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Duality for infinite horizon relaxed control problems

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1. Introduction

We consider control problems $(P)_\infty$ whose objective functional is in an economic context a utility functional,

$$J(x,u) \quad = \quad \int_0^\infty W(x(t),u(t))\, e^{-\varrho t}\, dt \quad \longrightarrow Max\,! \tag{1.1}$$

where W is an instantaneous utility function and ϱ is a positive or zero discount rate.
The objective can also be an energy functional in mechanical or quantum mechanical systems, or it can be chosen in such a way that the asymptotic and exponential controllability of the system is guaranteed,

$$J(x,u) \quad = \quad \int_0^\infty \frac{1}{2}\left(x(t)^T Q(t) x(t) + u(t)^T R(t) u(t)\right) e^{\beta t}\, dt \quad \longrightarrow Min\,! \tag{1.2}$$

where $\beta > 0$ assures together with the choice of suitable state spaces the exponential stability of the solution.

All the target functionals considered have in common that they are given on an a priori infinite horizon and a weight function occurs in the integrand of the objective.

We consider non-linear, non-autonomous dynamical systems. Consequently, one has to expect that convexity assumptions, which are usually required for existence results, are not fulfilled. We take this into account by passing to an optimal control problem with relaxed controls $(\bar{P})_\infty$. Dual-based methods for solving the problems are proposed. It turns out that $(P)_\infty$ and $(\bar{P})_\infty$ have a common dual problem. A Lotka-Volterra model is presented as an application.

2. Problem statement

The following problem $(\bar{P})_\infty$ is considered:

$$\begin{aligned} J(x,\mu) \quad &= \quad \int_0^\infty \int_U r(t,x(t),\mathrm{v})\, d\mu_t(\mathrm{v}) e^{-\varrho t}\, dt \longrightarrow Min\,! \\ x \quad &\in \quad W_2^{1,n}((0,\infty),\nu), \quad \mu \in \mathcal{M}_U,\ \ U \in comp(\mathbb{R}^m), \\ \dot{x}(t) \quad &= \quad \int_U f(t,x(t),\mathrm{v}) d\mu_t(\mathrm{v}) \text{ a.e. on } (0,\infty),\ x(t_0) = x^0. \end{aligned}$$

All integrals are to be understood in the Lebesgue sense. The control domain U is assumed to be compact. $W_2^{1,n}((0,\infty),\nu)$ is a weighted Sobolev-space and relaxed controls are taken from a family of probability measures $\mathcal{M}_U$, introduced in the next section.

3. Spaces of states and controls

3.1. Control spaces

The relaxed controls μ are taken from a regular family of probability measures $\mathcal{M}_U$, [4].

Definition 3.1 A relaxed control $\{\mu_t\}_{t\in\mathbb{R}_+}$ is a family of probability measures that has the following properties:

1. $\operatorname{supp} \mu_t \subseteq U$ a.e. on $\mathbb{R}_+$,
2. μ_t is a probability measure on U a.e. on $\mathbb{R}_+$,

3. For all continuous functions with compact support, $g \in C_c(\mathbb{R}_+ \times U)$, the function

$$t \longrightarrow \int_U g(t,v)\, d\mu_t(v)$$

is Lebesgue - measurable.

The motivation for introducing relaxed controls is given by the following arguments:

Remark 3.2 1. In general nonlinear systems cannot be stabilized using a continuous closed loop control $U(x)$, even if each state separately can be driven asymptotically to the origin.

2. Sometimes it can be stabilized with a *continuous closed loop relaxed control*.

3. Relaxed control-type stabilization is used both in theory and in practice; the method is known as *dithering*, see [1].

