for any $r\in[1,\infty]$; and by D_r^{θ} based on $\mathcal{J}$ and $\mathcal{H}_1$ and RD_r^{θ} based on $\mathcal{J}$ and $\mathcal{H}$, for any $r\in[1,\infty]$ and $\theta\in[0,\infty)$.*

Remark 4.11. The failure of $P1$ and its multivariate analog for some non-orthogonal matrices illustrates why 'lists of properties' are guides rather than axioms for depth functions. If the results of an analysis may be different depending on whether temperature values are expressed in the Celsius or Fahrenheit scale, one would like to ponder calmly whether it makes sense to use that method. Thus, failing affine invariance looks like an egregious violation for a depth function.

From the discussion of Property P1*, L^r-depths are rotation (and also translation) invariant, and only have problems with rescaling. Since both the function $x\mapsto(1+x)^{-1}$ and multiplication by a scalar are strictly monotonic, rescaling modifies the depth values but not their order. Therefore, as long as depth values are used as a ranking device (as opposed to important values in themselves) there will be no problem.

For instance, consider a depth-trimmed mean obtained by eliminating from the sample the 10% less L^r-deep points. Rescaling does not affect which sample points get trimmed and therefore the depth-trimmed mean will still be affinely invariant, even if the depth function itself is not. Similarly, a depth-based classification task will yield the same result regardless of rescaling.

Moreover, in some situations data are routinely standardized before the analysis, which makes the rescaling issue irrelevant. For instance, in cell studies like cancer diagnosis, cell measurements taken from tissue images need standardization since different images may not share the same scale.

Remark 4.12. In [27, Proposition 6.1], Sinova shows what amounts to stating that D_2^{θ} (in the real line) satisfies Property P1*. In that case, matrices are not involved since the only orthogonal transformations of $\mathbb{R}$ are the identity function id and its opposite $-id$. Although Sinova also states properties of monotonicity relative to the deepest point and vanishing at infinity, they are formulated in terms of the behavior of $E[d_{2,\theta}(\mathcal{X},A)]$ instead of A itself, following from the definition.

4.2.2. Maximality at the center of symmetry

While F-symmetry is suitable for D_r and RD_r, we will use (mid, spr)-symmetry for D_r^θ and RD_r^θ as the mid and spr functions are involved in their construction. We first focus on cases $r = 1, 2$, since some of our proofs employ arguments which are specific to those values.

For $r = 1$, the results are in Propositions 4.15 and 4.16, which require integrably bounded fuzzy random variables. These propositions rely on Lemmas 4.13 and 4.14, which ensure the existence of the expectation in the denominator of D_1 and D_1^θ, respectively. Note that for $r = 1$ one has $RD_1 = D_1$ and $RD_1^\theta = D_1^\theta$.

Lemma 4.13. *Let $r, s \in [1, \infty)$ and $\mathcal{X} \in L^r[\mathcal{F}_c(\mathbb{R}^p)]$. Then $E[\rho_s(I_{\{0\}}, \mathcal{X})^r] < \infty$.*

Lemma 4.14. *Let $r \in [1, \infty)$, $\theta \in [0, \infty)$ and $\mathcal{X} \in L^1[\mathcal{F}_c(\mathbb{R}^p)]$. Then $E[d_{r,\theta}(I_{\{0\}}, \mathcal{X})] < \infty$.*

Proposition 4.15. *Let $\mathcal{J} = \mathcal{F}_c(\mathbb{R}^p)$ and $\mathcal{H} \subseteq L^1[\mathcal{F}_c(\mathbb{R}^p)]$. Then D_1 (equivalently, RD_1) based on $\mathcal{J}$ and $\mathcal{H}$ satisfies Property P2 for F-symmetry.*

Proposition 4.16. *Let $\mathcal{J} = \mathcal{F}_c(\mathbb{R}^p)$, $\mathcal{H} \subseteq L^1[\mathcal{F}_c(\mathbb{R}^p)]$ and $\theta \in [0, \infty)$. Then D_1^θ (equivalently, RD_1^θ) based on $\mathcal{J}$ and $\mathcal{H}$ satisfies Property P2 for* (mid, spr)*-symmetry.*

For $r = 2$, the results are in Propositions 4.17 and 4.18.

Proposition 4.17. *Let $\mathcal{J} = \mathcal{F}_c(\mathbb{R}^p)$ and $\mathcal{H} \subseteq L^2[\mathcal{F}_c(\mathbb{R}^p)]$. Then, RD_2 based on $\mathcal{J}$ and $\mathcal{H}$ satisfies Property P2 for F-symmetry.*

Proposition 4.18. *Let $\mathcal{J} = \mathcal{F}_c(\mathbb{R}^p)$, $\mathcal{H} \subseteq L^2[\mathcal{F}_c(\mathbb{R}^p)]$ and $\theta \in [0, \infty)$. Then, RD_2^θ based on $\mathcal{J}$ and $\mathcal{H}$ satisfies Property P2 for* (mid, spr)*-symmetry.*

Fuzzy sets can be associated with their support functions in the function space $L^r(\mathbb{S}^{p-1} \times [0,1], \mathcal{V}_p \otimes \nu)$. Thus, it is possible to define a notion of symmetry in the fuzzy setting by using central symmetry in that function space (see [20]). Notice that this notion does not depend on the choice of $r \in [1, \infty)$.

Definition 4.19. Let $\mathcal{X}$ be a fuzzy random variable, we say that $\mathcal{X}$ is *functionally symmetric* with respect to a fuzzy set A if $s_{\mathcal{X}} - s_A$ is identically distributed as $s_A - s_{\mathcal{X}}$.

Theorem 4.20. *Let $r \in [1, \infty)$, $\mathcal{J} = \mathcal{F}_c(\mathbb{R}^p)$ and $\mathcal{H} \subseteq L^1[\mathcal{F}_c(\mathbb{R}^p)]$. Then, D_r based on $\mathcal{J}$ and $\mathcal{H}$ satisfies Property P2 for functional symmetry.*

4.2.3. Properties P3 and P4

We will now study properties P3 and P4 for L^r-type depths.

Lemma 4.13 guarantees the finiteness of the expectation in the denominator of D_r for each $r \in [1, \infty)$ and for every integrable bounded fuzzy random variable. In the case of RD_r, we consider fuzzy random variables $\mathcal{X} \in L^r[\mathcal{F}_c(\mathbb{R}^p)]$.

Theorem 4.21. *Let $r \in [1, \infty)$, $\mathcal{J} = \mathcal{F}_c(\mathbb{R}^p)$, $\mathcal{H}_1 \subseteq L^1[\mathcal{F}_c(\mathbb{R}^p)]$ and $\mathcal{H}_r \subseteq L^r[\mathcal{F}_c(\mathbb{R}^p)]$. Then D_r based on $\mathcal{J}$ and $\mathcal{H}_1$, and RD_r based on $\mathcal{J}$ and $\mathcal{H}_r$ both satisfy*

- *P3a and P4a,*
- *P3b for the ρ_s and $d_{s,\theta}$ metrics for any $s \in (1, \infty)$ and $\theta \in (0, \infty)$,*
- *P4b for the ρ_s and $d_{s,\theta}$ metrics for any $s \in [1, r]$ and $\theta \in (0, \infty)$.*

In general, P4b does not admit $s > r$, as shown for $r = 1$ in [13, Example 5.9]. Based on Lemma 4.14, the function D_r^θ is well defined for integrably bounded fuzzy random variables. For the case of RD_r^θ, we consider fuzzy random variables $\mathcal{X} \in L^r[\mathcal{F}_c(\mathbb{R}^p)]$.

Theorem 4.22. *Let $r \in [1, \infty)$, $\theta \in [0, \infty)$, $\mathcal{J} = \mathcal{F}_c(\mathbb{R}^p)$, $\mathcal{H}_1 \subseteq L^1[\mathcal{F}_c(\mathbb{R}^p)]$ and $\mathcal{H}_r \subseteq L^r[\mathcal{F}_c(\mathbb{R}^p)]$. Then D_r^θ based on $\mathcal{J}$ and $\mathcal{H}_1$ and RD_r^θ based on $\mathcal{J}$ and $\mathcal{H}_r$ satisfy*

- *P3a,*
- *P4a if $\theta \in (0, \infty)$,*
- *P3b for the ρ_s and $d_{s,\theta}$ metrics for any $s \in (1, \infty)$ and $\theta \in (0, \infty)$,*
- *P4b for the ρ_s and $d_{s,\theta}$ metrics for any $s \in [1, r]$ and $\theta \in (0, \infty)$.*

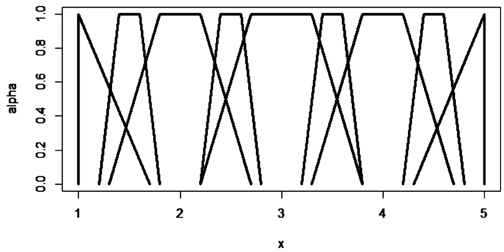

Fig. 3. Display of the dataset *Trees*.

Table 1
Sample frequencies and depths for each quality value.

Quality	T_1	T_2	T_3	T_4	T_5	T_6	T_7	T_8	T_9
Frequency	22	16	39	36	85	22	35	12	12
D_{FP}	.2333	.2917	.3889	.4737	1	.4737	.3889	.2917	.2333
D_1	.3726	.4149	.4887	.5488	.5790	.5163	.4493	.3781	.3814
D_2	.4530	.4751	.5214	.5506	.5903	.5287	.5001	.4545	.4564
D_1^5	.2979	.3036	.3761	.3695	.3814	.3545	.3307	.2838	.2522
D_1^{10}	.2295	.2390	.2972	.2780	.2839	.2694	.2682	.2265	.2014

Remark 4.23. The case $s = 1$ is special for property P3b in Theorems 4.21 and 4.22 because the space $L^1(\mathbb{S}^{p-1} \times [0,1], \mathcal{V}^p \otimes \nu)$ is not strictly convex. This results in that P3a and P3b are not necessarily equivalent for $s = 1$, that equivalence being used to prove P3b for $s > 1$.

