Beta equilibrium reaction rates at strong coupling using Gauge/Gravity Duality

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Chapter 1

Introduction

Quantum chromodynamics (QCD) is the fundamental theory of strong interactions. It has been subject of examination with great success in high-energy hadronic phenomenology based on perturbation theory [1]. However, it can't explain every physical system governed by the strong interaction. One of these physical systems, which is the main subject of this dissertation, is the long-studied strongly interacting matter present in the core of compact neutron stars. Some other approaches have been made to examine such systems in different temperature and density regimes, such as lattice QCD, where theory can describe nonzero temperature behavior and small baryonic chemical potential compared to temperature, or Chiral Effective Theory at small temperature and small baryonic density with respect to the saturation density. When the conditions in the core of heavy neutron stars result in a baryonic density higher than the saturation density, lattice QCD and Chiral Effective Theory aren't able to explain those regimes. Therefore, this work focuses on using holography to explore beta equilibrium conditions at high baryonic density, where matter must undergo weak decays to maintain its chemical composition against the density oscillations that neutron stars can experience. The regime of temperature in which these processes must be considered using a non-perturbative approach can be studied through gauge/gravity duality in a theoretical framework known as Holographic QCD [2].

The beta equilibrium condition that a neutron star must satisfy includes equalities in the chemical potentials for the quark content, which can be perturbed by density fluctuations in the star composition or in binary mergers [3]. The primary mechanism by which such a star maintains equilibrium is through weak decays of the quark flavors. The rates at which these changes occur are studied using thermal field theory techniques to study contributions to flavor current two point functions as a result of including flavor symmetry breaking terms due to weak interactions to the QCD Lagrangian and then obtaining non-conserved quark currents by the Ward Identities involving non-leptonic decays. The rates for these processes are interesting by themselves since they directly feed into the cooling behaviour of the neutron star [4].

This work is organized into basic concepts and the main part. The basic concepts are related to elements of Quantum Field Theory (QFT), specifically the quantum mechanics

partition functions and the field theory formalism used to define the path integral formulation. These are necessary to express the two-point correlators at non-zero temperature and non-zero chemical potential in the modern approach with quantum field theory, where the Schwinger-Keldysh formalism is required in the imaginary time approach.Then, Ward identities and QCD comments are discussed, which play an important role in the rate calculations, starting from one side of the holographic model. Chapter 3 is about gauge/gravity duality, first conjectured by Maldacena [5], in which a bottom-up model is used as a tool for non-perturbative calculations through a five-dimensional holographic QCD model based on the AdS5/CFT4 correspondence. There are also comments on Lie algebra to provide the correct background for the discussions in the Conformal Field Theory section. It then discusses the existence of a relationship based on the Holographic dictionary, which establishes connections between the field content of both approaches.This relationship is necessary for calculating the correlators defined in Chapter 4, which is part of the main section dealing with the beta equilibrium condition that the quark chemical potentials must satisfy, and the formulas for the quark density changes as functions of the quark decay rates. It then addresses flavor symmetry breaking through the addition of an electroweak term to the QCD action for massless quarks. Finally, using the theory developed in Chapter 2, the two-point functions for the electroweak currents derived from the Ward identities will be presented. There is also a mention of the bulk viscosity calculation that can be made through the quark density changes and the chemical potentials.

Chapter 5 concerns the holographic calculation for the electroweak rates. First, the holographic model is presented along with the corresponding duals for the quantities involved. It then deals with the calculation for the fluctuation equation of the left-handed gauge field, considering a Schwarzschild-AdS metric. The solutions to these fluctuations contain the form of the retarded correlators needed to calculate the reaction rates, as explained in Chapter 4. As a final note, it is important to mention the significance of new approaches in non-perturbative QCD as a means to perform calculations that provide insights into new aspects of field theory and its relation to the gravitational theory perspective. It is also concluded that this work can be extended by considering semi-leptonic weak decays and the finite temperature regime.

Part I

Basic concepts

Chapter 2

Elements of Quantum Field Theory

The main goal of this chapter is to provide context on elements of Thermal Quantum Field Theory. It begins with a quantum mechanical definition of the partition function and then establishes how thermal correlator functions can be computed from this definition within an imaginary-time theoretical framework. Subsequently, the chapter transitions to Quantum Field Theory through the introduction of the Schwinger-Keldysh formalism. It then defines correlators by the spectral function, which aligns with the final expression for the electroweak rates. The topics discussed here were taken from THERMAL FIELD THEORY by Michel Le Bellac [6]. After this development, the theory of Ward identities, as well as general aspects of QCD, are discussed due to their importance in defining current correlator functions, with specific references mentioned within the text.

2.1 Generating Functional

When a given process can take place in more than a single way in a time interval, its probability amplitude for transitions between an initial position $X_a(t_i)$ and a final one $X_b(t_f)$, points that belong to an space-time configuration with one spatial dimension, is defined as the total contribution of individual amplitudes for all possible paths between the extreme points. But first it is better to show how the formalism arises from the Quantum Statistical Mechanics theory and then look how temperature and chemical potential can be introduced in a quantum field theory, being important to derive the time-ordered propagator later.

2.1.1 Path Integral and Imaginary time approach

There are two ways of understanding quantum field theory with temperature and chemical potentials introduced as operator formalism and the path integral, where the last is better to quantize gauge theories. Both mathematical frames are useful for a zero temperature theory but when there is a finite theory is better to know the relationship between one another.

Let's start saying it is convenient to perform an analytical continuation from real to imaginary time as: $t \to -i\tau$ with $\tau \in \mathbb{R}$. There is also a correspondence with the momentum

space as $k_0 \to -ik_4$ and $k_4 \in \mathbb{R}$ as well. Now define the probability amplitude for a particle to make a transition between $(q,t) \to (q',t')$ as

$$F\left(q',t';q,t\right) = \langle q'|\, e^{-iH(t'-t)}\, |q\rangle \tag{2.1}$$

where the temporal evolution is described by a Hamiltonian with a time-independent potential $V(q)$. From now on, it has been taken a linear motion in $\mathbb{R}$ and with natural units ($\hbar = 1$). If one wants to make the analytical continuation of F, mentioned above, to imaginary time it just need a change like

$$F\left(q',t';q,t\right) \to F\left(q',-i\tau';q,-i\tau\right) = \langle q'|\, e^{-H(\tau'-\tau)}\, |q\rangle\ . \tag{2.2}$$

Now is when the path integral formalism can be proposed for the F quantity. Define the time interval $[\tau,\tau']$ and the length of each successive division as $\epsilon = (\tau-\tau')/(n+1)$ when $n \to \infty$ and take the well-known classical Hamiltonian for the time evolution in 2.2 with the momentum P and position q operators given by

$$e^{-(\tau'-\tau)H} = exp\left(-(\tau'-\tau)\left(\frac{\hat{P}^2}{2m} + V(q)\right)\right) \tag{2.3}$$

when trying to reach the bracket form of 2.2 let's introduce a complete set of position operator eigenstates at times $\tau_1 \dots \tau_n$

$$\begin{aligned}\langle q_{l+1}|\, e^{-(\tau'-\tau)H}\, |q_l\rangle &= \langle q_{l+1}|\, exp\left(-(\tau'-\tau)\left(\frac{\hat{P}^2}{2m} + V(q)\right)\right)|q_l\rangle \\ &= \langle q_{l+1}|\, exp\left(-\epsilon(n+1)\left(\frac{\hat{P}^2}{2m} + V(q)\right)\right)|q_l\rangle\ .\end{aligned} \tag{2.4}$$

At this point it is convenient to introduce the Lie product formula for A and B bounded operators

$$\lim_{n\to\infty}\left(e^{A/n}e^{B/n}\right)^n = e^{A+B} \tag{2.5}$$

and then turning back to the computation in (2.4), it can be followed by

$$\begin{aligned}\langle q_{l+1}|\, exp\left(-\epsilon(n+1)\left(\frac{\hat{P}^2}{2m} + V(q)\right)\right)|q_l\rangle = \langle q_{l+1}|\, exp\left(\frac{-\epsilon V(q)}{2}\right) \\ exp\left(\frac{-\epsilon\hat{P}^2}{2m}\right) exp\left(\frac{-\epsilon V(q)}{2}\right)|q_l\rangle\end{aligned}$$

with $\epsilon \to 0$. Reaching now the new form of the probability amplitude by

$$F\left(q', -i\tau'; q, -i\tau\right) = \lim_{\epsilon\to 0} \int \prod_{l=1}^{n} dq_l \left\langle q_{l+1}\right| exp\left(\frac{-\epsilon V(q)}{2}\right) exp\left(\frac{-\epsilon \hat{P}^2}{2m}\right) exp\left(\frac{\epsilon V(q)}{2}\right) \left|q_l\right\rangle \,. \tag{2.6}$$

The action of the momentum operator must be cleared with its matrix element, for which one introduces a complete set of eigenstates $|p_l\rangle \langle p_l|$ alongside the operator in the previous equation and putting all together in the probability amplitude one gets

$$F\left(q', -i\tau'; q, -i\tau\right) = \lim_{\epsilon\to 0} \left(\frac{m}{2\pi\epsilon}\right)^{\frac{1}{2}} \int \prod_{l=1}^{n} dq_l \left(\frac{m}{2\pi\epsilon}\right)^{\frac{1}{2}} \times exp\left(-\epsilon \left(\sum_{l=0}^{n} \frac{m(q_{l+1}-q_l)^2}{\epsilon^2}\right) + \sum_{l=0}^{n} V\left(\frac{q_{l+1}+q_l}{2}\right)\right) \,. \tag{2.7}$$

From this last expression its convenient to take a new definition which is nothing else than the one for the differential element for the path integral

$$Dq(\tau'') = \lim_{\epsilon\to 0} \left(\frac{m}{2\pi\epsilon}\right)^{\frac{1}{2}} \prod_{l=1}^{n} \left(\frac{m}{2\pi\epsilon}\right)^{\frac{1}{2}} dq_l \,. \tag{2.8}$$

The argument of the exponential term in (2.7) is related with the Riemann sum such that the Euclidean Action can be defined as

$$S_E(\tau' - \tau) = \int_{\tau}^{\tau'} d\tau'' \left(\frac{1}{2} m\dot{q}^2(\tau'') + V(q(\tau''))\right) \tag{2.9}$$

and finally the path integral formalism gives the probability amplitude

$$F\left(q', -i\tau'; q, -i\tau\right) = \int Dq(\tau'') exp\left(-S_E(\tau' - \tau)\right) \tag{2.10}$$

a quantity tied to the boundary conditions on the path $q(\tau'')$ defined by $q(\tau) = q$ and $q(\tau') = q'$.

2.1.2 Partition Function in Quantum Mechanics

With all this computation the main objective is now to get a path integral definition for the Partition function related with quantum statistical mechanics. First, one must take the expression

$$Z(\beta) = Tr\left(e^{-\beta H}\right) \to \sum_{n} \left(e^{-\beta E_n}\right) \tag{2.11}$$

with evaluated trace over the eigenvectors of the Hamiltonian H, $\beta = T^{-1}$ and units as $k_B = 1$. But the trace has another expression dealing with a complete set of the position operator eigenstates like

$$Z(\beta) = \int dq \, \langle q | \, e^{-\beta H} \, | q \rangle \ . \tag{2.12}$$

It is possible to make a connection with the probability amplitude in (2.1) such that

$$Z(\beta) = \int dq F \left(q, -i\beta; q, 0 \right) \tag{2.13}$$

and with (2.10) one have reached the partition function as a path integral like

$$Z(\beta) = \int Dq(t) exp \left(-S_E(\beta) \right) \tag{2.14}$$

where the exponential term will have an integral over the Hamiltonian from 0 to β in imaginary time and the paths in the partition function are restricted to the boundary condition $q(\beta) = q(0)$ as it was said with $\mathrm{Im}\, t) = \beta$.

With the path integral formalism, like in QFT, the mathematical object known as **generating functional** can be defined by the Euclidean action with an additional term for some external source

$$Z(\beta, j) = \int Dq(\tau) exp \left(S_E(\beta) + \int_0^\beta d\tau j(\tau) q(\tau) \right) \tag{2.15}$$

such that the previous object can be understood as a functional which by its differentiation defined as

$$\frac{1}{Z(\beta)} \frac{\delta^2 Z(\beta, j)}{\delta j(\tau_1) \delta j(\tau_2)} \bigg|_{J=0} = \frac{1}{Z(\beta)} \int Dq(\tau) q(\tau_1) q(\tau_2) exp \left(S_E(\beta) \right) \tag{2.16}$$

2.2 Thermal Correlators

The generating functional is now the key to understand the nature of some important quantities in QFT with finite temperature formalism. Indeed, the thermal average of a time-ordered product of position operators is written like

$$\left\langle T \left(q(-i\tau_1) q(-i\tau_2) \right) \right\rangle_\beta = \frac{1}{Z(\beta)} Tr[e^{-\beta H} T \left(q(-i\tau_1) q(-i\tau_2) \right)] \tag{2.17}$$

where, it is useful to describe the position operator in term of the Heisenberg picture by

$$\begin{aligned} q(t) &= e^{iHt} q e^{-iHt} \\ q(-i\tau) &= e^{H\tau} q e^{-H\tau} \ . \end{aligned} \tag{2.18}$$

By definition, the thermal average of an arbitrary operator $\hat{A}$ is

$$\left\langle \hat{A} \right\rangle_\beta = \frac{1}{Z(\beta)} Tr\left(\hat{A} e^{-\beta H}\right) . \tag{2.19}$$

Moreover, the time-ordered product was defined in the context of imaginary time, that is

$$T\left(q(-i\tau_1)q(-i\tau_2)\right) = \begin{cases} q(-i\tau_1)q(-i\tau_2), & \tau_1 > \tau_2 \\ q(-i\tau_2)q(-i\tau_1), & \tau_2 > \tau_1 \\ . \end{cases} \tag{2.20}$$

So, if one needs the operator form of the generating functional it can be seen by

$$Z(\beta, j) = Tr\left(e^{-\beta H} T\left(\int_0^\beta d\tau j(\tau) q(-i\tau)\right)\right) \tag{2.21}$$

and using the relation (2.17) together with the cyclic property of the trace and the periodicity of the path integral in (2.14) which is $q(\beta) = q(0)$, one can derive an important property for the propagator which is

$$\begin{aligned} \left\langle T\left(q(-i\beta)q(-i\tau)\right)\right\rangle_\beta &= \frac{1}{Z(\beta)} Tr[e^{-\beta H} T\left(q(-i\beta)q(-i\tau)\right)] \\ &= \left\langle T\left(q(0)q(-i\tau)\right)\right\rangle_\beta \end{aligned} \tag{2.22}$$

where one can notice the same argument of cyclic trace property seen before and then let's define a new function as

$$\Delta(\tau) = \left\langle T\left(q(-i\tau)q(0)\right)\right\rangle_\beta \tag{2.23}$$

also being periodic in imaginary time meaning $\Delta(\tau - \beta) = \Delta(\tau)$ for $\tau \in [0, \beta]$. It is now time to move on and giving the explicit formulation for the potential term in the Hamiltonian. Let's use the **Harmonic Oscillator** which implies

$$V(q) = \frac{1}{2}\omega^2 q^2 . \tag{2.24}$$

Now one may focus on knowing the time-ordered product in (2.17). But first it is necessary to include the explicit computation of $Z(\beta, j)$ and for that purpose, using the path integral seen in (2.15) one has

$$\begin{aligned} Z(\beta, j) &= \int Dq(\tau) exp\left(-\int_0^\beta \frac{1}{2} q(\tau)\left(-\frac{d^2}{d\tau^2} + \omega^2\right) q(\tau) - j(\tau)q(\tau) d\tau\right) \\ Z(\beta, j) &= Z(\beta) exp\left(\frac{1}{2}\int d\tau d\tau' j(\tau) K(\tau, \tau') j(\tau')\right) \end{aligned} \tag{2.25}$$

such that the new function $K(\tau, \tau')$ is just the Green function which also satisfy

$$\left(-\frac{d^2}{d\tau^2} + \omega^2\right) K(\tau, \tau') = \delta(\tau - \tau') . \tag{2.26}$$

Using some of the equations derived so far it is possible to make a correspondence between the Green function and the $\Delta(\tau)$ function. Specifically, the relations (2.16), (2.23) and (2.25) are manipulated to obtain

$$\Delta_F(\tau - \tau') = K(\tau, \tau') \tag{2.27}$$

where the subscript F is used to make it clear when taking the Harmonic Oscillator formalism on the free field approach. There is also another way to describe the $\Delta_F(\tau - \tau')$ function when $\tau \in [0, \beta]$ by

$$\Delta_F(\tau) = \frac{1}{2\omega}\Big\{(1 + n(\omega))e^{-\omega\tau} + n(\omega)e^{\omega\tau}\Big\} \tag{2.28}$$

with the new function $n(\omega)$ as the Bose-Einstein distribution

$$n(\omega) = \frac{1}{e^{\beta|\omega|} - 1} \tag{2.29}$$

2.2.1 Two-point function formalism

The next logical step is to define correlators for real values of time component. For this, let's define the following two-point functions, as in QFT by

$$\begin{aligned} D^{>}(t, t') &= \big\langle \big(q(t)q(t')\big)\big\rangle_\beta \\ D^{<}(t, t') &= \big\langle \big(q(t')q(t)\big)\big\rangle_\beta \end{aligned} \tag{2.30}$$

and then checking by symmetry that it is possible to associate one another through $D^{<}(t, t') = D^{>}(t', t)$. Another useful expression for the previous functions can be found inserting a complete set of eigenstates of the Hamiltonian, $|n\rangle\langle n|$ and $|m\rangle\langle m|$ between the exponential terms and $q(0)$ and using the equation (2.22) it is possible to write

$$\begin{aligned} D^{>}(t, t') &= \frac{1}{Z(\beta)} Tr[\Big(q(t)q(t')e^{-\beta H}\Big)] \\ D^{>}(t, t') &= \frac{1}{Z(\beta)} Tr\left(e^{-\beta H} q(0) e^{iH(t'-t)} q(0) e^{iH(t-t')}\right) \end{aligned}$$

$$\begin{aligned} D^{>}(t, t') = \big\langle \big(q(t)q(t')\big)\big\rangle_\beta = \frac{1}{Z(\beta)} \sum_{m,n} & e^{-\beta E_n} e^{iE_m(t'-t)} \\ & e^{-iE_n(t'-t)} |\langle n|\, q(0)\, |m\rangle|^2 . \end{aligned} \tag{2.31}$$

If is needed to see the convergence of the previous definition, this is given by the exponential terms, so the two-point functions are define in

$$\begin{cases} -\beta \leq \operatorname{Im} t - t') \leq 0, & D^{>}(t, t') \\ \beta \geq \operatorname{Im} t - t') \geq 0, & D^{<}(t, t') \end{cases} \tag{2.32}$$

where one notice that $e^{-\beta H}$ is an imaginary time evolution operator such that

$$q(t + i\beta) = e^{-\beta H} q(t) e^{\beta H} . \tag{2.33}$$

Now on it is convenient to mention the **Kubo-Martin-Schwinger** relation (KMS) which is important in understanding the periodicity properties of both two-point functions

$$D^{>}(t,t') = D^{<}(t+i\beta,t') \tag{2.34}$$

and for the previous interval $\tau \in [0,\beta]$, there is an equation between the Delta and the two-point functions

$$\begin{aligned} D^{>}(-i\tau,0) &= \left\langle q(-i\tau)q(0)\right\rangle_{\beta} \\ D^{>}(-i\tau,0) &= \Delta(\tau)\,. \end{aligned} \tag{2.35}$$

The time ordered propagator when $(t,t') \in \mathbb{R}$, must be defined by its importance in the next sections as

$$\begin{aligned} D(t,t') =& \left\langle T\left(q(t)q(t')\right)\right\rangle \\ =& \theta(t-t')D^{>}(t,t') + \theta(t'-t)D^{<}(t,t') \end{aligned} \tag{2.36}$$

which may remember the definition of the Feynman propagator in QFT.

2.2.2 Definition of the Spectral Function

Let's start with the introduction of a new label for the two-point functions, which will be useful later, by

$$\begin{aligned} D^{>}(t) &= D^{>}(t,0) \\ D^{<}(t) &= D^{<}(t,0) \end{aligned} \tag{2.37}$$

and the Fourier transform for both written as

$$\begin{aligned} D^{>}(k_0) &= \int_{-\infty}^{\infty} e^{ik_0 t} D^{>}(t)\,dt \\ D^{<}(k_0) &= \int_{-\infty}^{\infty} e^{ik_0 t} D^{<}(t)\,dt = \int_{-\infty}^{\infty} e^{ik_0\,t}\,D^{>}(t-i\beta)\,dt \end{aligned} \tag{2.38}$$

where the second equation holds from KMS relation. It is also possible to show an important property following from the previous relations with

$$D^{<}(k_0) = D^{>}(-k_0) = e^{-\beta k_0} D^{>}(k_0) \tag{2.39}$$

what this means is that the two-point function are real-valued functions with traslational invariance and remember the position operators q which are Hermitic. It is possible to define a commutation relation for the position operators by introducing the "spectral function" as

$$\rho(k_0) = D^{>}(k_0) - D^{<}(k_0) \tag{2.40}$$

which contains indeed a commutator by

$$\begin{aligned}\rho(k_0) &= D^{>}(k_0) - D^{<}(k_0) \\ &= \langle q(k_0)q(0)\rangle_\beta - \langle q(0)q(k_0)\rangle_\beta \\ &= \langle [q(k_0), q(0)]\rangle_\beta\end{aligned} \tag{2.41}$$

which means that the spectral function is a thermal average value. There is another way to write the two-point Green functions by the introduction of a new quantity such that

$$f(k_0) = \left(e^{\beta k_0} - 1\right)^{-1} \tag{2.42}$$

and with the relations in (2.39) one has

$$\begin{aligned}D^{<}(k_0) &= e^{-\beta k_0} D^{>}(k_0) + e^{-\beta k_0} D^{<}(k_0) - e^{-\beta k_0} D^{<}(k_0) \\ &= e^{-\beta k_0}\rho(k_0) + e^{-\beta k_0} D^{<}(k_0) \\ D^{<}(k_0) &= f(k_0)\rho(k_0)\end{aligned} \tag{2.43}$$

and with an analogous procedure

$$D^{>}(k_0) = (1 + f(k_0))\rho(k_0)\,. \tag{2.44}$$

An explicit version of the spectral function can be computed, using the relations in (2.39) and the equation (2.31), by the expression

$$\begin{aligned}\rho(k_0) = \frac{2\pi}{Z(\beta)} \sum_{n,m} e^{\beta E_n} \big(&\delta(k_0 + E_n - E_m) - \delta(k_0 + E_m - E_n) \\ &\times |\langle n|\, q(0)\, |m\rangle|^2\end{aligned} \tag{2.45}$$

from this one can highlight the following properties

- the spectral function $\rho(k_0)$ is an odd real-valued function of k_0; $\rho(k_0) = -\rho(-k_0)$
- the spectral function satisfy the positivity condition described by $\epsilon(k_0)\rho(k_0) > 0$ with ϵ as the sign function.

