French-German-Spanish Conference on Optimization
Gijón, June 18-21, 2024
(pp. 79-86)

# Double control problem: domains and coefficients for elliptic equations

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**Abstract**

In this work we are interested in a bi-optimal control problem for a linear elliptic state equation with homogeneous boundary Dirichlet condition. The two controls variables correspond to the coefficient of the diffusion term of the equation and the open set where the it is posed. From the practical point of view, this problem can be interpreted as finding materials from the mixture of other ones with different diffusion properties and on optimal shape. We analyze a relaxation process, optimality conditions, and finally we provide a numerical algorithm and we show some numerical experiments.

## 1. Introduction

Let $\Omega$ be a bounded open set of $\mathbb{R}^N$ considered as the domain of reference, a typical optimal design problem consists in finding the optimal layout of two materials in order to minimize a certain cost functional ( [1], [11], [14]). In this sense, in the case of two isotropic materials with diffusion constants $0 < \alpha < \beta$ the problem can be formulated from the mathematical point of view:

$$\min_{\omega\subset\Omega \text{ measurable}} \int_\omega F(x,u)\,dx \qquad (1.1)$$
$$\begin{cases} -\mathrm{div}\big((\alpha\chi_\omega + \beta\chi_{\Omega\setminus\omega})\nabla u\big) = f \ \text{ in } \Omega \\ u = 0 \ \text{ on } \partial\Omega \end{cases}$$

where $f$ is a given source. The control variable $\omega \subset \Omega$ measurable determines where the material $\alpha$ is placed.

Another typical problem in optimal design appears when we only dispose of one material and the control variable corresponds to the place where the material is or not posed, i.e., the control variable determines the shape of the optimal domain $\omega \subset \Omega$ with the presence of possible holes. From the mathematical point of view the problem can written by

$$\min_{\omega\subset\Omega \text{ open}} \int_\omega F(x,u)\,dx \qquad (1.2)$$
$$\begin{cases} -\Delta u = f \ \text{ in } \omega \\ u = 0 \ \text{ on } \partial\omega. \end{cases}$$

In this work we are interested in considering the couple problem where as in (1.1), we look for the optimal distribution of two conductive materials and, similarly to (1.2), we search the set where the diffusion equations is posed. If we consider a constraint on the amounts of the materials used in the mixture, the problem can be formulated as

$$\min_{\omega^\alpha,\omega^\beta} \int_{\omega^\alpha\cup\omega^\beta} F(x,u)\,dx \qquad (1.3)$$
$$\begin{cases} -\mathrm{div}\big((\alpha\chi_{\omega^\alpha} + \beta\chi_{\omega^\beta})\nabla u\big) = f \ \text{ in } \omega^\alpha\cup\omega^\beta \\ u = 0 \ \text{ on } \partial(\omega^\alpha\cup\omega^\beta) \\ \omega^\alpha,\omega^\beta \subset \Omega \ \text{measurable}, \ \omega^\alpha\cup\omega^\beta \ \text{open}, \ |\omega^\alpha| \le \kappa^\alpha, \ |\omega^\beta| \le \kappa^\beta, \end{cases}$$

with $\kappa^\alpha, \kappa^\beta$ two positive constants.

The lack of classical solutions of (1.1) and (1.2) is well-known ( [10]). In this work, we obtain a relaxed formulation of (1.3), system of optimality conditions, and we provide a numerical algorithm to solve it. We show some numerical experiments ( [6]).

## 2. Statement of the problem and relaxation

We are interested in the optimal design problems of the kind of (1.3) with $\Omega \subset \mathbb{R}^N$ a bounded open set, $f \in H^{-1}(\Omega)$, $\alpha, \beta, \kappa^\alpha, \kappa^\beta$, four positive constants with $\alpha < \beta$, and $F : \Omega \times \mathbb{R} \to \mathbb{R}$ such that

$$F(\cdot, s) \text{ is measurable in } \Omega,\ \forall\, s \in \mathbb{R}, \tag{2.1}$$

$$F(x, \cdot) \text{ is continuous in } \mathbb{R},\ \text{ a.e. } x \in \Omega, \tag{2.2}$$

$$\exists\, r \in L^1(\Omega),\ \gamma > 0,\ \text{ such that } |F(x,s)| \le r(x) + \gamma |s|^2, \quad \forall\, s \in \mathbb{R},\ \text{ a.e. } x \in \Omega. \tag{2.3}$$

Since as we said in the introduction the problem has no solution in general, we look for a relaxed formulation, it will be obtained using the homogenization theory. In this way we will use the following classical result due to S. Spagnolo ( [12]). See also [11].