For $\theta = 0$, Properties P4a and P4b are not satisfied, as shown next. Note that the distance function $d_{2,\theta}$ is defined in [30] for $\theta \in (0,1]$. It is not a distance for $\theta = 0$, as mentioned in Section 4.1.

Proposition 4.24. *D_r^0 and RD_r^0 can fail P4a for any $r \in [1,\infty)$ and P4b with the ρ_s metric for any $s \in (1,\infty)$ and $r \in [1,\infty)$.*

5. Real data example

In this section, we compute the depth of the elements in a real datasets composed of trapezoidal fuzzy sets. Subsection 5.1 focuses on computing our depth proposals, specifically tailored for fuzzy sets. To apply them, it suffices to take $\mathcal{H} = L^0[\mathcal{F}_c(\mathbb{R}^p)]$ and $\mathcal{J} = \mathcal{F}_c(\mathbb{R}^p)$. Since trapezoidal fuzzy sets are characterized by four real numbers, in Subsection 5.2 we compare the obtained results with those of the multivariate projection depth in $\mathbb{R}^4$. In Subsection 5.3, we make use of out fuzzy proposals to perform depth based classification procedures.

5.1. Fuzzy depth functions

In order to compare the behavior of projection and L^r-type depths we use the dataset *Trees* from the SAFD (Statistical Analysis of Fuzzy Data) R package [5]. It comes from a reforestation study at the INDUROT forest institute in Spain. The study considers the *quality* of the tree, a fuzzy random variable whose observations are trapezoidal fuzzy numbers. To define it, experts took into account different aspects of the trees, including leaf structure and height-diameter ratio. The x-axis represents quality, in a scale from 0 to 5, where 0 means null quality and 5 perfect quality. The y-axis represents membership. The dataset contains a random sample (size: 279) of 9 possible fuzzy values (see Fig. 3 and Table 1). There, the trapezoidal fuzzy numbers are represented by T_i, $i = 1, \ldots, 9$ from left to right, for which projection depth and some L^r-type depths were computed.

In Fig. 3, we appreciate a certain symmetry in the data representation. Beyond this fact, we can not discard any metric *a priori*, thus we compute the L^r-type dephts for $r = 1, 2$, the most common cases in the literature. It is clear that the median of the sample (in the sense of [28]) is the maximizer of D_1 and thus is T_5. This fact, together with the fact of symmetry, makes us suppose that the projection depth will give a symmetric ordering, that is T_1 will have the same depth of T_9, T_2 the same depth of T_8 and so on. Table 1 shows that the projection depth represents the symmetry of the data. In the left panel of Fig. 4 it is represented the ordering which induce the projection depth. The ordering induced in the fuzzy numbers by the 1- and 2-natural depths is the same and it is represented in the right panel of Fig. 4.

Finally, we compute some examples of (r,θ)-location depth. The cases $r = 1, 2$ and $\theta = 1$ generate the same ordering as D_1 and D_2. If we take $\theta > 1$, we prioritize *shape* over *location* and we should expect a different ordering (see Fig. 5). Indeed, as θ increases, trapezoidal fuzzy sets with intermediate slopes become deeper than centrally located ones.

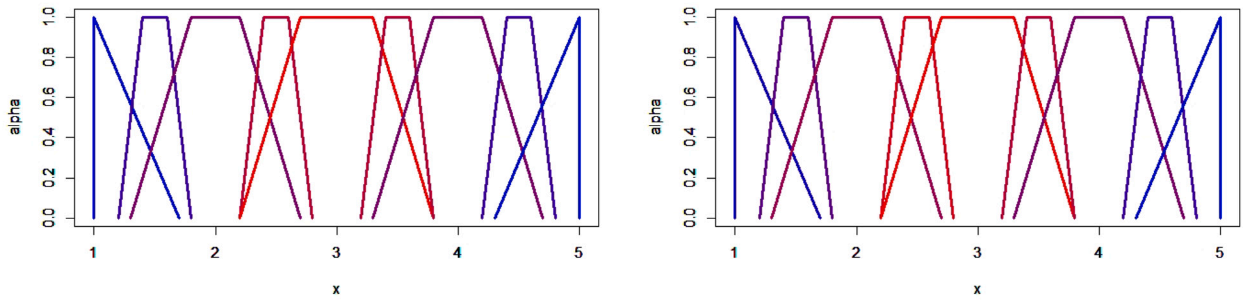

Fig. 4. Display of the dataset *Trees*. Color is assigned based on the projection depth (left panel) and on the 1-natural and 2-natural depths (right panel) of each fuzzy set in the empirical distribution. Colors range from red (high depth) to blue (low depth).

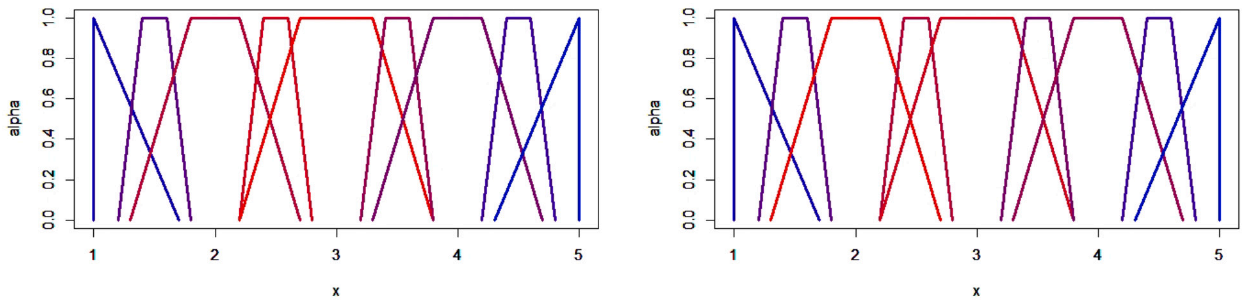

Fig. 5. Display of the dataset *Trees*. Color is assigned based on the $(1, 5)$-location depth (left) and $(1, 10)$-location depth (right), ranging from red (high depth) to blue (low depth).

Table 2
Sample frequencies and multivariate and fuzzy depths for each quality value.

Quality	T_1	T_2	T_3	T_4	T_5	T_6	T_7	T_8	T_9
Frequency	22	16	39	36	85	22	35	12	12
PD_R	.0124	.0044	.0357	.0371	.1971	.0317	.0070	.0109	.0042
D_{FP}	.2333	.2917	.3889	.4737	1	.4737	.3889	.2917	.2333

5.2. Multivariate depth functions

In this section, we compute the multivariate projection depth of each trapezoidal fuzzy set in the *Trees* dataset, in order to compare the ordering with the one given by the fuzzy depths, with a focus on the fuzzy projection depth. For that purpose, we consider each trapezoidal fuzzy set as an element of $\mathbb{R}^4$.

To compute the multivariate projection depth in practice, we employ a methodology akin to that of the random Tukey depth [6]. We approximate the O function by considering 1000 random directions on the sphere $\mathbb{S}^3$, evaluate the corresponding function and take the maximum over these directions. Table 2 presents the multivariate depth values, denoted as PD_R, and the fuzzy depth values of the different elements in the dataset; in addition to the frequency of each element in the dataset. Fig. 6 displays the ordering induced by the multivariate projection depth, using random directions on the sphere, (left panel) and by the fuzzy projection depth (right panel). It is evident that the ordering given by the fuzzy projection depth (or the other proposed fuzzy depths) is more coherent with a natural order. For instance, the fuzzy projection depth taks into account the symmetry among the trapezoidal fuzzy sets with respect to T_5. On the other hand, the ordering given by the multivariate depth gives to T_1 the fifth deepest value, a disposition that does not coincide in a natural way with the arrangement of the fuzzy sets; as T_1 is the fuzzy set with a farthest location to the left in the plots.

5.3. Real-world application of fuzzy depths

In this section, we compute the fuzzy projection and fuzzy L^r-type depths as the base of a supervised classification procedure. The dataset that we consider consists of 40 overall opinions of one expert and 38 of a second expert on the quality of the *Gamonedo* cheese from *Asturias, Spain*, see [26]. As the perception of the quality is a subjective process, the quality of the experts is expressed in terms of trapezoidal fuzzy sets. Here, the 0-level is the closed interval where the quality clearly lies and the 1-level is the closed

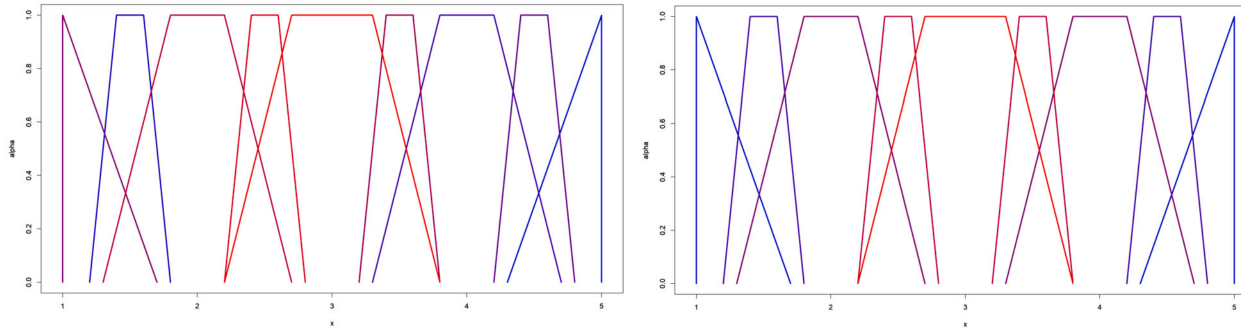

Fig. 6. Display of the dataset *Trees*. Color is assigned based on the multivariate projection depth (left panel) and on the fuzzy projection depth (right panel). Colors range from red (high depth) to blue (low depth).