At this point, it is convenient to go further in the Harmonic Oscillator formalism taking the position operators as a sum of the ladder ones by

$$q(t) = \frac{1}{\sqrt{2\omega}}\left(\hat{a}e^{-i\omega t} + \hat{a}^\dagger e^{i\omega t}\right) \tag{2.46}$$

so, with the commutation relation derived from (2.41) and the Harmonic Oscillator form in the previous equation, one has

$$\begin{aligned}D^{>}(t) - D^{<}(t) &= \langle [q(t), q(0)]\rangle_\beta \\ &= \frac{1}{2\omega}\langle e^{-i\omega t}[\hat{a}, \hat{a}^\dagger] + e^{i\omega t}[\hat{a}^\dagger, \hat{a}]\rangle_\beta\end{aligned} \tag{2.47}$$

and if $[\hat{a}, \hat{a}^\dagger] = 1$ one finally gets

$$D^>(t) - D^<(t) = \frac{1}{2\omega}\langle e^{-i\omega t} - e^{i\omega t}\rangle_\beta \,. \tag{2.48}$$

A different kind formalism inside Harmonic Oscillator theory can be related to the two-point function quantities through the canonical commutation rule between position operator and its derivative where

$$\begin{aligned}\int_{-\infty}^{\infty} dk_0 \frac{1}{2\pi} k_0 e^{ik_0 t}\left(D^>(k_0) - D^<(k_0)\right) &= i\frac{d}{dt}\left(D^>(t) - D^<(t)\right) \\ &= i\frac{d}{dt}\langle [q(t), q(0)]\rangle_\beta\end{aligned} \tag{2.49}$$

and if one takes the limit when $t \to 0$, a general sum rule expression over the spectral function can be written by

$$\int_{-\infty}^{\infty} dk_0 \frac{1}{2\pi}\rho(k_0) = 1\,. \tag{2.50}$$

Once again, in the free field approach one gets the free spectral, temperature-independent, function by a Fourier transform on the equation (2.48) and with a particular fact about δ-function on composition property such that

$$\begin{aligned}\rho(k_0) &= \int dt e^{ik_0 t}\frac{1}{2\omega}\langle e^{-i\omega t} - e^{i\omega t}\rangle_\beta \\ &= \frac{1}{2\omega}\left(\int e^{i(k_0-\omega)t}dt - \int e^{i(k_0+\omega)t}dt\right) \\ \rho_F(k_0) &= 2\pi\epsilon(k_0)\delta(k_0^2 - \omega^2)\end{aligned} \tag{2.51}$$

2.2.3 The Matsubara Propagator

There was a first approach to the Matsubara propagator, also named as Imaginary propagator, previously in this section by the introduction of delta function in equation (2.23). The purpose now is to define some properties starting from its Fourier transform in

$$\Delta(i\omega_n) = \int_0^\beta d\tau e^{i\omega_n \tau}\delta(\tau) \tag{2.52}$$

and the inverse transformation as

$$\Delta(\tau) = T\sum_n e^{-i\omega_n\tau}\Delta(i\omega_n) \tag{2.53}$$

where the periodicity property can be remembered from equation (2.23). The fact that Fourier transforming procedure is defined considering the $[0, \beta]$ time interval, the frequency becomes discrete given by the values

$$\omega_n = \frac{2\pi n}{\beta} \tag{2.54}$$

known as **Matsubara frequencies**. Now, if once again $\tau \in [0, \beta]$, together with the Fourier transform of two-point Green function in (2.38) one gets

$$\Delta(\tau) = D^{>}(-i\tau) = \int_{-\infty}^{\infty} dk_0 \frac{1}{2\pi} e^{-k_0\tau} D^{>}(k_0)\,. \tag{2.55}$$

There is also another path to describe the imaginary propagator in (2.52) by using the spectral function together with equation (2.44) to prove that

$$\Delta(i\omega_n) = -\int_{-\infty}^{\infty} dk_0 \frac{1}{2\pi} \frac{\rho(k_0)}{i\omega_n - k_0} \tag{2.56}$$

and using the free field approach in (2.51) together with the composition property of δ-function one gets

$$\begin{aligned} \Delta(i\omega_n) &= -\int_{-\infty}^{\infty} dk_0 \frac{1}{2\pi} \frac{2\pi\epsilon(k_0)\delta(k_0^2 - \omega^2)}{i\omega_n - k_0}; \\ \delta(x^2 - \alpha^2) &= \frac{1}{2|\alpha|}\left(\delta(x+\alpha) + \delta(x-\alpha)\right) \end{aligned} \tag{2.57}$$

and then one can finally write

$$\Delta_F(i\omega_n) = \frac{1}{\omega_n^2 + \omega^2} \tag{2.58}$$

where it is possible to see this expression as the Fourier transform of the previous equation for the free Delta Function in (2.23).

Another pair of operators must be defined in order to achieve a time-ordered propagator value. These ones are the retarded and advanced propagators like

$$\begin{aligned} D_R(t) &= \left\langle \theta(t)[q(t), q(0)] \right\rangle_\beta \\ D_A(t) &= -\left\langle \theta(-t)[q(t), q(0)] \right\rangle_\beta \end{aligned} \tag{2.59}$$

where the θ-function defined as

$$\theta(t) = i \int_{-\infty}^{\infty} dk_0' \frac{1}{2\pi} \frac{e^{-ik_0' t}}{{k_0'}^2 + i\eta}\,. \tag{2.60}$$

The equation (2.56) for the time-imaginary propagator owns an analytical continuation which is not unique. To know it, its enough to take in count the condition of convergence $|\Delta(z)| \to 0$ when $|z| \to \infty$ and also $\Delta(z)$ must satisfy being analytic beyond the real axis, all together joined in the expression

$$\Delta(z) = -\int_{-\infty}^{\infty} dk_0 \frac{1}{2\pi} \frac{\rho(k_0)}{z - k_0}\,. \tag{2.61}$$

It is also useful for the sections to come, performing a Fourier transform on the definitions

(2.59) of retarded propagator

$$D_R(k_0) = i\int_{-\infty}^{\infty} dk_0' \frac{1}{2\pi}\frac{\rho(k_0')}{k_0 - k_0' + i\eta} \tag{2.62}$$

and using relation (2.56) one gets a correspondence in momentum space between the retarded and advanced propagators with the Matsubara's by

$$\begin{aligned} D_R(k_0) &= -i\Delta(k_0 + i\eta) \\ D_A(k_0) &= i\Delta(k_0 - i\eta) \end{aligned} \tag{2.63}$$

So the analytic continuation in (2.61) is analogous to the complex contour used when defining the Feynman propagator in QFT, and for (2.62), it makes the retarded propagator analytic in the upper half of the k_0-complex plane

Until now, it has been possible to define a wide range of quantities in the momentum space through the Fourier transform and with the components of (2.36) one is able to do the same like

$$\begin{aligned} D(t) &= \theta(t-t')D^{>}(t,t') + \theta(t'-t)D^{<}(t,t') \\ D(k_0) &= \int dk_0 e^{ik_0 t}\left(\theta(t-t')D^{>}(t,t') + \theta(t'-t)D^{<}(t,t')\right) \end{aligned} \tag{2.64}$$

and finally, using the θ-function in (2.60) together with the Fourier transform in (2.43) and (2.44) one gets

$$D(k_0) = i\int dk_0 \frac{1}{2\pi}\frac{\rho(k_0')}{k_0 - k_0' + i\eta} + f(k_0)\rho(k_0)\,. \tag{2.65}$$

For the Green retarded and advanced functions, one notes a temperature-invariance given by the spectral function and joining this with the free field approach in (2.51) its possible to re-write (2.98) by

$$\begin{aligned} D_F(k_0) &= i\int dk_0' \frac{1}{2\pi}\frac{\rho_F(k_0')}{k_0 - k_0' + i\eta} + f(k_0)\rho_F(k_0) \\ &= i\int dk_0' \frac{\epsilon(k_0')\delta(k_0'^2 - \omega^2)}{k_0 - k_0' + i\eta} + 2\pi\epsilon(k_0)\delta(k_0^2 - \omega^2)f(k_0) \\ D_F(k_0) &= \frac{i}{k_0^2 - \omega^2 + i\eta} + 2\pi n(k_0)\delta(k_0^2 - \omega^2) \end{aligned} \tag{2.66}$$

where a new function given by $n(k_0) = \left(e^{\beta|k_0|} - 1\right)^{-1}$. Such function gives an important remark on temperature-dependence by the second term of the free field function D_F regarding the first temperature-invariant term.

2.2.4 Green Function in Quantum Field Theory

The main purpose now, once the formalism of the generating functional in quantum mechanics is established, is to find the generalized form in quantum field theory, considering, for simplicity, the scalar field operator in spacetime coordinates. Such a field, like the

position operators, evolves in the Heisenberg picture as

$$\hat{\phi}(x) = e^{it\hat{H}}\hat{\phi}(0)e^{-it\hat{H}} \tag{2.67}$$

such that the coordinates (x) belong to a flat spacetime in which one can identify $x^0 = t$ that could be a complex valued parameter, $t \in \mathbb{C}$. Previously in this section, it was defined the Green function depending on operators in quantum mechanics (2.36). What one needs now in quantum field theory is a definition of the thermal Green function given by

$$G_C(x_1, x_2, \ldots, x_N) = \left\langle T_C\left(\phi(x_1)\phi(x_2)\ldots\phi(x_N)\right)\right\rangle_\beta \tag{2.68}$$

where the time-ordered operator T_C is defined along the complex time path C. From now on the notation of field operators will be simplified by $\hat{\phi} \to \phi$.

The path C is essential for defining the path integral and the generating functional later on. However, there will be difficulties due to the fact that it is a complex path, and one needs to define real quantities. Therefore, it is possible to redefine the coordinate t from the previous path using a parameter map such that

$$t = z(v) \tag{2.69}$$

with the identification $v \in \mathbb{R}$ which increase in such way that the ordering along the path C will agree with the increasing ordered values of v. The quantum mechanical formalism required the definitions of the θ- and δ- functions to define the correlators like in (2.36), which will have a similar structure in this new formalism of the complex path C by

$$\begin{aligned} \theta_C(t - t') &= \theta(v - v') \\ \delta_C(t - t') &= \left(\frac{\partial z}{\partial v}\right)^{-1} \delta(v - v')\,. \end{aligned} \tag{2.70}$$

With these new definitions, one can define the time-ordered operator in the complex path C, in a similar way of (2.20), as

$$\begin{aligned} T_C\left(\phi(x)\,\phi(x')\right) &= \theta_C(t - t')\phi(x)\phi(x') + \theta_C(t' - t)\phi(x')\phi(x) \\ \partial_t T_C\left(\phi(x)\,\phi(x')\right) &= \delta_C(t - t')[\phi(x), \phi(x')] + T_C\left(\partial_t\phi(x)\phi(x')\right)\,. \end{aligned} \tag{2.71}$$

There is still the definition of the functional derivative that one needs to give a generalized version with spacetime coordinates by

$$\frac{\delta j(x)}{\delta j(x')} = \delta_C(t - t')\delta^{(3)}(x - x') \tag{2.72}$$

where the source functions $j(x)$ are defined along the path C.

As it was made in (2.16), one needs to construct the Generating functional, given by the notation along the C path by $Z_C(\beta, j)$, to get the Green Function from functional

differentiation depending on the sources from

$$G_C(x_1,\ldots,x_N) = \frac{1}{Z(\beta)} \frac{\delta^N Z_C(\beta,j)}{i\delta j(x_1)\ldots i\delta j(x_N)}\bigg|_{j=0} . \tag{2.73}$$

The specific structure of the new generating functional could be analogous to that in quantum mechanics given for example in (2.21), now along the C path and related to the possible results of the previous Green function, by

$$Z_C(\beta,j) = Tr\left[e^{-\beta H} T_C\left(exp\left(i\int_C d^4x j(x)\phi(x)\right)\right)\right] \tag{2.74}$$

in which the path C must cover all the arguments of the Green's function definition. One can also notice the boundary condition $Z_C(\beta, j=0) = Z(\beta) = Tr(exp(-\beta H))$, where it can be found the correspondence with quantum mechanics definition without considering the path C.

The structure for the propagator D in (2.36) can be brought to this formalism with the generalized expression for the complex path

$$D_C(x,x') = \theta_C(t-t')D_C^{>}(x,x') + \theta_C(t'-t)D_C^{<}(x,x') \tag{2.75}$$

and considering the definition for the $D^<$ and $D^>$ operators given in (2.30) by their new complex path formalism one finally gets

$$\begin{aligned} D_C^{>}(x,x') &= \left\langle \phi(x)\phi(x')\right\rangle_\beta \\ D_C^{<}(x,x') &= D_C^{>}(x',x) = \left\langle \phi(x')\phi(x)\right\rangle_\beta . \end{aligned} \tag{2.76}$$

The operators previously describes are both analytical while $-\beta < \mathrm{Im}\, t-t') < 0$ and $0 < \mathrm{Im}\, t-t') < \beta$ respectively and analogous to (2.31) and (2.32). From these requirements, looking at the propagator in (2.75), it is necessary to define the path C such that the imaginary part of $(t-t')$ doesn't increase when the parameter v in does because that will change the intervals of analytical definition, so one needs to add spacetime components and complex values of t looking for a new structure of (2.36).

The starting point guided for the previous purpose can be to postulate translation invariance for the propagator $D_C(x,x')$ as

$$D_C(x-x') = \theta_C(t-t')\left[D_C^{>}(x-x') - D_C^{<}(x-x')\right] + D_C^{<}(x-x') \tag{2.77}$$

and if also one uses the definition of the spectral function given in (2.40) with a generalization to 4-dimensional spacetime the new spectral function will be

$$\rho(k) = \rho(t,\vec{k}) \tag{2.78}$$

so, through the Fourier transform procedure one gets a 4-momentum expression for the

propagator in (2.77) with

$$D_C(x-x') = \int \frac{d^4k}{(2\pi)^4} e^{-ik(x-x')} \left(\theta_C(t-t') + f(k_0)\right) \rho(k) \tag{2.79}$$

where it was possible to find a symmetry rule by $D_C(x-x') = D_C(x'-x)$ and was preserved the form of $f(k_0)$ given in (2.42), also using a new relation such that

$$1 + f(k_0) + f(-k_0) = 0\,. \tag{2.80}$$

One can find some other relation in quantum mechanics formalism introducing again the "free" concept as for the free spectral function now in spacetime coordinates by

$$\rho_F(k) = 2\pi\epsilon(k_0)\delta(k^2 - m^2)\,. \tag{2.81}$$

At this point it is missing the path integral formalism to much used in the quantum mechanics section. For this reason, the following concentrates in developing a consistent formalism for the path integral to be incorporated to the generating functional and the field operator states. So, the field operator is given by $\phi(x) = \phi(t, \vec{x})$ in the Heisenberg picture and be $|\phi(\vec{x}); t\rangle$ the state vector when at a given time t is an eigenstate of the field operator with eigenvalue $\phi(\vec{x})$, formally

$$\phi(x)\,|\phi(\vec{x}); t\rangle = \phi(\vec{x})\,|\phi(\vec{x}); t\rangle\ . \tag{2.82}$$

Those state vectors evolve by the action of the Hamiltonian operator such that

$$|\phi(\vec{x}); t\rangle = e^{itH}\,|\phi(\vec{x}); t=0\rangle \tag{2.83}$$

where this states form a complete set for any value of time. These set will be the key to define the path integral later starting from the definition for the thermal average of an arbitrary operator in (2.19) and now in the field operator formalism

$$\begin{aligned} \left\langle \hat{A} \right\rangle_\beta &= \frac{1}{Z(\beta)} Tr\left(e^{-\beta H}\hat{A}\right) \\ &= \frac{1}{Z(\beta)} \int [d\phi]\,\langle\phi(\vec{x}); t|\, e^{-\beta H}\hat{A}\,|\phi(\vec{x}); t\rangle\ . \end{aligned} \tag{2.84}$$

The exponential term defined by the Hamiltonian acts on the dual state as an evolution operator on the time path and in a similar way like (2.22), with the integral over $[d\phi]$ which is counting all possible field configurations, the generating functional on the complex path C is now given by

$$Z_C(\beta, j) = \int [d\phi']\,\langle\phi'(\vec{x}); t - i\beta|\, T_C\left(exp\left(i\int_C d^4x j(x)\phi(x)\right)\right)|\phi'(\vec{x}); t\rangle \tag{2.85}$$

where the evolution operator makes clear the path followed by the time coordinate on C

starting from an initial value t_i ending in $t_i - i\beta$. Like in (2.15), the generating functional is described under the path integral formalism which later will be formalized but first, one needs to indicate the new domain of the integral in the exponential term by

$$Z_C(\beta, j) = \int D\phi exp\left(i \int_C d^4x\, \mathcal{L}(x) + j(x)\phi(x)\right) . \tag{2.86}$$

The Green functions G_C will be obtained from functional derivative on the previous definition and the difference between $[d\phi]$ and $D\phi$ is analogous to that in (2.8).

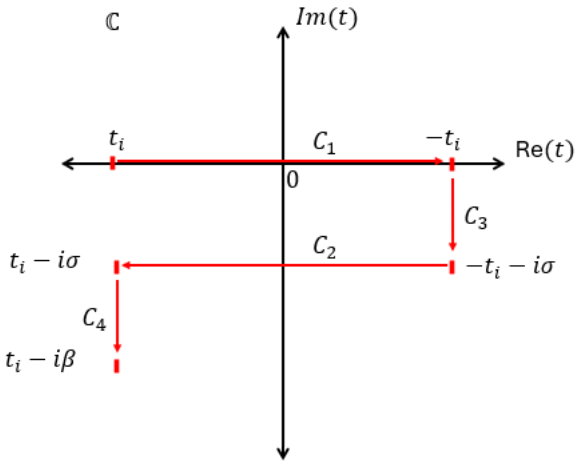

Figure 2.1: Followed paths which C is composed by

So far, the presented definitions are meant to be expressed in a complex path C formalism starting from an initial time t_i to a final one $t_i - i\beta$ and within this, the imaginary part of the time coordinate must be a non-increasing quantity evaluated by the parameter v in (2.2.4). The complex path C must be identified by some split rules depending on the Real or Complex axis that the coordinate t follows as described in figure 2.1 with $\sigma \in [0, \beta]$. Those partial paths will be useful for the physical interpretation of the quantities so far depending on the path followed by the formalism.

With the free field approach is possible to define again the Green functions D_C but taking the partial paths described in figure 2.1 by

$$D_C^F(x - x') = \theta_C(t - t')D_C^{>F}(x - x') + \theta_C(t' - t)D_C^{<F}(x - x') \tag{2.87}$$

where one can postulate a new free generating functional in which the followed path C can be factorized by

$$Z_C^F(\beta, j) = \mathcal{N}_1 Z_{C_{12}}^F(\beta, j) Z_{C_{34}}^F(\beta, j) \tag{2.88}$$

with a normalization constant and where the components had been taken splitting the horizontal and vertical paths in two different functions. It is not a formal physical interpretation, in such way that is formed by real and complex expressions of time.

In a similar way, one can do the same for the free Green function depending on the

followed path. Such components are given by the rules

$$\begin{aligned} D_{11}^F(t-t') =& D_F(t-t') \\ D_{22}^F(t-t') =& D_F^*(t-t') \\ D_{12}^F(t-t') =& D_C^F(t-(t'-i\sigma)) = D^<(t-t'+i\sigma) \\ D_{21}^F(t-t') =& D_C^F((t-i\sigma)-t') = D^>(t-t'-i\sigma) \end{aligned} \tag{2.89}$$

where one can notice that the last two equations satisfy the fact that "times" on the path C_2 are always later than those in C_1. Finally, one can review the Fourier transform in each of these parts in a similar procedure that was made on (2.66) with the introduction of $n(k_0)$ function. The results are

$$\begin{aligned} D_{11}^F(k) =& \frac{i}{k^2-m^2+i\eta} + n(k_0)2\pi\delta(k^2-m^2) \\ =& \left(D_{22}^F(k)\right)^* \\ D_{12}^F(k) =& e^{\sigma k_0} f(k_0)\rho_F(k) \\ D_{21}^F(k) =& e^{-\sigma k_0}\left(1+f(k_0)\right)\rho_F(k)\,. \end{aligned} \tag{2.90}$$

There are some useful symmetries depending on the choice of σ but the physical results don't depend on that. Those are first when $\sigma = \beta/2$ by

$$D_{12}^F(k) = D_{21}^F(k) = exp\left(\frac{\beta|k_0|}{2}\right) n(k_0)2\pi\delta(k^2-m^2) \tag{2.91}$$

and the second one when $\sigma = 0$, going back to reference (2.43) and (2.44) by

$$\begin{aligned} D_{12}^F(k) =& D_F^<(k_0) \\ D_{21}^F(k) =& D_F^>(k_0)\,. \end{aligned} \tag{2.92}$$

There is an easier way to see the functional derivative action on the partition function related to the time ordered operator by

$$Z\left[0\right] = \int \mathcal{D}\phi e^{iS} \tag{2.93}$$

where this definition make it is possible to define the Feynman Propagator for a two point function as

$$\begin{aligned} \langle\Omega|\, T\left[\phi(x)\phi(y)\right]|\Omega\rangle =& \frac{\int \mathcal{D}\phi\phi(x)\phi(y)e^{iS}}{\int \mathcal{D}\phi e^{iS}} \\ =& \frac{\int \mathcal{D}\phi\phi(x)\phi(y)e^{iS}}{Z\left[0\right]}\,. \end{aligned} \tag{2.94}$$

The **functional derivative** can be also re-defined with the following expression

$$\frac{\delta}{\delta J(x)} J(y) = \delta^{(4)}(x-y)$$
$$\frac{\delta}{\delta J(x)} \int J(y)\phi(y) d^4y = \phi(x)\,.$$

Once with these definitions, the structure of the numerator in equation (2.94) can be seen as functional derivatives over a generalized form of the generating functional

$$Z[J] = \int \mathcal{D}\phi e^{iS + i\int J(x)\phi(x)d^4x} \tag{2.95}$$

such that for the two point function the functional derivatives act like

$$\frac{\delta}{\delta J(x)} Z[J]\bigg|_{J=0} = \frac{\delta}{\delta J(x)} \int \mathcal{D}\phi e^{iS+i\int J(z)\phi(z)d^4z}\bigg|_{J=0} = i\int \mathcal{D}\phi\phi(x)e^{iS}$$
$$\frac{\delta}{\delta J(x)}\frac{\delta}{\delta J(y)} Z[J]\bigg|_{J=0} = i^2 \int \mathcal{D}\phi\phi(x)\phi(y)e^{iS}\bigg|_{J=0}\,. \tag{2.96}$$

This was an specific example but a general expression for the Feynman propagator in terms of generating functional can be written as

$$\begin{aligned}\left\langle T\left[\phi(x_1)\phi(x_2)\dots\phi(x_n)\right]\right\rangle &= \frac{1}{Z[0]}\int \mathcal{D}\phi\phi(x_1)\phi(x_2)\dots\phi(x_n)e^{iS} \\ &= \frac{1}{Z[0]}\left(\frac{1}{i}\right)^n \frac{\delta Z[J]}{\delta J(x_1)\delta J(x_2)\dots\delta J(x_n)}\bigg|_{J=0}\end{aligned} \tag{2.97}$$

2.3 Ward identities

In a quantum field theory the presence of a symmetry leads to an invariant action term under specific elements of a given transformation group. Take for example the Dirac Lagrangian and transformations under U(1) like $\psi \to e^{i\theta}\psi$ and promoting the transformation parameter to a space-time function $\theta \to \theta(x)$, so the Lagrangian transforms as

$$\mathcal{L} = i\bar{\psi}\gamma^\mu\partial_\mu\psi$$
$$\mathcal{L}' \to i\bar{\psi}'\gamma^\mu\partial_\mu\psi' = i\left(e^{-i\theta(x)}\bar{\psi}\right)\gamma^\mu\partial_\mu\left(e^{i\theta(x)}\psi\right)$$
$$\mathcal{L}' = \mathcal{L} + i\bar{\psi}\gamma^\mu\psi\partial_\mu\theta(x)\,.$$

Now, with the new Lagrangian term, the modified action is then given by

$$\begin{aligned}S' = \int \mathcal{L}' d^4x &= S + \int i\bar{\psi}\gamma^\mu\psi\partial_\mu\theta(x)d^4x \\ S' &= S - i\int \theta(x)\partial_\mu\left(\bar{\psi}\gamma^\mu\psi\right)d^4x\end{aligned} \tag{2.98}$$

where it was performed an integration by parts over the second term of the first line. Both action term and hence the generating functional must be invariant under the symmetry transformation U(1). With the expressions (2.93) and (2.98) it it possible to change the form of Z by

$$Z' = \int \mathcal{D}\psi\mathcal{D}\bar{\psi}e^{iS'} \, .$$

If θ is small enough it is also possible to expand the exponential term as

$$Z' = \int \mathcal{D}\psi\mathcal{D}\bar{\psi}e^{iS}\left(1 + \int \theta(x)\partial_\mu\left(\bar{\psi}\gamma^\mu\psi\right)d^4x + O(\theta^2)\right)$$
$$Z' = Z + \int \langle\partial_\mu\left(\bar{\psi}\gamma^\mu\psi\right)\rangle\theta(x)d^4x + \dots \, .$$

As it was said, the generating functional must remain invariant for a symmetry, then since $\theta \neq 0$ the expectation value term within the integral must be 0 for all $\theta(x)$, such relation its known as the **Ward Identity**, which also shows the conservation of the Dirac current in

$$\partial_\mu\langle\bar{\psi}\gamma^\mu\psi\rangle = \partial_\mu\langle j^\mu(x)\rangle = 0 \tag{2.99}$$

2.4 Comments on Quantum Chromodynamics

Connecting observations from neutron stars to properties of ultra-dense matter is an example of probing our understanding of fundamental theories such as Quantum Chromodynamics (QCD) with the help of astrophysics [4]. QCD is the SU(3) non-abelian gauge theory of color charge [7], describing the Strong Force which actually acts on quarks, not the leptons.