**Theorem 2.1** *Let $\Omega \subset \mathbb{R}^N$ be a bounded open set, and $A_n \in L^\infty(\Omega)^{N\times N}$ a sequence of symmetric matrix functions such that there exist $\alpha, \beta > 0$ satisfying*

$$\alpha |\xi|^2 \le A_n(x)\xi \cdot \xi \le \beta |\xi|^2, \quad \forall\, \xi \in \mathbb{R}^N,\ \text{ a.e. } x \in \Omega. \tag{2.4}$$

*Then, for a subsequence of n, still denoted by n, there exists a symmetric matrix function $A \in L^\infty(\Omega)^{N\times N}$, which also satisfies (2.4), such that for every $f \in H^{-1}(\Omega)$, the solution $u_n$ of*

$$\begin{cases} -\mathrm{div}(A_n \nabla u_n) = f \ \text{ in } \Omega \\ u_n \in H_0^1(\Omega), \end{cases}$$

*satisfies*

$$u_n \rightharpoonup u \ \text{ in } H_0^1(\Omega), \qquad A_n \nabla u_n \rightharpoonup A\nabla u \ \text{ in } L^2(\Omega)^N,$$

*with u the solution of*

$$\begin{cases} -\mathrm{div}(A \nabla u) = f \ \text{ in } \Omega \\ u \in H_0^1(\Omega). \end{cases}$$

*We say that $A_n$ H-converges to A and we write $A_n \overset{H}{\rightharpoonup} A$.*

We are interested in the case where the domains also varies. In this sense it is necessary to recall some results about capacity.

**Definition 2.2** *For a bounded open set $\Omega \subset \mathbb{R}^N$ and $E \subset \Omega$, we define the capacity of E in $\Omega$ as*

$$\mathrm{Cap}(E, \Omega) := \inf\left\{ \int_\Omega |\nabla \varphi|^2 dx \,:\ \varphi \in H_0^1(\Omega),\ \varphi \ge 1 \ \text{ a.e. in a neighbourhood of } E \right\}.$$

**Definition 2.3** *A set $U \subset \Omega$ is said to be quasi-open if for every $\varepsilon > 0$, there exists $G \subset \Omega$ open such that* $\mathrm{Cap}(U \Delta G, \Omega) < \varepsilon$. *The complementary in $\Omega$ of a quasi-open set U is said to be quasi-closed.*

*We define $M_0(\Omega)$ as the set of non-negative Borel measures which vanish on the null-capacity sets of $\Omega$ and satisfy*

$$\mu(E) = \inf\Big\{ \mu(U) \,:\ E \subset U,\ U \text{ quasi-open} \Big\}.$$

It is important to remark that the elements of $M_0(\Omega)$ are not necessarily Radon measures. They can take a infinity values in compact subets of $\Omega$. Namely, for every $\mu \in M_0(\Omega)$, there exists a unique quasi-closed set that we will note by $C_\mu$ such that

$$\mu = \infty_{C_\mu} \ \text{ in } C_\mu, \qquad \mu \ \text{ is } \sigma\text{-finite in } \Omega \setminus C_\mu,$$

where $\infty_{C_\mu}$ is the measure in $M_0(\Omega)$ defined as

$$\infty_{C_\mu}(E) = \begin{cases} \infty & \text{if } \mathrm{Cap}(E \cap C_\mu, \Omega) > 0 \\ 0 & \text{if } \mathrm{Cap}(E \cap C_\mu, \Omega) = 0. \end{cases}$$

An extension of Theorem 2.1 for the case where the open set $\Omega$ also varies is given by the following theorem due to G. Dal Maso and F. Murat ( [4] ).

