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Black holes. Theory and construction

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Contents

1 Introduction

General relativity (GR), formulated by Albert Einstein in 1915, revolutionized our understanding of gravity by describing it as the curvature of spacetime caused by mass and energy, replacing the Newtonian concept of gravity as a force between masses with a geometric interpretation: massive objects cause spacetime to curve, and this curvature dictates the motion of objects.

One of the most fascinating predictions of GR is the existence of black holes (BH), regions of spacetime where the gravitational field is so strong that nothing, not even light, can escape. The properties of the BH, namely charge, mass and spin, give rise to different conditions which we will analyze and review. The study of such phenomena has provided deep insights into the nature of gravity, quantum mechanics, and thermodynamics, prompting profound questions about the fundamental nature of spacetime, information, and the ultimate fate of matter.

While GR successfully describes the macroscopic behavior of gravity, it does not integrate well with the principles of quantum mechanics that govern the other fundamental forces of nature. This limitation has driven physicists to search for a more comprehensive theory that can unify gravity with the other forces. That theory is supergravity, which extends GR by incorporating the principles of supersymmetry. An important property is that, in such theories, each particle has a superpartner with differing spin properties.

We proceed as follows. First we make a review about the main topics of the theory, the metric, geodesics to end talking about maximally symmetric spaces. Then we introduce a new formalism, the p-forms, which makes both the theory and computations easier, to end up with the vielbein formalism and its relation with spinors. This covers section 2. In section 3 we review everything related to BHs, with more emphasis on the Reissner–Nordström (RN) type, since we will work with them later in section 5.

Section 4 covers a brief introduction to supergravity, starting from the free Rarita-Schwinger field and ending with $\mathcal{N} = 2$ minimal gauge supergravity, where we build the framework around the extreme RN BH, since its special stability makes it suitable for computations. Last in this section we review two of the simplest $\mathcal{N} = 2$ supergravity solutions.

And in section 5 we classify the supersymmetric RN solutions, explicitly constructing the Killing spinors in all cases, with special emphasis on intermediate calculations. The gauged version of $\mathcal{N} = 2$ supergravity, where the cosmological constant Λ is necessarily negative, provides the natural framework for analyzing supersymmetry. The easiest supersymmetry case is reviewed in subsection 5.3. In subsection 5.4, another supersymmetric solution is described, analogous to the extreme RN BH in flat space. When Λ is nonzero, the minimal coupling of the Maxwell field to the gravitini disrupts the duality symmetry between electric and magnetic fields, resulting in supersymmetry selecting the purely electric solution. The last class of supersymmetric RN solutions, referred to as "cosmic monopoles" and discussed in subsection 5.5, have no flat-space analogue because the magnetic charge becomes infinite in the flat space limit.

Last section contains a brief discussion about the results and some of their possible implications.

2 Review of General Relativity

2.1 What's General Relativity?

Newton's theory of gravity, which served us well for 250 years, needs replacing. We have realized that it fails when considering disturbances in the gravitational field. Imagine, for instance, that the Sun were to explode. What would we observe? For 8 minutes—the time it takes light to travel from the Sun to Earth—we would remain unaware of the impending doom, continuing to bask in sunlight. But what about Earth's orbit? If the Sun's mass distribution changes drastically, when would Earth start deviating from its elliptical path? Would this happen immediately, or would Earth continue its orbit for 8 minutes before sensing the change?

To answer that, we need special relativity. Since no signal can travel faster than the speed of light, Earth must continue its orbit for 8 minutes. But how is the information about the Sun's explosion transmitted? Does it travel at the speed of light? What medium carries this information?

Answering these questions requires rethinking our fundamental notions of space and time, leading us to some of modern physics' most profound ideas, including cosmology and black holes. The theoretical frame, more general than Special Relativity, where we encompass all of these ideas is General Relativity.

General relativity is the theory of space, time, and gravity. At its core, the theory posits that gravity is geometry: the effects we attribute to gravity result from the bending and warping of spacetime. This principle applies to phenomena ranging from falling objects to orbiting planets, and even to the motion of the cosmos on a grand scale.

We can view it as the high-energy and curvature Newton's gravity generalization, and it is that, whatever theory we come up to, if it's valid at high energies (usually that's a synonym of non-zero curvature), it has to reproduce the result of Newton's gravity theory, the first and most basic intuition about gravity and its effects.

2.2 Geodesics

Geodesics are one of the key topics in GR, since they are the path followed by a test particle, one on which no external forces are acting. More technically, they are the generalization to general manifolds of the notion of straight lines in flat space. To define them, first we have to introduce the concept of parallel transport.

2.2.1 Parallel transport

Given a curve γ defined in a manifold

$$\gamma : (a,b) \in \mathbb{R} \longrightarrow \mathcal{M} \tag{1}$$

$$\lambda \longrightarrow \gamma(\lambda) \tag{2}$$

which we take to the calculus world (the one we know how to calculate) through

$$\Psi \circ \gamma : (a,b) \in \mathbb{R} \longrightarrow \mathbb{R}^n \tag{3}$$

$$\lambda \longrightarrow x^\mu(\lambda) \tag{4}$$

In flat-spacetime, if a vector V^μ fulfills

$$\frac{dV^\mu}{d\lambda} = \frac{dx^\nu}{d\lambda}\frac{\partial V^\mu}{\partial x^\nu} = 0 \tag{5}$$

we say it's constant along γ. We say then that a vector is *paralleled transported* along the curve γ. If (5) $\neq 0$, we get a notion of the change of a vector field along a curve (compared to the parallel-transported vector). In curved spacetimes, parallel transport depends on the followed path. Now we can generalize this concept onto differentiable manifolds

$$\frac{\mathcal{D}V^\mu}{\mathcal{D}\lambda} = \frac{dx^\nu}{d\lambda}\nabla_v V^\mu = 0 \tag{6}$$

Now we can define strictly what a geodesic is. It's defined as a curve that transports parallel its own tangent vector. Since the tangent vector to the γ curve is (in a coordinate basis)

$$T = T^\mu \partial_\mu = \frac{dx^\mu}{d\lambda}\partial_\mu \tag{7}$$

the geodesic equation reads

$$\begin{aligned}\frac{D}{D\lambda}\frac{dx^\mu}{d\lambda} &= \frac{dx^\nu}{d\lambda}\nabla_\nu \frac{dx^\mu}{d\lambda} \\ &= \frac{dx^\nu}{d\lambda}\left(\frac{\partial}{\partial x^\nu}\frac{dx^\mu}{d\lambda}\right) + \frac{dx^\nu}{d\lambda}\Gamma^\mu_{\nu\rho}\frac{dx^\rho}{d\lambda} \\ &= \frac{d^2x^\mu}{d\lambda^2} + \Gamma^\mu_{\nu\rho}\frac{dx^\nu}{d\lambda}\frac{dx^\rho}{d\lambda} = 0\end{aligned} \tag{8}$$

A geodesic then gives us the *straightest possible curve*, tightly related concept to the *shortest possible path* between two points. How's that? Let's imagine the path a particle follows from point A to B. Since it's a particle (it's massive), its spacetime interval must be smaller than zero.

$$ds^2 = -g_{\mu\nu}\, dx^\mu\, dx^\nu = -d\tau^2 < 0 \tag{9}$$

where the τ parameter is the proper time, the one measured in the particle's reference frame. For a curve parameterised as $x^\mu(a) = A$ and $x^\mu(b) = B$

$$\frac{d\tau}{d\lambda} = \sqrt{-g_{\mu\nu}\frac{dx^\mu}{d\lambda}\frac{dx^\nu}{d\lambda}} = \sqrt{-g_{\mu\nu}\dot{x}^\mu\dot{x}^\nu} \tag{10}$$

where the total "distance" covered by the particle is given by the action

$$S[x(\lambda)] \equiv \int d\tau = \int_a^b d\lambda\sqrt{-g_{\mu\nu}\dot{x}^\mu\dot{x}^\nu} \tag{11}$$

If we extremize it under $\delta x^\mu(\lambda)$, we would find the the EOM are in fact the geodesic equation.

2.3 The metric $g_{\mu\nu}$

First of all, the convention used for the Minkowski flat-space metric is

$$\eta = diag(-1, 1, 1, 1) \tag{12}$$

in $D = 4$. In a nutshell, the metric is just a symmetric (0,2)-rank tensor field $g_{\mu\nu}(x)$ accounting for the form of spacetime. For further purposes, it's worth defining the inner product of two contravariant vectors $U^\mu(x)$ and $V^\nu(x)$ is $g_{\mu\nu}(x)U^\mu(x)V^\nu(x)$, which is a scalar field. The metric its usually taken to be non-singular, being the tool you use when you need to lower or raise an index, like $V_\mu(x) = g_{\mu\nu}(x)V^\nu(x)$, providing a natural isomorphism between the spaces of contravariant and covariant vectors and tensors.

Mathematically, the metric or inner product on a real vector space V is a non-degenerate bilinear map from $V \otimes V \to \mathbb{R}$. The inner product must satisfy the following properties:

1. *bilinearity*, $(u, c_1v_1 + c_2v_2) = c_1(u, v_1) + c_2(u, v_2)$ and $(c_1v_1 + c_2v_2, u) = c_1(v_1, u) + c_2(v_2, u)$;
2. *non-degeneracy*, if $(u, v) = 0$ for all $v \in V$, then $u = 0$;
3. *symmetry*, $(u, v) = (v, u)$.

It is convenient to summarize the properties of the metric by the line element

$$ds^2 = g_{\mu\nu}(x)dx^\mu dx^\nu. \tag{13}$$

which must also satisfy

$$g^{\mu\rho}g_{\rho\nu} = g_{\nu\rho}g^{\rho\mu} = \delta^\mu_\nu \tag{14}$$

As we shall see, the metric tensor contains all the information we need to describe the curvature of the manifold (at least in what is called Riemannian geometry).

A metric is characterized by its signature, which is the number of positive and negative eigenvalues it has. If all of the signs are positive, the metric is called *Euclidean* or *Riemannian* (or just positive definite), while if there is a single minus it is called *Lorentzian* or *pseudo-Riemannian*, and any metric with some +1's and some −1's is called indefinite. (So the word Euclidean sometimes means that the space is flat, and sometimes doesn't, but it always means that the canonical form is strictly positive; this is just a mess what we have to deal with.) The spacetimes of interest in GR have Lorentzian metrics.

2.4 Fields in GR

In GR, every function we work with can be classified into a general type of mathematical object, a tensor field, which is nothing more than a bunch of scalar functions related among them and numbered with indices, all put together.

A general form of a tensor is

$$T = T^{\mu_1\cdots\mu_k}{}_{\nu_1\cdots\nu_l} E_{\mu_1} \otimes \cdots \otimes E_{\mu_k} \otimes E^{\nu_1} \otimes \cdots \otimes E^{\nu_l} \tag{15}$$

We will usually take the shortcut of denoting the tensor T by its components $T^{\mu_1\cdots\mu_k}{}_{\nu_1\cdots\nu_l}$. A particularly useful basis is a corodinate basis $E_\mu = \partial_\mu$ and $E^\mu = dx^\mu$.

The action of the tensors on a set of vectors and dual vectors is defined as

$$T(\omega^{(1)}, \ldots, \omega^{(k)}, V^{(1)}, \ldots, V^{(l)}) = T^{\mu_1\cdots\mu_k}{}_{\nu_1\cdots\nu_l}\omega^{(1)}_{\mu_1}\cdots\omega^{(k)}_{\mu_k}V^{(1)\nu_1}\cdots V^{(l)\nu_l}. \tag{1.50}$$

where the order of the indices is important, since the tensor might not act in the same way on its various arguments.

One of the most important tensorial features is how do they transform under a general coordinate transformation. The answer is just what you would expect from index placement

$$\bar{T}^{\mu_1\ldots\mu_m}{}_{\nu_1\ldots\nu_n} = \left(\prod_{p=1}^{m}\frac{\partial x'^{\mu_p}}{\partial x^{\rho_p}}\prod_{q=1}^{n}\frac{\partial x^{\sigma_q}}{\partial x'^{\nu_q}}\right)T^{\rho_1\ldots\rho_m}{}_{\sigma_1\ldots\sigma_n} \tag{16}$$

Thus, each upper index gets transformed like a vector, and each lower index gets transformed like a dual vector.

2.5 Covariant derivative

It's natural to think of the notion of "curvature" as something that depends on the metric, but this turns out to be not quite true, or at least incomplete. In fact, there is one additional structure we need to introduce—a "connection"—which is characterized by the curvature. We will show how the existence of a metric implies a certain connection, whose curvature may be thought of as that of the metric.

The connection becomes necessary when we attempt to address the problem of the partial derivative not being a good tensor operator. We would like a *covariant derivative*; that is, an operator which reduces to the partial derivative in flat space with Cartesian coordinates but transforms as a tensor on an arbitrary manifold. The need is obvious; equations such as $\partial_\mu T^{\nu\rho} = 0$ have to be generalized to curved space somehow.

In flat space in Cartesian coordinates, the partial derivative operator ∂_μ is a map from (k,l) tensor fields to $(k,l+1)$ tensor fields, which acts linearly on its arguments and obeys the Leibniz rule on tensor products. All of this have to continue to be true in the more general situation we would now like to consider, and here's comes the problem. The map provided by the partial derivative depends on the coordinate system used. To solve it, we define a covariant derivative operator ∇ to perform the functions of the partial derivative, but in a way independent of coordinates. We require that ∇ be a map from (k,l) tensor fields to $(k,l+1)$ tensor fields which has these two properties:

1. linearity: $\nabla(T+S) = \nabla T + \nabla S$;
2. Leibniz (product) rule: $\nabla(T\otimes S) = (\nabla T)\otimes S + T\otimes(\nabla S)$.