3.2. State spaces

A weighted Sobolev space $W_2^{1,n}((0,\infty),\nu)$ with a suitable weight function ν is chosen as the state space. The introduction of the weighted Sobolev space $W_2^{1,n}((0,\infty),\nu)$ is motivated by the following facts. Density and weight functions appear naturally in the objective functionals. If a classical Sobolev space $W_2^{1,n}(0,T)$ is usually used as state space for control problems with bounded time interval $[0,T]$, the limit transition $T \to \infty$ leads in a natural way to an improper integral

$$\lim_{T\to\infty} \int_0^T f(x)\, dx$$

which, in general not coincides with the Lebesgue - integral, see [9].
While in the case of bounded intervals the elements of the Banach space $W_1^1((0,T))$ have a continuous representative and thus the space $W_1^1((0,T))$ can be identified with the space of absolutely continuous functions $AC((0,T))$, the continuation of this space to $AC_{loc}((0,\infty))$ loses the Banach space structure. This is an important theoretical motivation to switch to weighted Sobolev spaces as Banach spaces in the problem definition.

Definition 3.3 (weight function/density function) Let $\mathbb{R}_+ := [0,\infty)$. A continuous function $\nu : \mathbb{R}_+ \to \mathbb{R}_+$ is called *weight function* if ν and $\nu^{-1} \in L_{1,loc}(\mathbb{R}_+)$. If for a weight function ν also holds

$$\int_{\mathbb{R}_+} \nu(t)dt < \infty$$

we call this *density function*. Otherwise we name it *proper weight function*.

Definition 3.4 Let $M^n(\mathbb{R}_+)$ be the set of measurable vector functions on $\mathbb{R}_+$. By means of a weight function ν, we define the *weighted Lebesgue space*

$$L_2^n(\mathbb{R}_+,\nu) \quad = \quad \left\{ x \in M^n(\mathbb{R}_+) \mid \|x\|^2_{L_2^n(\mathbb{R}_+,\nu)} := \int_{\mathbb{R}_+} x^T(t)x(t)\nu(t)dt < \infty \right\} \tag{3.1}$$

the *weighted Sobolev space*

$$W_2^{1,n}(\mathbb{R}_+,\nu) = \left\{ x \in M^n(\mathbb{R}_+) \;\middle|\; x \in L_2^n(\mathbb{R}_+,\nu),\ \mathcal{D}x \in L_2^n(\mathbb{R}_+,\mu) \right\}. \tag{3.2}$$

where $\mathcal{D}x$ denotes the distributional derivative (shortly denoted by x'), see [8], p. 11 ff. With the introduced norm $L_2^n(\mathbb{R}_+,\nu)$ becomes a Hilbert space. With

$$\|x\|_{W_2^{1,n}(\mathbb{R}_+,\nu)} \;=\; \|x\|_{L_2^n(\mathbb{R}_+,\nu)} \;+\; \|\mathcal{D}x\|_{L_2^n(\mathbb{R}_+,\nu)}\,, \tag{3.3}$$

$W_2^{1,n}(\mathbb{R}_+,\nu)$ becomes a Hilbert space as well (this can be confirmed analogously to [8].

The following properties of functions in weighted Sobolev spaces should be mentioned here explicitly.

Remark 3.5 1. Let $x \in W_2^1(\mathbb{R}_+,\nu)$, $\|x\| \leq K$, $\nu(t) = e^{\beta t}$, $\beta > 0$, then x is exponentially stable,

$$|x(t)| \leq \big(|x(0)| + C\,K\,\sqrt{t}\big)e^{-\frac{\beta}{2}t}.$$

2. Let $x \in W_2^1(\mathbb{R}_+,\nu)$ and $y \in W_2^1(\mathbb{R}_+,\nu^{-1})$, $\nu(t) = e^{\beta t}$, $\beta > 0$, then $x\,y$ is asymptotically stable,

$$x\,y \in W_1^1(\mathbb{R}_+) \quad \text{and} \quad \lim_{t\to\infty} x(t)\,y(t) = 0.$$

For the proofs see [11].