An action principle for QCD is generally written as

$$S_{QCD} = \int d^4x[-\frac{1}{2}\mathrm{Tr}\left(G_{\mu\nu}G^{\mu\nu}\right) + i\sum_{i=1}\bar{q}_i\gamma^\mu D_\mu q^i - m_i\bar{q}_i q^i] \tag{2.100}$$

where i runs over the various quarks, fermions which carry charge under SU(3) named **color** charge (red, blue, green). The field strength associated to the flavored gauge potential is

$$G_{\mu\nu} = \partial_\mu A^f_\nu - \partial_\nu A^f_\mu - ig_s[A^f_\mu, A^f_\nu] \tag{2.101}$$

with g_s as the strong coupling constant and the field A^f_μ is the one associated to 8 independent gauge fields called **gluons** from the dimension of the gauge group where $dimSU(N) = N^2 - 1$ with N=3.

The gluons couple to the quarks through the covariant derivative

$$D_\mu q^a = \partial_\mu q^a - ig_s(A_\mu)^a{}_b q^b \tag{2.102}$$

where quarks also have the flavor index in Standard Model as: up, down, charm, strange,

top, bottom.

The QCD action describes massless gluons interacting with quarks, but this description is incomplete because quarks are not free in space. Unlike a massless gauge field associated with a classical force like $1/r^2$, the strong force is confined within the radius of a nucleus. Coupling constants are not truly constants, they depend on the energy or distance scale at which the theory is evaluated. At sufficiently long distances, the classical theory breaks down because the coupling becomes stronger. For systems containing a $q\bar{q}$ pair, the potential $V(r) = -\sigma\, r$ is observed, where σ is a constant known as the string tension. When quarks are forced into pairs or groups of three, the theory is said to be confining.

The strong interaction becomes weak at high energies, and the theory can rely on perturbative QCD, where quarks exhibit asymptotic freedom. At sufficiently high baryon density, matter is deconfined, and quarks and gluons become the relevant degrees of freedom. This phase of matter is called quark matter or, at high temperatures, quark-gluon plasma (QGP). In the context of neutron stars, quark matter is primarily composed of three quark flavors: up, down, and strange quarks; other quark flavors are too heavy to exist at the densities and temperatures typical for a neutron star [4].

Chapter 3

Gauge/Gravity Duality

The AdS/CFT correspondence is a theoretical framework in which the quantum physics of strongly correlated systems has a dual theory of gravitational dynamics in one additional dimension [8]. At this point, some important quantities are better calculated in one specific regime or the other. For this reason, there are rules contained in a dictionary where important objects are related to their dual counterparts. This chapter deals with the theoretical definition of Anti-de Sitter spaces and Conformal Field Theories related by the AdS/CFT correspondence, emphasizing the symmetry groups involved and exploring the relations between the field content on both sides. Following this, an example in scalar field theory is performed based on the background given in [9].

3.1 Overview on Lie Algebra

Group theory plays a crucial role in the development of theoretical physics. Understanding its definitions paves the way to defining symmetry transformations that act as representations of a given group on physical objects, depending on their nature. These representations associate a group element with a linear map (specifically, a matrix), which can be considered as elements of transformation. This section on Lie Theory is directly related to the following section on Conformal Field Theory, where the correct definition of generators is essential. With this purpose in mind, there are other statements to consider.

A Lie Group is a mathematical group, meaning algebraic structure, which satisfy two main properties

- It is a continuous group.
- It has a differentiable manifold structure associated, which means that its elements can be understood in a geometrical context as manifold points.

Formally speaking, a Lie group G is a differentiable manifold which is endowed with a group structure such that the group operations $G \times G \to G; (g_1, g_2) \to g_1.g_2$ and the inverse of an element $G \to G; g \to g^{-1}$ are differentiable [10]. Such differentiable manifold is called "Lie Group manifold" and must be smooth (C^∞) and without singularities. Being a differentiable

manifold implies an association between each point of the manifold with a tangent space that cover it locally.

A Tangent Space of a manifold **M** defined at a given point $p \in M$ denoted by $T_p(M)$ is the space of all tangent vectors to all curves at p. In the neighbourhood of every point p can be associated a space like $\mathbb{R}^n$ such that the point p is then described by the common coordinates used in $\mathbb{R}^n$. The dimension of the tangent space is $\mathbb{R}^{dimG}$ and its basis vectors set up the structure named as Lie Algebra. Such basis is also known as generators, t_A, of the Lie Algebra. So, like a vector field, an arbitrary element $\mathbb{X}$ of the Lie Algebra can be written by

$$\mathbb{X} = \sum_{A=1}^{dimG} \theta^A t_A \tag{3.1}$$

the θ^A elements are parameters or coordinates in the algebra. An important remark is that perturbative theory in theoretical physics deal with an algebra located at the neutral point of the associated group.

Given the definition of an element inside the algebra, one can ask about how such point can be related with its group, so given the algebra and one point like (3.1) the group element is given by the exponential map

$$g_{\mathbb{X}} = e^{\mathbb{X}}\,. \tag{3.2}$$

Within the Algebra it is possible to define the Commutation between two elements which allows one to find a third one, (by group properties). If $\mathbb{X}$ and $\mathbb{Y}$ are elements of the Algebra, then

$$\begin{cases} \mathbb{X} \to g_{\mathbb{X}} \\ \mathbb{Y} \to g_{\mathbb{Y}} \end{cases} \tag{3.3}$$

so the corresponding composition rule $g_{\mathbb{X}}.g_{\mathbb{Y}} = g_{\mathbb{Z}}$. $\mathbb{Z}$ as an element of the Algebra satisfy the Baker-Campbell-Hausdorff formula

$$\mathbb{Z} = \mathbb{X} + \mathbb{Y} + \frac{1}{2}[\mathbb{X}, \mathbb{Y}] + \dots\,. \tag{3.4}$$

The commutator operator must satisfy the following properties

- Anti-symmetric: $[\mathbb{X}, \mathbb{Y}] = -[\mathbb{Y}, \mathbb{X}]$
- Bilinear: $[a\mathbb{X} + b\mathbb{Y}, \mathbb{Z}] = a[\mathbb{X}, \mathbb{Z}] + b[\mathbb{Y}, \mathbb{Z}]$
- Jacobi Identity: $[\mathbb{X}, [\mathbb{Y}, \mathbb{Z}]] + [\mathbb{Z}, [\mathbb{X}, \mathbb{Y}]] + [\mathbb{Y}, [\mathbb{Z}, \mathbb{X}]] = 0\,.$

The commutation rules are very important to compute relations between the Algebra generators, t_A. Each rule defined by elements named as **structure constants** in such way that

$$[t_A, t_B] = \sum_{C=1}^{dimG} f_{AB}{}^C t_C \tag{3.5}$$

with ${f_{AB}}^C$ as those structure constants and t_C another Algebra generator. The structure constants determine the Algebra uniquely.

3.2 Conformal Field Theory

In Quantum Field Theory, one deals with Lorentz and later Poincaré symmetries and their contributions to the understanding of physical objects. Following this, there arises the question of a further generalization known as Scale Invariance. This type of symmetry relates physics at different scales, but it isn't always an intrinsic property of a given QFT; however, it can provide useful information, as is the case with QCD.

The Conformal Group involves Scale and Poincaré invariance transformations but a field theory not always satisfy both, only proven for dim=2 field theories.

3.2.1 Conformal Algebra

Related with previously said definition, the Conformal group is the transformation group that leaves the metric invariant up to an arbitrary space-time dependent scale factor [9]

$$g_{\mu\nu}(x) \to \Omega^{-2}(x)\, g_{\mu\nu}(x) \equiv e^{2\sigma(x)} g_{\mu\nu}(x) \tag{3.6}$$

where $\Omega^{-2}(x)$ is a smooth non-vanishing function of the spacetime coordinates. In other words, a conformal transformation is a local dilatation in which spacetime described by $\tilde{g}_{\mu\nu}$ is very different to the one by $g_{\mu\nu}$ and curvature tensors are not invariant under these but Conformal transformations do preserve the causal structure of the spacetime, meaning that space-like, null and time-like curves in $g_{\mu\nu}$ have their corresponding ones in $\tilde{g}_{\mu\nu}$.

If one takes the Euclidean signature in $\mathbb{R}^d$ Cartesian coordinates

$$ds^2 = \delta_{\mu\nu} dx^\mu dx^\nu \tag{3.7}$$

the conformal transformations leaves the form of the metric invariant up to a scale factor like (3.6)

$$\delta_{\mu\nu} \frac{\partial x'^\mu}{\partial x^\alpha} \frac{\partial x'^\nu}{\partial x^\beta} = \Omega^{-2} \delta_{\alpha\beta} \tag{3.8}$$

where one notice that a conformal transformation is a local dilatation [11].

Now, in a Minkowski spacetime where $g_{\mu\nu}(x) \to \eta_{\mu\nu}(x)$, an infinitesimal transformation on the coordinates given by $x^\mu \to x^\mu + \epsilon^\mu(x)$, the metric transforms as

$$\eta_{\mu\nu} \to \eta_{\mu\nu} + \partial_\nu \epsilon_\mu + \partial_\mu \epsilon_\nu \tag{3.9}$$

by the rules of tensor transformation. If one compares this with the general rule given in (3.6) one gets, expanding the exponential term,

$$2\sigma(x)\eta_{\mu\nu} = \partial_\mu \epsilon_\nu + \partial_\nu \epsilon_\mu \,. \tag{3.10}$$

If one then multiplies both sides by $\eta^{\mu\nu}$ contracting the indices, then one gets, in d dimensions, the expression

$$\partial \cdot \epsilon = d \cdot \sigma(x) \tag{3.11}$$

where it is then needed an infinitesimal transformation on the spacetime coordinates to give rise to a conformal change by the metric. This is achieved if the parameter $\epsilon(x)$ satisfy

$$\left(\eta_{\mu\nu}\partial^{\rho}\partial_{\rho} + (d-2)\partial_{\mu}\partial_{\nu}\right)\partial \cdot \epsilon = 0\,. \tag{3.12}$$

The Conformal group of the Minkowski space-time is generated by Poincaré transformations, scale transformations and special conformal transformations given by

$$\begin{aligned} x^{\mu} &\to \lambda x^{\mu} \\ x^{\mu} &\to \frac{x^{\mu} + a^{\mu}x^{2}}{1 + 2x^{\nu}a_{\nu} + a^{2}x^{2}}\,. \end{aligned} \tag{3.13}$$

For these finite conformal transformations it is also useful to introduce an inversion given by $x^{\mu} \to x^{\mu}/x^{2}$ mapping the point $x = 0$ to infinity which does not belong to Euclidean or Minkowski spacetime, something similar happens for the special conformal.

As seen in the Lie Algebra section, such group can also be described by the Algebra of its generators which satisfy the following commutation relations

$$\begin{aligned} [M_{\mu\nu}, P_{\rho}] &= -i\left(\eta_{\mu\rho}P_{\nu} - \eta_{\nu\rho}P_{\mu}\right) \\ [M_{\mu\nu}, M_{\rho\sigma}] &= -i\eta_{\mu\rho}M_{\nu\sigma} + \text{permutations} \\ [D, P_{\mu}] &= -iP_{\mu} \\ [M_{\mu\nu}, K_{\rho}] &= -i\left(\eta_{\mu\rho}K_{\nu} - \eta_{\nu\rho}K_{\mu}\right) \\ [M_{\mu\nu}, D] &= 0 \\ [D, K_{\mu}] &= iK_{\mu} \\ [P_{\mu}, K_{\nu}] &= 2iM_{\mu\nu} - 2i\eta_{\mu\nu}D \end{aligned} \tag{3.14}$$

where the generators are $M_{\mu\nu}$ for Lorentz transformations and P_{μ} for Translations, which are always present in any relativistic invariant quantum field theory, K_{μ} for those transformations as (3.13) and D for scaling transformations like $x^{\mu} \to \lambda x^{\mu}$.

There is an important isomorphism relation between this Algebra and the one for SO(d,2), (the special orthogonal group of Lorentz transformations adding up one space-like dimension and one time-like). This fact is a hint of a relation between d-dimensional Conformal field theory, a field theory on d-dimensional Minkowski space that is invariant under the Conformal group, and a gravity theory in d+1-dimensional Anti-de Sitter space [12]. The isomorphism put the conformal algebra in the standard form of SO(d,2) Algebra which

generators J_{ab}, with $a, b = 0, 1, \ldots, d+1$, are defined by

$$\begin{aligned} J_{\mu\nu} &= M_{\mu\nu} \\ J_{\mu d} &= \frac{1}{2}\left(K_\mu - P_\mu\right) \\ J_{\mu(d+1)} &= \frac{1}{2}\left(K_\mu + P_\mu\right) \\ J_{(d+1)d} &= D\,. \end{aligned} \tag{3.15}$$

Some applications are better studied within an Euclidean space and for this case the conformal group is isomorphic to SO(d+1,1).

3.2.2 Primary Operators

For physical applications, the most interesting representations of the conformal group are those which relate operators or fields as eigenfunctions of the scale operator D with an eigenvalue given by $-i\Delta$, with Δ known by the name of "scaling dimension" of the field. This means that under a scale transformation like $x^\mu \to \lambda x^\mu$, the fields transform as

$$\phi(x) \to \phi'(x) = \lambda^\Delta \phi(\lambda x)\,. \tag{3.16}$$

The transformations under the Conformal Algebra are done in irreducible representations on the fields, all done by a procedure called induced representations. The commutation rules given by the Algebra in (3.14) have, specially for the translation and special conformal transformation generators, an effect on the dimension of the field by the action of the scaling operator

$$\begin{aligned} D(P_\mu \phi) &= -i(1+\Delta)P_\mu \phi \\ D(K_\mu \phi) &= -i(\Delta - 1)K_\mu \phi \end{aligned} \tag{3.17}$$

so, P_μ raises the dimension of the field and K_μ lowers it. For the special case with the K_μ operator, there must by a lower limit for the dimension of a field which is annihilated by it, such limit is given by $\Delta \geq (d-2)/2$. From this statement, each representation of the conformal group must have some operator with such dimension that will be annihilated by K_μ in $(x = 0)$ and this kind of operators are called as primary operators. For this kind, one can postulate the scaling dimension of those fields from the operator D by

$$[D, \phi(0)] = -i\Lambda\phi(0)\,. \tag{3.18}$$

By the first rule in (3.13) a field which transforms covariantly under an irreducible representation of the conformal algebra has a fixed scaling dimension and is therefore an eigenstate of the dilatation operator D [9]. The alluded action of K_μ is then seen as

$$[K_\mu, \phi(0)] = 0 \tag{3.19}$$

thus, the conformal algebra has irreducible multiplets in which the conformal primary fields are those of lowest scaling dimension that satisfy (3.19) and all other fields are called **conformal descendants** of ϕ, obtained by raising the scaling dimension with the operator P_μ.

The conformal group generators acting on primary operators satisfy the following commutation rules

$$\begin{aligned}
[P_\mu, \phi(x)] &= i\partial_\mu \phi(x) \\
[M_{\mu\nu}, \phi(x)] &= [i\left(x_\mu \partial_\nu - x_\nu \partial_\mu\right) + \Sigma_{\mu\nu}]\phi(x) \\
[D, \phi(x)] &= i\left(-\Delta + x^\mu \partial_\mu\right)\phi(x) \\
[K_\mu, \phi(x)] &= [i\left(x^2\partial_\mu - 2x_\mu x^\nu \partial_\nu + 2x_\mu \Delta\right) - 2x^\nu \Sigma_{\mu\nu}]\phi(x)
\end{aligned} \tag{3.20}$$

where the matrices $\Sigma_{\mu\nu}$ are those representations for the Lorentz group acting on the indices of the field ϕ. So far, these last representations of the conformal group on the primary operators are classified by the Lorentz representation and the scaling dimension Δ. There is another way to classify those representations by the subgroup of the conformal one given by $SO(d) \times SO(2)$. The SO(2) generator is defined by $J_{0(d+1)} = 1/2(K_0 + P_0)$ together with the previously Lorentz generator can describe the conformal generators as well. This subgroup is useful for the radial quantization of a Conformal field theory in a space $S^{(d-1)} \times \mathbb{R}$ which will be related to the Anti-de Sitter space AdS in global coordinates.

In local field theories, there is a property called Operator Product Expansion, OPE, which creates a local perturbation at one specific point when two operators like $\mathscr{O}_1(x)$ and $\mathscr{O}_2(y)$ are brought to it. The general form of such expansion is given by

$$\mathscr{O}_1(x)\mathscr{O}_2(y) \to \sum_n C_{12}^n (x-y)\mathscr{O}_n(y) \tag{3.21}$$

which define the correlation functions expression and those coefficients C_{12}^n do not depend on other operators inside the function and, in a conformal theory, are given by just the conformal invariance

$$C_{12}^n(x-y) = \frac{c_{12}^n}{|x-y|^{\Delta_1+\Delta_2-\Delta_n}} \tag{3.22}$$

3.3 Anti-de Sitter Spacetime

It is known that Anti-de Sitter spacetime, AdS for short, is a maximally symmetric spacetime with a negative cosmological constant. These concepts are related to the homogeneous and isotropic nature of such a space, which is characterized by a constant curvature, and its metric satisfies the Einstein field equations with a cosmological constant.

If wanted a better understanding of the isometry group, a (d+1)-dimensional AdS, AdS_{d+1}, can be embedded in a higher dimensional flat spacetime, specifically a (d+2)-dimensional Minkowski spacetime described by coordinates $(X^0, X^1, \ldots, X^{d+1}) \in \mathbb{R}^{d,2}$ and a line element

given by

$$\begin{aligned} ds^2 =& -(dX^0)^2 + (dX^1)^2 + \cdots + (dX^d)^2 - (dX^{d+1})^2 \\ =& \tilde{\eta}_{MN} dx^M dx^N \end{aligned} \tag{3.23}$$

with the signature $\tilde{\eta}_{MN} = diag(-,+,\ldots,+,-)$ and the subscripts $M,N \in [0,1,\ldots,d,d+1]$. The AdS_{d+1} space is an hyperboloid, as said being embed in $\mathbb{R}^{d,2}$, described by

$$\tilde{\eta}_{MN} X^M X^N = -(X^0)^2 + \sum_{i=1}^{d} (X^i)^2 - (X^{d+1})^2 = -L^2 \tag{3.24}$$

where L is the radius of curvature of the AdS space. One can check an invariance under the group $O(d,2)$ by the previous hypersurface, acting on $\mathbb{R}^{d,2}$, which means that the isometry group of the AdS space is clearly $O(d,2)$ and also, later one can identify a conformal boundary. For large values of X^M, the hypersurface described by (3.24), can be related to a light-cone in $\mathbb{R}^{d,2}$ by

$$\tilde{\eta}_{MN} X^M X^N = -(X^0)^2 + \sum_{i=1}^{d} (X^i)^2 - (X^{d+1})^2 = 0 \tag{3.25}$$

whose boundary region will be described by the set of trajectories satisfying the asymptotic $|X^M| \to \infty$ behaviour of the lightcone below. In a simplified way, AdS and the lightcone have both d+1 dimensions and the boundary has d. Formally speaking, the boundary region of the AdS space is the set of points which satisfy

$$\partial AdS_{d+1} : \{[X] | X \in \mathbb{R}^{d,2}, X \neq 0, \tilde{\eta}_{MN} X^M X^N = 0\}\,. \tag{3.26}$$

Something important to mention is because of $\tilde{\eta}_{MN} X^M X^N = 0$ one can identify two sets of points related by a real number like $[X] = \lambda[\tilde{X}]$ and call ∂AdS_{d+1} as the conformal boundary of AdS_{d+1}. The set of points in ∂AdS_{d+1} satisfy (3.25) and one can then identify a conformal boundary with the $|X^M| \to \infty$ limit by

$$\begin{aligned} \sum_{i=1}^{d} (X^i)^2 &= 1 \\ (X^0)^2 + (X^{d+1})^2 &= 1 \end{aligned} \tag{3.27}$$

where the second equation is obtained from the first one in (3.25). So, the first equation represent a (d-1)-sphere S^{d-1} and the second one S^1, then one can define the topology of the conformal boundary ∂AdS_{d+1} by the coset $(S^1 \times S^{d-1})/\mathbb{Z}_2$, in which $\mathbb{Z}_2 : \{[X] \in \mathbb{R}^{d,2}\}$.

