French-German-Spanish Conference on Optimization
Gijón, June 18-21, 2024
(pp. 124-128)

Compatible TOSets with POSets: an application to additive manufacturing

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Abstract

Additive Manufacturing (AM) has become a widely used technique in 3D printing, but it has proven to be a very costly process, even when optimizing parameters in existing models. Due to the characteristics of AM, and in order to optimize its process, a new approach is introduced to the problem: the discretization of each layer to be printed. This involves establishing an order relation based on the sequence in which the layers should be printed. The valid orders for the execution of the process, referred to as compatible with the order relation, will be characterized. Additionally, algorithms will be provided to obtain new compatible orders from others that were already compatible, and strategies will be presented to optimally and efficiently reorder non-compatible orders, converting them into compatible ones.

1. Introduction

The presentation collects part of the ideas we developed to solve a problem presented to us by a company for optimizing the 3D-printing of an object. This process falls within the context of Additive Manufacturing (AM) in which, each object is created from a set of layers. The use of printing layers allows for the creation of objects with a virtually unlimited variety of geometries, adaptable to any requirements of the final product. Paired with the advantage of printing any imaginable geometry, it appears the drawback of the slowness and cost of this production process. The technology, energy and human resources employed have a very high cost, so minimizing processing time naturally becomes a desired goal for all companies using this production method.

As usual, the problem consists of two well-differentiated parts: Modeling and Resolution. The talk starts by explaining some results that have been found in modeling, and it will finish with others related to optimization. To model the problem were used binary relations, that means equivalence relations but more, Order Relations.To solve the problem, that is, to minimize the processing time, were used Genetic Algorithms.

Each of the layers of the object contains a large set of points. This set of points is the unique piece of information required to process the object, that means that having control over this set, turns into having control over the production of the object. To get this, it was necessary to order and classify these points in some way. The order in which the information is provided to the device is crucial since the execution time depends strongly on this arrangement.

After performing a series of classifications on the set of points, using certain order and equivalence relations, were obtained a Partially Ordered Set (POSet) with a computationally acceptable number (10-120) of elements (pieces). Observing the diagram associated with the POSet from the perspective of Graph Theory, the problem consists of a particular version of the Traveling Salesman Problem (TSP). This version is due to the idiosyncrasies of the machines we are working with; we might refer to it as the Constrained Traveling Salesman Problem (CTSP). To solve it, they are used genetic algorithm techniques.

2. Initial Definitions and Properties

Definition 2.1 A binary relation R defined on a set S is a subset of $S \times S$. If $(a, b) \in R$ it is said to be a is R-related to b. R is said to be an **order relation** or a **partial order relation** on S if it is:

- reflexive: $(a, a) \in R \; \forall a \in S$
- antisymmetric: $\forall a, b \in S$ if $(a, b) \in R$ and $(b, a) \in R$ then $a = b$
- transitive: $\forall a, b, c \in S$ if $(a, b) \in R$ and $(b, c) \in R$ then $(a, c) \in R$

A set S with a partial order relation is denoted by (S, R) and is known as **Partial Ordered Set** or **POSet**. If (S, R) is a POSet, then $a, b \in S$ are said to be **comparable** if aRb or bRa.
Let R be a binary relation defined on a set S, it is said to be a **total order relation** if it is an order relation and all the elements of S are comparable. If R is a total order (S, R) is said to be a **Totally Ordered Set**, or **TOSet**.

Definition 2.2 Given a set S with n elements, and a bijection from $I = \{1, 2, \ldots, n\}$ to S

$$\begin{array}{ccc} I & \rightarrow & S \\ i & \rightarrow & a_i \end{array}$$

This establishes an indexing by means I of the elements of S. Then, S is said to be an I**-indexed set** or an **indexed set**.