**Theorem 2.4** *Assume $\Omega \subset \mathbb{R}^N$ a bounded open set, $A_n \in L^\infty(\Omega)^{N\times N}$ symmetric, which satisfies (2.4) and $\mu_n \in M_0(\Omega)$. Then, for a subsequence of $n$ still denoted by $n$, there exits a symmetric matrix $A \in L^\infty(\Omega)^{N\times N}$ and a measure $\mu \in M_0(\Omega)$ such that $A_n$ $H$-converges to $A$ and for every $f \in H^{-1}(\Omega)$ the sequence of solutions of*

$$\begin{cases} -\operatorname{div}(A_n\nabla u_n) + \mu_n u_n = f \ \text{ in } \Omega \\ u_n \in H_0^1(\Omega) \cap L^2_{\mu_n}(\Omega), \end{cases} \tag{2.5}$$

*converges weakly in $H_0^1(\Omega)$ to the unique solution of*

$$\begin{cases} -\operatorname{div}(A\nabla u) + \mu u = f \ \text{ in } \Omega \\ u \in H_0^1(\Omega) \cap L^2_{\mu}(\Omega). \end{cases} \tag{2.6}$$

*We will write*

$$(A_n, \mu_n) \overset{H\gamma}{\rightharpoonup} (A, \mu). \tag{2.7}$$

**Definition 2.5** *For $p \in [0,1]$, we denote by $m^-(p)$ and $m^+(p)$ the harmonic and arithmetic mean values of $\alpha$ and $\beta$ with proportions $p$ and $1-p$ respectively, i.e.*

$$m^-(p) = \left(\frac{p}{\alpha} + \frac{1-p}{\beta}\right)^{-1}, \qquad m^+(p) = p\alpha + (1-p)\beta.$$

*We also define $K(p)$ as the set of symmetric matrices $M \in \mathbb{R}^{N\times N}$ such that their eigenvalues $\lambda_1 \leq \cdots \leq \lambda_N$ satisfy*

$$\begin{cases} m^-(p) \leq \lambda_i \leq m^+(p),\ 1 \leq i \leq N \\ \displaystyle\sum_{i=1}^N \frac{1}{\lambda_i - \alpha} \leq \frac{1}{m^-(p)-\alpha} + \frac{N-1}{m^+(p)-\alpha} \\ \displaystyle\sum_{i=1}^N \frac{1}{\beta - \lambda_i} \leq \frac{1}{\beta - m^-(p)} + \frac{N-1}{\beta - m^+(p)}. \end{cases}$$

**Remark 2.6** *The set $K(p)$ corresponds with the $H$-closure of two isotropic materials with fixed proportion $p$ and $1-p$, respectively, which was obtained in [13].*

Using Theorem 2.4 we have obtained in [6] the following result adapted to problem (1.3).

**Theorem 2.7** *Assume $\Omega \subset \mathbb{R}^N$ a bounded open set, $\mu_n \in M_0(\Omega)$, $\theta_n^\alpha, \theta_n^\beta \in L^\infty(\Omega;[0,1])$, and $A_n \in L^\infty(\Omega \setminus C_{\mu_n})^{N\times N}$ such that*

$$\theta_n^\alpha + \theta_n^\beta \leq 1 \ \text{a.e. in } \Omega, \quad \theta_n^\alpha + \theta_n^\beta = 1 \ \text{a.e. in } \Omega\setminus C_{\mu_n}, \quad A_n \in K(\theta_n^\alpha) \ \text{a.e. in } \Omega\setminus C_{\mu_n}. \tag{2.8}$$

*Then, there exist a subsequence of $n$, still denoted by $n$, $\mu \in M_0(\Omega)$, $\theta^\alpha, \theta^\beta \in L^\infty(\Omega,[0,1])$, and $A \in L^\infty(\Omega\setminus C_\mu)^{N\times N}$, satisfying*

$$\theta^\alpha + \theta^\beta \leq 1 \ \text{a.e. in } \Omega, \quad \theta^\alpha + \theta^\beta = 1 \ \text{a.e. in } \Omega\setminus C_{\mu}, \quad A \in K(\theta^\alpha) \ \text{a.e. in } \Omega\setminus C_{\mu}, \tag{2.9}$$

*such that*

$$\theta_n^\alpha \overset{*}{\rightharpoonup} \theta^\alpha, \quad \theta_n^\beta \overset{*}{\rightharpoonup} \theta^\beta \qquad \text{in } L^\infty(\Omega), \tag{2.10}$$

*and such that for every $f \in H^{-1}(\Omega)$, the sequence of solutions $u_n$ of (2.5) converges weakly in $H_0^1(\Omega)$ to the solution $u$ of (2.6).*

From Theorem 2.7 we can obtain the following relaxation version of (1.3).