If ∇ is going to obey the Leibniz rule, it can always be written as the partial derivative plus some linear transformation. Therefore we expect the covariant derivative to be the partial derivative plus a correction to make the result covariant.

As example, let's see what this means for the first non-trivial case, the covariant derivative of a vector

$$\nabla_\mu V^\nu = \partial_\mu V^\nu + \Gamma^\nu_{\mu\lambda}V^\lambda \tag{17}$$

where the $\Gamma^\nu_{\mu\lambda}$ are called **connection coefficients**. These components aren't tensors, since they do not transform in a proper way

$$\Gamma^{\nu'}_{\mu'\lambda'} = \frac{\partial x^\mu}{\partial x^{\mu'}}\frac{\partial x^\lambda}{\partial x^{\lambda'}}\frac{\partial x^{\nu'}}{\partial x^\nu}\Gamma^\nu_{\mu\lambda} - \underbrace{\frac{\partial x^\mu}{\partial x^{\mu'}}\frac{\partial x^\lambda}{\partial x^{\lambda'}}\frac{\partial^2 x^{\nu'}}{\partial x^\mu\partial x^\lambda}}_{\text{non-tensorial behaviour}} \tag{18}$$

This is not problematic, since we want the covariant derivative to be a tensor, not its separated components. Then, the connection coefficients purpose is to encode all of the information necessary to take the covariant derivative of a tensor of arbitrary rank. In the case of covectors, we have

$$\nabla_\mu\omega_\nu = \partial_\mu\omega_\nu - \Gamma^\lambda_{\mu\nu}\omega_\lambda \tag{19}$$

and the general case reads

$$\begin{aligned}\nabla_\sigma T^{\mu_1\mu_2\cdots\mu_k}_{\nu_1\nu_2\cdots\nu_l} = \partial_\sigma T^{\mu_1\mu_2\cdots\mu_k}_{\nu_1\nu_2\cdots\nu_l} \\ + \Gamma^{\mu_1}_{\sigma\lambda} T^{\lambda\mu_2\cdots\mu_k}_{\nu_1\nu_2\cdots\nu_l} + \Gamma^{\mu_2}_{\sigma\lambda} T^{\mu_1\lambda\cdots\mu_k}_{\nu_1\nu_2\cdots\nu_l} + \cdots \\ - \Gamma^{\lambda}_{\sigma\nu_1} T^{\mu_1\mu_2\cdots\mu_k}_{\lambda\nu_2\cdots\nu_l} - \Gamma^{\lambda}_{\sigma\nu_2} T^{\mu_1\mu_2\cdots\mu_k}_{\nu_1\lambda\cdots\nu_l} - \cdots\end{aligned} \tag{20}$$

For now we've defined everything about the connection but its explicit form. There are a large number of connections we could define on any manifold, with each of them implying a distinct notion of covariant differentiation. In general relativity this freedom is not a big concern, because it turns out that every metric defines a unique connection, which is the one used in GR. To start, we define the **torsion tensor**

$$T_{\mu\nu}{}^{\lambda} = \Gamma^{\lambda}_{\mu\nu} - \Gamma^{\lambda}_{\nu\mu} = 2\Gamma^{\lambda}_{[\mu\nu]} \tag{21}$$

valid for every connection. A connection symmetric in μ, ν indices is called *torsion-free*. This is one of the requirements we will always work with in GR. The other feature we can require is that

$$\nabla_\rho g_{\mu\nu} = 0 \tag{22}$$

By this, we are choosing a connection with the *metric compatibility* property, which, from all the possible connections, will make our computations easier. Then expanding (22) for three different index permutations

$$\nabla_\rho g_{\mu\nu} = \partial_\rho g_{\mu\nu} - \Gamma^{\lambda}_{\rho\mu} g_{\lambda\nu} - \Gamma^{\lambda}_{\rho\nu} g_{\mu\lambda} = 0 \tag{23}$$

$$\nabla_\mu g_{\nu\rho} = \partial_\mu g_{\nu\rho} - \Gamma^{\lambda}_{\mu\nu} g_{\lambda\rho} - \Gamma^{\lambda}_{\mu\rho} g_{\nu\lambda} = 0 \tag{24}$$

$$\nabla_\nu g_{\rho\mu} = \partial_\nu g_{\rho\mu} - \Gamma^{\lambda}_{\nu\rho} g_{\lambda\mu} - \Gamma^{\lambda}_{\nu\mu} g_{\rho\lambda} = 0 \tag{25}$$

Then

$$(23) - (24) - (25) \Longrightarrow \partial_\rho g_{\mu\nu} - \partial_\mu g_{\nu\rho} - \partial_\nu g_{\rho\mu} + 2\Gamma^{\lambda}_{\mu\nu} g_{\lambda\rho} = 0 \tag{26}$$

so

$$\Gamma^{\sigma}_{\mu\nu} = \frac{1}{2} g^{\sigma\rho} (\partial_\mu g_{\nu\rho} + \partial_\nu g_{\rho\mu} - \partial_\rho g_{\mu\nu}) \tag{27}$$

which we will refer to as the *Christoffel connection*.

2.6 Riemann curvature tensor

The curvature is quantified by the Riemann tensor, which is derived from the connection. It's defined as

$$R^{\rho}_{\sigma\mu\nu} = \partial_\mu \Gamma^{\rho}_{\nu\sigma} - \partial_\nu \Gamma^{\rho}_{\mu\sigma} + \Gamma^{\rho}_{\mu\lambda} \Gamma^{\lambda}_{\nu\sigma} - \Gamma^{\rho}_{\nu\lambda} \Gamma^{\lambda}_{\mu\sigma} \tag{28}$$

which satisfies

$$\begin{gathered} R^{\rho}_{\sigma\mu\nu} = -R^{\rho}_{\sigma\nu\mu} \\ R_{\rho\sigma\mu\nu} = -R_{\sigma\rho\mu\nu} \\ R_{\rho\sigma\mu\nu} = R_{\mu\nu\rho\sigma} \\ R_{\rho\sigma\mu\nu} + R_{\rho\mu\nu\sigma} + R_{\rho\nu\sigma\mu} = 0 \end{gathered} \tag{29}$$

using $R_{\rho\sigma\mu\nu} = g_{\rho\lambda} R^{\lambda}_{\sigma\mu\nu}$. From this we can derive a useful identity, the *Bianchi identity*

$$\nabla_\lambda R_{\rho\sigma\mu\nu} + \nabla_\rho R_{\sigma\lambda\mu\nu} + \nabla_\sigma R_{\lambda\rho\mu\nu} = 0 \tag{30}$$

It is frequently useful to consider contractions of the Riemann tensor. A useful one is the **Ricci tensor**:

$$R_{\mu\nu} = R^{\lambda}_{\mu\lambda\nu} \tag{31}$$

The connection usually used to make a contraction out of the Riemann tensor is usually the Christoffel connection, for which the Ricci tensor is the only independent contraction. It's symmetric

$$R_{\mu\nu} = R_{\nu\mu}, \tag{3.91}$$

as a consequence of the various symmetries of the Riemann tensor. We can take a last contraction to form the **Ricci scalar**:

$$R = R^{\mu}_{\mu} = g^{\mu\nu} R_{\mu\nu} \tag{32}$$

2.7 Einstein Hilbert action and field equations

Let's introduce the concept of action. It's the integral over spacetime of a Lagrange density (usually referred to as Lagrangian, even when it's not the same)

$$S = \int d^n x \mathcal{L} \tag{33}$$

This Lagrangian density is a tensorial object, which can be written as $\sqrt{-g}$ times a scalar. With this information, comes the question, what scalars can we make out of the metric?. To start with, we know that expanding about a point $p \in M$

$$g_{\mu\nu}(p) = \eta_{\mu\nu}(p) + \partial_\sigma g_{\mu\nu}\Big|_p (p) + \partial_\rho \partial_\sigma g_{\mu\nu}\Big|_p p^2 + \dots \tag{34}$$

where we can always set $\partial_\sigma g_{\mu\nu}\Big|_p = 0$ because of diffeomorphism invariance.

Therefore, any nontrivial scalar must be composed, at least, by second metric derivatives. We know one tensor which is made up of this stuff, the Riemann tensor, from which we can construct a scalar, the Ricci scalar R, being this the only independent one linear in the Riemann tensor. This is what Hilbert took as an ansatz to construct the action, since this is the simplest way possible, proposing

$$\mathcal{L} = \sqrt{-g} R \tag{35}$$

If you add other terms like $R_{\mu\nu}R^{\mu\nu}$, the field equations end up with higher-than-two order derivative terms, which is not like the equations describing other dynamical systems, ie., the KG equation $\Box\phi = m^2\phi$.

If we vary the action with respect to the metric, we should get the equations of motion (EOM), so

$$\delta S = \int d^n x \left[\sqrt{-g} g^{\mu\nu} \delta R_{\mu\nu} + \sqrt{-g} R_{\mu\nu} \delta g^{\mu\nu} + R \delta \sqrt{-g}\right] = (\delta S)_1 + (\delta S)_2 + (\delta S)_3. \tag{36}$$

where we've used that $R = g^{\mu\nu} R_{\mu\nu}$. In $(\delta S)_1$, we have the $\delta R_{\mu\nu}$. To compute this we can use the fact that the Ricci tensor is the contraction of the Riemann tensor, given by

$$R^{\rho}_{\mu\lambda\nu} = \partial_\lambda \Gamma^{\rho}_{\mu\nu} + \Gamma^{\rho}_{\lambda\sigma}\Gamma^{\sigma}_{\nu\mu} - (\lambda \leftrightarrow \nu) \tag{37}$$

We will first vary the Christoffel connection as

$$\delta\Gamma^{\rho}_{\mu\nu}(x) = \tilde{\Gamma}^{\rho}_{\mu\nu}(x') - \Gamma^{\rho}_{\mu\nu}(x) \tag{38}$$

which is itself a tensor. Taking the covariant derivative

$$\nabla_\lambda(\delta\Gamma^{\rho}_{\mu\nu}) = \partial_\lambda(\delta\Gamma^{\rho}_{\mu\nu}) + \Gamma^{\rho}_{\lambda\sigma}\delta\Gamma^{\sigma}_{\mu\nu} - \Gamma^{\sigma}_{\lambda\mu}\delta\Gamma^{\rho}_{\sigma\nu} - \Gamma^{\sigma}_{\lambda\nu}\delta\Gamma^{\rho}_{\mu\sigma} \tag{39}$$

which implies that

$$\delta R^{\rho}{}_{\mu\lambda\nu} = \nabla_\lambda(\delta\Gamma^{\rho}_{\nu\mu}) - \nabla_\nu(\delta\Gamma^{\rho}_{\lambda\mu}) \tag{40}$$

so (and using the fact that $\nabla_\sigma g_{\mu\nu} = 0$)

$$(\delta S)_1 = \int d^n x\sqrt{-g}\, g^{\mu\nu}\left[\nabla_\lambda(\delta\Gamma^{\lambda}_{\nu\mu}) - \nabla_\nu(\delta\Gamma^{\lambda}_{\lambda\mu})\right] \tag{41}$$

$$= \int d^n x\sqrt{-g}\,\nabla_\sigma\left[g^{\mu\sigma}(\delta\Gamma^{\lambda}_{\lambda\mu}) - g^{\mu\nu}(\delta\Gamma^{\sigma}_{\nu\mu})\right] \tag{42}$$

Using the Stokes theorem for a vector V^μ defined in a region Σ

$$\int_\Sigma \nabla_\mu V^\mu \sqrt{-g}\, d^n x = \int_{\partial\Sigma} n_\mu V^\mu \sqrt{|\gamma|}\, d^{n-1}x \tag{43}$$

for n_μ orthogonal to the boundary $\partial\Sigma$ and γ is the induced metric on $\partial\Sigma$, we see that this is equivalent to a boundary contribution at infinity, which we can assume to be zero since we are far away from the source.

For the $(\delta S)_3$ term, we can use the already well known identity

$$\delta\sqrt{-g} = -\frac{1}{2}\sqrt{-g}g_{\mu\nu}\delta g^{\mu\nu} \tag{44}$$

getting

$$\delta S = \int d^n x\sqrt{-g}\left[R_{\mu\nu} - \frac{1}{2}Rg_{\mu\nu}\right]\delta g^{\mu\nu} = \frac{\delta S}{\delta g^{\mu\nu}}\delta g^{\mu\nu} \tag{45}$$

which should vanish for arbitrary variations. We get the Einstein's vacuum equations

$$\frac{1}{\sqrt{-g}}\frac{\delta S}{\delta g^{\mu\nu}} = R_{\mu\nu} - \frac{1}{2}Rg_{\mu\nu} = G_{\mu\nu} = 0 \tag{46}$$

We can be interested in also getting the equations in a more general scenario. Considering the action as the sum of two terms, one for the gravity coupled to matter and the other for the additional matter, which for now we leave arbitrary.

$$S = \frac{1}{8\pi G}S_H + S_M \tag{47}$$

we get to

$$\frac{1}{\sqrt{-g}}\frac{\delta S}{\delta g^{\mu\nu}} = \frac{1}{8\pi G}\left(R_{\mu\nu} - \frac{1}{2}Rg_{\mu\nu}\right) + \frac{1}{\sqrt{-g}}\frac{\delta S_M}{\delta g^{\mu\nu}} = 0 \tag{48}$$

where if we identify

$$T_{\mu\nu} = -\frac{1}{\sqrt{-g}}\frac{\delta S_M}{\delta g^{\mu\nu}} \tag{49}$$

we get the field equations

$$G_{\mu\nu} = 8\pi G\, T_{\mu\nu} \tag{50}$$

Note that

$$\nabla_\mu G^{\mu\nu} = 0 \Longrightarrow \nabla_\mu T^{\mu\nu} = 0 \tag{51}$$

so the stress-energy tensor is conserved.