Table 3
Misclassification rates of different classification procedures based on projection and L^r-type depths.

Depth	D_{FP}	D_2	D_2^1	D_2^{40}
MD	.3589	.3333	.3333	.2307
$DTM_{0.1,0.1,d_{2,100}}$	.2692	.1923	.1923	.2307

interval where the experts think the quality lies. In this setting, we consider two fuzzy random variables, $\mathcal{X}_1$ and $\mathcal{X}_2$, where $\mathcal{X}_1$ is the fuzzy random variable that corresponds to the overall opinion of the first expert and, analogously, $\mathcal{X}_2$ to the second expert.

Given that the fuzzy sets are trapezoidal, determining the precise value of the L^r-type depths is straightforward. This ease arises from the fact that the metrics ρ_r and $d_{r,\theta}$ make use of integrals over the intervals $[0,1]$ and $\mathbb{S}^0$ and the structure of trapezoidal fuzzy sets is simple, being determined by 4 values. However, when calculating the exact value of the projection depth, it is required to compute the supremum of a function over an infinite number of α-levels. Here we estimate the projection depth values, by randomly selecting 200 values from the closed interval $[0,1]$, computing the function's value for each, and then taking the maximum among these 200 values.

Let us suppose we have two samples of fuzzy sets, $X_1,\dots,X_n$ and $Y_1,\dots,Y_m$. We use here the two following depth based supervised classification procedures.

1. *Maximum depth (MD)* [12]. In consists of adding each observation to the two training samples, computing its depth with respect to both samples, and classifying the observation into the group where its depth is the greatest.
2. *Distance to the trimmed mean ($DTM_{\alpha,\beta,d}$)* [21]. For this procedure we compute the depth of each observation of $X_1,\dots,X_n$ and select an $\alpha \in [0,1)$. Then, we compute the α-trimmed mean of the sample, $\mu_\alpha(X)$, which is the mean of the $n\times(1-\alpha)$ deepest points. We select $\beta \in [0,1)$ for the sample $Y_1,\dots,Y_m$ and compute, similarly, $\mu_\beta(Y)$. Let us consider a metric in the fuzzy space, d. Now, we classify a fuzzy set A in the first group if

$$d(A,\mu_\alpha(X)) < d(A,\mu_\beta(Y)).$$

We evaluate the accuracy of these procedures employing *one-leave-out cross-validation*. Table 3 shows the misclassification error rate for our depth proposals. We consider the projection depth, the 2-natural depth and the $(2,\theta)$-location depth, for $\theta \in \{1,40\}$. For the $DTM_{\alpha,\beta,d}$ procedure, we consider the family of metrics $d_{2,100}$. Different values of θ between 0 and 200 were used in a previous analysis on this dataset, which led to the selection of $\theta = 100$.

This dataset was used by [15] in a two-sample dispersion test. There it was obtained that there is no significant difference between the opinions of both experts. Despite this, the misclassification rate using DTM based on D_2 or D_2^1 is only 0.1923, indicating that we can accurately classify well over 80% of the observations.

6. Concluding remarks

Since the introduction of projection depth [35], it has been applied in multivariate analysis (see, e.g., [10] and [34]), measuring the worst case of the outlyingness of a point by comparing the projection of the point in every direction with respect to the univariate median of the projection in that direction. In the fuzzy case, as the support function of a fuzzy set considers the projection of every direction u and every α-level, we define the function D_{FP} replacing the inner product by the support function for every $(u,\alpha) \in \mathbb{S}^{p-1} \times [0,1]$.

The function D_{FP} is the natural generalization of the multivariate projection depth to the fuzzy setting (Proposition 3.4). Projection depth for fuzzy sets, as the Tukey depth defined in [13], is a semilinear depth function and also a geometric depth function for the ρ_r-distances with $r \in (1,\infty)$ (Corollary 3.6). It is also interesting that, being defined via medians, it imposes no integrability requirements on the fuzzy random variables. There is a limitation with respect to its implementation when working with general fuzzy sets, as the O function requires the use of every direction u on the sphere $\mathbb{S}^{p-1}$ and every level α on $[0,1]$. This is equivalent to the existing limitation in the multivariate setting. Both are easily solved by approximating the function randomly as in [6]: consider n random directions, m random levels, evaluate the functions and take the maximum. In summary, projection depth is a nice alternative to Tukey depth in the fuzzy setting.

For any $r \in [1,\infty)$, the L^r-type fuzzy depths satisfy the semilinear and the geometric depth notions under the assumption that the matrices considered in P1 are orthogonal (Proposition 4.10). Property P2 is satisfied by $D_1 = RD_1$ and RD_2 when F-symmetry is considered (see Proposition 4.15 and 4.17) and by D_r for $r \in [1,\infty)$ when *functional symmetry* is considered (see Theorem 4.20). It is also satisfied by $D_1^\theta = RD_1^\theta$ and RD_2^θ for $\theta \in [0,\infty)$ when $(\mathrm{mid},\mathrm{spr})$-symmetry is considered (see Proposition 4.16 and 4.18). The main shortcoming of the L^r-type depths is their use for general fuzzy sets. To compute L^r-type depths is necessary that the support functions of the sample of fuzzy sets are integrable functions, to determine the integrals related with the metrics. Although L^r-type depths are neither semilinear nor geometric depth functions, we can observe in Section 5 that their behavior can be similar to that of projection depth, which is in fact a semilinear and a geometric depth function.

For future work, it would be desirable to study the continuity or semicontinuity properties of these depth functions, as it is done in the multivariate case (see [35]) and in the functional case (see [24]). It is also open to find a geometric depth function for the ρ_1 metric or d_r metrics, or to show the impossibility of such functions. From the point of view of applied mathematics, it could be stimulating to develop algorithms to compute some fuzzy depth proposals, in order to generalize to fuzzy sets some nonparametric methods of multivariate and functional data analysis.

CRediT authorship contribution statement

Luis González-de la Fuente: Investigation, Software, Writing – original draft, Writing – review & editing. **Alicia Nieto-Reyes:** Conceptualization, Formal analysis, Funding acquisition, Investigation, Methodology, Supervision, Writing – original draft, Writing – review & editing. **Pedro Terán:** Conceptualization, Formal analysis, Writing – original draft, Investigation.

Declaration of competing interest

The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

Data availability

The dataset is already publicaly available, and we have cited it.

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Appendix A. Proofs

This appendix contains the mathematical proofs of the results in Sections 3 and 4.

The proof of Theorem 3.5 relies on Lemmas A.1 and A.2 below [13, Theorem 5.4 and Proposition 5.8]. Given a metric d in $\mathcal{F}_c(\mathbb{R}^p)$, these lemmas consider the following assumptions.

(A1) $d(\gamma \cdot A, \gamma \cdot B) = \gamma \cdot d(A,B)$ for all $A, B \in \mathcal{F}_c(\mathbb{R}^p)$ and $\gamma \in [0,\infty)$,
(A2) $d(A+W, B+W) = d(A,B)$ for all $A, B, W \in \mathcal{F}_c(\mathbb{R}^p)$.

Lemma A.1. *Let $(\mathbb{E}, \|\cdot\|)$ be a strictly convex Banach space, d a metric in $\mathcal{F}_c(\mathbb{R}^p)$ fulfilling A1 and A2, and $j : (\mathcal{F}_c(\mathbb{R}^p), d) \to (\mathbb{E}, \|\cdot\|)$ an isometry. Whenever $A, B, C \in \mathcal{F}_c(\mathbb{R}^p)$ are such that $d(A,B) = d(A,C) + d(B,C)$, the fuzzy set C has the form $(1-\lambda)\cdot A + \lambda \cdot B$ for some $\lambda \in [0,1]$.*

Lemma A.2. *Let $\mathcal{X}$ be a fuzzy random variable and $D(\cdot;\mathcal{X}) : \mathcal{F}_c(\mathbb{R}^p) \to [0,\infty)$ a function for which P4b holds with respect to a metric that fulfills A1 and A2. Then $D(\cdot;\mathcal{X})$ satisfies P4a.*

We introduce a basic result about symmetry of real random variables which is used in the proof of property P2 below.

Lemma A.3. *Let X be a real random variable symmetric with respect to $c \in \mathbb{R}$. Then $c = med(X)$ and also $c = E[X]$ provided $E[X] \in \mathbb{R}$ exists.*

Proof. If $\mathrm{E}[X] < \infty$, then $\mathrm{E}[X] - c = \mathrm{E}[X - c] = \mathrm{E}[c - X] = c - \mathrm{E}[X]$, where the second equality is due to the symmetric hypothesis ($X - c$ and $c - X$ are equally distributed). Thus, $\mathrm{E}[X] = c$.