Definition 2.3 Given σ a permutation of elements of I

$$\begin{array}{rcc} \sigma : I & \rightarrow & I \\ i & \rightarrow & \sigma(i) \end{array}$$

an **ordering** or **permutation** of elements of S can be generated as

$$\begin{array}{ccc} I & \rightarrow & S \\ i & \rightarrow & a_{\sigma(i)} \end{array}$$

We can represent the permutation σ by the images of the bijection that σ defines from I to itself as $(\sigma(1), \sigma(2), \ldots, \sigma(n))$.

Definition 2.4 Let S be an indexed set with $Card(S) = n$ and R an order relation defined on S. We say that a matrix $M = (m_{ij})_{n\times n}$, is the **adjacency matrix** of (S, R) if it satifies:

$$m_{R,ij} = \begin{cases} 1 & \text{if } a_i R a_j \\ 0 & \text{otherwise} \end{cases}$$

Obviously, the adjacency matrix depends on the ordering in which the elements are taken. Thus, for each permutation $(\sigma(1), \sigma(2), \ldots, \sigma(n))$ of elements of S, a matrix will be obtained, denoted by M_R^σ, and whose elements are:

$$m^\sigma_{R,ij} = \begin{cases} 1 & \text{if } a_{\sigma(i)} R a_{\sigma(j)} \\ 0 & \text{otherwise} \end{cases}$$

When there is no doubt about the order relation, the adjacency matrix for the permutation defined by σ can be denoted M^σ, and, for simplicity, we denote by M the adjacency matrix for the main permutation $(1, 2, 3, \ldots, n)$.
We denote by $MS(R)$ the set of the adjacency matrices that represent the relation R defined on the set S.

Proposition 2.5 *Let be an indexed set $S = \{a_1, a_2, \ldots, a_n\}$, the total order relation R such that*

$$a_i R a_j \textit{ if and only if } j \leq i$$

i.e., the elements ordered from highest to lowest index, and the adjacency matrix M^σ for R of a permutation $\sigma = (\sigma(1), \ldots, \sigma(n))$ of elements of S, then for all $i_0 \in \{1, \ldots, n\}$

$$\sum_{j=1}^{n} m^\sigma_{j\,i_0} = n - \sigma(i_0) + 1 \qquad\qquad \sum_{j=1}^{n} m^\sigma_{i_0\,j} = \sigma(i_0)$$

Proposition 2.6 *Given an indexed set S with n elements and an order relation R*

$$1 \leq Card(MS(R) \leq n!$$

If the order relation is total then $Card(MS(R)) = n!$

3. Compatibility

Definition 3.1 Given a permutation $\sigma = (\sigma(1), \sigma(2), \dots, \sigma(n))$ of elements of $I = \{1, 2, \dots, n\}$, we define the **relation induced by** σ **on** S and denote it by T_σ the relation defined as:

$$(a_i, a_j) \in T_\sigma \Leftrightarrow \sigma^{-1}(j) \leq \sigma^{-1}(i)$$

It is easy to see that, thus defined, this is a total order relation on S.

Definition 3.2 Let $S = \{a_1, a_2, \dots, a_n\}$ be an indexed set and let R be an order relation defined on S. A permutation σ of the elements of S is said to be **compatible with the relation** R if $R \subseteq T_\sigma$. We denote the set of permutations compatible with the relation R by $C(S, R)$.

Theorem 3.3 *Let (S, R) be an ordered indexed set and σ a permutation of elements of I; then*

$$\sigma \text{ is compatible with the relation } R \Leftrightarrow M_R^\sigma \text{ is lower triangular.}$$

Theorem 3.4 *Given an indexed POSet (S, R) with n elements, there is always a compatible permutation.*

Proof In a finite POSet, there always exist maximal elements. Let's assume there are μ_1 of these maximal elements.