We can question “What metrics obey Einstein’s equations?” Well, in the absence of some constraints on $T_{\mu\nu}$, any metric at all. For a given metric, just compute $G_{\mu\nu}$ and make it fulfill (49). It will automatically be conserved by the Bianchi identity, $\nabla_\mu G^{\mu\nu} = 0$. This is one of the cases where to think about Einstein’s equations without specifying the theory of matter from which $T_{\mu\nu}$ is derived is more useful, since this leaves us with a great deal of arbitrariness.

Nevertheless, our real concern is with the existence of solutions to Einstein’s equations in the presence of “realistic” sources of energy and momentum, whatever that means. To make sure that $T_{\mu\nu}$ makes sense, it’s usually demanded that $T_{\mu\nu}$ represent positive energy densities — that is, no negative masses are allowed. In a locally inertial frame this requirement can be stated as $\rho = T_{00} \geq 0$. To turn this into a coordinate-independent statement, we ask that

$$T_{\mu\nu}V^\mu V^\nu \geq 0, \quad \text{for all timelike vectors } V^\mu \tag{52}$$

This is known as the **Weak Energy Condition**, or WEC. Many of the important theorems about solutions to general relativity (such as the singularity theorems of Hawking and Penrose) rely on this condition or something very similar, that’s why it’s a fairly reasonable requirement. But this isn’t an absolute definition, since it’s possible to create otherwise respectable classical field theories which violate the WEC, and almost impossible to invent a quantum field theory which obeys it. Nevertheless, we will assume that the WEC holds in all but the most extreme conditions.

2.8 Cosmological constant

The cosmological constant is the simplest thing we can add to the Einstein-Hilbert action.

We refer to Λ as a cosmological constant; it’s the lowest order scalar we can create. On it’s own, a constant doesn’t lead to very interesting dynamics. Let’s try plugging it into the Einstein-Hilbert action, given by

$$S = \int d^n x \sqrt{-g}(R - 2\Lambda) \tag{53}$$

with the field equation being

$$R_{\mu\nu} - \frac{1}{2}R g_{\mu\nu} + \Lambda g_{\mu\nu} = 0 \tag{54}$$

Einstein realized that the vacuum case was not useful, since there was not solution to a non-vacuum static cosmology (a universe which doesn’t change with time over large scales). Here’s where the cosmological constant plays a crucial role. If the cosmological constant is finely tuned, it is possible to find a static solution, although it is unstable to small perturbations.

We can interpret the term $\propto \Lambda$ in (54) as an energy-momentum tensor with $T_{\mu\nu} = -\Lambda g_{\mu\nu}$ (automatically conserved due to metric compatibility). Thus, Λ can be viewed as the “energy density of the vacuum”, representing energy and momentum present even without matter fields. This interpretation is crucial because quantum field theory predicts a non-zero vacuum energy and momentum. This is why, even when the search for static solutions became less significant after Hubble’s discovery that the universe is expanding, leading Einstein to abandon his proposal, the cosmological constant has persistently reappeared in theoretical physics.

As we all know, in ordinary quantum mechanics, a harmonic oscillator with frequency ω and minimum classical energy $E_0 = 0$ has a ground state energy $E_0 = \frac{1}{2}\hbar\omega$ upon quantization. A quantized field can be considered an infinite collection of harmonic oscillators, with each mode contributing to the ground state energy. We end up with an infinite result, which must be appropriately regularized if we want our theory to hold. For that purpose we can introduce a cutoff at high energies or use some of the regularization methods, like *dimensional regularization* (DIMREG) + $\bar{M}S$.

The scale of the regularized sum of the energies of the ground state oscillations of all the fields of a theory, referred to as the final vacuum energy is expected to be or the order

$$\Lambda \sim m_P^4 \tag{55}$$

with m_P the Planck mass, with a value of the order of 10^{19} GeV. This theoretical value leads to one of the greatest discrepancies between theory and data, since the actual measured value (on large scales) is smaller than (55) by a factor of, at least, 10^{120}. This is why the "cosmological constant problem" is regarded as one of the most important unsolved problems today.

What we do know is that

$$\Lambda \sim 10^{-44}\ GeV \tag{56}$$

which is non-zero. In fact, it can take values that can significantly impact the evolution of the universe. Thus, Einstein's mistake continues to perplex both physicists, who aim to understand why Λ is so small, and astronomers, who seek to determine whether it is indeed small enough to be negligible.

2.9 Lie derivative

Given a vector field $V^\mu(x)$, we can define the *integral curves* of the vector field as

$$\frac{dx^\mu}{d\lambda} = V^\mu \tag{57}$$

for some curve $x^\mu(\lambda)$. We can ask the question, how fat does a tensor change when traveling along an integral curve? To answer that question, we first need to introduce the concept of diffeomorphism, which is no more than a spacetime translation, such as an infinitesimal general coordinate translation $x^\mu \to x'^\mu = x^\mu + \xi^\mu(x)$.

The *Lie derivative* is just diffeomorphisms way of acting on tensors along a contravariant vector field $\xi^\mu(x)$. It's defined as

$$\begin{aligned} \mathcal{L}_\xi T^{\mu_1\mu_2\cdots\mu_k}_{\nu_1\nu_2\cdots\nu_l} &= \xi^\sigma \nabla_\sigma T^{\mu_1\mu_2\cdots\mu_k}_{\nu_1\nu_2\cdots\nu_l} - (\nabla_\lambda \xi^{\mu_1}) T^{\lambda\mu_2\cdots\mu_k}_{\nu_1\nu_2\cdots\nu_l} - (\nabla_\lambda \xi^{\mu_2}) T^{\mu_1\lambda\cdots\mu_k}_{\nu_1\nu_2\cdots\nu_l} - \cdots \\ &\cdots + (\nabla_{\nu_1}\xi^\lambda) T^{\mu_1\mu_2\cdots\mu_k}_{\lambda\nu_2\cdots\nu_l} + (\nabla_{\nu_2}\xi^\lambda) T^{\mu_1\mu_2\cdots\mu_k}_{\nu_1\lambda\cdots\nu_l} + \cdots \end{aligned} \tag{58}$$

where ∇_μ represents any symmetric (torsion-free) covariant derivative. If we were to expand all the covariant derivatives, it would be like having replaced $\nabla_\mu \to \partial_\mu$, since all the connection coefficients would cancel. A particularly useful formula is for the Lie derivative of the metric:

$$\mathcal{L}_V g_{\mu\nu} = \xi^\sigma \nabla_\sigma g_{\mu\nu} + (\nabla_\mu \xi^\lambda) g_{\lambda\nu} + (\nabla_\nu \xi^\lambda) g_{\mu\lambda} = \nabla_\mu \xi_\nu + \nabla_\nu \xi_\mu \tag{59}$$

or equivalently

$$\mathcal{L}_\xi g_{\mu\nu} = 2\nabla_{(\mu}\xi_{\nu)} \tag{60}$$

For an scalar function it takes the form

$$\delta_\xi f = \mathcal{L}_\xi f = \xi^\mu \nabla_\mu f \tag{61}$$

which is just the directional derivative.

Let us consider the transformation properties of (7.3)-(7.5) for infinitesimal coordinate transformations, namely those for which $x'^\mu = x^\mu - \xi^\mu(x)$. To first order in $\xi^\mu(x)$, the transformation rules of everyday objects can be expressed in terms of Lie derivatives as

$$\delta\phi(x) \equiv \phi'(x) - \phi(x) = \mathcal{L}_\xi \phi, \tag{62}$$

$$\delta U^\mu(x) \equiv U'^\mu(x) - U^\mu(x) = \mathcal{L}_\xi U^\mu, \tag{63}$$

$$\delta\omega_\mu(x) \equiv \omega'_\mu(x) - \omega_\mu(x) = \mathcal{L}_\xi \omega_\mu, \tag{64}$$

$$\delta T^\mu{}_\nu(x) \equiv T'^\mu{}_\nu(x) - T^\mu{}_\nu(x) = \mathcal{L}_\xi T^\mu{}_\nu \tag{65}$$

Thus one of the useful roles of Lie derivatives is in the description of infinitesimal coordinate transformations.

2.10 Symmetries, isometries and Killing vectors

That GR is invariant under diffeomorphism can be seen at the level of EH action. Under a general change of metric we have

$$\delta S = \frac{1}{8\pi G}\int d^4x\sqrt{-g}G_{\mu\nu}\delta g^{\mu\nu} \tag{66}$$

up to boundary terms. Imposing $\delta S = 0$ for ANY $\delta g^{\mu\nu}$, then $G_{\mu\nu} = 0$.

On the other hand, a symmetry of the action are variations that leave $\delta S = 0$ for any choice of metric. For diffeomorphisms $x^\mu \to x^\mu + \delta x^\mu$ where $\delta x^\mu = -x^\mu$

$$\delta g_{\mu\nu} = 2\nabla_{(\mu} X_{\nu)} \tag{67}$$

so

$$\delta S = \frac{1}{8\pi G}\int d^4x\sqrt{-g}G^{\mu\nu}\nabla_\mu X_\nu \tag{68}$$

Integrating by parts leads to the Bianchi identity

$$\nabla_\mu G^{\mu\nu} = 0 \tag{69}$$

So it's invariant under diffeomorphism that lead to vacuum Einstein equations with $G^{\mu\nu}$ the conserved current.

Another instance in which δS is left invariant is when $\delta g_{\mu\nu} = 0$, ie

$$\delta g_{\mu\nu} = \mathcal{L}_K g_{\mu\nu} \Longleftrightarrow \nabla_{(\mu} K_{\nu)} = 0 \tag{70}$$

A $K = K^\mu \partial_\mu$ with this property is called a *Killing vector*, and are a possible feature of specific metrics, rather than GR in general.

A metric is said to be *form-invariant* under some coordinate transformation when

$$g'_{\mu\nu}(x')\frac{\partial x'^\rho}{\partial x^\mu}\frac{\partial x'^\sigma}{\partial x^\nu} = g_{\mu\nu}(x) \tag{71}$$

equivalently

$$g'_{\mu\nu}(x') = g_{\mu\nu}(x') \tag{72}$$

If a coordinate transformations leaves the metric form-invariant, we call that an *isometry.* This is equivalent to saying that the metric admits a Killing vector (we say then that the metric have an *isometry*). Killing vectors are important because they define conserved quantities along geodesics. Consider a massive particle following some trajectory $x^\mu(\tau)$, then

$$Q = K_\mu \frac{dx^\mu}{d\tau} \tag{73}$$

is a conserved charge along a geodesic. Explicitly

$$\frac{dQ}{d\tau} = \partial_\nu K_\mu \frac{dx^\nu}{d\tau}\frac{dx^\mu}{d\tau} + K_\mu \frac{d^2x^\mu}{d\tau^2} \tag{74}$$

$$= \partial_\nu K_\mu \frac{dx^\nu}{d\tau}\frac{dx^\mu}{d\tau} - K_\mu \Gamma^\mu_{\rho\sigma} \frac{dx^\rho}{d\tau}\frac{dx^\sigma}{d\tau} \tag{75}$$

$$= \nabla_\nu K_\mu \frac{dx^\nu}{d\tau}\frac{dx^\mu}{d\tau} = 0 \tag{76}$$

which follows from (70) and the geodesic equation.

The power of this resides in that, instead of finding all the symmetries of the metric, we can restrict ourselves to find the Killing vectors, which will, at the end of the day, simplify our computations.

2.11 Maximally symmetric spaces

Examples of homogeneous (translation invariant) and isotropic (Lorentz transformation invariant) spaces.

They're called *maximally symmetric* because they enjoy the maximal number of isometries possible in a given dimension given the sign of the Ricci scalar. This is equivalent to saying that it possesses the maximum number of Killing vectors, that is, $D(D+1)/2$ for a D-dimensional one.

They're characterized by a constant Ricci scalar R, in terms of which

- Riemann tensor

$$R_{\rho\sigma\mu\nu} = R(g_{\rho\mu}g_{\sigma\nu} - g_{\rho\nu}g_{\sigma\mu}) \tag{77}$$

- Ricci tensor

$$R_{\mu\nu} = \frac{R}{D} g_{\mu\nu} \tag{78}$$

where D is the dimension. Classified by their signature (**Euclidean** vs **Lorentzian**), their Ricci scalar R and discrete information relative to topology that will be important for us here. If we ignore their topology, there are

$$\text{Euclidean signature} \begin{cases} \mathbb{H}^D \ if \ R < 0 \text{ (Hyperboloid)} \\ \mathbb{R}^D \ if \ R = 0 \text{ (Flat)} \\ \mathbb{S}^D \ if \ R > 0 \text{ (Sphere)} \end{cases} \tag{79}$$

$$\text{Lorentzian signature} \begin{cases} AdS_D \ if \ R < 0 \text{ (Anti-de Sitter)} \\ Mink_D \ if \ R = 0 \text{ (Minkowski)} \\ dS_D \ if \ R > 0 \text{ (de Sitter)} \end{cases} \tag{80}$$

Maximally symmetric spaces are examples of spaces with a cosmological constant, ie, taking a Lorentzian $D = 4$ space

$$R_{\mu\nu} = \frac{R}{4} g_{\mu\nu} \tag{81}$$

$$G_{\mu\nu} = R_{\mu\nu} - \frac{1}{2} R g_{\mu\nu} = -\frac{R}{4} g_{\mu\nu} \tag{82}$$

so $\Lambda = \frac{R}{4}$.