**Theorem 2.8** *Let $\Omega \subset \mathbb{R}^N$ be a bounded open set, $f \in H^{-1}(\Omega)$ and $F$ satisfying 2.1, 2.2 and 2.3. Then a relaxed formulation of (1.3) is given by*

$$\min \int_\Omega F(x,u)dx$$

$$\begin{cases} -\operatorname{div}(A\nabla u) + \mu u = f \ \text{ in } \Omega, \quad u \in H_0^1(\Omega) \cap L^2_\mu(\Omega) \\ \mu \in M_0(\Omega), \quad \theta^\alpha, \theta^\beta \in L^\infty(\Omega;[0,1]), \quad A \in K(\theta^\alpha) \text{ a.e. in } \Omega\setminus C_\mu \\ \theta^\alpha + \theta^\beta = 1 \ \text{a.e. in } \Omega\setminus C_\mu,\ \theta^\alpha + \theta^\beta \leq 1 \ \text{in } \Omega,\ \displaystyle\int_\Omega \theta^\alpha dx \leq \kappa^\alpha,\ \int_\Omega \theta^\beta \leq \kappa^\beta. \end{cases} \tag{2.11}$$

**Remark 2.9** *The set $K(\theta^\alpha)$ has an explicit but complex identification, in this sense, having in mind that in the relaxed formulation it is necessary $A\nabla u$ only, we can replace this set by*

$$\mathrm{Sp}(A) \subset \left[m^-(\theta^\alpha), m^+(\theta^\alpha)\right] \text{ a.e. in } \Omega \setminus C_\mu.$$

*Then, an alternative formulation of Theorem 2.8 is the following.*

$$\min \int_\Omega F(x,u)dx$$

$$\left\{\begin{array}{l} -\mathrm{div}(A\nabla u) + \mu u = f \ \text{ in } \Omega, \quad u \in H_0^1(\Omega) \cap L^2_\mu(\Omega) \\ \mu \in M_0(\Omega), \quad \theta^\alpha \in L^\infty(\Omega \setminus C_\mu; [0,1]), \quad A \in L^\infty(\Omega \setminus C_\mu)^{N\times N} \text{ symmetric} \\ \mathrm{Sp}(A) \subset [m^-(\theta^\alpha), m^+(\theta^\alpha)] \text{ a.e. in } \Omega \setminus C_\mu, \quad |\Omega \setminus C_\mu| - \kappa^\beta \leq \displaystyle\int_{\Omega\setminus C_\mu} \theta^\alpha dx \leq \kappa^\alpha. \end{array}\right. \tag{2.12}$$

## 3. Numerical Algorithm

We propose a numerical algorithm to solve the relaxed problem (2.12). We have two controls in the problem, the matrix $A$ and the measure $\mu$, since it can take the value $+\infty$, in order to get an approximation let us use a truncation corresponding to take $\mu$ as a measurable function taking values in $[0,n]$ with $n$ a positive constant, large enough, we could identify the set $C_\mu$ with the set $\{\mu = n\}$. Then,

$$\int_{\Omega\setminus C_\mu} \theta^\alpha dx \leq \kappa^\alpha \text{ replaced by } \int_{\{\mu<n\}} \theta^\alpha dx \leq \kappa^\alpha \Leftrightarrow \int_\Omega \theta^\alpha \chi_{\{[0,n)\}}(\mu) dx \leq \kappa^\alpha.$$