Consider these maximal elements of (S, R),

$$M_{11}, M_{12}, \dots, M_{1\mu_1}$$

denoting by $M_{11} = a_{\sigma(1)}, M_{12} = a_{\sigma(2)}, \dots, M_{1\mu_1} = a_{\sigma(\mu_1)}$, we construct

$$(\sigma(1), \sigma(2), \dots, \sigma(\mu_1))$$

which verifies that if $i, j \in \{1, 2, \dots, \mu_1\}$, $a_{\sigma(i)}$ and $a_{\sigma(j)}$ are not comparable.

Let us now consider the set $S_1 = S - \{a_{\sigma(1)}, a_{\sigma(2)}, \dots, a_{\sigma(\mu_1)}\}$, and the restriction of R on S_1 that we denote R_1. As in the previous step, let's suppose that there are r_2 maximal elements, and, denoting $\mu_2 = \mu_1 + r_2$ and

$$M_{21} = a_{\sigma(\mu_1+1)}, M_{22} = a_{\sigma(\mu_1+2)}, \dots, M_{2r_2} = a_{\sigma(\mu_2)},$$

we add them to the previously constructed permutation, obtaining

$$(\sigma(1), \sigma(2), \dots, \sigma(\mu_1), \sigma(\mu_1 + 1), \dots, \sigma(\mu_2))$$

that verifies

- If $1 \leq i, j \leq \mu_1 \Rightarrow a_{\sigma(i)}$ and $a_{\sigma(j)}$ are not comparable
- If $\mu_1 < i, j \leq \mu_2 \Rightarrow a_{\sigma(i)}$ and $a_{\sigma(j)}$ are not comparable
- If $1 \leq i \leq \mu_1 < j \leq \mu_2 \Rightarrow$ as $a_{\sigma(j)}$ is maximal in S_1, $a_{\sigma(i)}$ is maximal in S and $S_1 \subseteq S$ therefore $a_{\sigma(i)}$ and $a_{\sigma(j)}$ are not comparable or $a_{\sigma(j)} R a_{\sigma(i)}$.

Repeating the process $k - 1$ times considering the set $S_k = S - \{a_{\sigma(1)}, \dots, a_{\sigma(\mu_k)}\}$ and taking the maximal elements of the poset (S_k, R_k) being R_k, the restriction of R to the set S_k, we will obtain, after a finite number of steps, a permutation of the n elements of S

$$\sigma = (\sigma(1), \sigma(2), .., \sigma(\mu_1), \sigma(\mu_1 + 1), .., \sigma(\mu_2), .., \sigma(\mu_k), \sigma(\mu_k + 1), .., \sigma(n))$$

which is compatible with the relation R by construction. □

Definition 3.5 Let be an indexed ordered set (S, R) with n elements, σ_1 and σ_2, permutations of elements of S and $k \in \{1, \dots, n-1\}$, we call the k-cut offspring permutation of σ_1 and σ_2 the permutation γ defined as:

$$\gamma = (\sigma_1(1), \sigma_1(2), \dots, \sigma_1(k), \sigma_2(i_1), \dots, \sigma_2(i_{n-k}))$$

where for all $h \in \{i_1, i_2, \dots, i_{n-k}\}$ such that $i_1 < i_2 < \dots < i_{n-k}$ then $\sigma_2(h) \notin \{\sigma_1(1), \dots, \sigma_1(k)\}$.

Theorem 3.6 *Given an indexed ordered set (S, R) with n elements and $k \in \{1, \dots, n-1\}$, the k-cut offspring permutation of two permutations, σ_1 and σ_2, compatible R is a permutation compatible with R.*

Proof Let be $\sigma_1 = (\sigma_1(1), \dots, \sigma_1(n)), \sigma_2 = (\sigma_2(1), \dots, \sigma_2(n))$ two compatible permutation and $k \in \{1, \dots, n-1\}$.

The k-cut offspring permutation is

$$\gamma = (\sigma_1(1), \sigma_1(2), \dots, \sigma_1(k), \sigma_2(i_1), \dots, \sigma_2(i_{n-k}))$$

where for all $h \in \{i_1, i_2, \dots, i_{n-k}\}$ such that $i_1 < i_2 < \dots < i_{n-k}$ then $\sigma_2(h) \notin \{\sigma_1(1), \dots, \sigma_1(k)\}$.