4. Optimality notions

In comparison to the literature, see [3], [5], where overtaking or weakly overtaking optimality is mainly used as optimality criterion, the classical comparison of Lebesgue integrals in the objective of $(\bar{P})$ is used here. The admissible domain $\mathcal{A}$ of $(\bar{P})_\infty$ is given by

$$\mathcal{A} := \left\{ (x,\mu) \in W_2^{1,n}(\mathbb{R}_+,\mu) \times \mathcal{M}_U \;\middle|\; \begin{array}{l} x'(t) = \int_U f(t,x(t),\mathrm{v})\mu_t(\mathrm{v}) \; a.e.\mathbb{R}_+, \\ x(0) = x^0,\; \mu \in \mathcal{M}_U \end{array} \right\}.$$

Definition 4.1 Let the processes (x,μ), $(x^*,\mu^*) \in \mathcal{A}$ be given. Then the pair (x^*,μ^*) is called *globally optimal in the sense of* **criterion L**, if $J(x^*,\mu^*) < \infty$ and for any pair $(x,\mu) \in \mathcal{A}$ we have

$$J(x^*,\mu^*) \leq J(x,\mu).$$

Under conditions that ensure the existence of the solution, cf. also the contribution by I. Dikariev at the FGS-Conference On Optimization, Gijon, Spain, (2024) , entitled

Existence Theorem for Relaxed Control Problems on Infinite Time Horizon Utilizing Weight Functions

we treat the problem $(\bar{P})$ with dual methods. Here, we mainly refer to the ideas of Carathéodory and Klötzler for the construction of a dual problem. This dual based approach has already been used for special optimal control problems with infinite horizon in [10, 13].

5. Duality

We use a very general scheme for the construction of a dual problem, which goes back to Klötzler, [6]:

Definition 5.1 Let real functionals $F : X \to \bar{\mathbb{R}} := \mathbb{R} \cup +\infty$ and $G : Y \to \bar{\mathbb{R}}$ with arbitrary sets X and Y be given. The problem

$$(\mathbf{D}) \quad G(y) \to \sup! \quad w.r.t.\; y \in Y$$

is called *dual program* to the primary program

$$(\mathbf{P}) \quad F(x) \to \inf! \quad w.r.t.\; x \in X,$$

if the inequality

$$G(y) \leq F(x) \;\; \forall\, x \in X, \forall\, y \in Y$$

or equivalently

$$\sup_{y\in Y} G(y) \leq \inf_{x\in X} F(x) \tag{5.1}$$

holds true. Relation (5.1) is called *weak duality relation*. If even the equality holds in (5.1), we say that *the strong duality relation* holds between both problems.

The construction is carried out in the following steps:

Step 1: Partition of the admissible set $\mathcal{A} = X_0 \cap X_1$
Step 2: Define a set Y and a real functional $\Phi(\cdot,\cdot) : X_0 \times Y \to \bar{\mathbb{R}}_+$ with

$$\begin{aligned} \inf_{(x,\mu)\in\mathcal{A}} J(x,\mu) &= \inf_{(x,\mu)\in X_0} \sup_{S\in Y} \Phi((x,\mu),S) \quad \text{(equivalence relation)} \\ &\geq \sup_{S\in Y} \inf_{(x,\mu)\in X_0} \Phi((x,\mu),S) \end{aligned}$$

Step 3: For a fixed element $S \in Y$ one sets

$$G(S) := \inf_{(x,\mu)\in X_0} \Phi((x,\mu),S).$$

We realize the scheme and construct a dual Program for $(\bar{\mathbf{P}})_\infty$, with $\nu(t) = e^{\beta t}, \beta > 0$

Step 1: Partition of the admissible set $\mathcal{A} = X_0 \cap X_1$

$$X_0 := \left\{(x,\mu) \in W_2^{1,n}(\mathbb{R}_+,\nu) \times \mathcal{M}_U \mid x(0) - x^0 = 0, \mu \in \mathcal{M}_U\right\}$$
$$X_1 := \left\{(x,\mu) \in W_2^{1,n}(\mathbb{R}_+,\nu) \times \mathcal{M}_U \mid x'(t) - \int_U f(t),x(t),\mathrm{v})d\mu_t(\mathrm{v}) = 0 \text{ a.e. on } (0,\infty)\right\}$$