Suppose for a contradiction that $c \notin \mathrm{Med}(X)$. Without loss of generality we assume $\mathbb{P}(X \le c) < 1/2$. Therefore, $\mathbb{P}(X \ge c) > 1/2$. By the symmetry hypothesis, $\mathbb{P}(X - c \le 0) = \mathbb{P}(c - X \le 0)$. Thus, we have that $1/2 > \mathbb{P}(X \le c) = \mathbb{P}(X \ge c) > 1/2$, which leads to a contradiction. Then,

$$c \in \mathrm{Med}(X). \tag{14}$$

If we restrict $\mathrm{Med}(X)$ to be a singleton, then $c = \mathrm{med}(X)$. If the set $\mathrm{Med}(X) = [m, M]$ is not a singleton, let us assume for a contradiction that $c \ne \mathrm{med}(X) = (M + m)/2$. Taking into account (14) we assume, without loss of generality,

$$\frac{(M+m)}{2} < c \le M. \tag{15}$$

That implies $M - c < c - m$. Then there exists some $\epsilon > 0$ such that $c - [(M - c) + \epsilon] > m$. As (15) also implies

$$M + \epsilon > c, \tag{16}$$

we get $c > c - [(M - c) + \epsilon] > m$. Then $c - [(M - c) + \epsilon] \in \mathrm{Med}(X)$ and

$$\mathbb{P}(X \le c - [(M - c) + \epsilon]) \ge \frac{1}{2}. \tag{17}$$

As $\mathbb{P}(X \le M + \epsilon) \ge \mathbb{P}(X \le c) \ge 1/2$, by (14) and (16), and $M + \epsilon \notin \mathrm{Med}(X)$,

$$\mathbb{P}(X \ge M + \epsilon) < \frac{1}{2}. \tag{18}$$

By the central symmetry of X, $\mathbb{P}(X - c \le t) = \mathbb{P}(c - X \le t)$ for each $t \in \mathbb{R}$. Setting $t = -[(M - c) + \epsilon]$ and taking into account (17) and (18),

$$1/2 \le \mathbb{P}(X \le c - [(M - c) + \epsilon]) = \mathbb{P}(X \ge c + (M - c) + \epsilon) < 1/2,$$

a contradiction. □

Proof of Theorem 3.5. *Property P1.* Let $M \in \mathcal{M}_{p\times p}(\mathbb{R})$ be a regular matrix and $A, B \in \mathcal{F}_c(\mathbb{R}^p)$. It suffices to prove $O(M \cdot A + B; M \cdot \mathcal{X} + B) = O(A; \mathcal{X})$. By translation invariance,

$$\mathrm{MAD}(s_{\mathcal{X}}(u,\alpha) + s_B(u,\alpha)) = \mathrm{MAD}(s_{\mathcal{X}}(u,\alpha))$$

for any $u \in \mathbb{S}^{p-1}$ and $\alpha \in [0,1]$, yielding $O(M \cdot A + B; M \cdot \mathcal{X} + B) = O(M \cdot A; M \cdot \mathcal{X})$. Now consider the function $g : \mathbb{S}^{p-1} \to \mathbb{S}^{p-1}$ defined by

$$g(u) = \left(\frac{1}{\|M^T \cdot u\|}\right) M^T \cdot u.$$

Then

$$O(M \cdot A; M \cdot \mathcal{X}) = \sup_{u \in \mathbb{S}^{p-1}, \alpha \in [0,1]} \frac{|s_A(g(u),\alpha) - \mathrm{med}(s_{\mathcal{X}}(g(u),\alpha))|}{\mathrm{MAD}(s_{\mathcal{X}}(g(u),\alpha))} = O(A; \mathcal{X}),$$

where the first identity uses (7) and the properties of the univariate median. The second identity holds because g is bijective, a consequence of M being regular.

Property P2. Let $\mathcal{X}$ be a fuzzy random variable, F-symmetric with respect to some $A \in \mathcal{F}_c(\mathbb{R}^p)$. It implies that the real random variable $s_{\mathcal{X}}(u,\alpha)$ is symmetric with respect to $s_A(u,\alpha)$ for every $u \in \mathbb{S}^{p-1}$ and $\alpha \in [0,1]$. By Lemma A.3, $s_A(u,\alpha) = \mathrm{med}(s_{\mathcal{X}}(u,\alpha))$ for all $u \in \mathbb{S}^{p-1}$ and $\alpha \in [0,1]$, thus $O(A; \mathcal{X}) = 0$. As $O(U; \mathcal{X}) \ge 0$ for all $U \in \mathcal{F}_c(\mathbb{R}^p)$, we obtain

$$D_{FP}(A; \mathcal{X}) = 1 \ge \sup_{U \in \mathcal{F}_c(\mathbb{R}^p)} D_{FP}(U; \mathcal{X}).$$

Property P3a. It is not hard to show that $O(C; \mathcal{X})$ is a convex function in C, i.e.

$$O((1-\lambda) \cdot U + \lambda \cdot V; \mathcal{X}) \le (1-\lambda) \cdot O(U; \mathcal{X}) + \lambda \cdot O(V; \mathcal{X})$$

for all $U, V \in \mathcal{F}_c(\mathbb{R}^p)$ and $\lambda \in [0,1]$, using the linearity of the support function, the triangle inequality and the fact that a sum of suprema majorizes the supremum of sums. Then, taking any $A, B \in \mathcal{F}_c(\mathbb{R}^p)$ such that A maximizes $D_{FP}(\cdot; \mathcal{X})$,

$$D_{FP}((1-\lambda)\cdot A+\lambda\cdot B;\mathcal{X}) = (1+O((1-\lambda)\cdot A+\lambda\cdot B;\mathcal{X}))^{-1} \geq$$
$$(1+(1-\lambda)\cdot O(A;\mathcal{X})+\lambda\cdot O(B;\mathcal{X}))^{-1} \geq$$
$$(1+O(B;\mathcal{X}))^{-1} = D_{FP}(B;\mathcal{X}).$$

Property P3b. By Lemma A.1, P3a and P3b are equivalent for all ρ_r with $r\in(1,\infty)$.

Property P4b. Let $r\in[1,\infty)$. Let $A\in\mathcal{F}_c(\mathbb{R}^p)$ maximize $D_{FP}(\cdot;\mathcal{X})$ and let $\{A_n\}_n$ be a sequence of fuzzy sets such that $\lim_n \rho_r(A,A_n)=\infty$. As $\rho_r(A,A_n)\leq d_\infty(A,A_n)$ for every $n\in\mathbb{N}$, we have $\lim_n d_\infty(A,A_n)=\infty$. By the triangle inequality,

$$\lim_n d_\infty(A_n,\mathrm{I}_{\{0\}})=\infty. \tag{19}$$

Let us denote by $A_{n,\alpha}$ the α-level of A_n. As $d_H(A_{n,\alpha},\{0\})=\sup\{\|x\| : x\in A_{n,\alpha}\}$ and $A_{n,\alpha}\subseteq A_{n,0}$ for all $\alpha\in[0,1]$ and $n\in\mathbb{N}$, we have

$$d_\infty(A_n,\mathrm{I}_{\{0\}})=d_H(A_{n,0},\{0\})=\sup\{\|x\| : x\in A_{n,0}\}. \tag{20}$$

Since the norm is continuous as a function and each $A_{n,0}$ is compact, the supremum is attained at some $x_n\in A_{n,0}$. Thus

$$\lim_n \|x_n\| = \lim_n d_\infty(A_n,\mathrm{I}_{\{0\}})=\infty.$$

In particular, some e_i in the standard basis $\{e_1,\dots,e_p\}$ of $\mathbb{R}^p$ is such that $\lim_n\langle e_i,x_n\rangle=\infty$. As $\langle e_i,x_n\rangle\leq s_{A_n}(e_i,0)$ for every $n\in\mathbb{N}$, we have $\lim_n s_{A_n}(e_i,0)=\infty$. Taking this into account, since $\mathrm{med}(s_{\mathcal{X}}(e_i,0))\in\mathbb{R}$ and $\mathrm{MAD}(s_{\mathcal{X}}(e_i,0))\in[0,\infty)$,

$$\lim_{n\to\infty} O(A_n;\mathcal{X}) \geq \lim_{n\to\infty}\frac{\left|s_{A_n}(e_i,0)-\mathrm{med}(s_{\mathcal{X}}(e_i,0))\right|}{\mathrm{MAD}(s_{\mathcal{X}}(e_i,0))}=\infty.$$

Then, $\lim_n D_{FP}(A_n;\mathcal{X})=0$, and D_{FP} satisfies P4b for the ρ_r metric for every $r\in[1,\infty)$, as well as for d_∞.

Now, let $\{A_n\}_n$ be a sequence such that $\lim_n d_r(A,A_n)=\infty$ for some $r\in[1,\infty)$. As $d_r(A,A_n)\leq d_\infty(A,A_n)$ for every $n\in\mathbb{N}$, the same proof establishes P4b for d_r.

Property P4a. By Lemma A.2, P4b for the ρ_r-metric implies P4a, for any $r\in[1,\infty)$. □

Proof of Proposition 4.8. Case 1 (ρ_r and ρ_r^r). For $r\in[1,\infty)$ and $\mathcal{X}\in L^r[\mathcal{F}_c(\mathbb{R}^p)]$,

$$\rho_r((1-\lambda)\cdot A+\lambda\cdot B,\mathcal{X}) = \|s_{(1-\lambda)\cdot A+\lambda\cdot B}-s_{\mathcal{X}}\|_r =$$
$$\|(1-\lambda)\cdot s_A+\lambda\cdot s_B-s_{\mathcal{X}}\|_r = \|(1-\lambda)\cdot(s_A-s_{\mathcal{X}})+\lambda\cdot(s_B-s_{\mathcal{X}})\|_r \leq$$
$$\|(1-\lambda)\cdot(s_A-s_{\mathcal{X}})\|_r+\|\lambda\cdot(s_B-s_{\mathcal{X}})\|_r = (1-\lambda)\cdot\rho_r(A,\mathcal{X})+\lambda\cdot\rho_r(B,\mathcal{X}),$$

for every $A,B\in\mathcal{F}_c(\mathbb{R}^p)$ and $\lambda\in[0,1]$, where the inequality is due to the triangle inequality and the second equality due to the linearity of the support function.