3.3.1 Conformal Compactifications on AdS space

The main topic this part is about try to establish the basics of conformal structure of flat spacetime, that will lead to a basic feature of AdS/CFT correspondence which relates the

isometry group of AdS_{p+2} with the conformal symmetry of flat Minkowski space $\mathbb{R}^{1,d}$ [13].

If one takes the flat metric with Euclidean signature $\mathbb{R}^d$, this can be compactified, (an extension of a topological space which is a compact topological space), to the n-sphere S^n, with infinity added by an additional point, in which a conformal theory can be defined. In a similar way, the (n+1)-dimensional hyperbolic space, which is the Euclidean form of the AdS space, can be conformally mapped to a (n+1)-dimensional disk D_{n+1}.

For example, with the Minkowski spacetime $\mathbb{R}^{1,1}$, the general metric, $ds^2 = -dt^2 + dx^2$, $(-\infty < t, x < \infty)$, can be changed by a coordinate transformation

$$\begin{aligned} ds^2 &= -du_+ du_- \\ ds^2 &= \frac{1}{4\cos^2\tilde{u}_+ \cos^2\tilde{u}_-}\left(-d\tau^2 + d\theta^2\right) \end{aligned} \tag{3.28}$$

where $(u_\pm = t \pm x = \tan\tilde{u}_\pm)$ and $\tilde{u}_\pm = (\tau + \theta)/2$. Thus, the Minkowski spacetime was conformally mapped inside the compact region defined by $|\tilde{u}_\pm| < \pi/2$, which has limits for the asymptotic values of (3.28) known as the Poincaré horizon

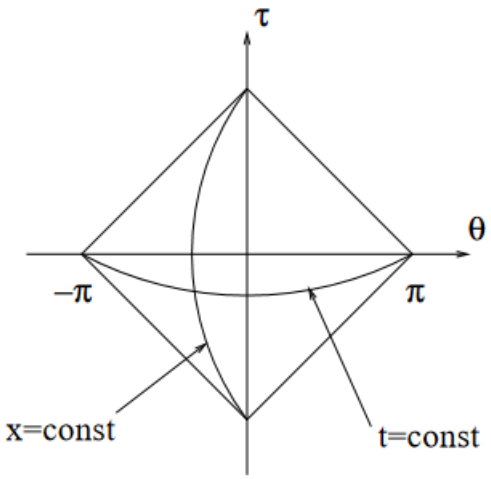

Figure 3.1: Compact region which the two-dimensional Minkowski spacetime was conformally mapped in

The line element that represent a null trajectory, satisfied by a light ray, is invariant under conformal transformation on the metric, so these kind are useful for describing the causal behaviour of $\mathbb{R}^{1,1}$. Those new coordinates in (3.28) are well defined with the asymptotic regions in figure 3.1, where the corners $(\tau, \theta) = (0, \pm\pi)$ belong to infinite points by the original coordinates when $x = \pm\infty$. Because of this, one can describe the asymptotic flatness by a compactification, which means that spacetime has the same boundary structure of a flat one after a conformal compactification.

Some procedure, which importance will be clear later with AdS spacetime, is to embed the space $\mathbb{R}^{1,1}$ in a cylinder described by $\mathbb{R} \times S^1$. It has been proved [13], that the correlation functions in a Conformal field theory $\mathbb{R}^{1,1}$ have an analytical continuation to the cylinder previously introduced.

When it is necessary to describe Minkowski higher dimensional spaces like $\mathbb{R}^{1,p}, p \geq 2$, the transformation on the line element is given by

$$ds^2 = -dt^2 + dr^2 + r^2 d\Omega_{p-1}^2 \tag{3.29}$$

where $d\Omega_{p-1}^2$ is for the line element in the unitary sphere S^{p-1}. And by a change of coordinates analogous to the one in (3.28), the new line element is

$$\begin{aligned} ds^2 &= -du_+ du_- + \frac{1}{4}(u_+ - u_-)^2 \, d\Omega_{p-1}^2 \\ ds^2 &= \frac{1}{4\cos^2\tilde{u}_+ \cos^2\tilde{u}_-}\left(-d\tau^2 + d\theta^2 + \sin^2\theta d\Omega_{p-1}^2\right) \end{aligned} \tag{3.30}$$

as well as before, one can see a conformal compactification of a Minkowski spacetime in which the region (t, r) is mapped inside a triangle similar to one described in 3.1 by the (τ, θ) plane.

An analogous procedure of conformal compactifications can be applied to AdS_{d+1} which will lead to understand ∂AdS_{d+1} as a compactification of the d-dimensional Minkowski spacetime. first, let's introduce the change of the hyperbolic coordinates

$$\begin{cases} X^0 = L\cosh\rho\cos\tau \\ X^{d+1} = L\cosh\rho\sin\tau \\ X^i = L\Omega_i \sinh\rho, \qquad i = 1, \ldots, d \end{cases} \tag{3.31}$$

with the element Ω_i satisfying $\sum_{i=1}^{d}\Omega_i^2 = 1$ so it parameterizes the (d-1)-sphere S^{d-1}. the other coordinates also belong to $\rho \in \mathbb{R}_+$ and $\tau \in [0, 2\pi)$. The full group of coordinates (τ, ρ, Ω_i) are called "Global Coordinates of AdS_{d+1}". The previous change of coordinates in (3.31) should be applied in the line element (3.23) to get

$$ds^2 = L^2\left(-\cosh^2\rho\, d\tau^2 + d\rho^2 + \sinh^2\rho\, d\Omega_{d-1}^2\right). \tag{3.32}$$

One can come up with another change in one specific coordinate such that $\sinh\rho = \tan\theta$, and then the new line element would be

$$ds^2 = \frac{L^2}{\cos^2\theta}\left(-d\tau^2 + d\theta^2 + \sin^2\theta\, d\Omega_{d-1}^2\right) \tag{3.33}$$

this one represent a static Einstein universe $\mathbb{R} \times S^d$ up to a positive squared scale factor. In fact, the isometry group of AdS_{d+1} embed in $\mathbb{R}^{d,2}$ is just $SO(2, d)$ whose Maximal compact subgroup is given by the product $SO(2) \times SO(d)$, where the translation on the coordinate τ are representations of $SO(2)$ and rotation on the (d-1)-sphere are those for $SO(d)$. It could be possible to ad such point that $\theta = \pi/2$ corresponding to the spatial infinity and in general $\theta \in [0, \pi)$, covering just the half of the Einstein space $\mathbb{R} \times S^d$.

When scaling the metric in (3.33) it is possible to avoid the overall factor without any change in the causal structure, as the line element for the light-cone obtained when $ds^2 = 0$

in the metric. Then the compactified spacetime is

$$ds^2 = -d\tau^2 + d\theta^2 + \sin^2\theta\, d\Omega_{d-1}^2 \tag{3.34}$$

with the coordinates described by their values in $\theta \in [0, \pi/2]$ and $\tau \in [0, 2\pi)$. When $\theta = \pi/2$ the metric is then changed by

$$ds^2 = -d\tau^2 + d\Omega_{d-1}^2\,. \tag{3.35}$$

After this procedure, one can see that the boundary $\theta = \pi/2$ in (3.35) of the conformally compactified AdS_{d+1} space is the same as the conformally compactified d-dimensional Minkowski spacetime, such feature is a basic pillar of the AdS/CFT duality.

The timelike coordinate τ is periodic by 2π and because of this the AdS space has closed timelike curves. To avoid inconsistencies, consider another version of the AdS denoted by $\widetilde{AdS}$ called Universal covering of AdS which unwraps the timelike circle τ to $\tau \in \mathbb{R}$.

The conformal compactification is an useful procedure to describe the asymptotic regions of AdS space. In general, if a spacetime can be compactified in a conformal way into a region which has the same boundary structure as the half static Einstein universe, such spacetime is called "Asymptotically AdS" [13].

It is also useful to introduce another change of coordinates with the purpose of describing the spacetime by the common quantities of a gravity theory. So, by using the new coordinates $t \in \mathbb{R}$, $\vec{x} = (x^1, \dots, x^d - 1)$ and $r \in \mathbb{R}_+$ one gets

$$\begin{aligned} X^0 =& \frac{L^2}{2r}\left(1 + \frac{r^2}{L^4}(\vec{x}^2 - t^2 + L^2)\right) \\ X^i =& \frac{r x^i}{L} \\ X^d =& \frac{L^2}{2r}\left(1 + \frac{r^2}{L^4}(\vec{x}^2 - t^2 - L^2)\right) \\ X^{d+1} =& \frac{rt}{L} \end{aligned} \tag{3.36}$$

with $i \in [1, d-1]$. The constraint $r > 0$ just covers the half AdS_{d+1} space. These local coordinates are called as the "Poincaré patch", and then the metric in (3.23) is given by

$$ds^2 \equiv \frac{L^2}{r^2}dr^2 + \frac{r^2}{L^2}\eta_{\mu\nu}dx^\mu dx^\nu \tag{3.37}$$

identifying $\mu, \nu \in [0, d]$, $x^0 = t$ and $\eta_{\mu\nu} = diag(-, +, \dots)$. For such spacetime, the Ricci's scalar is then

$$R = -\frac{d(d+1)}{L^2} \tag{3.38}$$

and, as said at the beginning of this section, the metric (3.37) satisfy the Einstein field equations with $T^{\mu\nu}$ and negative cosmological constant given by

$$\Lambda = -\frac{d(d-1)}{2L^2}\,. \tag{3.39}$$

As a final quote, the gauge theories of special interest are those in $3+1$-dimensional space which is related with the geometry above with AdS_5. This geometry is the one studied by the Maldacena's conjecture in which the dual QFT is a super Yang-Mills theory with four supersymmetries ($\mathcal{N} = 4$SYM) [8]

3.4 AdS/CFT correspondence and Holography

With Conformal Field Theory and Anti-de Sitter spaces described, it is now possible to delve into the AdS/CFT correspondence, also known as "Gauge/Gravity Duality". Two physical theories are considered dual to each other if there exists a mapping between parameters and observables in both theoretical frameworks, supported by precise mathematical tools and dynamic calculations bridging both sides.

In one side there could exist a quantum field theory in which is mandatory to define the following elements

- Some field operator set: $\hat{\phi}^i$
- An specific Lagrangian defined by the mass term and the coupling constants
- Define the point of the free field in which the frame can be developed by perturbation theory, for example around the trivial vacuum $< \phi^i >= 0$ or around a condensate $< \phi^i > \neq 0$.

There is also possible to give a QFT description by gauge invariant field operators labelled as $\hat{\mathcal{O}}^i(x)$ and including correlation functions of these operators with the vacuum state being characterized by the one point function. It will be clarified later that the coupling constants vary depending on the energy scale $g(\Lambda)$ and the renormalization procedure on those operators also depend on that scale.

Any dual representation of a QFT must give a complete procedure on how vacuum expectation values, gauge invariant operators and correlation functions are represented in the dual theory, with non gauge invariant operators representing non observable entities.

One of the most important tools in describing dual correspondence is the Gubser-Klebanov-Polyakov-Witten (GKP-W) relation between a gravitational theory in AdS_{d+1}, being the bulk, and a CFT in d dimension as the boundary. But first, it is convenient to define the rules of the correspondence by the following items

1. For each gauge invariant field operator $\hat{\mathcal{O}}^i(x)$ in a QFT, there is a field associated in the bulk theory ϕ^i
2. If $\varphi_{(0)}$ is the parameter condition in the boundary of the field ϕ, that will be the source of the operator $\hat{\mathcal{O}}$ in the dual QFT and also depend on the d-dimensional spacetime coordinates x^μ

Now the GKP-W relation establishes the correspondence between the partition function in the bulk as a function of the boundary condition $\varphi(0)$ with the generating functional of the

correlating functions in the QFT. In the Euclidean signature, the generating functional is given by the source $\phi_{(0)}$

$$Z[\phi_{(0)}] = e^{-W[\phi_{(0)}]} = \left\langle \exp\left\{ \left(\int d^d x\, \phi_{(0)}(x)\, \mathcal{O}(x) \right) \right\} \right\rangle_{CFT_d} \tag{3.40}$$

$$S_{grav}^{on-shell}[\varphi_{(0)}] = -W_{QFT}[\varphi_{(0)}] \tag{3.41}$$

where left hand side is related to the gravity action and a field equations solution evaluated by the boundary condition $\varphi_{(0)}$ and the right hand side is the QFT partition function in the strong coupling t'Hooft limit when $N \to \infty$, something that will be later explained.

The AdS/CFT correspondence is one of the most studied examples of a gauge/gravity duality by its simple way to notice symmetries between both sides. The correspondence establish the mathematical frame to describe physical quantities in the bulk of the AdS space by a field theory in one less dimension. Such task isn't trivial by the fact that not only the CFT has an infinite number of degrees of freedom but the area of the AdS boundary is also infinite.

3.4.1 Holographic Procedure

The starting point for dealing with this difficulty is to explain the known Bekenstein bound, which describes the maximum entropy within a region of a given space by

$$S_{max} = \frac{A_\partial}{4G_N} \tag{3.42}$$

Here, A_∂ denotes the area of the region's boundary. This expression suggests that the number of degrees of freedom distinguishing a region grows with the area of its boundary rather than its volume. According to the Holographic principle, in a quantum gravity theory, all physics within a volume can be described in terms of a theory on the boundary, which has fewer than one degree of freedom per Planck area [13]. In this sense, such entropy can satisfy the Bekenstein bound.

Let's propose a regularization for the QFT side by adding a regulator for both limits UV and IR. Taking a system as an spatial box of size R and introduce a lattice spacing ϵ. If the system is embed in a d-dimensional space, it has R^{d-1}/ϵ^{d-1} cells. If one also takes C_{QFT} as the number of degrees of freedom per lattice site, the QFT will have

$$N_{dof}^{QFT} = \left(\frac{R}{\epsilon}\right)^{d-1} C_{QFT} \tag{3.43}$$

as the total number of degrees of freedom. With a gauge theory SU(N), the fields are matrices of $N \times N$ in the adjoint representation, so for large N, the number C_{QFT} is proportional to N^2. A similar expression can be taken for the AdS_{d+1} space and it will be the same that (3.42) for a gravitational theory with S_{mx} as the total number of degrees of freedom in such space. If one now takes the integral of the volume element in AdS_{d+1} space at a slice $z = \epsilon$

one gets the area of the boundary given by

$$A_\partial = \left(\frac{RL}{\epsilon}\right)^{d-1} \tag{3.44}$$

where the system was put inside the same box of size R. Now, because of the previously idea about Planck area, it is necessary to introduce the Planck l_p and the Planck mass M_p for a gravitational theory in $(d+1)$-dimensions related to the gravitational constant by

$$G_N = (l_p)^{d-1} = \frac{1}{(M_p)^{d-1}}\,. \tag{3.45}$$

So, the number of degrees of freedom inside AdS space can now be defined with (3.42) as follows

$$N_{dof}^{AdS} = \frac{A_\partial}{4G_N} = \frac{1}{4}\left(\frac{R}{\epsilon}\right)^{d-1}\left(\frac{L}{l_p}\right)^{d-1}\,. \tag{3.46}$$

It can also be noticed that both number of degrees of freedom in a QFT by (3.43) and in AdS space in (3.46) are related by the same proportional factor $(R/\epsilon)^{d-1}$ which means that N_{dof}^{AdS} and N_{dof}^{QFT} scales in the same way with the cutoff values R and ϵ for the IR and UV limits respectively. The number C_{QFT} for (3.43) can be computed by

$$N_{dof}^{AdS} = N_{dof}^{QFT} \;\rightarrow\; \frac{1}{4}\left(\frac{L}{l_p}\right)^{d-1} = C_{QFT}\,. \tag{3.47}$$

To finish with these comments on degrees of freedom topic, if a QFT has a dual gravitational theory in AdS space it happens that when there are a large number of them by volume unit or, explicitly when

$$\left(\frac{L}{l_p}\right)^{d-1} \gg 1 \tag{3.48}$$

where the gravitational theory has a classical limit.

The following part of the procedure is to add a cutoff to the number of degrees of freedom in the QFT side and see the correspondence with the gravity theory. Such procedure can be seen starting from another way to write the AdS metric by

$$ds^2 = R^2\Big\{-\left(\frac{1+r^2}{1-r^2}\right)^2 dt^2 + \frac{4}{(1-r^2)^2}(dr^2 + r^2\, d\Omega^2)\Big\} \tag{3.49}$$

where the AdS boundary can be found when $r \to 1$. When one needs to compute the correlation functions the boundary conditions must be specified with $r = 1-\delta$ and take the limit when $\delta \to 0$. Close to that boundary one can also use the Metric form given by the Poincaré d-dimensional invariance that acts on the x^μ coordinates by

$$ds^2 = \frac{L^2}{z^2}\left(dz^2 + \eta_{\mu\nu}dx^\mu dx^\nu\right) = g_{mn}dx^m dx^n \tag{3.50}$$

with L as the curvature radius and the metric being solution of the Einstein field equation

with cosmological constant given in (3.39) and scale invariant under $z \to \lambda z$ and $x^\mu \to \lambda x^\mu$, so that the dual theory at the boundary is also expected as scale invariant. One can see (3.50) just being another expression for (3.37) and with a boundary located at $z \to 0$.

The action of the conformal group on the Poincaré coordinates is giving, to the radial coordinate z, the role of an energy scale since a conformal transformation is performed to locate objects in the CFT. Such energy scale relates the UV limit of the QFT when $z \to 0$ and the IR limit when $z \to \infty$. A cutoff at $z \sim \delta$, which is known as "UV-IR relation", corresponds to the UV cutoff in the field theory at distances δ[13].

The metric (3.50) is also invariant under special conformal transformations (3.13) such that

$$\begin{aligned} \delta x^\mu &= 2(\rho^\nu x_\nu)x^\mu - (z^2 + x^\nu x_\nu)\rho^\mu \\ \delta z &= 2(\rho^\nu x_\nu)z\,. \end{aligned} \tag{3.51}$$

The isometry group of the AdS_{d+1} space-time is $SO(2,d)$ which agrees with the conformal group of a d-dimensional QFT. By the scale transformation given in the first line of (3.13) and also with the same for the radial coordinate z, one can identify the last with the renormalization group flow direction of the dual QFT with the boundary conditions related to the energy scale through the UV-IR relation. For this purpose, it is convenient to transform the radial coordinate by

$$z = e^{-Ar}\, A^{-1} \tag{3.52}$$

with $A = L^{-1}$. Inserting this change in (3.50) one gets

$$ds^2 = dr^2 + e^{2Ar}\, \eta_{\mu\nu} dx^\mu dx^\nu\,. \tag{3.53}$$

The flat Minkowski space part of the metric is now modified by a global factor which is r-dependent. Then the theory identify the product Ar with the scale energy of the renormalization group flow

$$A\, r \sim \log \mu \tag{3.54}$$

It is understood that a field theory on the boundary of AdS space describes the physics within that volume. However, different regions of AdS space, defined by distinct radial coordinates, correspond to physical theories defined at different energy scales within the field theory. The fundamental correspondence here establishes that a quantum field theory at strong coupling is related to a gravity theory at weak coupling, characterized by the energy scale associated with the z coordinate in AdS space. Formally, one can posit that an AdS manifold in $(d+1)$ dimensions with negative curvature defines computations that asymptotically match those of a QFT in d dimensions.

Consider now the computation of the correlation functions from the AdS correspondence with another way to state the GKP-W prescription which link the calculation of the generating functional of the correlation functions in the dual CFT with the partition function in the

gravity-dependence side as

$$\left\langle \exp\left\{ \left(\int d^d x\, \phi_{(0)}(x)\, \mathcal{O}(x) \right) \right\} \right\rangle_{CFT_d} = \exp\{-S_{grav}[\phi(x,z)]\} \tag{3.55}$$

$$\phi(x,z)\big|_{z=0} = \phi_{(0)}(x)$$

with $\phi_{(0)}$ as the source of the CFT in the boundary like it was explained at the beginning of this section. The previous expression is only a classical approach to the gravitational side and the $\mathcal{O}(x)$ term are just the field operators of the CFT.

Note that the GKP-W relation pertains to a classical limit of a hypothetical path integral in quantum gravity. It is not certain that such a path integral exists, but it is believed that the right-hand side of (3.55) exists in this context within string theory, albeit without well-established details due to the difficulty of quantizing strings in the AdS background. It may suffice to consider the classical limit, which allows for the computation of CFT correlation functions from the classical equations of motion of bulk theories [14].

So far, the task of giving a well-formulated correspondence between scalar fields like $\phi(x,z)$ in the bulk and field operators $\hat{\mathcal{O}}(x)$ in the dual CFT, has been accomplished. But there are other fields that one can find in the bulk like the graviton by a $(d+1)$-dimensional metric and the gauge fields. Starting from the classical action principle

$$S_M = \int d^m x \sqrt{-g}\, \mathcal{L}_M \tag{3.56}$$

where one can take a perturbation on the metric by $g_{\mu\nu} \to g_{\mu\nu} + \delta g_{\mu\nu}$ and notice the variation on the action S_M by

$$\delta S_M = \frac{1}{2} \int d^m x \sqrt{-g}\, T^{\mu\nu}\, \delta g_{\mu\nu} \tag{3.57}$$

and finally by the GKP-W relation in (3.41) one can find the variation on the CFT side

$$\delta W[g] = \int d^d x \langle T^{\mu\nu} \rangle\, \delta g_{\mu\nu}\,. \tag{3.58}$$

Thus, it is possible to observe a perturbation in the metric that couples to the conserved energy-momentum tensor. Another example involves gauge fields that couple with conserved currents, which will be the central correspondence hypothesis in the next two sections. Based on these observations, one can propose a correspondence where a fluctuation in a $(d+1)$-dimensional metric corresponds analogously to the presence of a gauge field in $(d+1)$ dimensions coupled with a conserved current in d dimensions.

There are also divergent terms in the on-shell action in GKP-W because of the integration procedure near the limit $z \to 0$. One possible solution is to add a cutoff with $z = \epsilon$ and local counterterms in the boundary of the AdS spacetime. Those counterterms are given by the values of the metric or the scalar fields and its derivatives in the boundary. One needs

to impose boundary conditions to solve the equations of motions within the gravitational theory.

The isometries of AdS space act on the boundary by mapping points from it back to the boundary itself. This action corresponds precisely to the conformal group in dd-dimensions. The scale invariance described in (3.50) manifests as a dilation on the boundary [15]. The structure of a dual Quantum Field Theory (QFT) naturally emerges in large N gauge theories, characterized by a gauge group SU(N) with fields in the adjoint representation.

3.4.2 The Role of the Radial Coordinate

In a QFT one deals with typically massive particles, described by a quantum state of its position or momentum. But a scale invariant theory doesn't allow particles to have mass. So, all particles in such a theory are massless. However, with large N gauge theories there exist some weakly interactive excitations. These are object created from the action of single trace operators on the vacuum state. For example, given an operator like $Tr(F^{\mu\nu}F_{\mu\nu})$, the created state at zero coupling with a given 4-momentum is a gluons pair that sum up to the total momentum.

In a Conformal Field Theory (CFT), objects can be understood as having not only position but also size, which is treated as another continuous variable. Therefore, particles are identified not within the CFT itself but rather in AdS space. In other words, the size of a given state in a CFT is related to the position of a particle labeled by the radial coordinate in AdS space.