Thus

$$\begin{cases} AdS_4 \; has \; \Lambda < 0 \\ Mink_4 \; has \; \Lambda = 0 \\ dS_4 \; has \; \Lambda > 0 \end{cases} \tag{83}$$

Maximally symmetric spaces (with $R \neq 0$) can be defined by their embedding into $D+1$ dimensions in terms of embedding coordinates plus a constant.

- In the case of S^D, we can write the metric as

$$ds^2 = \sum_{i=1}^{D+1} dY_i^2 \tag{84}$$

subject to

$$\sum_{i=1}^{D+1} Y_i^2 = L^2 \tag{85}$$

We see that this metric has an $SO(D+1)$ invariance.

We can define the Y_i's in $D+1 = 3$ as

$$Y_i = L(\sin\theta\cos\phi, \sin\theta\sin\phi, \cos\theta) \tag{86}$$

so

$$ds^2(S^2) = L^2(d\theta^2 + \sin^2\theta d\phi^2) \tag{87}$$

- de Sitter dS_D

$$ds^2 = -dY_0^2 + \sum_{i=1}^{D} dY_i^2 \tag{88}$$

subject to

$$-Y_0^2 + \sum_{i=1}^{D} Y_i^2 = L^2 \tag{89}$$

We see that this metric has an $SO(1, D)$ invariance.

We can define the Y_i's as static coordinates

$$Y_0 = \sqrt{L^2 - r^2} \sinh\left(\frac{r}{L}\right) \tag{90}$$

$$Y_1 = \cosh\left(\frac{r}{L}\right) \tag{91}$$

$$Y_i = r y_i \tag{92}$$

such that

$$\sum_i y_i^2 = 1 \quad , \quad i = 2, \ldots, D \tag{93}$$

Then

$$ds^2(dS_D) = -\left(1 - \frac{r^2}{L^2}\right) dt^2 + \left(1 - \frac{r^2}{L^2}\right)^{-1} dr^2 + r^2 ds^2(S^{D-2}) \tag{94}$$

- Anti-de Sitter AdS_D

$$ds^2 = -dY_0^2 - dY_1^2 + \sum_{i=1}^{D-1} dY_i^2 \tag{95}$$

subject to

$$-Y_0^2 - Y_1^2 + \sum_{i=1}^{D-1} Y_i^2 = -L^2 \tag{96}$$

We see that this metric has an $SO(2, D-1)$ invariance.

We can define Global coordinates

$$Y_0 = L \sinh \rho \cos \tilde{t} \tag{97}$$

$$Y_1 = L \sinh \rho \sin \tilde{t} \tag{98}$$

$$Y_i = L \cosh \rho y_i \quad s.t \quad \sum_i y_i^2 = 1 \quad for \quad i = 2, \ldots, D \tag{99}$$

Then

$$ds^2 = L^2[-\sinh^2 \rho d\tilde{t}^2 + d\rho^2 + \cosh^2 \rho ds^2(S^{2-D})] \tag{100}$$

and changing coords as $r = L \cosh \rho$ and $t = L\tilde{t}$

$$ds^2(dS_D) = -\left(1 + \frac{r^2}{L^2}\right) dt^2 + \left(1 + \frac{r^2}{L^2}\right)^{-1} dr^2 + r^2 ds^2(S^{D-2}) \tag{101}$$

Maximally symmetric spaces inherit the isometries of their $D+1$ embedding space, the Killing vectors corresponding to these isometries are given by the independent components of the matrix

$$M_{AB} = (Y_A \partial_\mu Y_B - Y_B \partial_\mu Y_A) g^{\mu\nu} \partial_\nu \tag{102}$$

2.12 p-forms

In search of a better formalism in which to describe our theories, physicist/mathematicians have come up with an idea, the *p-forms*, which simplify calculus by making use of two tensorial properties, the antisymmetry of its indices and its absence. It's worth to make a quick review of it.

In an d-dimensional manifold (think of spacetime), we have a set of basis vectors E_a, which allows us to represent vectors as

$$u = \sum_\mu u^\mu E_\mu, \tag{103}$$

and one-forms (the dual of vectors) as

$$\omega = \sum_\mu \omega_\mu E^\mu \tag{104}$$

where $\langle \omega, u \rangle = \mathbb{R}$, $\langle E^\mu, E_\nu \rangle = \delta^\mu_\nu$, and its also linear. It is often useful to work in a coordinate basis where $E_\mu = \partial_\mu$ and $E^\mu = dx^\mu$. However, other choices of basis can be useful, for instance the vielbein basis as we shall see later.

A p-form is defined to be a tensor of type $(0, p)$ whose components are totally antisymmetric (in any basis):

$$T = T_{\mu_1 \ldots \mu_p} E^{\mu_1} \otimes \ldots \otimes E^{\mu_p} = T_{\mu_1 \ldots \mu_p} E^{[\mu_1} \otimes \ldots \otimes E^{\mu_p]} = \frac{1}{p!} T_{\mu_1 \ldots \mu_p} (E^{\mu_1} \wedge \ldots \wedge E^{\mu_p}) \tag{105}$$

Before going any further, let's see what $\wedge$ means. It's basically telling you to take the antisymmetric product:

$$E^\mu \wedge E^\nu = E^\mu \otimes E^\nu - E^\nu \otimes E^\mu \tag{106}$$

$$\begin{gathered} E^\mu \wedge E^\nu \wedge E^\rho = E^\mu \otimes E^\nu \otimes E^\rho + E^\nu \otimes E^\rho \otimes E^\mu + E^\rho \otimes E^\mu \otimes E^\nu - \\ - E^\mu \otimes E^\rho \otimes E^\nu - E^\nu \otimes E^\mu \otimes E^\rho - E^\rho \otimes E^\nu \otimes E^\mu \end{gathered} \tag{107}$$

As we can see, a general case $E^{a_1} \wedge \ldots \wedge E^{a_p}$ is antisymmetric under the interchange of any adjacent pair of indices. In d dimensions, the number of linearly independent p-forms objects is

$$\frac{d(d-1)\ldots(d-p+1)}{p!} = \frac{d!}{p!(d-p)!} = \binom{d}{p}. \tag{108}$$

This means one must have $p \leq d$, because one will get nothing otherwise.

2.12.1 Wedge product

We will also need to perform p-forms product. A p-form P can in any basis be written as

$$P = \frac{1}{p!} P_{\mu_1 \ldots \mu_p} E^{\mu_1} \wedge \ldots \wedge E^{\mu_p} \tag{109}$$

similarly

$$Q = \frac{1}{q!} Q_{\nu_1 \ldots \nu_q} E^{\nu_1} \wedge \ldots \wedge E^{\nu_q} \tag{110}$$

We define the *wedge product* of a p-form with a q-form to be

$$P \wedge Q = \frac{1}{(p+q)!} P_{\mu_1 \ldots \mu_p} Q_{\nu_1 \ldots \nu_q} E^{\mu_1} \wedge E^{\mu_2} \wedge \ldots \wedge E^{\mu_p} \wedge E^{\nu_1} \wedge E^{\nu_2} \wedge \ldots \wedge E^{\nu_q} \tag{111}$$

where $P \wedge Q$ is really equivalent to a tensor of type $(0, p+q)$ that is antisymmetric on all its $p+q$ indices.

2.12.2 Exterior derivative

We will also need to derivate. Starting from a p-form in a coordinate basis:

$$P = \frac{1}{p!} P_{\mu_1 \ldots \mu_p} \underbrace{dx^{\mu_1} \wedge dx^{\mu_2} \wedge \ldots \wedge dx^{\mu_p}}_{p\text{-form}} \tag{112}$$

We already know what d does on 0-forms (functions). We define

$$dP = \frac{1}{p!} \frac{\partial P_{\mu_1 \ldots \mu_p}}{\partial x^\nu} \; dx^\nu \wedge dx^{\mu_1} \wedge dx^{\mu_2} \wedge \ldots \wedge dx^{\mu_p}. \tag{113}$$

which we will call the *exterior derivative* d on a p-form. It has some useful properties, such as

- From (113) we see that d maps p-forms to $(p+1)$-forms.
- $d^2 = 0$
- The operator d is Leibnizian

$$dP_{\mu_1 \ldots \mu_p} \in dP = \frac{\partial P_{\mu_1 \ldots \mu_p}}{\partial x^\nu} dx^\nu \tag{114}$$

- $$d(P \wedge Q) = dP \wedge Q + (-1)^p P \wedge dQ. \tag{115}$$

We could argue that this doesn't hold in general, since we've used a coordinate basis to derive all these results. Its straightforward to see that this is inessential. The action of d is independent of a choice of co-ordinates.

2.12.3 Hodge dual

This maps a p-form into a $d-p$-form. The Hodge dual of P is defined as

$$*P = \frac{1}{(D-p)!}(*P)_{\mu_1 \dots \mu_{D-p}} E^{\mu_1} \wedge \dots \wedge E^{\mu_{D-p}} \tag{116}$$

where

$$(*P)_{\mu_1 \dots \mu_{D-p}} = \frac{1}{p!}\epsilon_{\mu_1 \dots \mu_{D-p}}{}^{\nu_1 \dots \nu_p} P_{\nu_1 \dots \nu_p} \tag{117}$$

As an example of the utility of this notations one can consider Maxwell theory in GR, defined by equations

$$\nabla_{[\mu} F_{\nu\rho]} = 0 \Longleftrightarrow \partial_{[\mu} F_{\nu\rho]} = 0 \tag{118}$$

$$\nabla_\mu F^{\mu\nu} = -J^\nu \tag{119}$$

in form notation this becomes

$$dF = 0 \quad , \quad *d*F = -J \tag{120}$$

for

$$F = \frac{1}{2} F_{\mu\nu} dx^\mu \wedge dx^\nu \quad , \quad J = J_\mu dx^\mu \tag{121}$$

2.13 Vielbein formalism

In GR we are used to working with line element

$$ds^2 = g_{\mu\nu} dx^\mu dx^\nu \tag{122}$$

where the dx^μ means we are working in a coordinate basis. General idea of vielbein formalism is to work in a non-coordinate basis such that the metric becomes flat. They're defined as

$$e^a = e^a_\mu dx^\mu \tag{123}$$

whose components are defined such that

$$e^a_\mu e^b_\nu \eta_{ab} = g_{\mu\nu} \quad , \quad e^a_\mu e^b_\nu g^{\mu\nu} = \eta^{ab} \tag{124}$$

where in $D=4$, $\eta_{ab} = \mathrm{diag}(-1,1,1,1)$ is the usual flat space Minkowski metric. We refer to "a" as a flat or tangent space index and μ as a curved index.

The line element can now be written as

$$ds^2 = \eta_{ab} e^a e^b \tag{125}$$

but notice that by doing this we have introduced an additional $O(1,3)$ symmetry, i.e.

$$\text{under} \quad e^a \to \Lambda^a e^b \quad \text{for} \quad \Lambda^\top \eta \Lambda = \eta \tag{126}$$

$$ds^2 \to \Lambda^a{}_c \Lambda^b{}_a \eta_{ab} e^c e^d = \eta_{ab} e^a e^b = ds^2 \tag{127}$$

which leaves the metric invariant.

So the vielbein is not unique, it is defined up to a local (i.e. $\Lambda = \Lambda(x)$) Lorentz transformation.

- Now curved indices can be raised/lowered with $g_{\mu\nu}$ and flat ones with η_{ab}. In particular, it follows that $e_a{}^\mu$ is the inverse of $e^a{}_\mu$ since

$$e^a_\mu e^\nu_a = \delta^\nu_\mu \quad , \quad e^a_\mu e^\mu_b = \delta^a_b \tag{128}$$

The vielbein can also be used to map between curved and flat space indices, i.e.

$$V^a = e^a_\mu V^\mu \quad , \quad V^\mu = e^\mu_a V^a \tag{129}$$

At the end of the day, what the vielbeins do is connect the flat spacetime with the curved one.

One thing that this formalism allow us to do (whose importance will be seen in later sections) is to introduce a curved space analog of the gamma matrices γ_a, i.e., if

$$\{\gamma_a, \gamma_b\} = 2\eta_{ab}\mathbb{1} \tag{130}$$

it follows that $\gamma_\mu = e^a_\mu \gamma_a$ and

$$\{\gamma_\mu, \gamma_\nu\} = 2e^a_\mu e^b_\nu \eta_{ab} = 2g_{\mu\nu}\mathbb{1} \tag{131}$$

which we can view as the "curved Clifford algebra". This is one step on the path to introducing spinors to GR, but we also need to generalize the covariant derivative to take account of local Lorentz transformations.

2.14 Spin connection and Covariant derivative

If one only has curved indices the covariant derivative of a vector is

$$\nabla_\mu V_\nu = \partial_\mu V_\nu - \Gamma^\lambda_{\mu\nu} V_\lambda \tag{132}$$

We need to introduce an analog of the affine connection for flat indices. This is the *spin connection* $(\omega_\mu)^{ab}$, so the covariant derivative with respect to flat indices becomes

$$\nabla_\mu V_a = \partial_\mu V_a + (\omega_\mu)_a{}^b V_b \tag{133}$$

Similarly to how we demand $\nabla_\mu g_{\alpha\beta} = 0 \Rightarrow \Gamma^\lambda_{\alpha\beta}$, one can define the spin connection through the *vielbein postulate*

$$\nabla_\mu e^a_\nu = 0 \quad , \quad \nabla_\mu \eta_{ab} = 0 \tag{134}$$

Expanding out

$$\begin{gathered} 0 = \nabla_\mu e^a_\nu = \partial_\mu e^a_\nu - \Gamma^\lambda_{\mu\nu} e^a_\lambda + (\omega_\mu)^a{}_b e^b_\nu \Longrightarrow \\ \Longrightarrow (\omega_\mu)^a{}_b = -e^\nu_b \partial_\mu e^a_\nu + \Gamma^\lambda_{\mu\nu} e^a_\lambda e^\nu_b \end{gathered} \tag{135}$$

using $\nabla_\mu \eta_{ab} = 0$ implies

$$(\omega_\mu)_{[ab]} = 0 \tag{136}$$

i.e., that the spin connection is antisymmetric.