Let's denote $S^1 = \{a_{\sigma_1(1)}, \dots, a_{\sigma_1(k)}\}$ and $S^2 = \{a_{\sigma_2(1)}, \dots, a_{\sigma_2(k)}\}$

- If $S^1 = S^2$, the elements belonging to S^1 are compatible with each other in the resulting permutation due to their presence in the compatible permutation σ_1, and the remaining elements $S - S^1$ with each other as well, because they are in σ_2.

  The elements belonging to S^1 are also compatible with those in $S - S^1$ by verifying the compatibility of σ_2.

  So, in this case we have a resulting permutation compatible with the relation.

- If $S^1 \neq S^2$
  - the elements of S^1 and those of $S - S^1$, due to the compatibility of σ_1 and σ_2, respectively, are compatible with each other in the resulting permutation;
  - if $a \in S^1$ and $b \in (S - S^1)$,

    $$a = a_{\sigma_1(j_a)} = a_{\sigma_2(i_a)} \text{ and } j_a < k \quad b = a_{\sigma_1(j_b)} = a_{\sigma_2(i_b)} \text{ and } j_b > k$$

    * if $b \notin S^2 \longrightarrow i_b > k$ then a and b are compatible in the resulting permutation;
    * if $b \in S^2 \longrightarrow i_b < k$
      - if $i_a < i_b$, they are in the same order in both permutations and are therefore compatible in the resulting permutation.
      - if $i_b < i_a$, as $j_a < k < j_b$, then they are interchanged in both permutations and therefore, by Proposition **??**, they are not comparable and, therefore, are compatible in the resulting permutation.

  So, in this case, we also have a resulting permutation compatible with the order relation.

Then we can conclude that the permutation resulting from two compatible permutations with the relation R is also a compatible one. □

Definition 3.7 The procedure described in Definition 3.5 can be extended recursively to the case of $m > 2$ permutations and a partition, $k = (k_1, \dots, k_m)$, of n, that is $\forall i \in \{1, \dots, m\} k_i \in \{1, \dots, n-1\}$ and $\sum_{i=1}^{m} k_i = n$. Given $\sigma_i = (\sigma_i(1), \sigma_i(2), \dots, \sigma_i(n)), i \in \{1, \dots, m\}$ permutations of elements of S and $k = (k_1, \dots, k_m = n - \sum_{i=1}^{m-1} k_i)$, we construct γ_m as follows

$$\begin{cases} \gamma_2 = k_1\text{-cut offspring permutation of } \sigma_1 \text{ and } \sigma_2 \\ \gamma_i = \left(\sum_{i=1}^{i-1} k_i\right)\text{-cut offspring permutation of } \gamma_{i-1} \text{ and } \sigma_i, \quad \text{if } i \in \{3, \dots m\} \end{cases}$$

and we call it $(k_1, k_2, \dots, k_{m-1})$-cut offspring permutation of $\sigma_1, \sigma_2, \dots, \sigma_m$. γ_m is that which the elements of the positions between $\sum_{j=1}^{i-1} k_j$ and $\sum_{j=1}^{i} k_j$ are the first k_i elements of permutation σ_i that are not in $\bigcup_{j=1}^{i-1} \{\sigma_j(i_{j_1}), \dots, \sigma_j(i_{j_{k_j}})\}$.

Theorem 3.8 *Given an indexed ordered set (S, R) with n elements, the resulting permutation of m permutations compatible with R, $k = (k_1, \dots, k_m)$ a partition of n, that is, as in the Definition 3.7, $\sum_{i=1}^{m} k_i = n$, is a permutation compatible with the relation R.*

Acknowledgements

This work has been partially supported by the collaborative project *FUO*-115-22.

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