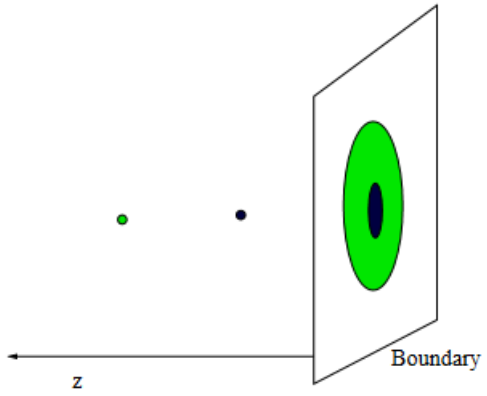

Figure 3.2: Size/radius correspondence between weakly interactive excitations in a CFT and particles of different radial position in the AdS space. [15]

The key idea linking these different perspectives is that in a Conformal Field Theory (CFT), excitations exhibit a specific size. The same object within the CFT can be described with two different sizes related to each other by dilation. These states correspond to a pair of particles with the same intrinsic size but located at different radial positions in AdS space, as illustrated in Figure 3.2. Essentially, these particles in AdS serve as a parameterization of representations of the conformal group. In other words, unitary representations of the conformal group correspond to a bijection between particles or fields in AdS space with

well-defined boundary conditions. These representations are also characterized by the scaling dimension Δ, which determines the mass of the corresponding particle or field.

3.4.3 Scalar Field example in AdS/CFT correspondence

There have been previous examples of correspondences between AdS quantities and operators on the CFT side, such as the scalar field, the conserved energy-momentum tensor, and the conserved currents. Up to now, these correspondences have been developed in a qualitative manner, emphasizing concepts like the GKP-W relation and the constant scaling dimension. Now, it is crucial to undertake a more detailed computation to illustrate the behavior of quantities on the gravity side and how they relate to boundary quantities in the AdS/CFT framework.

To begin with, let's consider the proposal given by GKP-W in equation (3.55), alongside insights from holographic calculations that emphasize boundary behavior. Specifically, let's focus on the scalar field ϕ, which is dual to a primary operator in the CFT. The gravitational action principle for such a field can be formulated as follows:

$$S_{grav} = -\frac{C}{2}\int dz\, d^d x\, \sqrt{-g}\, \left(g^{mn}\partial_m\phi\partial_n\phi + m^2\phi^2\right) \tag{3.59}$$

with $C \propto N^2$ in the large N limit by the gauge group SU(N). For a generic scalar field ϕ in the AdS_{d+1}/CFT_d correspondence, there is an equation which relates the fields mass with the scaling dimension of the dual primary operator, such relation belongs to the holographic dictionary and is given by

$$m^2L^2 = \Delta(\Delta - d) \tag{3.60}$$

with L as the AdS curvature radius. For the AdS spacetime in which one must define the gravity side, it is convenient to take the metric in Poincaré coordinates given by (3.50). The equation of motion, which are calculated starting from the action principle (3.59) by the Euler-Lagrange equations, show that the field ϕ satisfy the Klein-Gordon equation

$$\left(\Box_g - m^2\right)\phi = 0 \tag{3.61}$$

with $\Box_g$ as the D'Alembertian linked to the AdS metric in (3.50), which can be written as follows

$$\Box_g\phi = \frac{1}{\sqrt{-g}}\,\partial_m\left(\sqrt{-g}\,g^{mn}\,\partial_n\phi\right) \tag{3.62}$$

by another way, taking the Poincaré metric such D'Alembertian is more convenient to write by

$$\Box_g\big|_{AdS} = \frac{1}{L^2}\left(z^2\,\partial_z^2 - (d-1)\,z\,\partial_z + z^2\,\eta_{\mu\nu}\partial^\mu\partial^\nu\right)\,. \tag{3.63}$$

So far, equations have been described for a scalar field in the AdS spacetime, such field will take the form $\phi \to \phi(z, x)$. Now, it is possible to take a solution given by the Fourier

decomposition in the flat space and also considering a plane wave form by

$$\phi(z,x) = e^{ip^\mu x_\mu}\,\phi_p(z) \tag{3.64}$$

where the functions $\phi_p(z)$ are called as Fourier modes. One can then see if this solution satisfy the Klein-Gordon equation given in (3.61), considering that for the Fourier modes one gets

$$z^2\,\partial_z^2\,\phi_p(z) - (d-1)\,z\,\partial_z\,\phi_p(z) - (z^2\,p^2 + L^2 m^2)\,\phi_p(z) = 0\,. \tag{3.65}$$

The previous equation has two independent solutions given by their asymptotic behaviour in the boundary when $z \to 0$

$$\phi_p(z) \sim \begin{cases} z^{\Delta_+} \\ z^{\Delta_-} \end{cases} \tag{3.66}$$

where the first solution is called to be "normalisable" and the second one "non-normalisable", (such labels will be later explain) and the Delta-terms are given by the roots of equation (3.60) by

$$\Delta_\pm = \frac{d}{2} \pm \sqrt{\frac{d^2}{4} + m^2 L^2} \tag{3.67}$$

such that it can be notice that $\Delta_+ > \Delta_-$ and also useful $\Delta_+ + \Delta_- = d$. Near the boundary it is possible to make an expansion on $\phi(z,x)$ taking the form (3.66) by

$$\phi(z,x) \sim \phi_{(0)}(x)\,z^{\Delta_-} + \phi_{(+)}(x)\,z^{\Delta_+} + \dots \tag{3.68}$$

where one doesn't take into account the subleading terms in z by the expansion. The non-normalisable term define the associated field in the boundary that is dual to the source of the CFT where

$$\phi_{(0)}(x) \equiv \lim_{z\to 0} \phi(z,x)\,z^{-\Delta_-} = \lim_{z\to 0} \phi(z,x) z^{\Delta_+ - d} \tag{3.69}$$

and the normalisable term also define a duality in which the field $\phi_{(+)}(x)$ is dual to the vacuum expectation value for a dual scalar field operator with scaling dimension $\Delta = \Delta_+$.

As it was said above, the definitions of normalisable and non-normalisable in (3.66) deal with the finiteness of the action (3.59) when evaluated on those solutions. For example, a solution is normalisable if the action evaluated on that solution is finite. It can be proven taking the action term and the field solution depending on z by

$$\begin{aligned} S_{grav} &= -\frac{C}{2}\int dz\,d^dx\,\sqrt{-g}\,\left(g^{mn}\partial_m\phi(z)\partial_n\phi(z) + m^2\phi^2(z)\right) \\ &= -\frac{C}{2}(L)^{d-1}\int dz\,d^dx\,\frac{1}{z^{d+1}}\,\left(z^2\,(\partial_z\phi(z))^2 + L^2m^2\phi^2(z)\right) \end{aligned} \tag{3.70}$$

together with the condition of finiteness by the z dependence and then introducing a solution

with the form $\phi(z) \sim z^{\Delta}$ and taking the cutoff IR limit, one gets

$$\begin{aligned} S_{grav} &= -\frac{C}{2}(L)^{d-1} \int d^d x \int_{\epsilon}^{z_{IR}} dz \, \frac{1}{z^{d+1}} \left(z^2 \left(\Delta z^{\Delta - 1} \right)^2 + L^2 m^2 z^{2\Delta} \right) \\ &= -\frac{C}{2}(L)^{d-1} \int d^d x \, \frac{(\Delta^2 + L^2 m^2)}{2\Delta - d} \left((z_{IR})^{2\Delta - d} - \epsilon^{2\Delta - d} \right) \end{aligned} \tag{3.71}$$

where one can notice that the condition $\Delta > d/2$ must be satisfied for the convergence condition and also notice that such requirement agrees with $\Delta = \Delta_+$ in (3.67). This result then leads to a normalisable mode.

Part II

Main part

Chapter 4

Electroweak Rates and Bulk Viscosity

The primary objective of this dissertation is to explore a theoretical framework where developments in Quantum Field Theory (QFT) serve as a foundational approach for computing reaction rates under non-equilibrium conditions prevalent within the cores of compact neutron stars, known for their extreme matter properties. At high densities, it is hypothesized that matter exists in a deconfined state where quarks, rather than hadrons, become the natural dynamical degrees of freedom [16]. Accordingly, the calculations presented here primarily focus on non-leptonic electroweak processes, with some discussion on semi-leptonic processes, albeit their contributions are relatively minor.

An essential focus of this study is the divergence of the density current of constituent particles within the core region, which is analyzed in terms of chemical imbalances arising from deviations from beta-equilibrium conditions. Additionally, an important aspect discussed is the calculation of bulk viscosity, albeit briefly touched upon as it is not the primary focus of this work. Bulk viscosity in three-flavor quark matter is primarily generated by non-leptonic and semi-leptonic weak processes, manifesting as a function of thermodynamic variables, chemical potentials of constituents, and density oscillations [17].

4.1 Quark Matter Reaction Rates

The beta-equilibrium condition in the core of compact neutron stars governs the particle densities of quark and leptonic constituents. This condition restricts the range of thermodynamic processes inside the stars and plays a crucial role in determining their mass and radius properties. Governed by quark decay processes, the beta-equilibrium condition is expressed by the following relation for chemical potentials in deconfined hadronic matter:

$$\mu_u + \mu_d = \mu_u + \mu_s \tag{4.1}$$

Neutrons, protons, and leptonic species play pivotal roles in the beta-equilibrium condition, imposing relationships among their relative abundances in the core of compact neutron stars. When a density perturbation drives matter out of the equilibrium condition described in (4.1), weak processes are activated to restore equilibrium. These processes contribute to an

effective bulk viscosity.

As mentioned earlier, efforts are focused on determining rates for electroweak processes. However, due to the densities involved, perturbative treatments of strong interactions are not applicable. Therefore, this study employs non-perturbative methods such as holographic dual calculations. The rates of change of interest have been computed at tree-level, assuming massless up and down quarks and a strange quark mass smaller than its chemical potential [17].

For the densities expected in the interior of neutron stars one should be able to restrict to three quark flavors (u,d,s) and two lepton flavors (e,μ). The relevant weak processes in the out-of-beta-equilibrium conditions are produced by the exchange of a W boson like

$$\begin{aligned} u + d \leftrightarrow u + s \qquad & u + l^- \rightarrow d, s + \nu_l \\ d, s \rightarrow u + l^- + \bar{\nu}_l \qquad & l_1^- \rightarrow l_2^- + \nu_1 + \bar{\nu}_2 \,. \end{aligned} \tag{4.2}$$

The purely quark process will have the main technical development here, considering the other ones just for mentioning. Assuming neutrino-transparency regime, the beta equilibrium for quark matter, implied by the previous reactions, impose conditions on the quark and lepton chemical potentials by

$$\mu_u + \mu_\ell = \mu_d = \mu_s, \qquad \mu_\ell = \mu_e = \mu_\mu \,. \tag{4.3}$$

where it is necessary for the calculations present here that the chemical potential is larger than the mass of the corresponding particle.

If a perturbation on the density takes place, the chemical potential relations will suffer changes according to non-equilibrium processes that will be understood as

$$\begin{aligned} \mu_d = \mu_s \quad &\rightarrow \quad \delta\mu_d = -\delta\mu_s \\ \mu_u + \mu_l = \mu_{d,s} \quad &\rightarrow \quad \delta\mu_u + \delta\mu_l = -\delta\mu_{d,s} \,. \end{aligned} \tag{4.4}$$

One can think about these changes like a sum on the contribution for a special flavor and a subtraction for the decay of it. With this idea in mind, one can then postulate the density changes according to the weak rates, for the processes in (4.2) just for electrons, by

$$\frac{dn_u}{dt} = \Gamma_{d\to ue\bar{\nu}} - \Gamma_{ue\to d\nu} + \Gamma_{s\to ue\bar{\nu}} - \Gamma_{ue\to s\nu} \,, \tag{4.5a}$$

$$\frac{dn_d}{dt} = \Gamma_{us\to ud} - \Gamma_{ud\to us} + \Gamma_{ue\to d\nu} - \Gamma_{d\to ue\bar{\nu}} \,, \tag{4.5b}$$

$$\frac{dn_s}{dt} = \Gamma_{ud\to us} - \Gamma_{us\to ud} + \Gamma_{ue\to s\nu} - \Gamma_{s\to ue\bar{\nu}} \,, \tag{4.5c}$$

$$\frac{dn_e}{dt} = \Gamma_{d\to ue\bar{\nu}} - \Gamma_{ue\to d\nu} + \Gamma_{s\to ue\bar{\nu}} - \Gamma_{ue\to s\nu} \,. \tag{4.5d}$$

after which it is also possible to confirm the conservation of electric charge and baryon

number

$$\begin{aligned} n_B &= \frac{1}{3}\,(n_u + n_d + n_s) \\ n_Q &= \frac{2}{3}n_u - \frac{1}{3}n_d - \frac{1}{3}n_s - n_e \end{aligned} \tag{4.6}$$

through their time derivatives

$$\frac{dn_B}{dt} = \frac{dn_Q}{dt} = 0\,. \tag{4.7}$$

The set of equations for the divergence of the particle density current can be written at linear order in the chemical imbalances [17] or the same can be thought as assuming the deviation of the chemical potentials from their beta equilibrium values is small to the baryon chemical potential ($\mu_B = \mu_u + \mu_d + \mu_s$) the rates can be expanded as differences like [3]

$$\Gamma_{ud\to us} - \Gamma_{us\to ud} \approx \lambda_{ds}(\mu_s - \mu_d), \tag{4.8a}$$

$$\Gamma_{ue\to d\nu} - \Gamma_{d\to ue\bar{\nu}} \approx \lambda_{ude}(\mu_d - \mu_u - \mu_e), \tag{4.8b}$$

$$\Gamma_{ue\to s\nu} - \Gamma_{s\to ue\bar{\nu}} \approx \lambda_{use}(\mu_s - \mu_u - \mu_e)\,. \tag{4.8c}$$

Once with the weak rates, one can relate them to the divergence of the particle density current associated and for that purpose the strategy will be to find the lambda coefficients using Ward identities for the non-conservation flavor currents due to the weak sector coupling.

4.2 Flavor Symmetry breaking by electroweak processes

The electroweak sector gives the theory of the main processes studied here as $u+d \to u+s$ and $u+e \to d, s+\nu_e$ through an exchange of a W^+ boson. Enrico Fermi first model β decays by 1933 in the limit of mass to much higher than the momentum in the Feynman propagator formalism, which gives a correction term in the effective Lagrangian of the process, known as the 4-Fermi interaction [18]

$$\Delta\mathcal{L}_{ew} = -2\sqrt{2}G_F\left(J^{\mu}_{ch}\right)^{\dagger} J_{ch,\mu} \tag{4.9}$$

where G_F denotes the Fermi coupling constant and the term J_{ch} is for the left charged current. To see explicitly the form of this current it is necessary to take the three massless flavors of quarks $q^i = (u, d, s)$, as seen in the electroweak sector of interest, and one massless lepton flavor $l^a = (\nu_e, e)$, but it is not enough until the Cabbibo angle is introduce to give the change of basis between mass and flavor eigenstates. Now the form of the current is

$$J^{\mu}_{ch} = \bar{\nu}_{eL}\gamma^{\mu}e_L + \cos\theta_C\bar{u}_L\gamma^{\mu}d_L + \sin\theta_C\bar{u}_L\gamma^{\mu}s_L \tag{4.10}$$

but the expression is only valid for the particular reaction $u + e \to d, s + \nu_e$. Now it is convenient to introduce the following notation, referred to matrix elements as

$$\left(J_q^\mu\ \right)^i{}_j = \bar{q}_j \gamma^\mu q^i, \quad \left(J_l^\mu\ \right)^a{}_b = \bar{l}_b \gamma^\mu l^a \,. \tag{4.11}$$

Now the equation (4.10) can be written by

$$\begin{aligned} J^\mu_{ch} &= \bar{l}_1 \gamma^\mu l^2 + \cos\theta_C \bar{q}_1 \gamma^\mu q^2 + \sin\theta_C \bar{q}_1 \gamma^\mu q^3 \\ J^\mu_{ch} &= \left(J_l^\mu\right)^2{}_1 + \cos\theta_C \left(J_q^\mu\right)^2{}_1 + \sin\theta_C \left(J_q^\mu\right)^3{}_1 \,. \end{aligned} \tag{4.12}$$

As mentioned before, the currents are related with their transformation symmetry group. In this case, (J_l^μ) is the current for the $SU(2)_L$ lepton symmetry and (J_q^μ) is the current for the $SU(3)_L$ flavor symmetry. With the previous expressions, the electroweak term in equation (4.9) is then

$$\begin{aligned} \Delta\mathcal{L}_{ew} = &-2\sqrt{2} G_F \left\{ \left(J_l^\mu\right)^2{}_1{}^\dagger + \cos\theta_C \left(J_q^\mu\right)^2{}_1{}^\dagger + \sin\theta_C \left(J_q^\mu\right)^3{}_1{}^\dagger \right\} \\ &\times \left\{ \left(J_{\mu,l}\right)^2{}_1 + \cos\theta_C \left(J_{\mu,q}\right)^2{}_1 + \sin\theta_C \left(J_{\mu,q}\right)^3{}_1 \right\}, \end{aligned} \tag{4.13}$$

where the dagger operator change the form of the current matrix elements notation as

$$\begin{aligned} \left(J_l^\mu\right)^2{}_1{}^\dagger &= \left(J_l^\mu\right)^1{}_2 \\ \left(J_q^\mu\right)^2{}_1{}^\dagger &= \left(J_q^\mu\right)^1{}_2 \\ \left(J_q^\mu\right)^3{}_1{}^\dagger &= \left(J_q^\mu\right)^1{}_3 \end{aligned}$$

and putting these together and neglecting terms that include the currents associated with leptons and terms that mix lepton and quark currents, one can have

$$\begin{aligned} \Delta\mathcal{L}_{ew} = &-2\sqrt{2} G_F \Big(\cos^2\theta_C \left(J_q^\mu\right)^1{}_2 \left(J_{\mu,q}\right)^2{}_1 + \sin^2\theta_C \left(J_q^\mu\right)^1{}_3 \left(J_{\mu,q}\right)^3{}_1 \\ &+ \cos\theta_C \sin\theta_C \left\{ \left(J_q^\mu\right)^1{}_2 \left(J_{\mu,q}\right)^3{}_1 + \left(J_q^\mu\right)^1{}_3 \left(J_{\mu,q}\right)^2{}_1 \right\} \Big) \,. \end{aligned} \tag{4.14}$$

It will be also important for Section 4 the electroweak term considering leptonic interactions. Later, depending on the calculation purpose, $\Delta\mathcal{L}_{ew}$ will take a complete form given by

$$\begin{aligned} \Delta\mathcal{L}_{EW} = &-2\sqrt{2} G_F \Big[\cos^2\theta_C (J^\mu_{q\,L})^2{}_1 (J_{q\,L\,\mu})^1{}_2 + \sin^2\theta_C (J^\mu_{q\,L})^1{}_3 (J_{q\,L\,\mu})^3{}_1 \\ &+ \cos\theta_C \sin\theta_C \left((J^\mu_{q\,L})^2{}_1 (J^\mu_{q\,L})^1{}_3 + (J^\mu_{q\,L})^3{}_1 (J^\mu_{q\,L})^1{}_2 \right) \\ &+ \cos\theta_C \left((J^\mu_{l\,L})^2{}_1 (J_{q\,L\,\mu})^1{}_2 + (J^\mu_{l\,L})^1{}_2 (J_{q\,L\,\mu})^2{}_1 \right) \\ &+ \sin\theta_C \left((J^\mu_{l\,L})^2{}_1 (J_{q\,L\,\mu})^1{}_3 + (J^\mu_{l\,L})^1{}_2 (J_{q\,L\,\mu})^3{}_1 \right) \Big] \,. \end{aligned} \tag{4.15}$$

If one needs the expression (4.14), it is convenient to introduce the projectors $(P_{kl})^i{}_j =$

${\delta^i}_k {\delta^l}_j$ to make simple the writing of

$$\begin{aligned}\Delta\mathcal{L}_{ew} = &-2\sqrt{2}G_F\left[\cos^2\theta_C \mathrm{tr}\left(P_{11}J_q^\mu \;\; P_{22}J_{q\,\mu}\right) + \sin^2\theta_C \mathrm{tr}\left(P_{11}J_q^\mu \;\; P_{33}J_{q\,\mu}\right)\right.\\ &\left.+\cos\theta_C \sin\theta_C \mathrm{tr}\left(P_{11}J_q^\mu \;\; P_{32}J_{q\,\mu} + P_{11}J_q^\mu \;\; P_{23}J_{q\,\mu}\right)\right] .\end{aligned} \tag{4.16}$$

The main purpose now is to achieve the Ward identity expression for a transformation under the group $SU(3)$. As seen before, one can find the Ward identities starting from a transformed Lagrangian and then show how is the change over the generating functional, so it is necessary to find how the term $\Delta\mathcal{L}_{ew}$ transforms under $SU(3)_L \times SU(3)_R$ which is the representation where the flavor currents belong. Then the law of rotation by an equal angle is

$$q^i \to e^{-i\theta_A T^A} q^i \tag{4.17}$$

where T^A are the generators of the gauge group, with the normalization rule defined by $\mathrm{tr}(T^A T^B) = \frac{1}{2}\delta^{AB}$, and θ_A is the transformation parameter. Then the flavor current, explicitly defined as $J_q^\mu \;\; = J_{q,L}^\mu \;\; + J_{q,R}^\mu$, is

$$\begin{aligned} J_q'^\mu &= J_q^\mu \;\; + \delta_\theta J_q^\mu \\ J_q'^\mu &= J_q^\mu \;\; + i\theta_A[T^A, J_q^\mu \;\;] \end{aligned} \tag{4.18}$$

Notice that expression (4.15) can also be simplified using another kind of projectors such that

$$\begin{aligned} \left(\tilde{P}_{ai}\right)^b_{\;j} &= \delta^b_a\, \delta^i_j \\ \left(\tilde{P}^T_{ia}\right)^j_{\;b} &= \delta^a_b\, \delta^j_i \end{aligned} \tag{4.19}$$

with these projector relating quark and lepton currents each other by the following rules

$$\begin{aligned} \tilde{P} J_q^\mu \tilde{P}^T &\sim J_l^\mu \\ \tilde{P}^T J_l^\mu \tilde{P} &\sim J_q^\mu \end{aligned} \tag{4.20}$$

as it were made for the quark currents, the leptonic ones can also be transformed under the action of the $SU(2)_R \times SU(2)_L$ rotation group which leaves the vector currents as

$$\delta_\phi\, J_l^\mu = i\phi_\alpha[\tau^\alpha\,,\, J_l^\mu] \tag{4.21}$$

where the generators τ^α satisfy the normalization rule $tr(\tau^\alpha\,\tau^\beta) = \frac{1}{2}\delta^{\alpha\,\beta}$.