Now let us consider the function $f:[0,\infty)\to[0,\infty)$ defined by $f(x)=x^r$. The function f is convex and increasing, thus

$$\rho_r((1-\lambda)\cdot A+\lambda\cdot B,\mathcal{X})^r = f\left(\rho_r((1-\lambda)\cdot A+\lambda\cdot B,\mathcal{X})\right) \leq$$
$$f\left((1-\lambda)\cdot\rho_r(A,\mathcal{X})+\lambda\cdot\rho_r(B,\mathcal{X})\right) \leq (1-\lambda)\cdot f\left(\rho_r(A,\mathcal{X})\right)+\lambda\cdot f\left(\rho_r(B,\mathcal{X})\right) =$$
$$(1-\lambda)\cdot\rho_r(A,\mathcal{X})^r+\lambda\cdot\rho_r(B,\mathcal{X})^r,$$

for all $A,B\in\mathcal{F}_c(\mathbb{R}^p)$ and $\lambda\in[0,1]$.

Case 2 ($d_{r,\theta}$ and $d_{r,\theta}^r$). Let $r\in[1,\infty)$, $\theta\in[0,\infty)$ and $\mathcal{X}\in L^r[\mathcal{F}_c(\mathbb{R}^p)]$. The mapping

$$(\|\cdot\|_r^r+\theta\|\cdot\|_r^r)^{1/r} : L^r(\mathbb{S}^{p-1}\times[0,1],\mathcal{V}_p\otimes\nu)\oplus_r L^r(\mathbb{S}^{p-1}\otimes[0,1],\theta^{1/r}\cdot(\mathcal{V}_p\otimes\nu))\to[0,\infty)$$

is a norm. We identify each $A\in\mathcal{F}_c(\mathbb{R}^p)$ with the pair

$$(\mathrm{mid}(s_A),\mathrm{spr}(s_A))\in L^r(\mathbb{S}^{p-1}\times[0,1],\mathcal{V}_p\otimes\nu)\oplus_r L^r(\mathbb{S}^{p-1}\otimes[0,1],\theta^{1/r}\cdot(\mathcal{V}_p\otimes\nu))$$

Using the properties of mid, spr and support functions one obtains

$$h(s_{(1-\lambda)\cdot A})+h(s_{\lambda\cdot B})=h(s_{(1-\lambda)\cdot A+\lambda\cdot B})$$

for every $h \in \{\text{mid}, \text{spr}\}$, $A, B \in \mathcal{F}_c(\mathbb{R}^p)$ and $\lambda \in [0,1]$. Now

$$
\begin{aligned}
& d_{r,\theta}((1-\lambda)\cdot A + \lambda \cdot B; \mathcal{X}) = \\
& \left(\| \,\text{mid}(s_{(1-\lambda)\cdot A+\lambda\cdot B}) - \text{mid}(s_{\mathcal{X}})\|_r^r + \theta \cdot \| \,\text{spr}(s_{(1-\lambda)\cdot A+\lambda\cdot B}) - spr(s_{\mathcal{X}})\|_r^r \right)^{1/r} = \\
& \Big(\|(1-\lambda)\cdot(\text{mid}(s_A) - \text{mid}(s_{\mathcal{X}})) + \lambda\cdot(\text{mid}(s_B) - \text{mid}(s_{\mathcal{X}}))\|_r^r + \\
& \theta \cdot \|(1-\lambda)\cdot(\text{spr}(s_A) - \text{spr}(s_{\mathcal{X}})) + \lambda\cdot(\text{spr}(s_B) - \text{spr}(s_{\mathcal{X}}))\|_r^r \Big)^{1/r} \leq \\
& \left(\|(1-\lambda)\cdot(\text{mid}(s_A) - \text{mid}(s_{\mathcal{X}}))\|_r^r + \theta \cdot \|(1-\lambda)\cdot(\text{spr}(s_A) - \text{spr}(s_{\mathcal{X}}))\|_r^r \right)^{1/r} + \\
& \left(\|\lambda\cdot(\text{mid}(s_B) - \text{mid}(s_{\mathcal{X}}))\|_r^r + \theta \cdot \|\lambda\cdot(\text{spr}(s_B) - \text{spr}(s_{\mathcal{X}}))\|_r^r \right)^{1/r} = \\
& (1-\lambda)\cdot d_{r,\theta}(A,\mathcal{X}) + \lambda \cdot d_{r,\theta}(B,\mathcal{X})
\end{aligned}
$$

where the inequality is due to the triangle inequality for the norm $(\|\cdot\|_r^r + \theta\|\cdot\|_r^r)^{1/r}$.

The proof for $d_{r,\theta}^r$ is analogous to that of ρ_r^r. □

Proof of Proposition 4.10. Let $r \in [1,\infty]$ and let M, A and B be as in P1∗. For D_r and RD_r, it suffices to prove that for every $\omega \in \Omega$ we have $\rho_r(A, \mathcal{X}(\omega)) = \rho_r(M\cdot A, M\cdot\mathcal{X}(\omega))$, as, by (9), clearly $\rho_r(M\cdot A + B, M\cdot\mathcal{X}(\omega) + B) = \rho_r(M\cdot A, M\cdot\mathcal{X}(\omega))$. Since

$$
\rho_r(M\cdot A, M\cdot\mathcal{X}(\omega)) = \left(\int_{[0,1]} \int_{\mathbb{S}^{p-1}} |s_{M\cdot A}(u,\alpha) - s_{M\cdot\mathcal{X}(\omega)}(u,\alpha)|^r \, \mathrm{d}\mathcal{V}_p(u)\, \mathrm{d}\nu(\alpha) \right)^{1/r},
$$

using (7) and the orthogonality of M,

$$
\rho_r(M\cdot A, M\cdot\mathcal{X}(\omega)) = \left(\int_{[0,1]} \int_{\mathbb{S}^{p-1}} \left| s_A\left(M^T\cdot u, \alpha\right) - s_{\mathcal{X}(\omega)}\left(M^T\cdot u,\alpha\right)\right|^r \mathrm{d}\mathcal{V}_p(u)\, \mathrm{d}\nu(\alpha) \right)^{1/r}.
$$

With the change of variable $v = M^T u$ and the notation $M = (m_{i,j})_{i,j}$, $u = (u_1,\dots,u_p)$ and $v = (v_1,\dots,v_p)$, we have $u_i = \sum_{j=1}^p m_{i,j}\cdot v_j$. Thus, the domain of integration remains the same and the Jacobian determinant is $\det(J(Mv)) = \det(M)$. By the orthogonality, $\det(M) = \pm 1$ and $|\det(J(Mv))| = 1$. Thus

$$
\rho_r(M\cdot A, M\cdot\mathcal{X}(\omega)) = \left(\int_{[0,1]} \int_{\mathbb{S}^{p-1}} |s_A(v,\alpha) - s_{\mathcal{X}(\omega)}(v,\alpha)|^r \, \mathrm{d}\mathcal{V}_p(v)\, \mathrm{d}\nu(\alpha) \right)^{1/r} = \rho_r(A,\mathcal{X}(\omega)).
$$

The proof for D_r^θ and RD_r^θ, $\theta \in [0,\infty)$ follows similar ideas, as shown next. It suffices to prove that

$$
\begin{aligned}
&\| \,\text{mid}(s_{M\cdot\mathcal{X}(\omega)}) - \text{mid}(s_{M\cdot A})\|_r = \| \,\text{mid}(s_{\mathcal{X}(\omega)}) - \text{mid}(s_A)\|_r \text{ and} \\
&\| \,\text{spr}(s_{M\cdot\mathcal{X}(\omega)}) - \text{spr}(s_{M\cdot A})\|_r = \| \,\text{spr}(s_{\mathcal{X}(\omega)}) - \text{spr}(s_A)\|_r
\end{aligned}
$$

for any orthogonal matrix $M \in \mathcal{M}_{p\times p}(\mathbb{R})$ and $\omega \in \Omega$.

As before, by (7) and the orthogonality of M,

$$
\begin{aligned}
&\| \,\text{mid}(s_{M\cdot\mathcal{X}(\omega)}) - \text{mid}(s_{M\cdot A})\|_r \\
&= \left(\int_{[0,1]} \int_{\mathbb{S}^{p-1}} | \,\text{mid}(s_{M\cdot\mathcal{X}(\omega)})(u,\alpha) - \text{mid}(s_{M\cdot A})(u,\alpha)|^r \, \mathrm{d}\mathcal{V}_p(u)\, \mathrm{d}\nu(\alpha) \right)^{1/r} \\
&= \left(\int_{[0,1]} \int_{\mathbb{S}^{p-1}} \left|\text{mid}(s_{\mathcal{X}(\omega)})\left(M^T\cdot u,\alpha\right) - \text{mid}(s_A)\left(M^T\cdot u,\alpha\right)\right|^r \mathrm{d}\mathcal{V}_p(u)\, \mathrm{d}\nu(\alpha) \right)^{1/r}
\end{aligned}
$$

Again, with the change of variable $v = M^T \cdot u$ we obtain

$$\begin{aligned}
&\left\|\mathrm{mid}(s_{M\cdot\mathcal{X}(\omega)}) - \mathrm{mid}(s_{M\cdot A})\right\|_r \\
&= \left(\int_{[0,1]} \int_{\mathbb{S}^{p-1}} |\,\mathrm{mid}(s_{\mathcal{X}(\omega)})(v,\alpha) - \mathrm{mid}(s_A)(v,\alpha)|^r \, \mathrm{d}\mathcal{V}_p(v)\, \mathrm{d}\nu(\alpha) \right)^{1/r} \\
&= \|\,\mathrm{mid}(s_{\mathcal{X}(\omega)}) - \mathrm{mid}(s_A)\|_r
\end{aligned}$$