In form notation one can introduce

$$\omega_{ab} = -\omega_{ba} = (\omega_\mu)_{ab} dx^\mu \quad , \quad R^a{}_b = \frac{1}{2} R^a{}_{b\mu\nu} dx^\mu \wedge dx^\nu \tag{137}$$

Assuming no torsion one then has

$$de^a + \omega^a{}_b \wedge e^b = 0 \tag{138}$$

$$d\omega^{ab} + \omega^a{}_c \wedge \omega^{cb} = R^{ab} \tag{139}$$

These are **Cartan's first and second structure equations**, which give a useful way to compute the spin connection from the vielbein and curvature tensors from the spin connection.

2.15 Addition of fermions

In order to couple fermions to GR we also need to know how the covariant derivative acts on them. This can be derived from consistency with

$$\nabla_\mu V_a = \partial_\mu V_a + (\omega_\mu)_a{}^b V_b \tag{140}$$

For the choice $V_a = \bar{\Psi}\gamma_a\psi$ with ψ a spinor, the result is

$$\nabla_\mu \psi = \partial_\mu \psi + \frac{1}{4}(\omega_\mu)_{ab}\gamma^{ab}\psi \tag{141}$$

Thus the Dirac equation in GR becomes

$$(\gamma^\mu \nabla_\mu - m)\,\psi = 0 \tag{142}$$

where the generalization of the flat space equation is contained in $\gamma_\mu = e^a_\mu \gamma_a$ and the definition of the covariant derivative for spinors.

Under a local Lorentz transformation of the vielbeins like in (126), the corresponding action on a spinor is given by

$$\psi' = S(\Lambda)\psi, \tag{143}$$

where $S(\Lambda)$ is a spinor representation of the Lorentz group in the spinor space and satisfies:

$$S(\Lambda)\gamma^a S^{-1}(\Lambda) = \Lambda^a{}_b \gamma^b. \tag{144}$$

3 Black holes

3.1 The Chandrasekhar limit

We all know that a black hole starts as a star, but not a usual one. Something unusual has to happen if we want that everyday object to transform into one of, if not the most, mysterious object in the universe.

When a star is born, fusion in its nucleus begin to occur, giving the star its aspect, but this process eventually stops as the star runs out of fuel (hydrogen mainly). That's where the star cools and contracts, giving rise to a white dwarf, but is this always the case? To arrive to this state, its assumed the *non-relativistic* character of the electrons, so the electron degeneracy pressure can counteract the collapse. But, what if this approximation is not valid? What if electrons are relativistic? Well, this happens if we have a sufficiently large mass, therefore

$$E = E_{kin} + E_{grav} \tag{145}$$

where

$$E_{grav} \sim -\frac{GM^2}{R} \tag{146}$$

and

$$E_{kin} \sim nR^3\langle E\rangle \tag{147}$$

where $\langle E\rangle$ is the average kinetic energy of atoms. We have

$$\langle E\rangle = \langle p_e\rangle c = \hbar c n_e^{1/3} \tag{148}$$

then

$$E_{kin} \sim n_e R^3\langle E\rangle \sim \hbar c R^3 n_e^{4/3} \sim \hbar c R^3 \left(\frac{M}{m_p R^3}\right)^{4/3} \sim \hbar c \left(\frac{M}{m_p}\right)^{4/3} \frac{1}{R} \tag{149}$$

where we've used the fact that, since $m_e \ll m_p$, they will become degenerated first, occupying each one a cube of side of Compton wavelength

$$n_e^{-1/3} \sim \frac{\hbar}{\langle p_e\rangle} \tag{150}$$

and also because of the mass difference between protons and electrons,

$$M \approx n_e R^3 m_e, \Longrightarrow n_e \sim \frac{M}{m_p R^3} \tag{151}$$

Now we have that the energy goes as

$$E \sim -\frac{\alpha}{R} + \frac{\beta}{R} \tag{152}$$

To reach equilibrium, one must have $\alpha = \beta$, getting

$$M \sim \frac{1}{m_p^2}\left(\frac{\hbar c}{G}\right)^{3/2} \tag{153}$$

If we increase the mass unlimitedly, the radius must also decrease, making impossible for electrons pressure degeneracy to support the star. There's a critical mass (and a radius),

$$M_C \sim \frac{1}{m_p^2}\left(\frac{\hbar c}{G}\right)^{3/2} \Longrightarrow R_C \sim \frac{1}{m_e m_p}\left(\frac{\hbar^3}{Gc}\right)^{1/2} \tag{154}$$

above (under) the initial star cannot end as a white dwarf. This is known as the *Chandrasekar limit*, of about $\approx 1.4 M_\odot$ (solar masses). The solution is a more extreme type of star, a neutron star.

3.2 Neutron stars

Just as in the white dwarfs, were the electron pressure was the force holding everything stable, if we surpass M_C, we need to account for the *neutron-degeneracy pressure*. That's because when we go above that limit, a process that was previously impossible begins to happen, inverse β-decay

$$e^- + p^+ \to n + \nu_e \tag{155}$$

which cannot be countered with the reactions

$$n + \nu_e \to e^- + p^+ \qquad or \qquad n \to e^- + p^+ + \bar{\nu}_e \tag{156}$$

because neutrinos escape the star and energy levels to where electrons of β-decay would go are occupied. Here we see that, at the end of the day, we are effectively removing electrons (therefore losing their degeneracy pressure) and getting neutrons instead.

But, how far can we go? The limit for a stable neutron star is $\sim 3M_\odot$. Beyond that, is there any other state to go to? The answer is yes, there are two options, a new (and unobserved) ultra-high density state or a black hole. Even if the first one exist, (quark star), we will assume the latter, since for large enough M you will still end up with a BH.

3.3 The Schwarzschild Black Hole

Let's start by reviewing the easiest case of a black hole. It emerges as a solution of the vacuum Einsteins equations, $R_{\mu\nu} = 0$. It's the only spherically symmetric solution in vacuum (which is proved by Birkhoff's theorem). Its metric is given by

$$ds^2 = -\left(1 - \frac{2M}{r}\right) dt^2 + \left(1 - \frac{2M}{r}\right)^{-1} dr^2 + r^2 ds^2(S^2) \tag{157}$$

where we are working in natural units, that's $G = c = 1$, $0 < r < \infty$ and $ds^2(S^2)$ is the two-sphere metric

$$ds^2(S^2) = d\theta^2 + \sin^2\theta\, d\phi^2 \tag{158}$$

We see a problem with the metric, at $r = 2M$ it blows up. That value of the radius is what we call a *horizon*, a point which is not a physical singularity (as we will see), but a frontier between "our" universe and what lies beyond. Another problems comes when $r = 0$. This is a physical singularity, and a good way to check this is with scalars, since they're invariant under coordinate transformations, making them independent of the observer's frame of reference. Using the *Kretschmann scalar*, a measure of the curvature of spacetime

$$K = R_{\mu\nu\rho\sigma} R^{\mu\nu\rho\sigma} = \frac{48M^2}{r^6} \tag{159}$$

which blows up at $r = 0$.

In order to make further computations easier, let's put the metric in the form

$$ds^2 = -e^{2\alpha(r)} dt^2 + e^{2\beta(r)} dr^2 + r^2 ds^2(S^2) \tag{160}$$

where

$$e^{2\alpha} = e^{-2\beta} = \left(1 - \frac{2GM}{r}\right) \tag{161}$$

For now, we will assume that the metric is continuous, so it must be valid on the surface of the star. Let's consider a trajectory followed by freely-falling particles (geodesic). For simplicity we will assume that it follows a path along the radial direction, that is $d\theta = d\phi = 0$, at the same time we neglect the counter reaction on the metric. Its action is given by

$$S = \int \mathcal{L} d\tau = \frac{1}{2m} \int \left(g_{\mu\nu} \frac{dx^\mu}{d\tau} \frac{dx^\nu}{d\tau} - m^2 \right) d\tau \tag{162}$$

where τ denotes the proper time. The trajectory followed by the particle can be parameterized by

$x^\mu(\tau) = (t(\tau), r(\tau), \theta_0, \phi_0)$. Using the geodesic equation (8) we have

$$(\mu = t)\frac{d^2t}{d\lambda^2} + 2(\partial_r\alpha)\frac{dr}{d\lambda}\frac{dt}{d\lambda} = 0 \tag{163}$$

$$(\mu = r)\frac{d^2r}{d\lambda^2} + e^{2(\alpha-\beta)}(\partial_r\alpha)\left(\frac{dt}{d\lambda}\right)^2 + (\partial_r\beta)\left(\frac{dr}{d\lambda}\right)^2 - re^{-2\beta}\left[\left(\frac{d\theta}{d\lambda}\right)^2 + \sin^2\theta\left(\frac{d\phi}{d\lambda}\right)^2\right] = 0 \tag{164}$$

$$(\mu = \theta)\frac{d^2\theta}{d\lambda^2} + \frac{2}{r}\frac{d\theta}{d\lambda}\frac{dr}{d\lambda} - \sin\theta\cos\theta\left(\frac{d\phi}{d\lambda}\right)^2 = 0 \tag{165}$$

$$(\mu = \phi)\frac{d^2\phi}{d\lambda^2} + \frac{2}{r}\frac{d\phi}{d\lambda}\frac{dr}{d\lambda} + 2\cot\theta\frac{d\theta}{d\lambda}\frac{d\phi}{d\lambda} = 0 \tag{166}$$

If we take $\theta_0 = \pi/2$, the θ equation is automatically satisfied. The t and ϕ equations simplify to

$$(\mu = t)\left(\frac{d\lambda}{dt}\right) \times \frac{d^2t}{d\lambda^2} + 2(\partial_r\alpha)\frac{dt}{d\lambda}\cdot\frac{dr}{d\lambda} = 0 \tag{167}$$

$$(\mu = \phi)\left(\frac{d\lambda}{d\phi}\right) \times \frac{d^2\phi}{d\lambda^2} + \frac{2}{r}\frac{d\phi}{d\lambda}\frac{dr}{d\lambda} = 0 \tag{168}$$

Massaging a little bit, we end up with two conservation laws

$$\frac{dt}{d\lambda}\left(1 - \frac{2M}{r}\right) = E \tag{169}$$

$$\frac{d\phi}{d\lambda}r^2 = J \tag{170}$$

The radial geodesic reads

$$\frac{d^2r}{d\lambda^2} + e^{2(\alpha-\beta)}(\partial_r\alpha)\left(\frac{dt}{d\lambda}\right)^2 + (\partial_r\beta)\left(\frac{dr}{d\lambda}\right)^2 - re^{-2\beta}\left(\frac{d\phi}{d\lambda}\right)^2 = 0 \tag{171}$$

Using the conservation laws (169) and (170), and multiplying by $2e^{2\beta}\dot{r}$, we end up with the conservation equation

$$\frac{1}{2}\left(\frac{dr}{d\lambda}\right)^2 + \underbrace{\frac{1}{2}\left(1 - \frac{2GM}{r}\right)\left(\frac{J^2}{r^2} + \chi\right)}_{V(r)} = \underbrace{\frac{1}{2}E^2}_{\varepsilon} \tag{172}$$

Where the χ parameter covers the range $\chi \geq 0$ for massless ($\chi = 0$) or massive particles. The radial equation is precisely the equation of a particle of unit mass and energy ε in a one-dimensional potential. In the case of a massive particle, it's usual to choose $\lambda = \tau$ (proper time).

As we've seen, the metric has two problematic points, $r = 0$ and the *Schwarzschild radius* $r = 2M$. In reality, nothing special happens in the second case, since it's just a point where our coordinate system fails, but not spacetime!, as we see from the fact that curvature is finite at that point. We fix that by performing a coordinate change. It's usual to start with the one adapted to infalling observers , then the one adapted to outgoing observers, and then kind of merge both of them into the Kruskal-Szekeres coordinates.