If one wish to know the Ward identities it is necessary the procedure of section 2.3 in which one transforms the Lagrangian under $SU(3)_R \times SU(3)_L$. The general computation, without details, must include

$$\mathcal{L}' = (\mathcal{L}_0 + \Delta\mathcal{L}_{EW}) + (\delta\mathcal{L}_0 + \delta\Delta\mathcal{L}_{EW}) \tag{4.22}$$

where the generating functional varies depending on small perturbations as

$$\begin{aligned} Z' &= \int DA_\mu DqD\bar{q}e^{i\int(\mathcal{L}_0+\Delta\mathcal{L}_{EW})+(\delta\mathcal{L}_0+\delta\Delta\mathcal{L}_{EW})d^4x} \\ &= \int DA_\mu DqD\bar{q}e^{iS}\left(1+i\int\delta\mathcal{L}_0 d^4x\right)\left(1+i\int\delta\Delta\mathcal{L}_{EW}d^4x\right) \end{aligned} \tag{4.23}$$

and finally, because of the invariance on Z, the form of Ward identities involve the form of $\delta\mathcal{L}_0$ which is the QCD Lagrangian with no mass and no kinetic term, under the transformation group as well as $\delta\Delta\mathcal{L}_{EW}$. Based on these, the Ward identities relate $\langle\delta\mathcal{L}_0\rangle$ with $\langle\delta\Delta\mathcal{L}_{EW}\rangle$ through

$$\delta\mathcal{L}_0 = 2\theta_A\left(D_\mu J^\mu_{q,L}\right)T^A \tag{4.24}$$

where the covariant derivative has the form

$$D_\mu J^\mu_{q,L} = \partial_\mu J^\mu_{q,L} - i[A^f_\mu, J^\mu_{q,L}] \tag{4.25}$$

for a background flavor gauge potential with no coupling constant (g_s)by the fact that it isn't a color-charged field. The rule defined in (4.18) change the electroweak term as

$$\begin{aligned} \delta\Delta\mathcal{L}_{EW} = -2\sqrt{2}G_F(i\theta_A)\Big\{&\cos^2\theta_C\left(\mathrm{tr}\left(P_{11}J^\mu_{q,L}\ P_{22}[T^A,J_{q,L,\mu}]\right.\right. \\ &\left.+P_{11}[T^A,J^\mu_{q,L}\ ]P_{22}J_{q,L,\mu}\right)\Big)+\sin^2\theta_C\left(\mathrm{tr}\left(P_{11}J^\mu_{q,L}\ P_{33}\right.\right. \\ &\left.[T^A,J_{q,L,\mu}]+P_{11}[T^A,J^\mu_{q,L}\ ]P_{33}J_{q,L,\mu}\right)\Big) \\ &+\cos\theta_C\sin\theta_C\left(\mathrm{tr}\left(P_{11}J^\mu_{q,L}\ P_{23}[T^A,J_{q,L,\mu}]+P_{11}\right.\right. \\ &[T^A,J^\mu_{q,L}\ ]P_{23}J_{q,L,\mu}+P_{11}[T^A,J^\mu_{q,L}\ ]P_{32}J_{q,L,\mu} \\ &\left.\left.+P_{11}J^\mu_{q,L}\ P_{32}[T^A,J_{q,L,\mu}]\right)\right)\Big\}. \end{aligned} \tag{4.26}$$

At this point the ward identities can be defined from the previous elements starting with $\delta\mathcal{L}_0 = \delta\Delta\mathcal{L}_{EW}$ and taking the trace at both sides using the cyclic properties one finally gets

$$\begin{aligned} D_\mu J^\mu_q = &i\sqrt{2}G_F\times \\ &\Big(\cos^2\theta_C\left[J_{q\,L\,,\mu}\,,\,P_{11}\,J^\mu_{q\,L}\,P_{22}+P_{22}\,J^\mu_{q\,L}\,P_{11}\right] \\ &+\sin^2\theta_C\left[J_{q\,L\,,\mu}\,,\,P_{11}\,J^\mu_{q\,L}\,P_{33}+P_{33}\,J^\mu_{q\,L}\,P_{11}\right] \\ &+\cos\theta_C\,\sin\theta_C\left(\left[J_{q\,L\,,\mu}\,,\,P_{11}\,J^\mu_{q\,L}\,P_{23}+P_{23}\,J^\mu_{q\,L}\,P_{11}\right]\right. \\ &\left.\left.+\left[J_{q\,L\,,\mu}\,,\,P_{11}\,J^\mu_{q\,L}\,P_{32}+P_{32}\,J^\mu_{q\,L}\,P_{11}\right]\right)\right). \end{aligned} \tag{4.27}$$

Just remember that the quark masses wasn't included in the calculations with the QCD Lagrangian, those terms would have represented another kind of contributions to the expression above.

4.3 Formulas for the Weak Quark Decay Rate

The electroweak rates could be calculated starting from the Ward identities in (4.27) for each one of the quark and lepton currents.

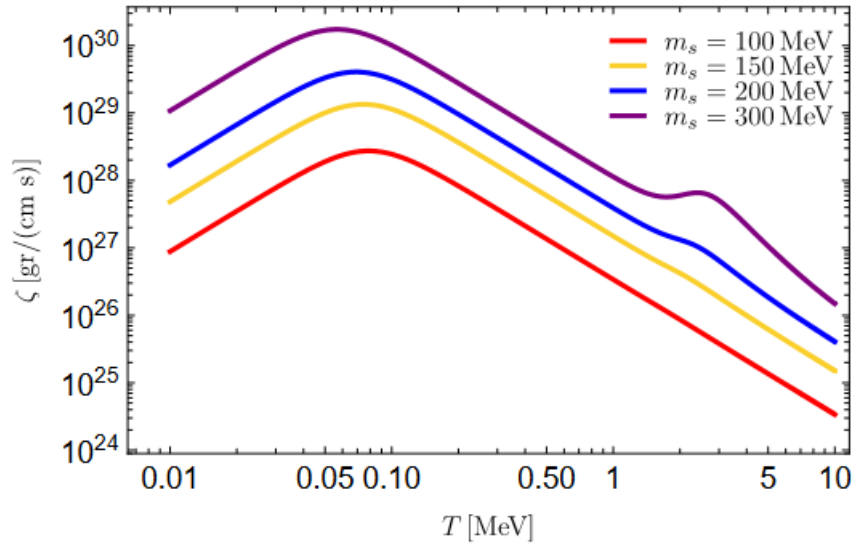

Figure 4.1: Bulk viscosity of three-flavor quark matter for different values of the strange quark mass at $\omega/2\pi = 1$ kHz in the neutrino-free regime [17]

As it was said above, this work only concentrates in the quark weak decays, just to mention that this statement holds because lepton contributions for the complete set of rates represent a small deviation for the bulk viscosity behaviour related to temperature as it can be seen in Figure 4.1, Where the first peak at $T \in [0.05\,,\,0.1]MeV$, which is the range of temperatures typical of a neutron star, is determined by the rate of the non-leptonic process, and the lowest peaks are due to the leptonic contribution to the rates. One should highlight the non-changing shape of the curve around the temperatures of interest if leptonic processes are not considered.

The rate calculations will consider non-vanishing chemical potentials, which will be introduced by a time component in a 4-spacetime background flavor gauge potential with a diagonal matrix notation such that

$$(A_0^q)^i{}_j = \mu_i\,\delta^i_j \tag{4.28}$$

from which one can identify the chemical potentials of the three main quarks as $\mu_1 = \mu_u\,,\,\mu_2 = \mu_d\,,\,\mu_3 = \mu_s$. It is also possible to identify the quark currents by its diagonal components in matrix notation as $(J_q^\mu)^1{}_1 = J_u^\mu\,,\,(J_q^\mu)^2{}_2 = J_d^\mu\,,\,(J_q^\mu)^3{}_3 = J_s^\mu\,.$

Starting from the previous notation, one can identify the quark and lepton densities from the time component of the currents by

$$(J_q^0)^1{}_1 = n_u\,,\,(J_q^0)^2{}_2 = n_d\,,\,(J_q^0)^3{}_3 = n_s\,,\,(J_l^0)^l{}_l = n_l\,. \tag{4.29}$$

Now, one can identify such currents with the Ward identities mentioned above with not only the quark contribution but better with the complete form of the electroweak term in (4.15). This procedure reflects interest in the flavor-changing contributions introduced by Fermi's interaction. Leaving the details in such calculation, one can arrive to the following Ward identities

$$\begin{aligned}
\partial_\mu \langle J^\mu_u \rangle &= i\sqrt{2}G_F\eta_{\mu\nu}\Big[\cos\theta_C \left\langle (J^\mu_{q\,L})^u{}_d (J^\nu_{l\,L})^e{}_{\nu_e} - (J^\mu_{q\,L})^d{}_u (J^\nu_{l\,L})^{\nu_e}{}_e \right\rangle \\
&\quad + \sin\theta_C \left\langle (J^\mu_{q\,L})^u{}_s (J^\nu_{l\,L})^e{}_{\nu_e} - (J^\mu_{q\,L})^s{}_u (J^\nu_{l\,L})^{\nu_e}{}_e \right\rangle\Big], \\
\partial_\mu \langle J^\mu_d \rangle &= i\sqrt{2}G_F\eta_{\mu\nu}\Big[\cos\theta_C \sin\theta_C \eta_{\mu\nu} \left\langle (J^\mu_{q\,L})^u{}_s (J^\nu_{q\,L})^d{}_u - (J^\mu_{q\,L})^s{}_u (J^\nu_{q\,L})^u{}_d \right\rangle \\
&\quad - \cos\theta_C \left\langle (J^\mu_{q\,L})^u{}_d (J^\nu_{l\,L})^e{}_{\nu_e} - (J^\mu_{q\,L})^d{}_u (J^\nu_{l\,L})^{\nu_e}{}_e \right\rangle\Big], \\
\partial_\mu \langle J^\mu_s \rangle &= i\sqrt{2}G_F\eta_{\mu\nu}\Big[-\cos\theta_C \sin\theta_C \eta_{\mu\nu} \left\langle (J^\mu_{q\,L})^u{}_s (J^\nu_{q\,L})^d{}_u - (J^\mu_{q\,L})^s{}_u (J^\nu_{q\,L})^u{}_d \right\rangle \\
&\quad - \sin\theta_C \left\langle (J^\mu_{q\,L})^u{}_s (J^\nu_{l\,L})^e{}_{\nu_e} - (J^\mu_{q\,L})^s{}_u (J^\nu_{l\,L})^{\nu_e}{}_e \right\rangle\Big], \\
\partial_\mu \langle J^\mu_\nu \rangle &= i\sqrt{2}G_F\eta_{\mu\nu}\Big[-\cos\theta_C \left\langle (J^\mu_{q\,L})^u{}_d (J^\nu_{l\,L})^e{}_{\nu_e} - (J^\mu_{q\,L})^d{}_u (J^\nu_{l\,L})^{\nu_e}{}_e \right\rangle \\
&\quad - \sin\theta_C \left\langle (J^\mu_{q\,L})^u{}_s (J^\nu_{l\,L})^e{}_{\nu_e} - (J^\mu_{q\,L})^s{}_u (J^\nu_{l\,L})^{\nu_e}{}_e \right\rangle\Big], \\
\partial_\mu \langle J^\mu_e \rangle &= i\sqrt{2}G_F\eta_{\mu\nu}\Big[\cos\theta_C \left\langle (J^\mu_{q\,L})^u{}_d (J^\nu_{l\,L})^e{}_{\nu_e} - (J^\mu_{q\,L})^d{}_u (J^\nu_{l\,L})^{\nu_e}{}_e \right\rangle \\
&\quad + \sin\theta_C \left\langle (J^\mu_{q\,L})^u{}_s (J^\nu_{l\,L})^e{}_{\nu_e} - (J^\mu_{q\,L})^s{}_u (J^\nu_{l\,L})^{\nu_e}{}_e \right\rangle\Big].
\end{aligned} \tag{4.30}$$

where the numbered notation of (4.11) was changed by the flavored one identifying $q^i = (u, s, d)$ with $q^i = (1, 2, 3)$ and the same for the first leptonic family with $l^a = (\nu_e, e)$ with $l^a = (1, 2)$.

Now, one needs to find such two point functions value developing a theory in real time but starting from a general point of view involving complex path and finite temperature formalism as seen in Section 2. A thermal correlator in such theory is defined by (2.68) with a complex path in the Schwinger-Keldysh formalism identified by labeled sections as in figure 2.1. From now on, it is more convenient to take another notation when $(t_i \to -\infty)$ and $(t_f \to +\infty)$ by $C_1 \to (t_i, t_f) \cup C_3 \to (t_f, t_f - i\sigma) \cup C_2 \to (t_f - i\sigma, t_i - i\sigma) \cup C_4 \to (t_i - i\sigma, t_i - i\beta)$.

The two point correlators in (2.68) will have two possible contributions depending on the location of the coordinates. If one locates the x_1 time component along C_1 path, corresponding to the real axis, the correlator is equal to the Feynman two point function by

$$\left\langle T_C\left(\phi(x_1)\phi(x_2)\right)\right\rangle = \left\langle T\left(\phi(x_1)\phi(x_2)\right)\right\rangle = G^F_{1,2}(t_1 - t_2, \vec{x_1} - \vec{x_2}) \tag{4.31}$$

when the time component of x_2 is in the same C_1 path. On the other hand, if such component is located on the C_3 path, the thermal correlator is equal to the Wightman correlator given by

$$\left\langle T_C\left(\phi(x_1)\phi(x_2)\right)\right\rangle = G^<_{1,2}(t_1 - t_2 + i\sigma, \vec{x_1} - \vec{x_2}) \tag{4.32}$$

in the same way which it was described the thermal correlator by the third equation in (2.89). Inside the framework of Fermi electroweak theory, the correlator function as a thermal

mean value, can be written, analogously to (2.22), in a perturbative way by the following expression

$$\begin{aligned}\langle \mathcal{O}(x)\rangle_{G_F} &= \left\langle T_C\left[\mathcal{O}(x)e^{i\int_C d^4x'\,\Delta\mathcal{L}_{EW}(x')}\right]\right\rangle_0 \approx \left\langle T_C\left[\mathcal{O}(x)\left(1+i\int_C d^4x'\,\Delta\mathcal{L}_{EW}(x')\right)\right]\right\rangle_0 \\ &\approx \langle\mathcal{O}(x)\rangle_0 + i\int_C d^4x'\,\langle T_C\left[\mathcal{O}(x)\Delta\mathcal{L}_{EW}(x')\right]\rangle_0\,.\end{aligned} \tag{4.33}$$

where the subscript "0" means that the expectation value is calculated without Fermi interaction considerations. The electroweak term in the right hand side of the previous equation can be submitted to the same formalism described in equations (4.31) and (4.32) with the Feynman and Wightman contributions. So, if the x -time coordinate in (4.33) is on the real axis path and x' is defined along the complex path C, the integral term will be

$$\begin{aligned}\int_C d^4x'\,\langle T_C\left[\mathcal{O}(x)\Delta\mathcal{L}_{EW}(x')\right]\rangle_0 = \int d^4x'\,\Big(&G^F_{\mathcal{O}\Delta\mathcal{L}_{EW}}(t-t',\vec{x}-\vec{x'}) \\ &+G^<_{\mathcal{O}\Delta\mathcal{L}_{EW}}(t-t'+i\sigma,\vec{x}-\vec{x'})\Big)\,.\end{aligned} \tag{4.34}$$

At this point, one has the proper mathematical tools to compute the correlators for the current expressions in (4.30), but after some calculations it it necessary to apply another approximation, this time on the terms which mix the current correlators and those in the $\Delta\mathcal{L}_{EW}$ term, considering if one needs only the quark-dependent expression in (4.14) or the mixed one in (4.15).

So far, the equations for the rates in (4.30) are defined by the expectation value of the non-diagonal components of current quadratic terms appearing in the right hand side of the Ward identities. Such correlators, as seen before, can be computed starting from the expression (4.34) considering the absence of flavor-breaking terms where the most general form of current correlators would be

$$\left\langle (J^\mu_{f\,L})^a{}_b\,(J^\nu_{f\,L})^c{}_d\right\rangle = G_1\,\delta^a_b\,\delta^c_d + G_2\,\delta^a_d\,\delta^c_b\,. \tag{4.35}$$

and if for instance, the action has not only the electroweak term but also mass term or chemical potential introduced by a gauge field, the δ-term previously introduced must be changed for a matrix one associated to these quantities, something that could add new terms in the Ward identities but if such matrices are diagonal, (4.35) holds with the index notation.

Once with the perturbative way to compute the current thermal correlators, one will have four-point expressions for the rates. Such new correlators can be treated by an approximation with the factorization procedure where single trace operators are basically classical objects in the large N limit $\langle\mathcal{O}\,\mathcal{O}\rangle \approx \langle\mathcal{O}\rangle\,\langle\mathcal{O}\rangle + O(\frac{1}{N^2})$ [19], also considering that could be some non-factorizable contributions but they are assumed to be small. With this in mind, the

4-point correlator functions for the currents as a result of (4.33) can be written by

$$\begin{aligned}\left\langle\left[(J_f^\mu)^a{}_b(x)(J_f^\nu)^c{}_d(x)\right]\left[(J_{f'}^\alpha)^{a'}{}_{b'}(x')(J_{f'}^\beta)^{c'}{}_{d'}(x')\right]\right\rangle_0 \approx \\ \delta_{ff'}\left(\left\langle(J_f^\mu)^a{}_b(x)(J_f^\alpha)^{a'}{}_{b'}(x')\right\rangle_0\left\langle(J_f^\nu)^c{}_d(x)(J_f^\beta)^{c'}{}_{d'}(x')\right\rangle_0 + \right. \\ \left.\left\langle(J_f^\mu)^a{}_b(x)(J_f^\beta)^{c'}{}_{d'}(x')\right\rangle_0\left\langle(J_f^\nu)^c{}_d(x)(J_f^\alpha)^{a'}{}_{b'}(x')\right\rangle_0\right).\end{aligned} \tag{4.36}$$

After the theory described below and some long calculations, one can find for example the approximated expression for the Ward identity current associated to the d-quark in (4.30) as

$$\begin{aligned}\partial_\mu\left\langle J_d^\mu(x)\right\rangle \approx 4G_F^2\cos^2\theta_C\sin^2\theta_C\eta_{\mu\nu}\eta_{\alpha\beta}\int_C d^4x' \\ \left\{\left\langle T_C\left[(J_{q\,L}^\mu)^u{}_s(x)(J_{q\,L}^\alpha)^s{}_u(x')\right]\right\rangle_0\left\langle T_C\left[(J_{q\,L}^\nu)^d{}_u(x)(J_{q\,L}^\beta)^u{}_d(x')\right]\right\rangle_0\right. \\ \left.-\left\langle T_C\left[(J_{q\,L}^\mu)^s{}_u(x)(J_{q\,L}^\alpha)^u{}_s(x')\right]\right\rangle_0\left\langle T_C\left[(J_{q\,L}^\nu)^u{}_d(x)(J_{q\,L}^\beta)^d{}_u(x')\right]\right\rangle_0\right\}.\end{aligned} \tag{4.37}$$

The next step, as seen above, is to use the formalism in (4.34) and later performing a Fourier transform on it, such that the correlators will take the following form

$$\begin{aligned}\partial_\mu\left\langle J_d^\mu(x)\right\rangle \approx 4G_F^2\cos^2\theta_C\sin^2\theta_C\eta_{\mu\nu}\eta_{\alpha\beta} \\ \left[\int\frac{d^4q}{(2\pi)^4}\left[G_{us,su}^{F\,\mu\alpha}(q)G_{du,ud}^{F\,\nu\beta}(-q)-G_{su,us}^{F\,\mu\alpha}(q)G_{ud,du}^{F\,\nu\beta}(-q)\right]\right. \\ \left.+\int\frac{d^4q}{(2\pi)^4}\left[G_{us,su}^{<\,\mu\alpha}(q)G_{du,ud}^{<\,\nu\beta}(-q)-G_{su,us}^{<\,\mu\alpha}(q)G_{ud,du}^{<\,\nu\beta}(-q)\right]\right].\end{aligned} \tag{4.38}$$

where the Fourier transform expression used on the Feynman and Wightman correlators was given by

$$G_{ab,a'b'}^{F,<\,\mu\alpha}(x-x') = \int\frac{d^4q}{(2\pi)^4}e^{iq\cdot(x-x')}G_{ab,a'b'}^{F,<\,\mu\alpha}(q)\,. \tag{4.39}$$

Both equations (4.37) and (4.38) were derived using the only quark-dependent expression of $\Delta\mathcal{L}_{EW}$ in (4.14) and there is still some new change of notation to make both expression shorter with

$$\begin{aligned}\gamma_{us\to ud} &= \gamma_{us\to ud}^F + \gamma_{us\to ud}^< \\ &= \eta_{\mu\nu}\,\eta_{\alpha\beta}\left(\int\frac{d^4q}{(2\pi)^4}\left[G_{us,su}^{F\,\mu\alpha}(q)G_{du,ud}^{F\,\nu\beta}(-q)-G_{su,us}^{F\,\mu\alpha}(q)G_{ud,du}^{F\,\nu\beta}(-q)\right]\right. \\ &\left.+\int\frac{d^4q}{(2\pi)^4}\left[G_{us,su}^{<\,\mu\alpha}(q)G_{du,ud}^{<\,\nu\beta}(-q)-G_{su,us}^{<\,\mu\alpha}(q)G_{ud,du}^{<\,\nu\beta}(-q)\right]\right)\end{aligned} \tag{4.40}$$

and once with these new quantities, one can define the rate of change of the d-quark current in (4.38) by

$$\partial_\mu\left\langle J_d^\mu\right\rangle = 4G_F^2\sin\theta_C^2\cos^2\theta_C\,\gamma_{us\to ud}\,. \tag{4.41}$$

Once with all the mathematical development described above for the d-quark rate of change with all the non-vanishing contributions and without specifying the calculation

details, it is natural now to give the expressions for the other rates of change in (4.30) as follow

$$\partial_\mu \langle J_s^\mu \rangle = -4G_F^2 \, \sin\theta_C^2 \cos^2\theta_C \, \gamma_{us\to ud} = -\partial_\mu \langle J_d^\mu \rangle \tag{4.42}$$

where one can see the relation between currents due to to first process in (4.4).