The proof for the spread function is analogous. □

Proof of Lemma 4.13. For any $\omega \in \Omega$ and $\alpha \in [0,1]$ we have $\mathcal{X}_\alpha(\omega) \subseteq \mathcal{X}_0(\omega)$, which implies $|s_{\mathcal{X}(\omega)}(u,0)| \geq |s_{\mathcal{X}(\omega)}(u,\alpha)|$ for each $u \in \mathbb{S}^{p-1}$. Thus

$$\|\mathcal{X}_0(\omega)\|^r = \sup_u |s_{\mathcal{X}(\omega)}(u,0)|^r = \sup_{u,\alpha} |s_{\mathcal{X}(\omega)}(u,\alpha)|^r \geq \rho_r(\mathcal{X}(\omega), \mathrm{I}_{\{0\}})^r.$$

The inequality holds because the integrand in the definition of $\rho_s(\mathcal{X}(\omega), \mathrm{I}_{\{0\}})$ is precisely $|s_{\mathcal{X}(\omega)}(u,\alpha)|$. Taking expectations in both sides,

$$\mathrm{E}[\rho_s(\mathrm{I}_{\{0\}}, \mathcal{X}(\omega))^r] \leq \mathrm{E}[\|\mathcal{X}_0\|^r] < \infty$$

because $\mathcal{X} \in L^r[\mathcal{F}_c(\mathbb{R}^p)]$. □

Proof of Lemma 4.14. Fix $\theta \in [0,\infty)$ and $r \in [1,\infty)$. It suffices to prove $\mathrm{E}[d_{r,\theta}(\mathcal{X}, \mathrm{I}_{\{0\}})] < \infty$. By [30, Proposition 4.2],

$$\left(\int_{\mathbb{S}^{p-1}} |\,\mathrm{mid}(s_A)(u,\alpha)|^r + \theta \cdot |\,\mathrm{spr}(s_A)(u,\alpha)|^r \, \mathrm{d}\mathcal{V}_p(u) \right)^{1/r} \leq d_H(A_\alpha, \{0\}) \leq \|A_0\|,$$

for any $A \in \mathcal{F}_c(\mathbb{R}^p)$ and $\alpha \in [0,1]$. From this and (13) we obtain

$$\mathrm{E}[d_{r,\theta}(\mathrm{I}_{\{0\}}, \mathcal{X})] \leq \quad \mathrm{E}\left[\left(\int_{[0,1]} \|\mathcal{X}_0\|^r \, \mathrm{d}\nu(\alpha) \right)^{1/r} \right] = \mathrm{E}[\|\mathcal{X}_0\|] < \infty$$

because $\mathcal{X}$ is integrably bounded. □

Proof of Proposition 4.15. Let $\mathcal{X} \in \mathcal{H}$ be F-symmetric with respect to $A \in \mathcal{F}_c(\mathbb{R}^p)$. As stated in (10), $s_A(u,\alpha) \in \mathrm{Med}(s_{\mathcal{X}}(u,\alpha))$ for all $u \in \mathbb{S}^{p-1}$ and $\alpha \in [0,1]$. Because of that and since the medians of the integrable random variable $s_{\mathcal{X}}(u,\alpha)$ minimize the expected absolute deviation,

$$s_A(u,\alpha) \in \mathrm{argmin}_{x\in\mathbb{R}}\, \mathrm{E}(|s_{\mathcal{X}}(u,\alpha) - x|) \tag{21}$$

for each $u \in \mathbb{S}^{p-1}$ and $\alpha \in [0,1]$. Consider $E[\rho_1(U,\mathcal{X})]$ for any fixed $U \in \mathcal{F}_c(\mathbb{R}^p)$. Since the function $s_{\mathcal{X}}$ is jointly measurable in its three arguments [18, Lemma 4], by Fubini's theorem and (9)

$$\mathrm{E}[\rho_1(U,\mathcal{X})] = \int_{[0,1]} \int_{\mathbb{S}^{p-1}} \mathrm{E}[|s_{\mathcal{X}}(u,\alpha) - s_U(u,\alpha)|] \, \mathrm{d}\mathcal{V}_p(u)\, \mathrm{d}\nu(\alpha).$$

Applying (21) now,

$$\mathrm{E}[\rho_1(U,\mathcal{X})] \geq \int_{[0,1]} \int_{\mathbb{S}^{p-1}} \mathrm{E}[|s_{\mathcal{X}}(u,\alpha) - s_A(u,\alpha)|] \, \mathrm{d}\mathcal{V}_p(u)\, \mathrm{d}\nu(\alpha) = \mathrm{E}[\rho_1(\mathcal{X}, A)].$$

Then $D_1(U;\mathcal{X}) \leq D_1(A;\mathcal{X})$. By the arbitrariness of U, property P2 is satisfied. □

Proof of Proposition 4.16. Let $\mathcal{X} \in \mathcal{H}$ be (mid, spr)-symmetric with respect to $A \in \mathcal{F}_c(\mathbb{R}^p)$. Applying the same reasoning in the proof of Proposition 4.15, but using (11) instead of (10), to the mid and spr functions separately, we obtain $D_1^\theta(A;\mathcal{X}) \geq D_1^\theta(U;\mathcal{X})$ for all $U \in \mathcal{F}_c(\mathbb{R}^p)$ and $\theta \in [0,\infty)$. □

Proof of Proposition 4.17. Let $\mathcal{X} \in \mathcal{H}$ be F-symmetric with respect to some $A \in \mathcal{F}_c(\mathbb{R}^p)$. This implies $\mathrm{E}[\|\mathcal{X}_0\|] < \infty$ and hence $\mathrm{E}[s_{\mathcal{X}}(u,\alpha)] < \infty$ for all $u \in \mathbb{S}^{p-1}$ and $\alpha \in [0,1]$. By the definition of F-symmetry, the real random variable $s_{\mathcal{X}}(u,\alpha)$ is centrally symmetric with respect to $s_A(u,\alpha)$ for all $u \in \mathbb{S}^{p-1}$ and $\alpha \in [0,1]$. By Lemma A.3, $s_A(u,\alpha) = \mathrm{E}[s_{\mathcal{X}}(u,\alpha)]$ for all $u \in \mathbb{S}^{p-1}$ and $\alpha \in [0,1]$. For any square integrable random variable, $\mathrm{E}[X] = \mathrm{argmin}_{y\in\mathbb{R}} \mathrm{E}[|X-y|^2]$. Then, since $\mathcal{X} \in L^2[\mathcal{F}_c(\mathbb{R}^p)]$,

$$s_A(u,\alpha) = \mathrm{argmin}_{U\in\mathcal{F}_c(\mathbb{R}^p)} \mathrm{E}[|s_{\mathcal{X}}(u,\alpha) - s_U(u,\alpha)|^2] \tag{22}$$

for each $u \in \mathbb{S}^{p-1}$ and $\alpha \in [0,1]$. Like in Proposition 4.15, applying Fubini's theorem and (22), we obtain $RD_2(U;\mathcal{X}) \le RD_2(A;\mathcal{X})$ for all $U \in \mathcal{F}_c(\mathbb{R}^p)$. Thus RD_2 satisfies P2. $\square$

Proof of Proposition 4.18. Let $\theta \in [0,\infty)$ and let $\mathcal{X} \in \mathcal{H}$ be (mid, spr)-symmetric with respect to $A \in \mathcal{F}_c(\mathbb{R}^p)$. By applying the same reasoning in the proof of Proposition 4.17 but taking into account $\mathrm{mid}(s_A)(u,\alpha) = \mathrm{E}[\mathrm{mid}(s_{\mathcal{X}})(u,\alpha)]$ and $\mathrm{spr}(s_A)(u,\alpha) = \mathrm{E}[\mathrm{spr}(s_{\mathcal{X}})(u,\alpha)]$ for every $u \in \mathbb{S}^{p-1}$ and $\alpha \in [0,1]$, one obtains $RD_2^\theta(A;\mathcal{X}) \ge RD_2^\theta(U;\mathcal{X})$ for all $U \in \mathcal{F}_c(\mathbb{R}^p)$ and $\theta \in [0,\infty)$. $\square$

Proof of Theorem 4.20. Let $\mathcal{X} \in \mathcal{H}$ be functionally symmetric with respect to $A \in \mathcal{F}_c(\mathbb{R}^p)$ and $r \in [1,\infty)$. By Lemma 4.13, $D_r(\cdot;\mathcal{X})$ is well defined. To reach

$$D_r(A;\mathcal{X}) \ge \sup_{U\in\mathcal{F}_c(\mathbb{R}^p)} D_r(U;\mathcal{X})$$

it suffices to prove

$$\mathrm{E}[\|s_{\mathcal{X}} - s_A\|_r] \le \inf_{U\in\mathcal{F}_c(\mathbb{R}^p)} \mathrm{E}[\|s_{\mathcal{X}} - s_U\|_r]. \tag{23}$$

Let us denote by $\mathcal{B}$ the Banach space $L^r(\mathbb{S}^{p-1}\times[0,1], \mathcal{V}_p \otimes \nu)$. As the norm is a convex function, for every $f \in \mathcal{B}$

$$\mathrm{E}[\|s_{\mathcal{X}} - s_A\|_r] \le \frac{1}{2}\cdot \mathrm{E}[\|s_{\mathcal{X}} - s_A + f\|_r] + \frac{1}{2}\cdot \mathrm{E}[\|s_{\mathcal{X}} - s_A - f\|_r]. \tag{24}$$