In the infalling radial case we define (massless case for convenience)

$$dt^2 = \frac{dr^2}{\left(1 - \frac{2M}{r}\right)^2} \equiv (dr^*)^2 \tag{173}$$

with

$$r^* = r + 2M \ln \left| \frac{r - 2M}{2M} \right| \tag{174}$$

This is the *Regge-Wheeler radial coordinate*, better know as the *tortoise coordinate*, since its behaviour is "slower" than r. What we've done is send the problematic behaviour from $r = 2M$ to $r^* = -\infty$, so $-\infty < r^* < \infty$. Now for the ingoing radial null coordinate, we define

$$v = t + r^*, \qquad -\infty < v < \infty \tag{175}$$

and rewrite the Schwarzschild metric in *ingoing Eddington-Finkelstein coordinates* (v, r, θ, ϕ).

$$ds^2 = \left(1 - \frac{2M}{r}\right)\left(-dt^2 + dr^{*2}\right) + r^2 ds^2(S^2) = -\left(1 - \frac{2M}{r}\right) dv^2 + 2dr dv + r^2 ds^2(S^2) \tag{176}$$

Now we can analytically continue the metric to all $r > 0$. We see that the $drdv$ cross-term in EF coordinates is non-singular at $r = 2M$, so this singularity in Schwarzschild coordinates was in fact a coordinate singularity. There is nothing at $r = 2M$ to prevent the star collapsing at that point. To illustrate this a little bit, let's analyze the associated *Finkelstein diagram*

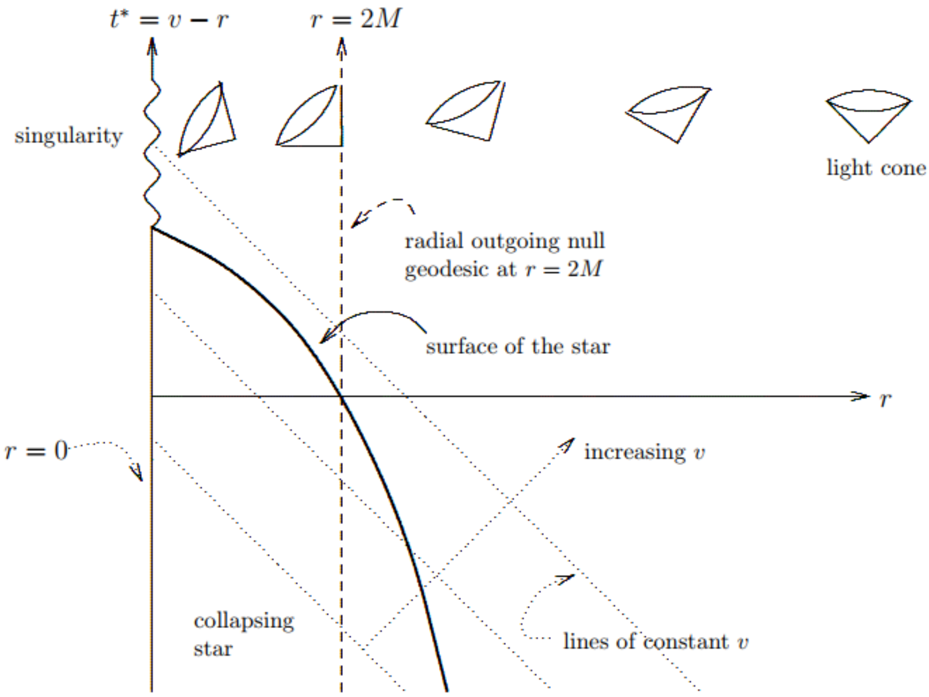

Figure 1: Ingoing coordinates Finkelstein diagram.

In Figure (1), as we approach the Schwarzschild radius, $r = 2M$, the light cone distorts, making that no real thing can reach $r > 2M$ starting from $r \leq 2M$. In fact, all geodesics that cross the horizon end up at the singularity at $r = 0$. This is translated in mathematical form as, if we are in $r \leq 2M$,

$$2dr\, dv = -\left[-ds^2 + \left(\frac{2M}{r} - 1\right) dv^2 + r^2 d\Omega^2\right] \leq 0 \tag{177}$$

when $ds^2 \leq 0$.

for all timelike or null worldlines $dr\, dv \leq 0$. $dv > 0$ for future-directed worldlines, so $dr \leq 0$ with equality when $r = 2M$, $d\Omega = 0$ (i.e., ingoing radial null geodesics at $r = 2M$).

3.3.1 Outgoing geodesics

The hypersurface $r = 2M$ is like a one-way membrane. This may seem paradoxical in view of the time-reversibility of Einstein's equations. Define the outgoing radial null coordinate u by

$$u = t - r^*, \quad -\infty < u < \infty \tag{178}$$

and rewrite the Schwarzschild metric in the*outgoing Eddington-Finkelstein coordinates* (u, r, θ, ϕ).

$$ds^2 = -\left(1 - \frac{2M}{r}\right) du^2 - 2dr\, du + r^2 ds^2(S^2) \tag{179}$$

Just as in the previous case, metric can be analytically continued to all $r > 0$. However the $r < 2M$ region in outgoing EF coordinates is NOT the same as the $r < 2M$ region in ingoing EF coordinates. They represent different regions of spacetime.

As before, for $r \leq 2M$

$$2dr\, du = -ds^2 + \left(\frac{2M}{r} - 1\right) du^2 + r^2 d\Omega^2 \geq 0 \tag{180}$$

when $ds^2 \leq 0$.

i.e. $dr\, du \geq 0$ on timelike or null worldlines. But $du > 0$ for future-directed worldlines so $dr \geq 0$, with equality when $r = 2M$, $d\Omega = 0$, and $ds^2 = 0$. In this case, a star with a surface at $r < 2M$ must *expand* and explode through $r = 2M$, as illustrated in (2).

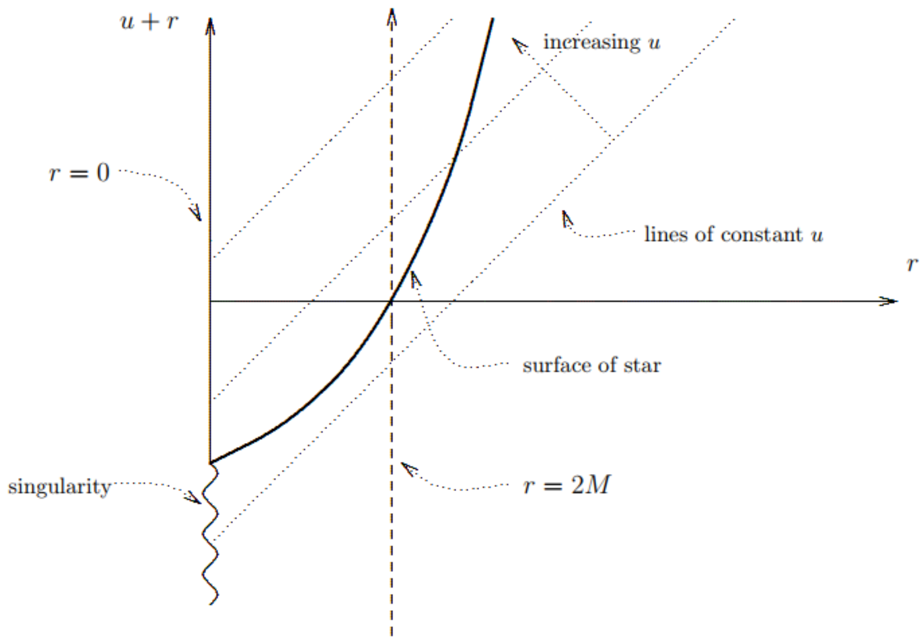

Figure 2: Outgoing coordinates Finkelstein diagram.

This corresponds with a white hole, the time reverse of a black hole. Both black and white holes are allowed by GR, but white holes are believed to not be physical, since they require very special initial conditions near the singularity and, unlike black holes, they cannot form from the gravitational collapse of stars.

3.4 Kruskal-Szekeres Coordinates

The exterior region $r > 2M$ is covered by both ingoing and outgoing Eddington-Finkelstein coordinates. We may write the Schwarzschild metric in this terms (u, v, θ, ϕ)

$$ds^2 = -\left(1 - \frac{2M}{r}\right) du\, dv + r^2 ds^2(S^2) \tag{181}$$

We can pack all the information about Schwarzschild BH into a Penrose diagram, like the one in Figure (3). It's worth explaining this diagram in more detail.

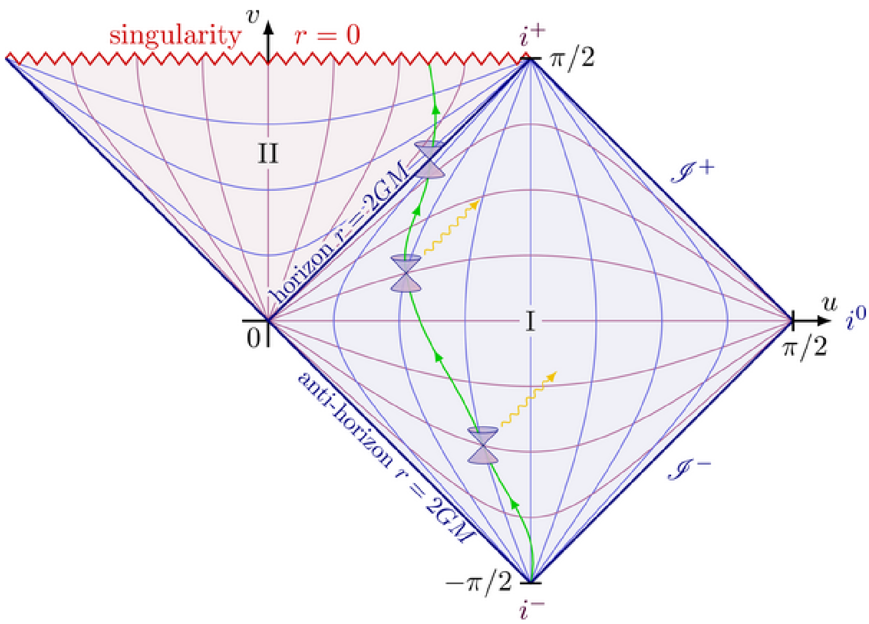

Figure 3: Penrose diagram for Schwarzschild black hole, derived via Kruskal-Szekeres coordinates. The horizon is at $r = 2GM$ ($v = \pm u$), singularity at $r = 0$.

Horizons and Singularities. The wavy lines at the edges labeled $r = 0$ represent the singularities inside the black hole where the curvature becomes infinite, while the diagonal lines running from top left to bottom right (or vice versa) represent the event horizons $r = 2M$ (in natural units, where M is the mass of the black hole). These horizons separate the regions from which no escape is possible from the black hole's gravitational pull.

Future and Past Infinity. The points at the top and bottom of the diagram represent future and past timelike infinity (i^+ and i^-), where worldlines (geodesics) of particles end up as $t \to +\infty$ and $t \to -\infty$, respectively.

The lines going off to the right and left edges represent future and past null infinity ($\mathcal{I}^+$ and $\mathcal{I}^-$), where light rays end up as $t \to \pm\infty$.

The vertical curved blue lines represent surfaces where the radial coordinate r is constant, while the horizontal curved magenta lines represent surfaces where the time is constant.

Causal Structure.

- **Light Cones**: The 45-degree lines (diagonal lines in the diagram) represent the paths that light rays would take. The causal structure of spacetime can be inferred by following these lines.

- **Future and Past**: Movement towards the top of the diagram represents moving forward in time, while movement towards the bottom represents moving backwards in time.

And introduce the new coordinates (U, V) defined (for $r > 2M$) by

$$U = -e^{-u/4M} \quad , \quad V = e^{v/4M} \tag{182}$$

in terms of which the metric is

$$ds^2 = -\frac{32M^3}{r}e^{-r/2M}dU\,dV + r^2 ds^2(S^2) \tag{183}$$

where $r(U, V))$ is given implicitly by $UV = -e^{r^*/2M}$ or

$$UV = -e^{r^*/2M} = -\left(\frac{r - 2M}{2M}\right)e^{r/2M} \tag{184}$$

Initially the metric is defined for $U < 0$ and $V > 0$, but it can be extended by analytic continuation to $U > 0$ and $V < 0$. Note that $r = 2M$ corresponds to $UV = 0$, i.e. either $U = 0$ or $V = 0$. The singularity at $r = 0$ corresponds to $UV = 1$.

If we were to plot this description of spacetime in a diagram, we will end up with Figure (4).

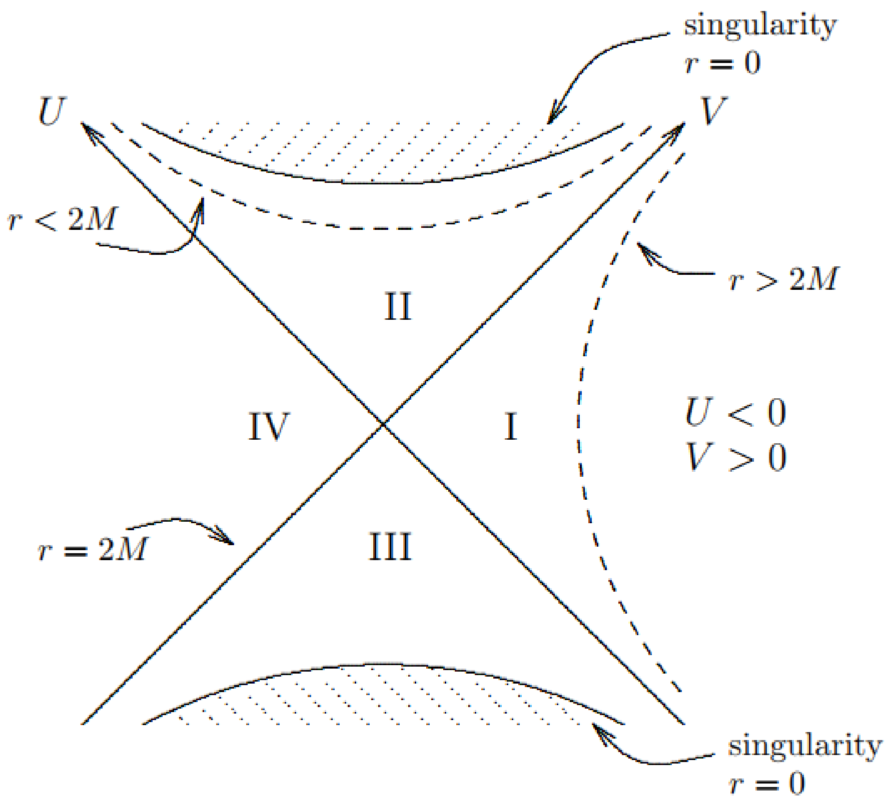

Figure 4: Kruskal spacetime

Since the Kruskal-Szekeres coordinates resolve the issues with the problematic points of the Schwarzschild metric by extending the Schwarzschild solution beyond the event horizon, thus providing a complete and nonsingular description of the black hole's spacetime, this is the maximum analytic continuation of the Schwarzschild metric. We see that there are four regions

- Region I: The external regions where $r > 2M$, representing the universe outside the black hole.