However the function $(s,\mu) \in [0,1]\times[0,\infty) \to s\chi_{\{[0,n)\}}(\mu)$ is not convex. Thus, it is more convenient to use its convex hull given by

$$(s,\mu) \in [0,1]\times[0,\infty) \to \left(s - \frac{\mu}{n}\right)^+.$$

Thus, we replace (2.12) by

$$\min \int_\Omega F(x,u)dx$$

$$\left\{\begin{array}{l} -\mathrm{div}(A\nabla u) + \mu u = f \ \text{ in } \Omega, \quad u \in H_0^1(\Omega) \\ \mu \in L^\infty(\Omega;[0,n]), \quad \theta \in L^\infty(\Omega;[0,1]), \quad A \in L^\infty(\Omega)^{N\times N} \text{ symmetric} \\ \mathrm{Sp}(A) \subset [m^-(\theta), m^+(\theta)] \text{ a.e. in } \Omega \\ \displaystyle\int_\Omega \left(\theta - \frac{\mu}{n}\right)^+ dx \leq \kappa^\alpha, \quad \int_\Omega \left(1 - \theta - \frac{\mu}{n}\right)^+ dx \leq \kappa^\beta. \end{array}\right. \tag{3.1}$$

The following theorem is proved in [6].

**Theorem 3.1** *Problem (3.1) has at least one solution for every $n \in \mathbb{N}$. Moreover, for every sequence of solutions $(\theta_n, A_n, \mu_n)$ of (3.1), there exist a subsequence, still denoted by $n$, and a solution $(\hat\theta^\alpha, \hat\theta^\beta, \hat A, \hat\mu)$ of (2.11) such that denoting by $u_n$ and $\hat u$ the solutions of the respective state equations, we have*

$$\left\{\begin{array}{l} u_n \rightharpoonup \hat u \ \text{ in } H_0^1(\Omega), \quad (A_n, \mu_n) \overset{H\gamma}{\rightharpoonup} (\hat A, \hat\mu) \\ \left(\theta_n - \dfrac{\mu_n}{n}\right)^+ \overset{*}{\rightharpoonup} \hat\theta^\alpha, \quad \left(1 - \theta_n - \dfrac{\mu_n}{n}\right)^+ \overset{*}{\rightharpoonup} \hat\theta^\beta \quad \text{ in } L^\infty(\Omega). \end{array}\right. \tag{3.2}$$

*Moreover*

$$\lim_{n\to\infty} \int_\Omega F(x,u_n)\,dx = \int_\Omega F(x,\hat u)\,dx. \tag{3.3}$$

Having in mind the convexity of the set of controls, for a given set of controls $(\theta_k, A_k, \mu_k)$ we search some new controls

$$\left\{\begin{array}{l} \theta_{k+1} = \theta_k + \varepsilon_k(\hat\theta - \theta_k), \\ A_{k+1} = A_k + \varepsilon_k(\hat A - A_k), \\ \mu_{k+1} = \mu_k + \varepsilon_k(\hat\mu - \mu_k), \end{array}\right. \tag{3.4}$$

such that the cost function decreases.

We propose to use a gradient descent method where the volume constraints are introduced by Lagrange multipliers (to determine) in the cost functional, these Lagrange multipliers are obtained using the Uzawa method. For more details for the algorithm see [6].

We put $u_k$ the solutin of

$$\begin{cases} -\operatorname{div}(A_k\nabla u_k) + \mu_k u_k = f \text{ in } \Omega \\ u_k \in H_0^1(\Omega). \end{cases} \tag{3.5}$$

We introduce the adjoint state $p_k$ as follow:

$$\begin{cases} -\operatorname{div}(A_k\nabla p_k) + \mu_k p_k = \partial_s F(x, u_k) \text{ in } \Omega \\ p_k \in H_0^1(\Omega), \end{cases} \tag{3.6}$$

and the functions

$$\begin{cases} E_k^+ = \dfrac{|\nabla u_k||\nabla p_k| + \nabla u_k \cdot \nabla p_k}{2}, \\ E_k^- = \dfrac{|\nabla u_k||\nabla p_k| - \nabla u_k \cdot \nabla p_k}{2}. \end{cases} \tag{3.7}$$

We fix a number $n \in \mathbb{N}$, large enough and note $I_k = \int_\Omega F(x, u_k)\,dx$. The algorithm is the following:

- Initialization: consider $\lambda_{0,1}, \lambda_{0,2} \geq 0$, $\theta_0 \in L^\infty(\Omega; [0,1])$, $A_0 \in L^\infty(\Omega)^{N\times N}$, $\mathrm{Sp}(A_0) \subset [m^-(\theta), m^+(\theta)]$, $\mu_0 \in L^\infty(\Omega; [0,n])$, $\rho > 0$ small and $\bar{j} \in \mathbb{N}$.
- for $k \geq 0$, iterate until convergence as follow:
  - We compute the solutions $u_k, p_k$ of (3.5) and (3.6) respectively, and later $E_k^+, E_k^-$ defined by (3.7).
  - We denote $\lambda_{k,1}^0 = \lambda_{k,1}, \lambda_{k,2}^0 = \lambda_{k,2}$, then for $j \leq \bar{j} - 1$, we define $(\lambda_{k,1}^{j+1}, \lambda_{k,2}^{j+1})$ by

$$\begin{cases} \lambda_{k,1}^{j+1} = \left(\lambda_{k,1}^j + \rho\Big(\displaystyle\int_\Omega \Big(\theta_k^j - \frac{\mu_k^j}{n}\Big)^+ dx - \kappa^\alpha\Big)\right)^+ \\ \lambda_{k,2}^{j+1} = \left(\lambda_{k,2}^j + \rho\Big(\displaystyle\int_\Omega \Big(1 - \theta_k^j - \frac{\mu_k^j}{n}\Big)^+ dx - \kappa^\beta\Big)\right)^+, \end{cases} \tag{3.8}$$

    with $\theta_k^j, \mu_k^j$ are defined by Proposition 4.2 in [6].
  - We take $\lambda_{k,1} = \lambda_{k,1}^{\bar{j}}, \lambda_{k,2} = \lambda_{k,2}^{\bar{j}}, \hat{\theta} = \theta_k^{\bar{j}}, \hat{\mu} = \mu_k^{\bar{j}}$ and $\hat{A}$ as a symmetric matrix function in $L^\infty(\Omega)^{N\times N}$ such that

$$\begin{cases} \hat{A}\nabla u_k = \dfrac{m^+(\hat{\theta}) + m^-(\hat{\theta})}{2}\nabla u_k + \dfrac{m^+(\hat{\theta}) - m^-(\hat{\theta})}{2}\dfrac{|\nabla u_k|}{|\nabla p_k|}\nabla p_k \quad \text{a.e. in } \{\nabla p_k \neq 0\} \\ \hat{A}\nabla p_k = \dfrac{m^+(\hat{\theta}) + m^-(\hat{\theta})}{2}\nabla p_k + \dfrac{m^+(\hat{\theta}) - m^-(\hat{\theta})}{2}\dfrac{|\nabla p_k|}{|\nabla u_k|}\nabla u_k \quad \text{a.e. in } \{\nabla u_k \neq 0\}. \end{cases} \tag{3.9}$$

    with $\mathrm{Sp}(\hat{A}) \subset [m^-(\hat{\theta}), m^+(\hat{\theta})]$, a.e. in $\Omega$ where $m^-(\hat{\theta})$ and $m^+(\hat{\theta})$ the harmonic and arithmetic mean values of $\alpha$ and $\beta$ with proportions $\hat{\theta}$ and $1 - \hat{\theta}$ respectively.
  - For $\varepsilon_k \in (0,1]$, we update $\theta_{k+1}, A_{k+1}, \mu_{k+1}$ by (3.4).
- Stop if convergence: $\frac{|I_k - I_{k-1}|}{|I_0|} < tol$, for $tol > 0$ small.

We finish this section showing some numerical experiments based in the algorithms described above. The computation has been carried out using the free software FreeFem++ v4.5 ( [8], available in http://www.freefem.org). The figures are obtained using Paraview 5.10.1 (available at https://www.kitware.com/open-source/# paraview), which is free also.

We use $P_1$-Lagrange finite element approximations for $u_k$ and $p_k$, solutions of the state and costate equations respectively, and $P_0$-Lagrange finite element approximations for control variables, $(\theta_k, A_k, \mu_k)$. For all simulations we consider $\Omega = [0,1]^2, \alpha = 1$ and $\beta = 2$.