Step 2: One possible choice for Φ is a Lagrange - functional

$$\Phi_1((x,\mu),S) \quad = \quad J(x,\mu) + \left\langle \underbrace{x'(\cdot) - \int_U f(t,x(\cdot),v)d\mu_t(v)}_{\in L_2^n((0,\infty),e^{\beta t})}, \underbrace{\nabla_\xi S(\cdot,x(\cdot))}_{\in L_2^n((0,\infty),e^{-\beta t})} \right\rangle$$

where$\langle\cdot\,,\,\cdot\rangle$ is the scalar product in $L_2^n(\mathbb{R}_+)$, which satisfies

$$\langle \zeta\,, p\,\rangle \le \|\zeta\|_{L_2^n((0,\infty),e^{\beta t})}\|p\|_{L_2^n((0,\infty),e^{-\beta t})}.$$

Then we define the set Y by the following setting:

$$S \in Y \Leftrightarrow S(t,\xi) = y^T(t)\xi \quad \text{and} \quad y \in L_2^n(\mathbb{R}_+,\nu^{-1}), \tag{5.2}$$

$$\Phi_1((x,\mu),y) = J(x,\mu) + \left\langle x'(\cdot) - \int_U f(\cdot,x(\cdot),\mathrm{v})d\mu_t(\mathrm{v}), y(\cdot)\right\rangle_{L_2^n(\mathbb{R}_+)} \tag{5.3}$$

Step 3: Formulation of a dual program (integated version):

$$(\mathbf{D_1}): \qquad G(y) := \inf_{(x,\mu)\in X_0} \Phi_1((x,\mu),y) \to \max! \;\; \text{w.r.t.} \;\; y \in L_2^n(\mathbb{R}_+,\nu^{-1}).$$

We can identify the idea of choosing a suitable functional Φ from Carathéodory's approach as well, see [2,3].

It consists of adding an invariant integral to the integral in the objective. Invariance means that the added integral depends on the values of the function S on the boundary of $[0,\infty)$, i.e. on $S(0,x^0)$, only. More precisely, by choosing the function space Y it must be ensured that

$$\int_0^\infty [\int_U r(t,x(t),\mathrm{v})\, d\mu_t(\mathrm{v})\, e^{\beta t} \quad - \quad \frac{d}{dt}S(t,x(t))]\, dt \; = J(x,\mu) + S(0,x^0)$$
$$= \int_0^\infty [\int_U r(t,x(t),\mathrm{v})\, d\mu_t(\mathrm{v})\, e^{\beta t} \quad - \quad [\nabla_\xi^T S(t,x(t)) \int_U f(t,x(t),\mathrm{v})d\mu_t(\mathrm{v}) + S_t(t,x(t))]dt.$$

for all $(x,\mu) \in \mathcal{A}$. Then we conclude

$$J(x,\mu) + S(0,x_0) \quad \ge \quad -\int_0^\infty [\mathcal{H}(t,x(t),\nabla_\xi^T S(t,x(t))) + S_t(t,x(t))]\, dt$$

with the Hamiltonian function

$$\mathcal{H}(t,\xi,\nabla_\xi S(t,\xi)) \quad = \quad \sup\{H(t,\xi,\mathrm{v},S_\xi(t,\xi)) \mid \mathrm{v} \in U\}, \tag{5.4}$$

and

$$H(t,\xi,\mathrm{v},S_\xi(t,\xi)) = -r(t,\xi,\mathrm{v})e^{\beta t} + \nabla_\xi^T S(t,\xi) f(t,\xi,\mathrm{v}).$$