- Region II: The regions inside the event horizons but outside the singularity, known as the black hole interior.
- Region III: The regions inside another event horizon in the white hole part of the extended solution, which is a time-reversed black hole.
- Region IV: The external regions to the other side of the white hole.

What's the physical interpretation?

- **Event Horizon (EH)**: The boundaries between regions I and II (and similarly, regions III and IV) are the event horizons. Particles or light that cross these boundaries can no longer escape to the region I (the external universe).
- **Singularity**: The wavy lines labeled $r = 0$ indicate the singularities where the spacetime curvature becomes infinite. These are inside the black holes.
- **Multiple Universes**: Regions I and IV can be interpreted as separate asymptotically flat universes connected by a black hole and a white hole.

Its usual to plot lines of constant U and V (outgoing or ingoing radial null geodesics) as 45 axis. There are four regions which depend on the signs of U and V. However, the only regions relevant to gravitational collapse are the I and II because the other regions are then replaced by the star's interior, e.g. for collapse of homogeneous ball of pressure-free fluid, represented in Figure (5). Regions I and II are also covered by the ingoing Eddington-Finkelstein coordinates. Similarly, regions I and III are those relevant to a white hole.

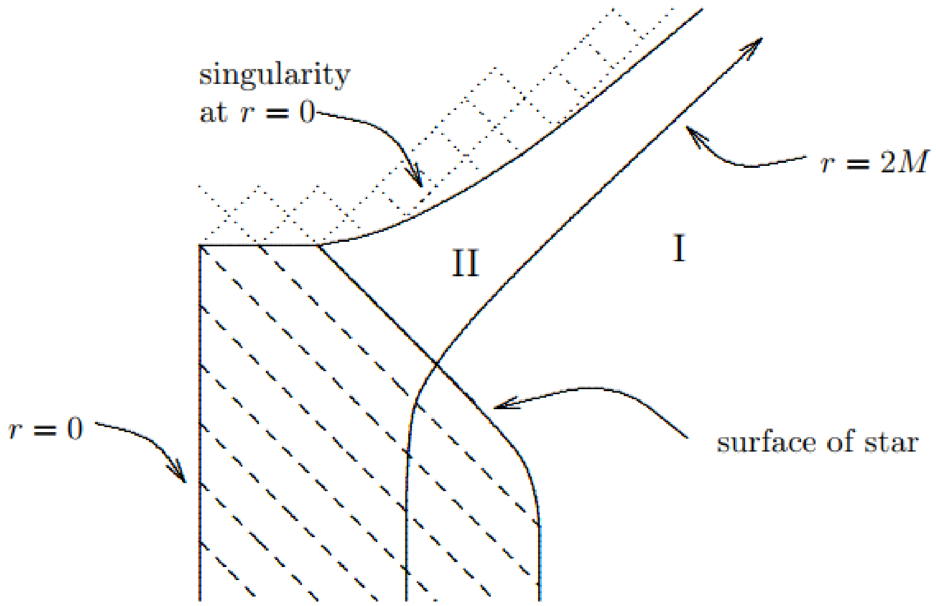

Figure 5: Kruskal spacetime

So the white hole and second universe have disappeared, but the horizon and singularity at $r = 0$ remain!

The Schwarzschild black hole is only the simplest among a number of black hole solutions to the Einstein equations. In fact, the astrophysical black holes for which we have observational evidence appear to be rotating and have charge, while the Schwarzschild BH doesn't account for any of these features. In this section we review two further, more general, black hole solutions.

3.5 Charged Black Holes

Among the properties a black hole can have, one of them is charge. This is the case we now focus on, which receives the name of **Reissner-Nordström** black hole. To start with, let's use the Einstein-Maxwell action, accounting for both the mass (encoded in the curvature, encoded in R) and electric charge (electromagnetic tensor), that is, gravity coupled to the electromagnetic field

$$S = \frac{1}{16\pi G} \int d^4x \sqrt{-g} \left[R - F_{\mu\nu} F^{\mu\nu} \right] \tag{185}$$

where $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$.

The Maxwell term appearing in the action is normalized such that the Coulomb force between two charges Q_1 and Q_2 separated by a (large enough) distance is

$$\frac{G|Q_1 Q_2|}{r^2} \tag{186}$$

This corresponds to *geometrized units of charge.*
The equations of motion are

$$G_{\mu\nu} = 2F_{\mu\lambda} F^{\lambda}{}_{\nu} - \frac{1}{2} g_{\mu\nu} F_{\rho\sigma} F^{\rho\sigma} \tag{187}$$

$$\partial_\mu F^{\mu\nu} = 0 \tag{188}$$

which admit the usual spherically symmetric solution

$$ds^2 = -V(r)dt^2 + V(r)^{-1} dr^2 + r^2 ds^2(S^2) \tag{189}$$

with

$$V(r) = 1 - \frac{2M}{r} + \frac{Q^2}{r^2} \tag{190}$$

which is known as the Reissner-Nordström solution. The electric potential is defined as

$$A_t = \frac{Q}{r} \qquad , \qquad A_i = 0 \tag{191}$$

We interpret Q as the charge of the black hole (by analogy with the electric potential of a point charge) and M as its mass. We assume that $Q > 0$. Just like in the Scharzchild case, here also exist a theorem like Birkhoff's theorem, which guarantees that the Reissner-Nordström solution is the unique asymptotically flat, spherically symmetric solution to the Einstein-Maxwell equations. It's often refered to as *Reissner-Nordström no-hair theorem.*

Now, instead of just one problematic radius, we have two, where $V(r) = 0$, which are

$$r_\pm = M \pm \sqrt{M^2 - Q^2} \tag{192}$$

It is convenient to introduce the function

$$\Delta = Q^2 - 2Mr + r^2 = (r - r_+)(r - r_-) \tag{193}$$

allowing us to rewrite the metric as

$$ds^2 = -\frac{\Delta}{r^2}dt^2 + \frac{r^2}{\Delta}dr^2 + r^2 ds^2(S^2) \tag{194}$$

There are three separate cases to look at: $Q > M$, $Q < M$, and $Q = M$. Let's consider them in turn.

3.5.1 Super-Extremal RN: $Q > M$

If $Q > M$ then $r_\pm \in \mathbb{C}$ and the metric is regular for all $r > 0$. However, there's still a curvature singularity at $r = 0$, which doesn't lie behind a horizon. This is why these kind of clack holes are believed to be non-physical, since it would mean that a black hole have formed with with $GM2 < Q^2$, stating that the total energy of the hole is less than the contribution to the energy from the electromagnetic fields alone[1].

For that reason we assume that the **Cosmic Censorship Conjecture** is true.

This conjecture was formulated by Roger Penrose, stating that "*Nature abhors a naked singularity*", that is, naked singularities (except for the Big Bang) are unphysical and do not occur in the real world because the gravitational collapse of physical matter configurations would never produce such a thing[2]. In fact, we should not ever expect to find

As a curiosity, could it be that an electron is just a charged black hole? The answer is no, because the electron is a quantum mechanical object, whose Compton wavelength $\lambda = \frac{h}{mc} = 2.4 \times 10^{-12}$ m is much larger than its Schwarzschild radius $r_s = \frac{2Gm_e}{c^2} = 1.4 \times 10^{-57}$ m.

3.5.2 Sub-Extremal RN: $Q < M$

Now Δ has two real roots $r_+ > r_-$ creating two coordinate singularities. As always, we can avoid them if we find a suitable coordinate system. Following our strategy with the Schwarzschild metric, let us define a tortoise coordinate r_*

$$\frac{\Delta}{r^2}dr_*^2 = \frac{r^2}{\Delta}dr^2, \tag{195}$$

ithen

$$ds^2 = -\frac{\Delta}{r^2}(dt^2 - dr_*^2) + r^2 ds^2(S^2). \tag{196}$$

Radial null geodesics are follow the equation $t \pm r_* = \text{const}$ with $\theta = \phi = \text{const}$. A solution of (2.6) with a convenient choice of sign and integration constant is

$$r_* = r + \frac{1}{2\kappa_+}\ln\left(\frac{r - r_+}{r}\right) + \frac{1}{2\kappa_-}\ln\left(\frac{r - r_-}{r}\right), \tag{197}$$

where

$$\kappa_+ = \frac{r_+ - r_-}{2r_+^2} > 0 \quad \text{and} \quad \kappa_- = \frac{r_- - r_+}{2r_-^2} < 0. \tag{198}$$

If we define the null coordinates $u = t - r_*$ and $v = t + r_*$ and ingoing Eddington-Finkelstein coordinates (v, r, θ, ϕ). In terms of the latter, the metric becomes

$$ds^2 = -\frac{\Delta}{r^2}dv^2 + 2dvdr + r^2 ds^2(S^2), \tag{199}$$

[1] That is, the mass of the matter which carried the charge would have had to be negative.

[2] Of course, it's just a conjecture, and it may not be right; there are some claims from numerical simulations that collapse of spindle-like configurations can lead to naked singularities.

which is regular for all $r > 0$, including $r = r_+$ and $r = r_-$. Since it has an abstract form, in order to understand the spacetime structure close to $r = r_\pm$ we can use two different sets of Kruskal-type coordinates at each of the two radii:

$$U_\pm = -\exp(-\kappa_\pm u) \quad \text{and} \quad V_\pm = \exp(\kappa_\pm v). \tag{200}$$

This gives rise to the Penrose diagram shown in Figure (6).

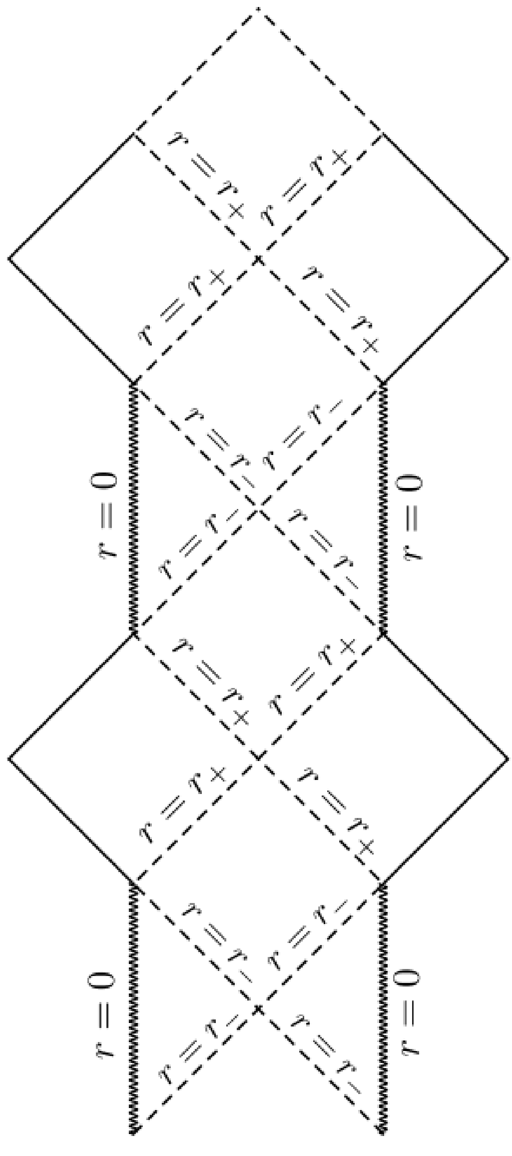

Figure 6: Penrose diagram for the sub-extremal Reissner-Nordström solution.

The diagram shows an infinite extension of the Schwarzschild solution, being able to see each diamond-shaped section as a separate patch of spacetime. The diagram is symmetric around the vertical axis, indicating that the spacetime can be extended into an infinite series of black holes and white holes.

Notice that a timelike trajectory can avoid $r = 0$, since the $r = 0$ singularity is timelike itself. In fact, to hit $r = 0$, one must accelerate toward it (this time it is like a position in space).

3.5.3 Extremal RN: $Q = M$

The metric of the extremal Reissner-Nordström solution is

$$ds^2 = -\left(1 - \frac{M}{r}\right)^2 dt^2 + \left(1 - \frac{M}{r}\right)^{-2} dr^2 + r^2 ds^2(S^2), \tag{201}$$

which has one coordinate singularity at $r = r_+ = r_- = M$. To get rid of it, define the tortoise coordinate $dr_* = \left(1 - \frac{M}{r}\right)^{-2} dr$ so

$$ds^2 = -\left(1 - \frac{M}{r}\right)^2 (dt^2 - dr_*^2) + r^2 ds^2(S^2), \tag{202}$$

and change to ingoing Eddington-Finkelstein coordinates (v, r, θ, ϕ). We get

$$ds^2 = -\left(1 - \frac{M}{r}\right) dv^2 + 2dvdr + r^2 ds^2(S^2), \tag{203}$$

which is regular at $r = M$. The inner and outer horizons have now coalesced. The result is the Penrose diagram shown in Figure (7).