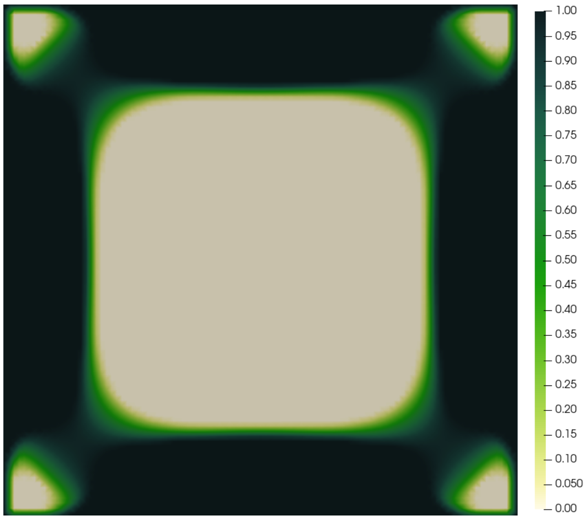


**Fig. 1** Example 1: $\kappa^\alpha = \kappa^\beta = \frac{1}{2}$: optimal $\theta$.

**Example 1.** We consider $F(x,u) = -u$, $f = 1$ and $\kappa^\alpha = \kappa^\beta = 0.5$. This problem has been solved by several authors in the case where we only optimize the matrix $A$ and fixed $\mu \equiv 0$ ( [1], [5], [7], [9]). We have considered $n = 10^4$, and we recover the optimal measure $\mu = 0$ and $(\theta^\alpha, A)$ given by the previous works, see Figure 1.

**Example 2.** We consider $F(x,u) = \dfrac{1}{2}\displaystyle\int_\Omega |u-1|^2\,dx$, $f = 1$ and different values of $\kappa^\alpha$ and $\kappa^\beta$. For the first simulation we consider $\kappa^\alpha = 0.35$ and $\kappa^\beta = 0.3$, in this case there is not enough material to fill out all the domain $\Omega$, thus we expect that the optimal $\mu \not\equiv 0$ defines a smaller domain, see Figure 2.

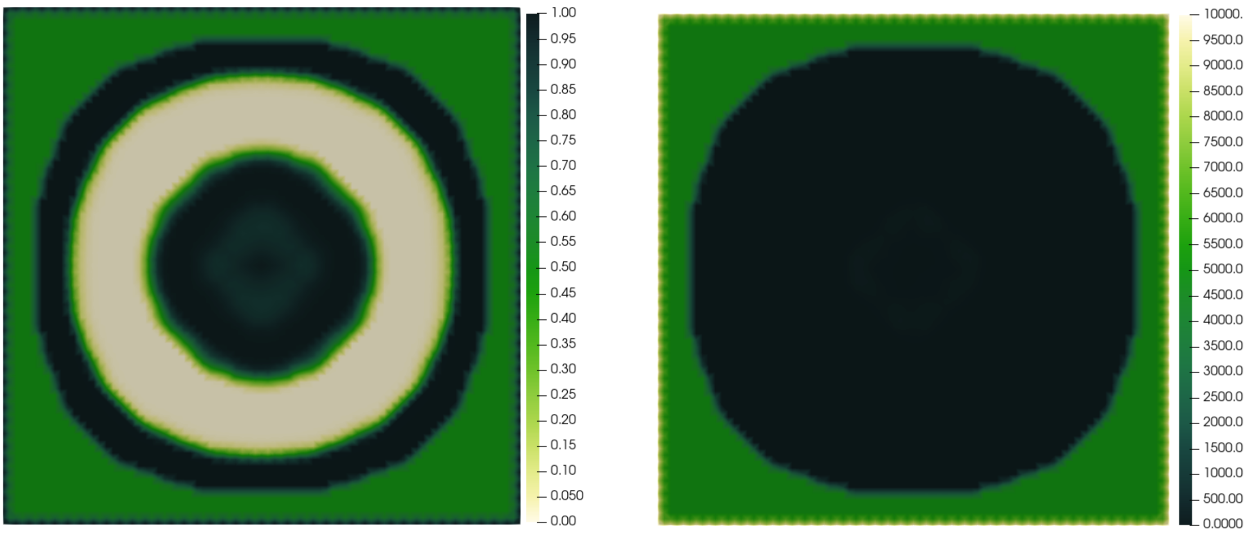


**Fig. 2** Example 2, $\kappa^\alpha = 0.35$ and $\kappa^\beta = 0.3$: computed optimal $\theta$ (left), computed optimal $\mu$ (right).