Since $\mathcal{X}$ is functionally symmetric with respect to A, we know $s_{\mathcal{X}} - s_A$ and $s_A - s_{\mathcal{X}}$ are identically distributed. Thence $\|s_{\mathcal{X}} - s_A + f\|_r$ and $\|-s_{\mathcal{X}} + s_A + f\|_r$ are identically distributed and the right-hand side of (24) equals

$$\frac{1}{2}\cdot \mathrm{E}[\|-(s_{\mathcal{X}} - s_A - f)\|_r] + \frac{1}{2}\cdot \mathrm{E}[\|s_{\mathcal{X}} - s_A - f\|_r] = \quad \mathrm{E}[\|s_{\mathcal{X}} - s_A - f\|_r].$$

Therefore

$$\mathrm{E}[\|s_{\mathcal{X}} - s_A\|_r] \le \mathrm{E}[\|s_{\mathcal{X}} - s_A - f\|_r]$$

for each $f \in \mathcal{B}$ and

$$\mathrm{E}[\|s_{\mathcal{X}} - s_A\|_r] \le \inf_{g\in\mathcal{B}} \mathrm{E}[\|s_{\mathcal{X}} - g\|_r] \le \inf_{U\in\mathcal{F}_c(\mathbb{R}^p)} \mathrm{E}[\|s_{\mathcal{X}} - s_U\|_r]$$

taking all possible $g = s_A + f \in \mathcal{B}$ and using the inclusion $\{s_U : U \in \mathcal{F}_c(\mathbb{R}^p)\} \subseteq \mathcal{B}$. $\square$

Proof of Theorem 4.21. Let $r \in [1,\infty)$, $\mathcal{J} = \mathcal{F}_c(\mathbb{R}^p)$, $\mathcal{H}_1 \subseteq L^1[\mathcal{F}_c(\mathbb{R}^p)]$ and $\mathcal{H}_r \subseteq L^r[\mathcal{F}_c(\mathbb{R}^p)]$.

Property P3a. By Proposition 4.8, the mappings $d_r(\cdot,\cdot)$ and $d_r^r(\cdot,\cdot)$ are convex in their first argument. Lemma 4.7 yields D_r based on $\mathcal{J}$ and $\mathcal{H}_1$, as well as RD_r based on $\mathcal{J}$ and $\mathcal{H}_r$, satisfy P3a. Notice Lemma 4.13 ensures that the integrability assumption in Lemma 4.7 is satisfied, for the classes $\mathcal{H}_1$ and $\mathcal{H}_r$ in the statement.

Property P3b. By Lemma A.1, P3a and P3b are equivalent for the ρ_s metric for every $s \in (1,\infty)$.

Now, for the $d_{s,\theta}$-metrics we want to apply Lemma A.1 as well, with $s \in (1,\infty)$ and $\theta \in (0,\infty)$. The mapping

$$j : \mathcal{F}_c(\mathbb{R}^p) \to L^s(\mathbb{S}^{p-1}\otimes[0,1], \mathcal{V}_p\otimes\nu) \oplus_s L^s(\mathbb{S}^{p-1}\otimes[0,1], \theta^{(1/r)}\cdot(\mathcal{V}_p\otimes\nu))$$

defined by $j(A) = (\mathrm{mid}(s_A), \mathrm{spr}(s_A))$ is an isometry, considering in $\mathcal{F}_c(\mathbb{R}^p)$ the metric $d_{s,\theta}$ and in $L^s(\mathbb{S}^{p-1}\otimes[0,1], \mathcal{V}_p\otimes\nu) \oplus_s L^s(\mathbb{S}^{p-1}\otimes[0,1], \theta^{(1/s)}\cdot(\mathcal{V}_p\otimes\nu))$ the distance induced by its norm $(\|\cdot\|_s^s + \theta\cdot\|\cdot\|_s^s)^{1/s}$. It is clear from its definition that $d_{s,\theta}$ fulfills A1 and A2. In order to use the lemma, we need to show that the Banach space

$$\left(L^s(\mathbb{S}^{p-1}\otimes[0,1], \mathcal{V}_p\otimes\nu) \oplus_s L^s(\mathbb{S}^{p-1}\otimes[0,1], \theta^{(1/s)}\cdot(\mathcal{V}_p\otimes\nu)), (\|\cdot\|_s^s + \theta\cdot\|\cdot\|_s^s)^{1/s}\right)$$

is strictly convex.

Let us define the mapping $\psi : [0,1] \to [0,1]$ with

$$\psi(t) = \left((1-t)^s + \theta\cdot t^s\right)^{1/s}.$$

It is easy to show

$$\left(\|f\|_s^s + \theta\cdot\|g\|_s^s\right)^{1/s} = \left(\|f\|_s + \|g\|_s\right)\cdot\psi\left(\frac{\|g\|_s}{\|f\|_s + \|g\|_s}\right)$$

for every $(f,g) \in L^s(\mathbb{S}^{p-1} \otimes [0,1], \mathcal{V}_p \otimes \nu) \oplus_s L^s(\mathbb{S}^{p-1} \otimes [0,1], \theta^{(1/s)} \cdot (\mathcal{V}_p \otimes \nu))$. By [29, Theorem 6], the Banach space $L^s(\mathbb{S}^{p-1} \otimes [0,1], \mathcal{V}_p \otimes \nu) \oplus_s L^s(\mathbb{S}^{p-1} \otimes [0,1], \theta^{(1/s)} \cdot (\mathcal{V}_p \otimes \nu))$ will be strictly convex if and only if $L^s(\mathbb{S}^{p-1} \otimes [0,1], \mathcal{V}_p \otimes \nu)$ and $L^s(\mathbb{S}^{p-1} \otimes [0,1], \theta^{1/s} \cdot (\mathcal{V}_p \otimes \nu))$ are strictly convex and the function ψ is strictly convex. For $s \in (1,\infty)$, L^s-spaces are always strictly convex (e.g., [2, p. 114]), and Ψ is strictly convex as $\Psi''(t) > 0$ for $t \in (0,1)$. Therefore, by Lemma A.1, P3b for $d_{s,\theta}$ is equivalent to P3a, which has already been established.

Property P4b. Let $\mathcal{X} \in \mathcal{H}_1$ be a fuzzy random variable and $A \in \mathcal{J}$ a fuzzy set maximizing $D_r(\cdot;\mathcal{X})$. Let us first prove the case $s = r$. Let $\{A_n\}_n$ be a sequence of fuzzy sets in $\mathcal{J}$ such that

$$\lim_n \rho_r(A_n, A) = \infty. \tag{25}$$

As $r \geq 1$, by Lemma 4.13 $\mathrm{E}[\rho_r(I_{\{0\}}, \mathcal{X})]$ is finite. As $\rho_r(I_{\{0\}}, A)$ is a constant, applying the triangle inequality to $\rho_r(A, \mathcal{X})$, we obtain

$$\mathrm{E}[\rho_r(A, \mathcal{X})] < \infty. \tag{26}$$

Using again the triangle inequality,

$$\mathrm{E}[\rho_r(A_n, \mathcal{X})] \geq \mathrm{E}[\rho_r(A_n, A) - \rho_r(A, \mathcal{X})] = \rho_r(A_n, A) - \mathrm{E}[\rho_r(A, \mathcal{X})] \to \infty, \tag{27}$$

where the limit is obtained from (25) and (26). Accordingly, $D_r(A_n, \mathcal{X}) \to 0$.

For the general case, notice $\rho_s \leq \rho_r$ whenever $s \leq r$. Thus, $\rho_s(A_n, A) \to \infty$ implies $\rho_r(A_n, A) \to \infty$ and therefore $D_r(A_n; \mathcal{X}) \to 0$ by the former case.

That establishes the result for D_r under the ρ_s-metrics. Let us prove it now for RD_r.

Let $\mathcal{X} \in \mathcal{H}_r$. Like before, we will prove first the case $s = r$. By Jensen's inequality,

$$\mathrm{E}[\rho_r(A_n, \mathcal{X})^r] \geq \mathrm{E}[\rho_r(A_n, \mathcal{X})]^r. \tag{28}$$

From (27),

$$\lim_{n\to\infty} \mathrm{E}[\rho_r(A_n, \mathcal{X})^r] = \infty.$$

Consequently, $RD_r(A_n, \mathcal{X}) \to 0$. The general case follows as with D_r.

Now let us consider the $d_{s,\theta}$-metrics. Let $s = r$ and $\theta \in (0,\infty)$. Given a fuzzy random variable $\mathcal{X} \in \mathcal{H}_1$, a fuzzy set $A \in \mathcal{J}$ maximizing $D_r(\cdot;\mathcal{X})$ and a sequence $\{A_n\}_n$ in $\mathcal{J}$ such that

$$\lim_n d_{s,\theta}(A_n, A) = \infty. \tag{29}$$

By Lemma 4.14, $\mathrm{E}[d_{s,\theta}(\mathrm{I}_{\{0\}}, \mathcal{X})] < \infty$. By (29), $\lim_n d_{s,\theta}(A_n, A)^r = \infty$, whence

$$\lim_n \| \mathrm{mid}(s_{A_n}) - \mathrm{mid}(s_A)\|_s^r = \infty$$

or

$$\lim_n \| \mathrm{spr}(s_{A_n}) - \mathrm{spr}(s_A)\|_s^r = \infty$$

Since the other case is analogous, we assume without loss of generality $\| \mathrm{mid}(s_{A_n}) - \mathrm{mid}(s_A)\|_s^r \to \infty$. Moreover,

$$\begin{aligned}
&\| \mathrm{mid}(s_{A_n}) - \mathrm{mid}(s_A)\|_s \\
&= \left(\int_{[0,1]} \int_{\mathbb{S}^{p-1}} | \mathrm{mid}(s_{A_n})(u,\alpha) - \mathrm{mid}(s_A)(u,\alpha)|^s \, \mathrm{d}\mathcal{V}_p(u) \, \mathrm{d}\nu(\alpha) \right)^{1/s} \\
&= \frac{1}{2} \cdot \left(\int_{[0,1]} \int_{\mathbb{S}^{p-1}} |(s_{A_n}(u,\alpha) - s_A(u,\alpha)) + (s_A(-u,\alpha) - s_{A_n}(-u,\alpha))|^s \, \mathrm{d}\mathcal{V}_p(u) \, \mathrm{d}\nu(\alpha) \right)^{1/s} \\
&\leq \frac{1}{2} \cdot \left(\|s_{A_n} - s_A\|_s + \|s_{A_n} - s_A\|_s \right) = \rho_s(A, A_n) \leq \rho_r(A, A_n)
\end{aligned}$$

whence $\lim_n \rho_r(A_n, A) = \infty$. Thus, using the previous proof, the depth function D_r based on $\mathcal{J}$ and $\mathcal{H}_1$ fulfills P4b for $d_{s,\theta}$.