This leads together with defect function

$$\Lambda_S(t,\xi) := \mathcal{H}(t,\xi,\nabla_\xi S(t,\xi)) + S_t(t,\xi)$$

in the Hamilton- Jacobi equation and

$$\Lambda_S(t, x^*(t)) = 0 \text{ on } [0, \infty)$$

to the following variant of the dual problem (pointwise version):

$$\begin{array}{lrcll} (\mathbf{D_2}) & \mathbf{G_2(S)} & = & -S(0,x^0) & \longrightarrow \max! \\ & \text{with respect to } \quad S & \in & Y & \\ & \Lambda_S(t,\xi) & \leq & 0 & \forall t \in [0,\infty),\ \forall \xi \\ & \Lambda_S(t,x^*(t)) & = & 0 & \forall t \in [0,\infty). \end{array}$$

Remark 5.2 The Hamiltonian for $(P)_\infty$ and $(\bar{P})_\infty$ coincide, since

$$\begin{aligned} \mathcal{H}(t,\xi,\nabla_\xi S(t,\xi)) &= \max\{H(t,\xi,v,S_\xi(t,\xi)) \mid v \in U\} \\ &= \max\{\int_U H(t,\xi,\text{v},S_\xi(t,\xi))d\mu_t(\text{v}) \mid \mu_t \in P_U\} \end{aligned}$$

where P_U is the set of probability measure concentrated on U, see [4]. We conclude that both problems, $(\bar{P})_\infty$ and $(P)_\infty$, have a same dual problem $(\mathbf{D_2})$,

$$\sup((\mathbf{D_2})) \leq \inf((\bar{\mathbf{P}})_\infty) \leq \inf((\mathbf{P})_\infty).$$

6. Applications

The uncontrolled bilinear Lotka-Volterra model considered is

$$\begin{aligned} x_1'(t) &= x_1(t)\,[a - bx_2(t)] \\ x_2'(t) &= x_2(t)\,[-c + dx_1(t)]\,. \end{aligned}$$

6.1. A linearized Lotka-Volterra model

First we transform the non-trivial equilibrium $(\frac{c}{d}, \frac{a}{b}) = (\bar{x}_1, \bar{x}_2)$ of the uncontrolled equilibrium to the zero point. Then we linearize the system around the uncontrolled steady state and look for a bounded control (u_1, u_2) which stabilizes the system exponentially. We arrive at a problem of the following type.

$$(\mathbf{Q}): \quad J(x,u) \quad = \quad \int_0^\infty \frac{1}{2}\left\{(x^T(t)x(t) + u^T(t)u(t))\right\} e^{\beta t} dt \longrightarrow \min\,!$$

with respect to

$$\begin{aligned} (x,u) &\in W_2^{1,2}(\mathbb{R}_+, e^{\beta t}) \times L_2^2(\mathbb{R}_+, e^{\beta t}),\ \beta > 0 \\ x'(t) &= Ax(t) + u(t) \text{ a. e. on } \mathbb{R}_+\,,\ x(0) = x^0\,, \\ u(t) &\in U := [-1,1] \times [-1,1] \text{ a. e. on } \mathbb{R}_+\,. \end{aligned}$$

For the detailed assumptions and settings see [7,13]. The corresponding dual problem $(\mathbf{D_Q})$ (integrated version) is

$$(\mathbf{D_Q}): \quad G(y) := -\int_0^\infty \left\{\frac{1}{2}\left[y'(t) + A^T y(t)\right]^T \left[y'(t) + A^T y(t)\right] + \theta(t, y(t))\right\} e^{-\beta t} dt - x_0^T y(0) \longrightarrow \max\,!$$

w. r. t.

$$y \in W_2^{2,2}(\mathbb{R}_+, e^{-\beta t}) \quad \text{with} \quad x^0 = y'(0) + A^T y(0),$$

with

$$\theta(t,y(t)) = \sum_{i=1}^{2} -\frac{1}{2}\sigma_i^2(t,y(t)) + \sigma_i(t,y(t))y_i(t) \quad \text{and}$$

$$\sigma_i(t,y(t)) = \min\left\{\max\left\{-1, y_i(t)e^{-\beta t}\right\}, 1\right\} e^{\beta t}$$

Remark 6.1 1. In the general construction of the dual problem, (5.2), i.e. $S(t,\xi) = y^T(t)\xi$ and $y \in W_2^{2,2}(\mathbb{R}_+, \nu^{-1})$ is used.