4.3.1 Rates in Thermal Field Theory at finite Chemical Potential

The best way to compute the gamma function described in (4.42) deals with the Thermal correlator formalism seen in Chapter 2. A good starting point is to re-write the correlator components of gamma function in (4.40) by its spacetime coordinates, taking the example in (4.31) and (4.32), by

$$\begin{aligned} G^{F\,\mu\alpha}_{us,su}(q) \to G^{F\,\mu\alpha}_{us,su}(t-t',\vec{x}-\vec{x'}) &= \langle T\left[(J^\mu_{q\,L})^u{}_s(x)\,(J^\alpha_{q\,L})^s{}_u(x')\right]\rangle_\beta \\ G^{<\mu\alpha}_{us,su}(q) \to G^{<\mu\alpha}_{us,su}(t-t'+i\sigma,\vec{x}-\vec{x'}) &= \langle T_C\left[(J^\mu_{q\,L})^u{}_s(x)\,(J^\alpha_{q\,L})^s{}_u(x')\right]\rangle_\beta \end{aligned} \tag{4.43}$$

and with an analogous procedure for the other correlators in (4.40). Now, one has to remember the definition of a thermal correlator given in (2.84) to see the previous expression by the time dependence with a general form

$$\begin{aligned} G^{<\mu\nu}_{ab,cd}(t,t') &= Tr\left(\rho (J^\nu_q)^c{}_d(t')\,(J^\mu_q)^a{}_b(t)\right) \\ G^{>\mu\nu}_{ab,cd}(t,t') &= Tr\left(\rho (J^\mu_q)^a{}_b(t)\,(J^\nu_q)^c{}_d(t')\right) \end{aligned} \tag{4.44}$$

where the definition agrees with that in (2.30) and with a new quantity given by the ρ-function as

$$\rho = \frac{e^{-\beta H_\mu}}{Z(\beta)} \tag{4.45}$$

with the expression for the partition function illustrated in (2.11). For correct use of this last expression in the ρ-function one has to state the Hamiltonian operator H_μ as the one where there exist a chemical potential μ conjugate to a charge Q associated with the corresponding currents. So, the new Hamiltonian gives the grand partition function by [6]

$$Z(\beta) \to Z(\beta\,,\mu) = Tr\left(\exp\Big\{-\beta\,(\hat{H}-\mu\,\hat{Q})\Big\}\right). \tag{4.46}$$

Notice that the previous Hamiltonian has turned the time component of the background flavor gauge potential on, like shown in (4.28), and as seen in (2.67), one has to introduce the adjoint representation of the currents to get the eigenvalues of such effective Hamiltonian of the form $H_\mu = H - \mu\,Q$ with

$$\begin{aligned} (J^\mu_q)^a{}_b(t) &= e^{i\,t\,\hat{H}}\,(J^\mu_q)^a{}_b(0)\,e^{-i\,t\,\hat{H}} \\ &= e^{i\,t\,(\hat{H}_\mu+\mu Q)}\,(J^\mu_q)^a{}_b(0)\,e^{-i\,t\,(\hat{H}_\mu+\mu Q)} \\ &= e^{i\,t\,\hat{H}_\mu}\,e^{i\,t\,\mu_a}\,(J^\mu_q)^a{}_b(0)\,e^{-i\,t\,\hat{H}_\mu}\,e^{-i\,t\,\mu_b} = e^{i\,t\,(\mu_a-\mu_b)}\,(\hat{J}^\mu_q)^a{}_b(t) \end{aligned} \tag{4.47}$$

then, in a similar procedure like (2.31) introducing evolution operators, one can find the correlator functions in (4.44) by the following expressions

$$\begin{aligned} G^{>\,\mu\nu}_{ab,cd}(t,t') &= e^{i\mu_{ab}t+i\mu_{cd}t'}\, Tr\left(e^{-\beta H_\mu}(\hat{J}^\mu_q)^a{}_b(t)\,(\hat{J}^\nu_q)^c{}_d(t')\right) = e^{i\mu_{ab}t+i\mu_{cd}t'}\,\widehat{G}^{>\,\mu\nu}_{ab,cd}(t,t') \\ G^{<\,\mu\nu}_{ab,cd}(t,t') &= e^{i\mu_{ab}t+i\mu_{cd}t'}\, Tr\left(e^{-\beta H_\mu}(\hat{J}^\nu_q)^c{}_d(t')\,(\hat{J}^\mu_q)^a{}_b(t)\right) = e^{i\mu_{ab}t+i\mu_{cd}t'}\,\widehat{G}^{<\,\mu\nu}_{ab,cd}(t,t')\,. \end{aligned} \tag{4.48}$$

where, for simplicity, was introduced the notation ($\mu_{ab} = \mu_a - \mu_b$). Also notice that the new correlators labelled by $\hat{G}$ do not depend on the chemical potentials and their thermal dependence is related to the new Hamiltonian H_μ. Starting from these equations, one can figure out, by the KMS formula in (2.34), a relation between the correlators $G^{<\,\mu\nu}_{ab,cd}$ and $G^{>\,\mu\nu}_{ab,cd}$ through a temperature dependent exponential term given by

$$\begin{aligned} G^{<\,\mu\nu}_{ab,cd}(t+i\beta,\,t') &= e^{-\mu_{ab}\,\beta}\, e^{i(\mu_{ab}\,t+\mu_{cd}\,t')}\,\widehat{G}^{<\,\mu\nu}_{ab,cd}(t+i\beta,t') \\ G^{<\,\mu\nu}_{ab,cd}(t+i\beta,\,t') &= e^{-\mu_{ab}\beta}\,G^{>\,\mu\nu}_{ab,cd}(t,t') \end{aligned} \tag{4.49}$$

This last equation is very important in the thermal correlators formalism because after some Fourier transform procedures one can derive relations that depend on the chemical potential in the framework of Bose-Einstein distribution. Starting from a Fourier transform, that follows from the KMS condition like (2.38), on this equation, one can find that both correlators in 4-momentum space are related analogously as (2.39) with the (k_0) exponential term. But, one also must notice that such correlators have been already related by another exponential quantity given by ($e^{-\mu_{ab}\beta}$) which is non dependent on the coordinates if one takes the condition of interest ($cd \to ba$), that takes place in the rate of change equations, then one can see

$$G^{<\,\mu\nu}_{ab,ba}(k_0) = e^{-\beta(k_0\,+\,\mu_{ab})}\,G^{>\,\mu\nu}_{ab,ba}(k_0) \tag{4.50}$$

where such correlators must be defined inside a Bose-Einstein distribution with ($k_0 \to k_0 + \mu_{ab}$)e. Taking the formalism of the spectral function seen in Chapter 2, specifically in (2.40), it is possible to write the correlators in a similar way than (2.43) and (2.44) by

$$G^{>\,\mu\nu}_{ab,ba}(k_0) - G^{<\,\mu\nu}_{ab,ba}(k_0) = \rho^{\mu\nu}_{ab,ba}(k_0) \tag{4.51}$$

and each of one correlators with their own expressions written as

$$\begin{aligned} G^{>\,\mu\nu}_{ab,ba}(k_0) &= (1+f_{ab}(k_0))\;\rho^{\mu\nu}_{ab,ba}(k_0) \\ G^{<\,\mu\nu}_{ab,ba}(k_0) &= f_{ab}(k_0)\,\rho^{\mu\nu}_{ab,ba}(k_0) \end{aligned} \tag{4.52}$$

where the f-function is defined by the modified Bose-Einstein distribution mentioned above

$$f_{ab}(k_0) = \frac{1}{e^{\beta(k_0+\mu_{ab})}-1}\,. \tag{4.53}$$

All of the mathematical formalism so far for the correlators defined by the ($<,>$) labels will state the time-ordered correlator function, which is needed for the rate of change expressions.

Starting from the same time-ordered expression for a generic function given in (2.36), one has

$$G^{F\,\mu\alpha}_{ab,ba}(t,t') = \theta(t-t')\,G^{>\,\mu\nu}_{ab,ba}(t,t') + \theta(t'-t)\,G^{<\,\mu\nu}_{ab,ba}(t,t') \tag{4.54}$$

with the usual definition of the θ-function. But it is not ended here when the expression needed is such that defined by the Fourier transform in the 4-momentum space like in (2.65). So, the new time-ordered correlator function is given by

$$G^{F\,\mu\nu}_{ab,ba}(k_0) = i\int \frac{dk_0'}{2\pi}\,\frac{\rho^{\mu\nu}_{ab,ba}(k_0')}{k_0-k_0'} \;+ f_{ab}(k_0)\,\rho^{\mu\nu}_{ab,ba}(k_0)\,. \tag{4.55}$$

This last expression is useful to get the Time-reversal symmetry formula for the time-ordered correlator function. Applying (2.80) and considering that the correlator must be a real-valued function, then it is satisfy the following property

$$G^{F\,\mu\nu}_{ab,ba}(-k_0,-\vec{k}) = G^{F\,\nu\mu}_{ba,ab}(k_0,\vec{k})\,. \tag{4.56}$$

The theoretical framework developed so far, states the prerequisites to find the contributions of the gamma functions in the rate of change equations (4.42). Starting from the Feynman contributions in (4.38) for the quark current components and using the previous symmetry property, one has

$$\begin{aligned}
\gamma_{us\to ud} &= \eta_{\mu\nu}\,\eta_{\alpha\beta}\int\frac{d^4q}{(2\pi)^4}\left[G^{F\,\mu\alpha}_{us,su}(q)G^{F\,\nu\beta}_{du,ud}(-q) - G^{F\,\mu\alpha}_{su,us}(q)G^{F\,\nu\beta}_{ud,du}(-q)\right]\\
&= \eta_{\mu\nu}\,\eta_{\alpha\beta}\int\frac{d^4q}{(2\pi)^4}\,G^{F\,\mu\alpha}_{us,su}(q)G^{F\,\nu\beta}_{du,ud}(-q) - \int\frac{d^4q}{(2\pi)^4}\,G^{F\,\mu\alpha}_{su,us}(q)G^{F\,\nu\beta}_{ud,du}(-q)\\
&= \eta_{\mu\nu}\,\eta_{\alpha\beta}\int\frac{d^4q}{(2\pi)^4}\,G^{F\,\mu\alpha}_{us,su}(q)G^{F\,\nu\beta}_{du,ud}(-q) - \int\frac{d^4q}{(2\pi)^4}\,G^{F\,\mu\alpha}_{su,us}(-q)G^{F\,\nu\beta}_{ud,du}(q)\\
&= \eta_{\mu\nu}\,\eta_{\alpha\beta}\int\frac{d^4q}{(2\pi)^4}\,G^{F\,\mu\alpha}_{us,su}(q)G^{F\,\nu\beta}_{ud,du}(q) - \int\frac{d^4q}{(2\pi)^4}\,G^{F\,\mu\alpha}_{us,su}(q)G^{F\,\nu\beta}_{ud,du}(q)\\
&= 0\,.
\end{aligned} \tag{4.57}$$

Notice that it was taken a flexible notation, ignoring the metric terms $\eta_{\mu\nu}\,\eta_{\alpha\beta}$ and because the correlators depend on a 4-momentum argument. One can separate the integral in two parts given by (dq^0) and the other for (d^3q), as the same for the argument of the correlators where $(q\to q^0,\vec{q})$ but the result shown in (4.57) holds, considering the following property satisfied by the Wightman correlators

$$G^{>\,\mu\nu}_{ab,ba}(t,\vec{x}\,) = G^{<\,\nu\mu}_{ba,ab}(-t,-\vec{x}\,)\,. \tag{4.58}$$

Now one has to look for the Wightman correlators contribution, associated to the gamma function described above in the second term of (4.40). For this purpose, it is convenient to use the spectral function equation in (4.52), considering

$$G^{<\,\mu\nu}_{ba,ab}(k_0) = f_{ba}(k_0)\,\rho^{\mu\nu}_{ba,ab}(k_0) \tag{4.59}$$

where one must remember that the function f_{ba} only have the time-component dependence $f_{ba} \to f_{ba}(k_0)$ and the spectral function will be $\rho^{\mu\nu}_{ba,ab} \to \rho^{\mu\nu}_{ba,ab}(k_0, \vec{k}\,)$. With such details one is now able to find the contribution given by the Wightman correlators, focusing on the time dependence by

$$\begin{aligned}
\gamma^{<}_{us\to ud} &= \eta_{\mu\nu}\,\eta_{\alpha\beta}\int \frac{d^4q}{(2\pi)^4}\left[G^{<\mu\alpha}_{us,su}(q)G^{<\nu\beta}_{du,ud}(-q) - G^{<\mu\alpha}_{su,us}(q)G^{<\nu\beta}_{ud,du}(-q)\right] \\
&= \eta_{\mu\nu}\,\eta_{\alpha\beta}\int \frac{d^4q}{(2\pi)^4}\Big(f_{us}(q^0)\,\rho^{\mu\alpha}_{us,su}(q)\,f_{du}(-q^0)\,\rho^{\nu\beta}_{du,ud}(-q) \\
&\qquad\qquad -f_{su}(q^0)\,\rho^{\mu\alpha}_{su,us}(q)\,f_{ud}(-q^0)\,\rho^{\nu\beta}_{ud,du}(-q)\Big) \\
&= \eta_{\mu\nu}\,\eta_{\alpha\beta}\int \frac{d^4q}{(2\pi)^4}\Big(f_{us}(q^0)\,\rho^{\mu\alpha}_{us,su}(q)\,f_{du}(-q^0)\,\rho^{\nu\beta}_{du,ud}(-q) \\
&\qquad\qquad -f_{su}(-q^0)\,\rho^{\mu\alpha}_{su,us}(-q)\,f_{ud}(q^0)\,\rho^{\nu\beta}_{ud,du}(q)\Big) \\
&= \eta_{\mu\nu}\,\eta_{\alpha\beta}\int \frac{d^4q}{(2\pi)^4}\Big(f_{us}(q^0)\,\rho^{\mu\alpha}_{us,su}(q)\,(1+f_{ud}(q^0))\,\rho^{\nu\beta}_{ud,du}(q) \\
&\qquad\qquad -(1+f_{us}(q^0))\,\rho^{\mu\alpha}_{us,su}(q)\,f_{ud}(q^0)\,\rho^{\nu\beta}_{ud,du}(q)\Big) \\
&= \eta_{\mu\nu}\,\eta_{\alpha\beta}\int \frac{d^4q}{(2\pi)^4}\,\left(f_{us}(q^0) - f_{ud}(q^0)\right)\,\rho^{\mu\alpha}_{us,su}(q)\,\rho^{\nu\beta}_{ud,du}(q)\,.
\end{aligned} \tag{4.60}$$

Finally, one gets the expression for the rate of change for the d-quark current with only quark dependence in (4.41) by

$$\begin{aligned}
\partial_\mu\left\langle J^\mu_d\right\rangle = 4G_F^2\sin\theta_C^2\cos^2\theta_C\,\eta_{\mu\nu}\eta_{\alpha\beta} \\
\times\int \frac{d^4q}{(2\pi)^4}\,\left(f_{us}(q^0) - f_{ud}(q^0)\right)\,\rho^{\mu\alpha}_{us,su}(q)\,\rho^{\nu\beta}_{ud,du}(q)\,.
\end{aligned} \tag{4.61}$$

By the comments on the introduction, and by direct comparison, the divergence of the particle density current is equivalent to the function ($\Gamma_{ud\to us}$) in (4.8) for the selected process. The result of such identification is given by

$$4G_F^2\,\sin\theta_C^2\cos^2\theta_C\,\gamma_{us\to ud} \approx \Gamma_{ud\to us} - \Gamma_{us\to ud} \tag{4.62}$$

with the γ-factor as seen in (4.60) with the Bose-Einstein distribution and the spectral function. Just to mention that one can check in beta equilibrium condition (4.2) that if ($\mu_u - \mu_s = \mu_u - \mu_d$), the Bose-Einstein function which depends on the chemical potential difference (4.53), losses such difference and both f_{us} and f_{ud} would be the same in (4.60) and for that reason the $\gamma_{us\to ud}$-function will vanish and then the rate of change in (4.61) will be zero, as it would be expected from beta equilibrium condition.

Previous discussions have addressed how fluctuations within the quark matter region of compact neutron stars can disrupt the beta equilibrium condition. When deviations occur from this equilibrium, the primary quantities that adjust are the chemical potentials. These changes are necessary to restore the system to its equilibrium state, following a small

perturbation around the equilibrium values during such processes.

$$\mu_a = \mu_a^{eq} + \delta\mu_a \,. \tag{4.63}$$

Now, one can introduce such perturbation formula in (4.61) which will only affect the Bose-Einstein function within the exponential factor. So, the γ- factor can be approximated by

$$\gamma_{us\to ud} \approx (\delta\mu_s - \delta\mu_d)\,\Lambda_{us\to ud}\,, \tag{4.64}$$

using (4.63) in (4.53) with Λ representing the integral factor with the spectral functions involved. Its expression will take place after the procedure described above in which the remnant part, after the change on the Bose-Einstein functions, is given by

$$\Lambda_{us\to ud} = \eta_{\mu\nu}\eta_{\alpha\beta}\int \frac{d^4q}{(2\pi)^4}\frac{\rho^{eq\,\mu\alpha}_{us,su}(q)\rho^{eq\,\nu\beta}_{ud,du}(q)}{4T\sinh^2\left(\frac{q^0+\Delta\mu_{sd}^{eq}}{2T}\right)}. \tag{4.65}$$

where $\Delta\mu_{sd}^{eq}$ is defined by the difference of equilibrium chemical potential values for the quarks involved ($\Delta\mu_{sd}^{eq} = \mu_s^{eq} - \mu_d^{eq}$).

After all this theoretical framework, one can finally get, by the correspondence between (4.62) and the linear expansion in (4.8) for the down and strange quarks process, the expression for the Lambda coefficients after going through the calculation in holography by

$$\lambda_{ds} \approx 4G_F^2 \sin^2\theta_C \cos^2\theta_C\ \Lambda_{us\to ud}\,. \tag{4.66}$$

The next chapter will deal with the holographic model dual calculation of this lambda coefficient.

4.4 Bulk viscosity from electroweak processes

Bulk viscosity, also known as volume or dilatational viscosity, encompasses a comprehensive set of equations governing the dynamics of viscous fluids. It quantifies the resistance to compression and expansion within a fluid element in a given system. In neutron stars, changes in volume due to radial pulsations lead to fluctuations in baryon density [3].

Bulk viscosity in neutron stars is predominantly influenced by chemical re-equilibration, particularly through flavor-changing processes. Electroweak reactions contribute significantly to bulk viscosity because their timescales align with the oscillation periods of the star, which are typically comparable to its rotation period [4].

As discussed earlier, bulk viscosity is intricately linked to electroweak rates and beta equilibrium conditions, which are the focus of this and the subsequent sections. This part aims to outline this relationship, emphasizing how the reaction rates of weak processes and externally induced volume oscillations are computed to determine the bulk viscosity of dense hadronic matter [4].

One has to notice how the baryon or better the quark density changes from its equilibrium

value by

$$n_{B/q} \simeq n_{B/q}^{eq} + \Delta n_{B/q} \sin(\omega t)\,. \tag{4.67}$$

considering that such deviation value is defined by a small homogeneous oscillation of period $\tau = 2\pi/\omega$. By the existent relation between density and volume, such change in the density number for a given quark flavor can be seen as a change in the specific volume like $V_B = n_B^{-1}$ and then if one takes the temporal derivative, both quantities will be coupled by

$$\begin{aligned} \frac{dV_B}{dt} &= -(n_B)^{-2}\,\frac{dn_B}{dt} \\ &\simeq -\frac{\omega\,\Delta n_B}{(n_B^{eq})^2}\,\cos(wt) + \ldots \end{aligned} \tag{4.68}$$

where by definition, the equilibrium density number doesn't change in time ($dn_B/dt = 0$). The baryon density also satisfy the continuity equation

$$\dot{n}_B + \nabla \cdot (n_B \vec{v}) = 0 \tag{4.69}$$

where the divergence of the velocity of the fluid is then given by the derivative on the baryon density in (4.68) as

$$\nabla \cdot \vec{v} = -\frac{\dot{n}_B}{n_B} \simeq -\omega \frac{\Delta n_B}{n_B^{eq}} \cos(\omega t)\,. \tag{4.70}$$

The bulk viscosity coefficient (ζ) is responsible for dissipation in the presence of a nonzero divergence [4], like the previous equation. So, if one takes the dissipated energy ($\dot{\mathcal{E}}$) of the star rotation, averaging over an oscillation period, the equation will be dependent on the baryon density by

$$\begin{aligned} \left\langle \dot{\mathcal{E}} \right\rangle_\tau &= -\frac{\zeta}{\tau} \int_0^\tau dt (\nabla \cdot \vec{v})^2 \\ &\simeq -\frac{\zeta \omega^2}{2} \left(\frac{\Delta n_B}{n_B^{eq}} \right)^2 \end{aligned} \tag{4.71}$$

There is also another way to evaluate the energy loss with an expression that involve the mechanical work induced by pressure oscillations which is

$$\begin{aligned} \left\langle \dot{\mathcal{E}} \right\rangle_\tau &= \frac{n_B^{eq}}{\tau} \int_0^\tau dt\, P(t) \frac{dV_B}{dt} \\ &= \frac{1}{n_B^{eq}} \left(-\omega \Delta n_B\right) \int_0^\tau dt\, P(t)\, \cos(\omega t)\,. \end{aligned} \tag{4.72}$$

With equation (4.68) governing the volume derivative and the ability to describe pressure in terms of chemical potentials as thermodynamic variables within the context of the grand canonical potential, this theory necessitates expressing non-equilibrium pressure through deviations of chemical potentials via the Gibbs-Duhem equation [17].

$$P = P_0 + \sum_a \frac{\partial P}{\partial \mu_a} \delta \mu_a\,. \tag{4.73}$$

The quark flavor density is another quantity that has an expansion due to the dependence on chemical potentials as thermodynamic variables by

$$n_a = n_a^0 + \sum_b \frac{\partial n_a}{\partial \mu_b} \delta\mu_b \,. \tag{4.74}$$

in which the derivative term is associated to the susceptibilities as

$$\chi_a{}^b = \frac{\partial n_a}{\partial \mu_b} \,. \tag{4.75}$$

If one considers the relationship between density and the rates in (4.5) and their linear expansion in (4.8), the time-dependent variations of the chemical potentials must be included in the pressure expansion (4.73) and in the number density (4.74). One can then incorporate these expressions into the dissipated energy formula to obtain the bulk viscosity coefficients, resulting in

$$\zeta = \frac{\lambda_1 A_1^2}{\omega^2 + \lambda_1^2 C_1^2} \tag{4.76}$$

where $\lambda_1 = \lambda_{ds}$ in (4.8) counts for the quark decay without leptonic contribution and the other factors will relate the susceptibilities and density number by

$$C_1 = \frac{1}{\chi_{dd}} + \frac{1}{\chi_{ss}} \,, \quad A_1 = \frac{n_d}{\chi_{dd}} - \frac{n_s}{\chi_{ss}} \,. \tag{4.77}$$

Chapter 5

Holographic calculation of the Electroweak Rate

This final chapter encompasses observables derived from gauge/gravity duality, particularly focusing on correlators of gauge-invariant operators. The computation of flavor currents commences with the action principle of the holographic QCD model, based on its field content. Subsequently, equations of motion are derived, followed by the introduction of the Schwarzschild-AdS metric to describe the asymptotic behavior of the radial coordinate. Solutions for the left gauge field are evaluated, which are connected to the two-point functions of the flavor currents through the spectral function.

5.1 Holographic Model of Correspondence and Action Term

As it was seen in Section 3 for the scalar field example in the AdS/CFT correspondence, specifically in (3.68), the expansion for a scalar field in the bulk theory must satisfy the IR and UV boundary conditions, which for this model will be identified with a quark mass matrix as the source and another matrix composed by the quark condensate in the IR condition satisfying the vacuum expectation value of the dual scalar field.