The case of RD_r based on $\mathcal{J}$ and $\mathcal{H}_r$ is done in an analogous way as in the case of ρ_s.

Property P4a. As ρ_r and $d_{r,\theta}$ metrics fulfill assumptions $A1$ and $A2$, property P4b implies P4a (Lemma A.2). □

Proof of Theorem 4.22. *Property P3a.* Like in the proof of Property P3a in Theorem 4.21, the mapping $(\|\cdot\|_r^r+\theta\cdot\|\cdot\|_r^r)^{1/r}$ is convex (because it is a norm) and, by Lemma 4.7, D_r^θ and RD_r^θ satisfy P3a for any $r\in[1,\infty)$ and $\theta\in[0,\infty)$.

Property P3b. By Lemma A.1, P3b is equivalent to P3a for the ρ_s metric if $s\in(1,\infty)$. In the proof of Theorem 4.21 it was shown that P3b is equivalent to P3a for the $d_{s,\theta}$ metric.

Property P4b. Let $\theta\in(0,\infty)$. Let $\mathcal{X}\in\mathcal{H}_1$ and let $A\in\mathcal{F}_c(\mathbb{R}^p)$ be a fuzzy set that maximizes $D_r^\theta(\cdot,\mathcal{X})$. We consider a sequence $\{A_n\}_n$ of fuzzy sets such that $\rho_r(A_n,A)\to\infty$. By the triangle inequality, for any $h\in\{\text{mid},\text{spr}\}$,

$$\begin{aligned}\mathrm{E}[\|h(s_{\mathcal{X}})-h(s_{A_n})\|_r]&\geq E[\|h(s_{A_n})-h(s_A)\|_r-\|h(s_{\mathcal{X}})-h(s_A)\|_r]\\&=\|h(s_A)-h(s_{A_n})\|_r-\mathrm{E}[\|h(s_{\mathcal{X}})-h(s_A)\|_r].\end{aligned}\tag{30}$$

On the other hand, as $\lim_n\rho_r(A,A_n)=\infty$ and ρ_r is a metric, the triangle inequality yields $\lim_n\rho_r(A_n,\mathrm{I}_{\{0\}})=\infty$. By the decomposition given in (5),

$$\rho_r(A_n,\mathrm{I}_{\{0\}})=\left(\int_{[0,1]}\int_{\mathbb{S}^{p-1}}|\,\text{mid}(s_{A_n})(u,\alpha)+\text{spr}(s_{A_n})(u,\alpha)|^r\,\mathrm{d}\mathcal{V}_p(u)\,\mathrm{d}\nu(\alpha)\right)^{1/r}$$
$$=\|\,\text{mid}(s_{A_n})+\text{spr}(s_{A_n})\|_r\leq\|\,\text{mid}(s_{A_n})\|_r+\|\,\text{spr}(s_{A_n})\|_r.$$

Therefore $\lim_n\|\,\text{mid}(s_{A_n})\|_r=\infty$ and/or $\lim_n\|\,\text{spr}(s_{A_n})\|_r=\infty$. Since the other case is analogous, without loss of generality assume

$$\lim_n\|\,\text{mid}(s_{A_n})\|_r=\infty.\tag{31}$$

Because $\mathcal{X}$ is integrably bounded, by Lemma 4.14 we have $\mathrm{E}[d_{r,\theta}(\mathcal{X},\mathrm{I}_{\{0\}})]<\infty$, which implies

$$\mathrm{E}[\|\,\text{mid}(s_{\mathcal{X}})-\text{mid}(s_A)\|_r]<\infty.\tag{32}$$

Then

$$\begin{aligned}&E[d_{r,\theta}(A_n,\mathcal{X})]\geq\mathrm{E}[\|\,\text{mid}(s_{\mathcal{X}})-\text{mid}(s_{A_n})\|_r]\\&\geq\|\,\text{mid}(s_A)-\text{mid}(s_{A_n})\|_r-\mathrm{E}[\|\,\text{mid}(s_{\mathcal{X}})-\text{mid}(s_A)\|_r]\\&\geq\|\,\text{mid}(s_{A_n})\|_r-\|\,\text{mid}(s_A)\|_r-\mathrm{E}[\|\,\text{mid}(s_{\mathcal{X}})-\text{mid}(s_A)\|_r]\to\infty,\end{aligned}\tag{33}$$

where the first inequality is due to (13), the second one to (30) and the limit to (31) and (32). Consequently, $D_r^\theta(A_n;\mathcal{X})\to 0$. That proves the case $s=r$. The case $s<r$ follows like in the proof of Theorem 4.21.

Let us prove it now for RD_r^θ and the ρ_s-metrics.

Let $\theta\in(0,\infty)$. Let $\mathcal{X}\in\mathcal{H}_r$ and let $A\in\mathcal{F}_c(\mathbb{R}^p)$ maximize $RD_r^\theta(\cdot,\mathcal{X})$. Let $\{A_n\}_n$ be a sequence of fuzzy sets such that $\rho_r(A_n,A)\to\infty$. By Jensen's inequality,

$$\mathrm{E}[d_{r,\theta}(A_n,\mathcal{X})^r]\geq\mathrm{E}[d_{r,\theta}(A_n,\mathcal{X})]^r.$$

By (33),

$$\lim_{n\to\infty}\mathrm{E}[d_{r,\theta}(A_n,\mathcal{X})^r]=\infty.$$

Thus $RD_r^\theta(A_n,\mathcal{X})\to 0$. That establishes the case $s=r$. The case $s<r$ is deduced like in the proof of Theorem 4.21.

The proof of P4b for D_r^θ and RD_r^θ with $d_{s,\theta}$ is analogous to that of P4b for D_r and RD_r with respect to the ρ_s-metrics (see Theorem 4.21), taking into account the inequality $d_{s,\theta}\leq d_{r,\theta}$ for $s\in[1,r]$.

Property P4a. By Lemma A.2, property P4b for ρ_r implies P4a. □

Proof of Proposition 4.24. Let $(\Omega,\mathcal{A},\mathbb{P})$ be a probabilistic space such that $\Omega=\{\omega_1\}$, $\mathcal{A}=\mathcal{P}(\Omega)$ and let $r\in[1,\infty)$. We consider the fuzzy random variable $\mathcal{X}$ defined by $\mathcal{X}(\omega_1):=\mathrm{I}_{[-1,1]}$. Let $A=\mathcal{X}(\omega_1)$ and $A_n:=\mathrm{I}_{[-n,n]}$ for all $n\in\mathbb{N}$. It is clear that A maximizes $D_r^0(\cdot;\mathcal{X})$ with $D_r^0(A;\mathcal{X})=1$, and that $\text{mid}(s_B)(u,\alpha)=0$ for $B\in\{A,A_n\}$, $\text{spr}(s_A)(u,\alpha)=1$, and $\text{spr}(s_{A_n})(u,\alpha)=n$ for all $u\in\mathbb{S}^0,\alpha\in[0,1]$ and $n\in\mathbb{N}$. By the mid / spr decomposition (5),

$$\lim_{n\to\infty}\rho_r(A_n,A)=\lim_{n\to\infty}\Big(\int_{[0,1]}|n-1|^r d\alpha\Big)^{1/r}=\lim_{n\to\infty}|n-1|=\infty.$$

Taking into account $\mathrm{E}[d_{r,0}(A_n,\mathcal{X})]=0$ for all $n\in\mathbb{N}$, whence $D_r^0(A_n;\mathcal{X})=1$, i.e., D_r^0 fails P4b for ρ_r. In the case $r=1$, we have $RD_1^0(A_n;\mathcal{X})=D_1^0(A_n;\mathcal{X})$ so RD_r^0 can fail P4b as well.

To prove that D_r^0 and RD_r^0 violate P4a, we use $B:=\mathrm{I}_{[-2,2]}$. Let $r\in[1,\infty)$. Note $A+nB=\mathrm{I}_{[-1-2n,2n+1]}$ for all $n\in\mathbb{N}$. Clearly,

$$\text{mid}(s_{\mathcal{X}(\omega_1)})(u,\alpha)=0=\text{mid}(s_{A+nB})(u,\alpha)$$

for all $u \in \mathbb{S}^0$ and $\alpha \in [0,1]$. Thus $\mathrm{E}[d_{r,0}(A+nB,\mathcal{X})] = 0$ and

$$D_r^0(A+nB;\mathcal{X}) = 1 = RD_r^0(A+nB;\mathcal{X})$$

for all $n \in \mathbb{N}$ whence D_r^0 and RD_r^0 violate P4a.

A fortiori, by Lemma A.2, this is also a counterexample to property P4b for ρ_r, for any $r \in (1,\infty)$. □

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