2. The duality construction is carried out with the Lagrange functional (5.3).
3. It can be shown that the Hamilton function (5.4) is smooth.
4. In the dual problem, the inverse weight function appears in the objective functional as well as in the weighted Sobolev space.
5. $(\mathbf{D_Q})$ has an optimal solution.
6. Spectral methods can be applied to approximate the solution.

6.2. A controlled bi-linear Lotka-Volterra model

We transform the steady state of the uncontrolled equilibrium $(\frac{c}{d}, \frac{a}{b}) = (\bar{x}_1, \bar{x}_2)$ to the zero point and look for a bounded control (u_1, u_2) which stabilizes the non-linear system exponentially. We arrive at the following problem:

$$(\tilde{\mathbf{Q}}): \quad J(\tilde{x},u) = \int_0^\infty \frac{1}{2}\left(\tilde{x}(t)^T Q(t)\tilde{x}(t) + u(t)^T R(t) u(t)\right) e^{\beta t}\, dt \quad \longrightarrow Min!$$

w.r.t.

$$\left(\tilde{x},u\right) \in W_2^{1,2}(\mathbb{R}_+, e^{\beta t}) \times L_2^2(\mathbb{R}_+, e^{\beta t}), \quad \beta > 0$$

with

$$\begin{aligned}
\tilde{x}_1'(t) &= \left[\tilde{x}_1(t) + \frac{c}{d}\right][-b\tilde{x}_2(t) - u_1(t)] \quad \text{a.e. on } \mathbb{R}_+,\\
\tilde{x}_2'(t) &= \left[\tilde{x}_2(t) + \frac{a}{b}\right][d\tilde{x}_1(t) - u_2(t)] \quad \text{a.e. on } \mathbb{R}_+,\\
\tilde{x}_1(0) &= x_1^0 - \frac{c}{d}, \quad \tilde{x}_2(0) = x_2^0 - \frac{a}{b}
\end{aligned}$$

For the duality construction we now use a nonlinear ansatz for S,

$$S(t,\xi) := y_1(t)\ln\left(\xi_1 + \frac{c}{d}\right) + y_2(t)\ln\left(\xi_2 + \frac{a}{b}\right), \quad y \in W_2^{1,2}(\mathbb{R}_+, e^{-\beta t}) \tag{6.1}$$

Then

$$\begin{aligned}
\Phi_2((\tilde{x},u),S) = J(x,u) &+ \int_0^\infty \left(\tilde{x}_1'(t) - \left[\tilde{x}_1(t) + \frac{c}{d}\right][-b\tilde{x}_2(t) - u_1(t)]\right) S_{\xi_1}(t,\tilde{x}(t))\, dt\\
&+ \int_0^\infty \left(\tilde{x}_2'(t) - \left[\tilde{x}_2(t) + \frac{a}{b}\right][d\tilde{x}_1(t) - u_2(t)]\right) S_{\xi_2}(t,\tilde{x}(t))\, dt\\
= J(\tilde{x},u) &+ \int_0^\infty \left((\ln(\tilde{x}_1 + \frac{c}{d}))'(t) - [-b\tilde{x}_2(t) - u_1(t)]\right) y_1(t)\, dt\\
&+ \int_0^\infty \left((\ln(\tilde{x}_2 + \frac{a}{b}))'(t) - [d\tilde{x}_1(t) - u_2(t)]\right) y_2(t)\, dt
\end{aligned} \tag{6.2}$$

is well defined and all integrals exist. The final construction of the dual problem in integrated form is similar to that introduced in [7, 11] and [13].

Remark 6.2 1. In the general construction of the dual problem, the nonlinear ansatz of S, (6.1), is used.

2. The duality construction is carried out with the Lagrange functional Φ_2 in (6.2).
3. In the dual problem, the inverse weight function ν^{-1} appears in the objective functional as well as in the weighted Sobolev space.
4. Similar to [13] spectral methods can be applied to approximate the solution of the dual problem.

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