4D: $\mathcal{O}(x)$	5D: $\phi(x,z)$	p	Δ	$(m_5)^2$
$\bar{q}_L\gamma^\mu t^a q_L$	$A^a_{L\mu}$	1	3	0
$\bar{q}_R\gamma^\mu t^a q_R$	$A^a_{R\mu}$	1	3	0
$\bar{q}^\alpha_R q^\beta_L$	$(2/z)X^{\alpha\beta}$	0	3	-3

Figure 5.1: Operator/Field correspondence in AdS_5/CFT_4 space bottom-up model [20]

With the mass dimension in figure 5.1, for the quark condensate and using this value in the scaling dimension and mass equation (3.60) one gets two solutions for Δ through (3.67), which give the exponential number for the holographic dimension (z). So, the scalar field

would be given by its asymptotic expansion by

$$X_{ab} \sim \frac{1}{2} M_{ab} z + \frac{1}{2} \Sigma_{ab} z^3 \,. \tag{5.1}$$

where, as it was said, (M) will be the quark mass matrix given as

$$M = \begin{pmatrix} m_u & 0 & 0 \\ 0 & m_d & 0 \\ 0 & 0 & m_s \end{pmatrix} \tag{5.2}$$

and (Σ) shall be considered as an input parameter of the model linked to the quark condensate of the form

$$\Sigma^{\alpha\beta} = \left\langle q^\alpha \, \bar{q}^\beta \right\rangle \,. \tag{5.3}$$

In addition, not only the scalar field correspondence is needed but also the model must include two additional massless gauge fields which are dual to the current operators $(\bar{q}_L \, \gamma^\mu \, q_L)$ and $(\bar{q}_R \, \gamma^\mu \, q_R)$. This correspondence was made by including the electroweak action term in (4.9) into the holographic model, knowing that such is a double-trace term introduced by the boundary conditions described above. Assembling all these important quantities, the considered action for the flavor fields in a commonly used Holographic QCD model will be given by

$$S_f = \frac{1}{16\pi G_5} \frac{N_f}{N_c} \int dx^5 \sqrt{-g} \, Tr \left(-|DX|^2 + 3|X|^2 - \frac{1}{4g_s^2} \left(F_{(R)}^2 + F_{(L)}^2 \right) \right) \tag{5.4}$$

where the complex scalar field (X) is dual to the quark bi-linear in the bi-fundamental representation and also couples to the $(A_{(L)})$ and $(A_{(R)})$ gauge fields in representation $SU(N_f)_L \times SU(N_f)_R$ with flavor fields fixed by $(N_f = 3)$, through the covariant derivative as

$$\begin{aligned} D^N X &= \partial^N X - i A^N_{(L)} X + i X A^N_{(R)} \\ \left(D^N X \right)^\dagger &= \partial^N X^\dagger + i X^\dagger A^N_{(L)} - i A^N_{(R)} X^\dagger \end{aligned} \tag{5.5}$$

and the field strength defined as usual in a Yang-Mills theory by

$$F_{AB} = \partial_A \, A_B - \partial_B \, A_A + i \left[A_A \, , \, A_B \right] . \tag{5.6}$$

Just to simplify the notation in subsequent calculations, one shall refer the left and right components of the gauge field by the association $(A^N_{(R)} = R^N)$ and $((A^N_{(L)} = L^N))$.

5.2 Equations of Motion and Field Fluctuations

In classical field theory, the action principle reveals a coupling between gravity and the field content of the theory. For the model of interest, the action is given by (5.4), incorporating dynamical fields associated with the AdS metric, scalar field, and gauge fields. Using the

constructed holographic dictionary, one can derive the expression for the retarded correlator of the two-point function of left flavor currents from the solutions to the fluctuation equations. Applying the variational principle to these fields and neglecting the back-reaction onto the metric, the Euler-Lagrange formalism yields the equations of motion

$$\begin{aligned}
\frac{1}{\sqrt{-g}}\, D_M \left(\sqrt{-g}\, F_{(R)}^{MN}\right) &= j_{(R)}^N \\
\frac{1}{\sqrt{-g}}\, D_M \left(\sqrt{-g}\, F_{(L)}^{MN}\right) &= j_{(L)}^N \\
\frac{1}{\sqrt{-g}}\, D_M \left(\sqrt{-g}\, D^M X\right) &= -3X
\end{aligned} \tag{5.7}$$

where the currents at the right hand side are expressions in terms of the scalar field and its complex conjugate by the following equations

$$\begin{aligned}
j_{(R)}^N &= i\left(\left(D^N X\right)^\dagger\, X - X^\dagger\, \left(D^N X\right)\right) \\
j_{(L)}^N &= i\left(\left(D^N X\right)\, X^\dagger - X\, \left(D^N X\right)^\dagger\right)
\end{aligned} \tag{5.8}$$

so one can notice the Hermitian rule in ($j_{(R,L)}^N = (j_{(R,L)}^N)^\dagger$). Consider now a simple model of gravity in five dimensional space hold by the AdS_5/QFT_4 approach. These models are conjectured to be dual to confining gauge theories [9], less motivated by string theory also called bottom-up models.

From this point, a massless quark formalism will be developed by the reasons exposed in the previous section and by this reason the right hand side of (5.7) will vanish.

$$\frac{1}{\sqrt{-g}}\, \partial_M \left(\sqrt{-g}\, F_{(L)}^{MN}\right) = 0\,. \tag{5.9}$$

The Einstein action principle for a gravitational theory with negative cosmological constant is given by

$$S_c = \frac{1}{16\pi G_5} \int d^5x\, \sqrt{-g}\, (R - 2\Lambda) \tag{5.10}$$

where the Schwarzschild black hole is a solution for the field equations. Then, it is possible to consider the asymptotic regime of such space when the radial component tends to infinity, such that the Schwarzschild solution tends to AdS spacetime. By combining this with the approximation ($N_f/N_c \ll 1$), the solution for the field equations is given by the Schwarzschild-AdS metric

$$ds^2 = \frac{1}{z^2}\left(\frac{dz^2}{f(z)} - f(z)dt^2 + d\vec{x}^2\right); \quad f(r) = 1 - \frac{z^4}{z_H^4} \tag{5.11}$$

where r_H is known as the horizon radius. As it was shown in (4.28), the time component of the gauge field is related to the equilibrium value of the chemical potential for each quark flavor. In order to make a correspondence with the treatment of those chemical potentials out of the beta equilibrium as well as the perturbation on the electroweak term ($\Delta\mathcal{L}_{EW}$),

through some fluctuations in quark density, it is now needed to examine the fluctuations on the field content in the holographic model around spatially homogeneous equilibrium values expanding such fluctuations in Fourier modes as

$$\begin{aligned} R_\mu(x,z) &\to \int \frac{d\omega}{2\pi} \int \frac{d^3k}{(2\pi)^3}\, e^{-i\omega t + i\vec{k}\cdot\vec{x}}\, R_\mu(\omega,\vec{k},z) \\ L_\mu(x,z) &\to \int \frac{d\omega}{2\pi} \int \frac{d^3k}{(2\pi)^3}\, e^{-i\omega t + i\vec{k}\cdot\vec{x}}\, L_\mu(\omega,\vec{k},z) \end{aligned} \tag{5.12}$$

where one can notice the oscillatory feature of the fluctuations, related to the compact neutron stars out of the beta equilibrium phenomena described in the previous section.

From now on, the solution to the fluctuation equations will be given in terms of the Fourier modes as mentioned in (5.12) and in the subsequent procedures. Let the following be the fluctuation equations for the left transverse gauge field in (5.11) by

$$\partial_z^2 (L_i)_b^a - \frac{1}{z}\,\partial_z (L_i)_b^a + (\omega^2 - \boldsymbol{k}^2)\,(L_i)_b^a = 0 \tag{5.13}$$

The solutions of such differential equation are given by Bessel functions depending on the existing relation between (ω) and $(\vec{k})$. If is satisfied the condition $(\omega^2 > \boldsymbol{k}^2)$, then the solutions are given by the sum of Bessel J and Bessel Y functions like

$$(L_i)_b^a = c_1\, z\, J_1\left[z\,\sqrt{\omega^2 - \boldsymbol{k}^2}\right] + c_2\, z\, Y_1\left[z\,\sqrt{\omega^2 - \boldsymbol{k}^2}\right] \tag{5.14}$$

but if the condition satisfied is $(\omega^2 < \boldsymbol{k}^2)$, then the solutions will be given by

$$(L_i)_b^a = c_1\, z\, I_1\left[z\,\sqrt{\boldsymbol{k}^2 - \omega^2}\right] + c_2\, z\, K_1\left[z\,\sqrt{\boldsymbol{k}^2 - \omega^2}\right]. \tag{5.15}$$

With the Bessel functions it is also possible to give solutions for the left transverse gauge potential with the two kinds of Hankel functions if $(\omega^2 > \boldsymbol{k}^2)$, by

$$(L_i)_b^{a\,(\alpha)} = C_i^{(\alpha)}\, z\, H_1^{(\alpha)}\left[\left|\omega^2 - \boldsymbol{k}^2\right|^{1/2} z\right]\, e^{-i\omega\, t + i\,\vec{k}\cdot\vec{x}} \tag{5.16}$$

where this set of solutions is related to the Fourier modes in (5.12) transformation. The complete set of solutions will be a superposition of plane waves in which on has to distinguish the ingoing and outgoing solutions. Taking the asymptotic form of the first and second kind Hankel functions when $(z \to \infty)$ one gets

$$\begin{aligned} H_1^{(1)}\left(\left|\omega^2 - \boldsymbol{k}^2\right|^{\frac{1}{2}} z\right) &\sim e^{i\left|\omega^2 - \boldsymbol{k}^2\right|^{1/2} z} \\ H_1^{(2)}\left(\left|\omega^2 - \boldsymbol{k}^2\right|^{\frac{1}{2}} z\right) &\sim e^{-i\left|\omega^2 - \boldsymbol{k}^2\right|^{1/2} z} \end{aligned} \tag{5.17}$$

from which one must notice for the first kind, inserted (5.16), the ingoing solution for $(\omega > 0)$ and the outgoing one for $(\omega < 0)$. The same must be done for the second kind where the opposite assignment takes place with the ingoing solution for $(\omega < 0)$ and the outgoing solution for $(\omega > 0)$. One is particularly interested in both ingoing solutions for the (5.16)

expression closing to the limit $(z \to \infty)$

$$(L_i)_b^{a\,(\alpha)} = \begin{cases} (L_i)_b^{a\,(1)}, & \omega > 0 \\ (L_i)_b^{a\,(2)}, & \omega < 0\,. \end{cases} \tag{5.18}$$

One can fix the value of the $C_i^{(\alpha)}$ coefficient in (5.16) so that the $(z\,H_1^{(\alpha)})$ solution will have the appropriate normalization proportional to the source. After this, both solutions for (5.18) will be given in terms of the second-order series expansion in (z) for the two kinds of Hankel functions and the source $\mathcal{E}_i$, resulting in

$$\begin{aligned} L_i^{(1)} &\approx \mathcal{E}_i \left(1 - \frac{1}{4}\,(\omega^2 - \boldsymbol{k}^2)\left(-1 + 2\gamma_E - i\pi + 2\log\frac{(\omega^2 - \boldsymbol{k}^2)^{1/2}}{2} + 2\log z\right) z^2\right) \\ L_i^{(2)} &\approx \mathcal{E}_i \left(1 - \frac{1}{4}\,(\omega^2 - \boldsymbol{k}^2)\left(-1 + 2\gamma_E + i\pi + 2\log\frac{(\omega^2 - \boldsymbol{k}^2)^{1/2}}{2} + 2\log z\right) z^2\right). \end{aligned} \tag{5.19}$$

with (γ_E) as the Euler constant.

It is easy to notice that both solutions only differ by the sign of the imaginary term according to the ingoing solutions in (5.18), so one can join them in a general expression with the sign function referred to the frequency by

$$L_i \approx \mathcal{E}_i \left(1 - \frac{1}{4}\,M^2\left(-1 + 2\gamma_E - i\pi\ \mathrm{sign}\,(\omega) + 2\log\frac{M}{2} + 2\log z\right) z^2\right). \tag{5.20}$$

where the quantity (M) was taken as $M = \left|\omega^2 - \boldsymbol{k}^2\right|^{1/2}$ just for simplicity.

5.3 Correlator Solutions and Final Calculation for the Rate

Once with the solutions for the left gauge field, one needs to compute the associated left current one-point function which is extracted from the boundary value of

$$\begin{aligned} \left\langle J^i_{q,\,L}\right\rangle &= -\frac{1}{16\pi G_5}\,\frac{N_f}{N_c}\,\lim_{z\to 0}\,\sqrt{-g}\,g^{ii}\,g^{zz}\,F_{zi} \\ &= -\frac{1}{16\pi G_5}\,\frac{N_f}{N_c}\,\lim_{z\to 0}\,\frac{1}{z}\,\left(\eta^{ii}\,\partial_z\,L_i\right). \end{aligned} \tag{5.21}$$

The next logical step is to compute the derivative for the solution in (5.20), which is given by

$$\partial_z\,L_i = -M^2\,z\left(\gamma_E + \log\frac{M}{2} + \log z - \frac{1}{2}\,i\pi\,\mathrm{sign}\,(\omega)\right)\mathcal{E}_i\,. \tag{5.22}$$

So, following the expression for the current in (5.21) one gets

$$\left\langle J^i_{q,L}\right\rangle = \frac{1}{16\pi G_5}\,\frac{N_f}{N_c}\,M^2\left(\gamma_E + \log\frac{M}{2} - \frac{1}{2}\,i\pi\,\mathrm{sign}\,(\omega)\right)\mathcal{E}^i\,. \tag{5.23}$$

The two-point function, which is the main focus of the model, is obtained by taking the

variation of the correlator in (5.23) respect to the source and due to the current conservation because there is no flavor symmetry braking term and then

$$\begin{aligned}\left\langle J^i_{q,L}\, J^j_{q,L}\right\rangle = \frac{1}{16\pi G_5}\,\frac{N_f}{N_c}\,M^2 \\ \times\left(\delta^{ij}-\frac{k^i\,k^j}{\boldsymbol{k}^2}\right)\left(\gamma_E+\log\frac{M}{2}-\frac{1}{2}\,i\pi\,\mathrm{sign}\,(\omega)\right)\,.\end{aligned} \tag{5.24}$$

It is possible to relate the previous expression with the retarded Green function ($G_R(\omega,\vec{k})$). This relationship is critical for obtaining the spectral functions needed in equation (4.65) to find the Λ-factor for the associated rate. As mentioned, the expression for the spectral function is given by the imaginary part of the retarded correlator, and then

$$\begin{aligned}\rho^{ij} &= -2\,Im\left(G_R^{i\,j}(\omega,\vec{k})\right) \\ &= \frac{1}{16G_5}\,\frac{N_f}{N_c}\,M^2\,\mathrm{sign}\,(\omega)\left(\delta^{ij}-\frac{k^i\,k^j}{\boldsymbol{k}^2}\right)\end{aligned} \tag{5.25}$$

where only the transverse component of the left currents contributes. This corresponds to an incomplete calculation for the correlator, in which not only the transverse components contribute but also the temporal and longitudinal ones. However, it is possible to derive the general expression using the current conservation theorem, as follows

$$\partial_\mu\,\langle J^\mu(x)\,J^\nu(0)\rangle = 0 \quad\rightarrow\quad q_\mu\,\langle J^\mu\,J^\nu\rangle = 0 \tag{5.26}$$

with ($q^\mu=(\omega,\vec{k})$) and then the two point function as well as its transverse component can both be written by

$$\begin{aligned}\langle J^\mu\,J^\nu\rangle &= \left(q^2\,\eta^{\mu\nu}-q^\mu\,q^\nu\right)\,\Pi(\omega,\vec{k}) \\ \langle J^i\,J^j\rangle_\perp &= \left(\delta^{ij}-\frac{k^i\,k^j}{\boldsymbol{k}^2}\right)\,G_R(\omega,\vec{k})\end{aligned} \tag{5.27}$$

so, one can find an equation that relates both Π and G_R terms through

$$\Pi(\omega,\vec{k}) = \frac{1}{q^2}\,G_R(\omega,\vec{k})\,. \tag{5.28}$$

All the theory developed so far is written in terms of the frequency and momentum positivity relation ($\omega^2>\boldsymbol{k}^2$). This is precisely the main contribution to the rates since the opposite relation ($\omega^2<\boldsymbol{k}^2$), which gives the solution (5.15), leads to a real-valued expression for the transverse component of the left gauge field, and then the spectral function will vanish. As seen in (5.28), the Π-function only differs from the retarded Green function by a constant factor so the general spectral function will vanish too. It is then necessary to provide a new equation for the spectral function that takes this difference into account,

which will be given by adding the unit step function as

$$\rho^{ij} \cong \frac{1}{16G_5}\,\frac{N_f}{N_c}\,\Theta(\omega^2-\boldsymbol{k}^2)\,M^2\,\text{sign}\,(\omega)\,\left(\delta^{ij}-\frac{k^i\,k^j}{\boldsymbol{k}^2}\right)\,. \tag{5.29}$$

Finally, based in (5.27) the expression for the spectral function is then the imaginary part in

$$\rho^{\mu\nu} = \left(q^2\,\eta^{\mu\nu}-q^\mu\,q^\nu\right)\times\left(-2\,\text{Im}\,\Pi(\omega,\boldsymbol{k})\right)\,\Theta(\omega^2-\boldsymbol{k}^2) \tag{5.30}$$

so the contracted form which is the contribution for the Λ-factor in (4.65) will be given by

$$\begin{aligned}\eta_{\mu\nu}\,\eta_{\alpha\beta}\,\rho^{\mu\alpha}(q)\rho^{\nu\beta}(q) &= \eta_{\mu\nu}\,\eta_{\alpha\beta}\,\left(q^2\,\eta^{\mu\alpha}-q^\mu\,q^\alpha\right)\,\left(q^2\,\eta^{\nu\beta}-q^\nu\,q^\beta\right)\\ &\quad\times\left(-2\,\text{Im}\,\Pi(\omega,\boldsymbol{k})\right)^2\,\Theta(\omega^2-\boldsymbol{k}^2)^2\\ &= 9\,(q^2)^2\,\left(-2\,\text{Im}\,\Pi(\omega,\boldsymbol{k})\right)^2\,\Theta(\omega^2-\boldsymbol{k}^2)^2\\ &= 36\,\left(\text{Im}\,G_R(\omega,\vec{k})\right)^2\,\Theta(\omega^2-\boldsymbol{k}^2)^2\end{aligned} \tag{5.31}$$

where one can introduce the expression (5.25) getting

$$\eta_{\mu\nu}\,\eta_{\alpha\beta}\,\rho^{\mu\alpha}(q)\rho^{\nu\beta}(q) = \frac{9}{256}\,\frac{1}{G_5^2}\,\left(\frac{N_f}{N_c}\right)^2\,(\omega^2-\boldsymbol{k}^2)^2\,\Theta(\omega^2-\boldsymbol{k}^2)^2 \tag{5.32}$$

As mentioned before, this last expression must be introduced in (4.65) where the difference between the involved chemical potentials in equilibrium vanish, and then to obtain the rates the computation was made using the Mathematica software [21], resulting in the following value:

$$\begin{aligned}\Lambda_{us\to ud} &= \eta_{\mu\nu}\eta_{\alpha\beta}\int\frac{d^4q}{(2\pi)^4}\frac{\rho^{eq\,\mu\alpha}_{us,su}(q)\rho^{eq\,\nu\beta}_{ud,du}(q)}{4T\sinh^2\left(\frac{q^0+\Delta\mu^{eq}_{sd}}{2T}\right)}\\ \Lambda_{us\to ud} &= \frac{9}{256}\,\frac{1}{G_5^2}\,\left(\frac{N_f}{N_c}\right)^2\,\frac{\pi}{T\,(2\pi)^4}\,\int_{-\infty}^{\infty}d\omega\,\int_0^{\infty}dk\,k^2\,\frac{(\omega^2-k^2)^2\,\Theta(\omega^2-k^2)}{\sinh^2\frac{\omega}{2T}}\\ \Lambda_{us\to ud} &= \frac{9}{256}\,\frac{1}{G_5^2}\,\left(\frac{N_f}{N_c}\right)^2\,\frac{1}{T\,(2\pi)^3}\,\int_0^{\infty}d\omega\,\int_0^{|\omega|}dk\,k^2\,\frac{(\omega^2-k^2)^2}{\sinh^2\frac{\omega}{2T}}\\ \Lambda_{us\to ud} &= \frac{27\,\zeta[7]}{4\pi^3\,G_5^2}\,\left(\frac{N_f}{N_c}\right)^2\,T^7\,.\end{aligned} \tag{5.33}$$

Finally the Lambda coefficient in (4.66) will be approximated by

$$\begin{aligned}\lambda_{ds} &\approx 4G_F^2\sin^2\theta_C\cos^2\theta_C\,\Lambda_{us\to ud}\\ \lambda_{ds} &\approx \frac{27\,\zeta[7]}{\pi^3\,G_5^2}\,\left(\frac{N_f}{N_c}\right)^2\,G_F^2\,\sin^2\theta_C\,\cos^2\theta_C\,T^7\,.\end{aligned} \tag{5.34}$$

Then, the rates difference in (4.8) for the calculation of quark density change according to

(4.5) will give the following relation

$$\begin{aligned} -\frac{dn_d}{dt} &= \frac{dn_s}{dt} = \Gamma_{ud\to us} - \Gamma_{us\to ud} = \lambda_{ds}\,(\mu_s - \mu_d) \\ -\frac{dn_d}{dt} &= \frac{dn_s}{dt} \approx \frac{27\,\zeta[7]}{\pi^3\,G_5^2}\left(\frac{N_f}{N_c}\right)^2 G_F^2\,\sin^2\theta_C\,\cos^2\theta_C\,T^7\,. \end{aligned} \tag{5.35}$$

For that purpose, this dissertation has accomplished the development of an expression for the equilibrium rates, which turn out to be only temperature-dependent.

Chapter 6

Conclusions and Further Aims

This work has been done with the perspective of gaining insight into quantum chromodynamics for systems in which perturbative methods are not suitable. Holography was proposed as a new method to perform calculations in such regimes, focusing on physical systems that meet these conditions, such as compact neutron stars with high baryonic density.

The theoretical background involved here contains a wide variety of tools and concepts related to physical systems. Seeing all of them naturally come together to calculate physical quantities present in real environments, such as neutron stars, is highly interesting. Especially when such different theories as QFT and gravity are brought closer by the Maldacena conjecture, allowing one to compute quantities on both sides of the holographic correspondence.

The neutron star condition known as beta equilibrium was examined through the density changes in quark flavors and the calculation of such changes using the holographic model presented here. One should notice from the final expression in (5.35) that the quark density changes depend only on temperature, with other factors being constants of the model involved.

There are reasons to conduct further analysis in the developed work, given that certain approximations were made, such as massless quarks and low-temperature behavior of the spectral function. These hypotheses can be altered by introducing mass into the holographic model in the QFT dual, and also by incorporating semi-leptonic processes as discussed in Section 4. These processes involve the non-conservation of electroweak currents due to flavor symmetry breaking, as indicated by the Ward identities, and the associated rates affecting the leptonic density change.

In [4], a result for the lambda factor is provided in (5.34), where the authors consider non-leptonic processes and also express the quark number density as a linear expansion. The results presented there depend on the chemical potential and also on the square of the temperature, which contrasts with the calculation in this work. Therefore, this work can be extended by possibly examining how to calculate (5.34) considering the back-reaction of the charge density.

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