For a second simulation we consider $\kappa^\alpha = 0.43$ and $\kappa^\beta = 0.62$. In this case, as we expect all the domain is filled out using both materials and holes do not appears, and $\mu \equiv 0$, see Figure 3.

Finally, in Figure 4 we show the convergence of the algorithm for Example 2 in the case $\kappa^\alpha = 0.43$ and $\kappa^\beta = 0.62$. For the rest of the numerical simulations the convergence evolution is similar.

## Acknowledgements

The authors were partially supported by Grant PID2020-116809GB-I00 of the Government of Spain.

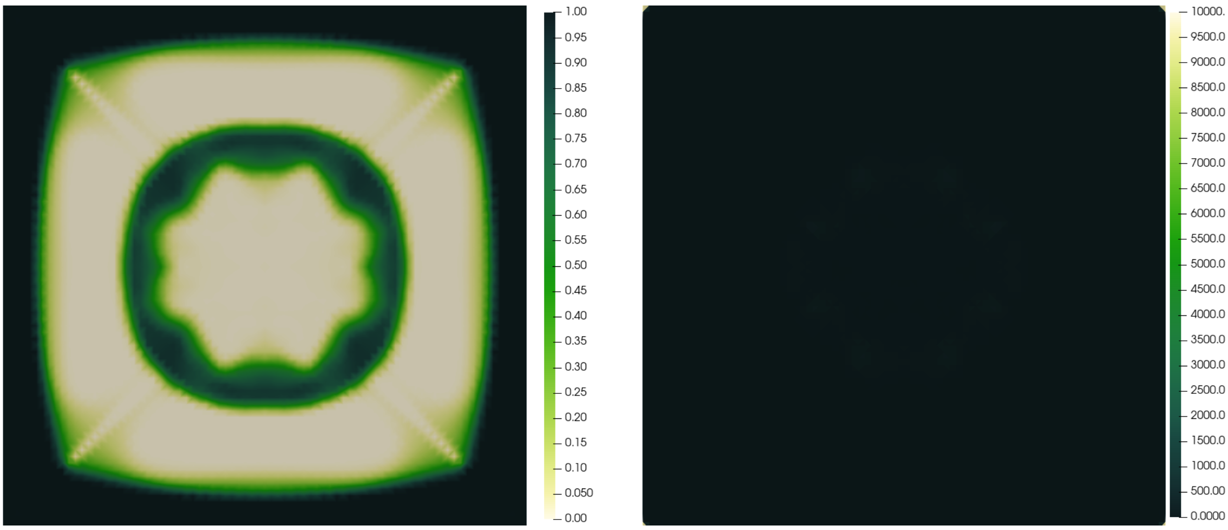


**Fig. 3** Example 2, $\kappa^\alpha = 0.43$ and $\kappa^\beta = 0.62$: computed optimal $\theta$ (left), computed optimal $\mu$ (right).

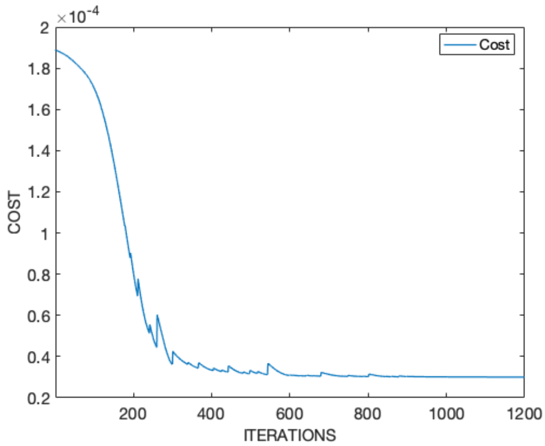


**Fig. 4** Example 2, $\kappa^\alpha = 0.43$ and $\kappa^\beta = 0.62$: cost evolution.

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