An interesting feature of the extremal case is that the near horizon limit of BH is now $\mathrm{AdS}_2 \times S^2$. To see this we express the BH geometry as

$$\begin{aligned} ds^2 &= -\left(1 - \frac{MG}{r}\right)^2 dt^2 + \left(1 - \frac{MG}{r}\right)^{-2} dr^2 + r^2 ds^2(S^2) \\ &= -\left(1 + \frac{MG}{v}\right)^{-2} dt^2 + \left(1 + \frac{MG}{v}\right)^2 \left[dv^2 + v^2 ds^2(S^2)\right] \end{aligned} \tag{204}$$

where $v = r - MG$. Near the horizon, $v \to 0$, so

$$\begin{aligned} \left(1 + \frac{MG}{v}\right)^{-2} &= \left(\frac{MG}{v}\right)^{-2} \frac{1}{\left(\frac{v}{MG} + 1\right)^2} \simeq \frac{v^2}{(MG)^2} \cdot \frac{1}{1 + \frac{v}{MG}} \\ &\simeq \frac{v^2}{(MG)^2} \left(1 - \frac{v}{MG}\right) \simeq \frac{v^2}{(MG)^2} \end{aligned} \tag{205}$$

so

$$ds^2 \simeq -\frac{v^2}{(MG)^2} dt^2 + \frac{(MG)^2}{v^2} dv^2 + (MG)^2 ds^2(S^2) \tag{206}$$

and defining $z = \frac{(MG)^2}{v} \quad \Longrightarrow \quad dz = -\frac{(MG)^2}{v^2} dv = -\frac{z^2}{(MG)^2} dv$

$$\begin{aligned} ds^2 &\approx -\frac{1}{z^2}(MG)^2 dt^2 + \frac{z^2}{(MG)^2} dv^2 + (MG)^2 ds^2(S^2) \\ &= -\frac{1}{z^2}(MG)^2 dt^2 + \frac{z^2}{(MG)^2} \cdot \frac{(MG)^4}{z^4} dz^2 + (MG)^2 ds^2(S^2) \\ &= \frac{GM^2}{z^2}\left[-dt^2 + dz^2\right] + GM^2 ds^2(S^2) \\ &= GM^2[ds^2\left(AdS_2\right) + ds^2\left(S^2\right)] \end{aligned} \tag{207}$$

where we see that near the horizon is indeed $\mathrm{AdS}_2 \times S^2$.

3.6 Rotating Black Holes

We've only discussed solutions with spherical symmetry. Let's study the Kerr-Newman solution to the Einstein-Maxwell equations, which describes a rotating charged black hole of mass M, charge Q and angular momentum $J = Ma$ with a being the angular momentum per unit mass. This type of BH is

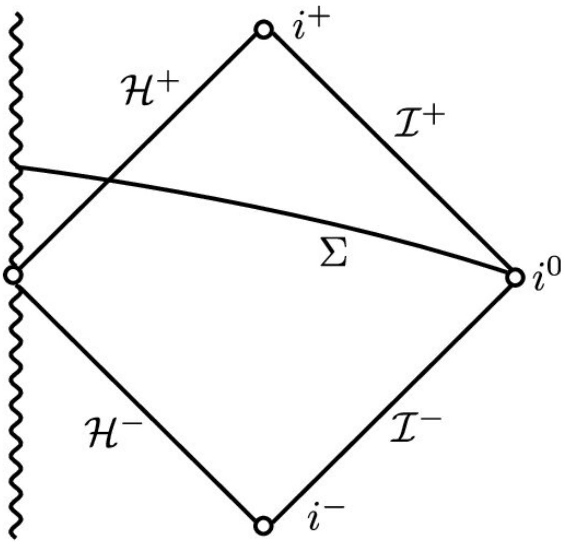

Figure 7: Penrose diagram for extreme RN black hole.

astrophysically important, since it is a good approximation to the metric outside of a rotating star at large distances. In Boyer-Lindquist coordinates (t, r, θ, ϕ), in which the black hole rotates about the polar axis, the metric reads

$$ds^2 = -\left(\frac{\Delta - a^2\sin^2\theta}{\Sigma}\right)dt^2 + \frac{\Sigma}{\Delta}dr^2 + \Sigma d\theta^2 + \left(\frac{(r^2+a^2)^2 - \Delta a^2\sin^2\theta}{\Sigma}\right)\sin^2\theta d\phi^2 - \\ -\frac{2a\sin^2\theta(r^2+a^2-\Delta)}{\Sigma}dtd\phi \tag{208}$$

where

$$\Sigma^2 = r^2 + a^2\cos^2\theta, \quad \Delta = r^2 - 2Mr + a^2 + Q^2. \tag{209}$$

The components of the electromagnetic potential are

$$A_t = \frac{Qr}{\Sigma}, \quad A_\phi = \frac{Qar\sin^2\theta}{\Sigma}, \quad A_r = A_\theta = 0. \tag{210}$$

This contains the previous BH geodesics as limiting cases. So for $a = Q = 0$, we recover the Schwarzschild solution. For $a = 0$, we recover the Reissner-Nordström solution. Finally, the solution is symmetric under the simultaneous replacements $\phi \to -\phi$ and $a \to -a$, so we can set $a \geq 0$ without loss of generality.

We cannot use the same reasoning as in the spherically symmetric case during gravitational collapse with rotating matter to argue that, on the surface of the collapsing matter, the metric should be of the form given above when a black hole is rotating, since there is no analogue of Birkhoff's theorem. All we can say is that, after enough time has passed and matter and spacetime have "settled down" to equilibrium, they will be described by the Kerr-Newman solution.

It's worth investigating the structure of the simple special case of a rotating black hole with zero charge $Q = 0$. The metric then reduces to the *Kerr solution*

$$ds^2 = \Sigma\left(\frac{dr^2}{\Delta} + d\theta^2\right) + (r^2+a^2)\sin^2\theta d\phi^2 + \frac{2Mr}{\Sigma}\left(a\sin^2\theta d\phi - dt\right)^2 - dt^2, \tag{211}$$

where Δ and Σ are redefined accordingly. This metric is a solution to the vacuum Einstein equations. The coordinate singularities happen when

$$\Delta = 0 \Longrightarrow r = r_{\pm} = M \pm \sqrt{M^2 - a^2} \tag{212}$$

Now we also have a curvature singularity (a real one, not just from the coordinate system used) at

$$\Sigma = 0 \Longrightarrow r = 0 \quad and \quad \cos\theta = 0 \tag{213}$$

This is interesting. It's telling us that only when we approach with an angle $\theta = \pi/2$ the singularity conditions are fulfilled and therefore it exists, i.e. when $r = 0$ is approached along the equator. When approached from any other angle, there is no singularity at $r = 0$!, i.e, it has a "ring" like singularity.

From the three cases to consider: $M < a$, $M = a$ and $M > a$ we will concentrate on the $M > a$ case, for which there are two coordinate singularities at r_+ (the "outer" horizon) and r_- (the "inner" horizon). To remove them, we do a coordinate transformation to ingoing Kerr coordinates (v, r, θ, χ), where $v = t + r^*$ and r^* and χ are defined by

$$dr^* = \frac{r^2 + a^2}{\Delta} dr, \quad d\chi = d\phi + \frac{a}{\Delta} dr. \tag{214}$$

We see from the definition of χ that a constant ϕ angle doesn't correspond to $\chi =$ const. For example, in order to stay at $\chi =$ const. as you fall inwards ($dr < 0$), you need to rotate to: $d\phi = -\frac{a}{\Delta} dr$. In terms of ingoing Kerr coordinates the metric becomes

$$\begin{aligned} ds^2 = &-\left(\frac{\Delta - a^2 \sin^2\theta}{\Sigma}\right) dv^2 + 2dvdr - 2\frac{a \sin^2\theta (r^2 + a^2 - \Delta)}{\Sigma} dv d\chi \\ &- 2a \sin^2\theta d\chi dr + \left[\frac{(r^2 + a^2)^2 - \Delta a^2 \sin^2\theta}{\Sigma}\right] \sin^2\theta d\chi^2 + \Sigma d\theta^2 \end{aligned} \tag{215}$$

Like before, we've solved the coordinate singularities problem, since there are no more factors of Δ in the numerators and the metric is regular at r_+ and r_-. We still have, of course, the curvature singularity at $\rho^2 = 0$.

To draw the Penrose diagram is more difficult because the metric is not spherically symmetric. Since the curvature singularity at $r = 0$ only appears when $\theta = \pi/2$, the Penrose diagram should look very different for $\theta \neq \pi/2$ and $\theta = \pi/2$. In order to represent both cases, it is customary to draw a Penrose diagram that is an amalgam of the Penrose diagram for an observer falling in from the north pole ($\theta = 0$) and of that for an observer falling in in the equatorial plane ($\theta = \pi/2$) at fixed $\tilde{\phi}$. Notice that $\tilde{\phi} =$ const. means that ϕ is not constant, so the observer falling in at $\theta = \pi/2$ rotates about the polar axis.

The procedure is very similar to that for the sub-extremal Reissner-Nordström solution. First, perform a coordinate transformation to coordinates $(u, v, \theta, \tilde{\phi})$ where $u = t - r^*$ and $v = t + r^*$ with r^* as defined in (2.20). Then, define Kruskal-type coordinates $U_{\pm}$ and $V_{\pm}$ close to $r = r_{\pm}$, respectively, and draw the Penrose diagram. This leads to the infinitely sequence of spacetime regions we saw in Figure 14. Up to this point, the analysis is identical for $\theta = 0$ and $\theta = \pi/2$. The only difference is that the Penrose diagram for $\theta = 0$ has a curvature singularity at $r = 0$, whereas the Penrose diagram for $\theta = \pi/2$ has none. In the amalgam Penrose diagram for the Kerr spacetime, we indicate this by drawing an interrupted wavy line at $r = 0$. The result is shown in Figure (8).

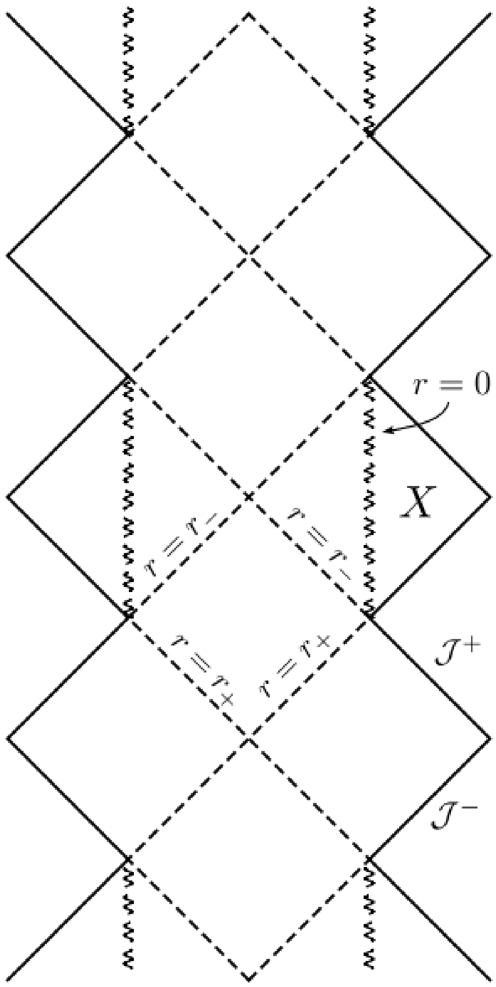

Figure 8: Penrose diagram for the sub-extremal Kerr black hole.

3.6.1 The ergoregion

In Schwarzschild spacetime, you (or a test particle) can travel along integral curves of the Killing vector $k = \partial_t$ anywhere outside the horizon, appearing stationary[3] to observers at infinity, since your position in space is not changing. This was possible because g_{00} is negative everywhere for $r > 2M$, so that $k^2 = g_{00}$ is negative, meaning that the integral curves of k are timelike.

And here's the interesting part; this is not the case in Kerr spacetime: there is a region around the outer horizon, called the *ergosphere*[4] or *ergoregion*, in which it is impossible for anything to remain stationary with respect to observers at infinity. That is, everything rotates. It's represented in Figure (9). This happens because

$$g_{00} = -\left(1 - \frac{2Mr}{\Sigma}\right) \tag{3.25}$$

becomes positive in the region

$$\frac{2Mr}{\Sigma} > 1 \implies \xi(r) \equiv \Sigma - 2Mr = r^2 + a^2\cos^2\theta - 2Mr < 0, \tag{3.26}$$

part of which lies outside the outer horizon $r = r_+$ when $a \neq 0$. This is easy to see by noting that

[3]Not moving in space from the point of view of the an external observer.
[4]It's usully not used because it's not a sphere, but a oblate spheroidal.

the equation for $\xi(r)$ is a parabola with roots at $\tilde{r}_\pm = M \pm \sqrt{M^2 - a^2\cos^2\theta}$ with $\tilde{r}_+$ being bigger than $r_+ = M + \sqrt{M^2 - a^2}$ for $\theta \neq 0, \pi$. Hence g_{00} is positive in the ellipsoidal region $r_+ < r < \tilde{r}_+$, which has a maximum extent on the equator $\theta = \frac{\pi}{2}$ where $\tilde{r}_+ = M + \sqrt{M^2 + a^2}$.

In the ergoregion, the orbits of $k = \partial_t$ are not timelike, preventing anything from traveling along them while remaining stationary relative to distant observers. For a curve $x^\mu = (t, r, \theta, \phi)$ to be timelike, its tangent vector $u^\mu = \frac{dx^\mu}{d\tau}$ must satisfy $u^2 = -1$. However, within the ergoregion, every term in $u^2 = g_{\mu\nu}u^\mu u^\nu$ is positive, except for $g_{t\phi}u^t u^\phi$, which implies that $u^\phi = \frac{d\phi}{d\tau}$ cannot be zero. Furthermore, since $u^t > 0$ for a future-directed worldline and $g_{t\phi} < 0$, u^ϕ must be positive. Consequently, any timelike worldline is dragged in the direction of the black hole's rotation, a phenomenon known as *frame dragging*.

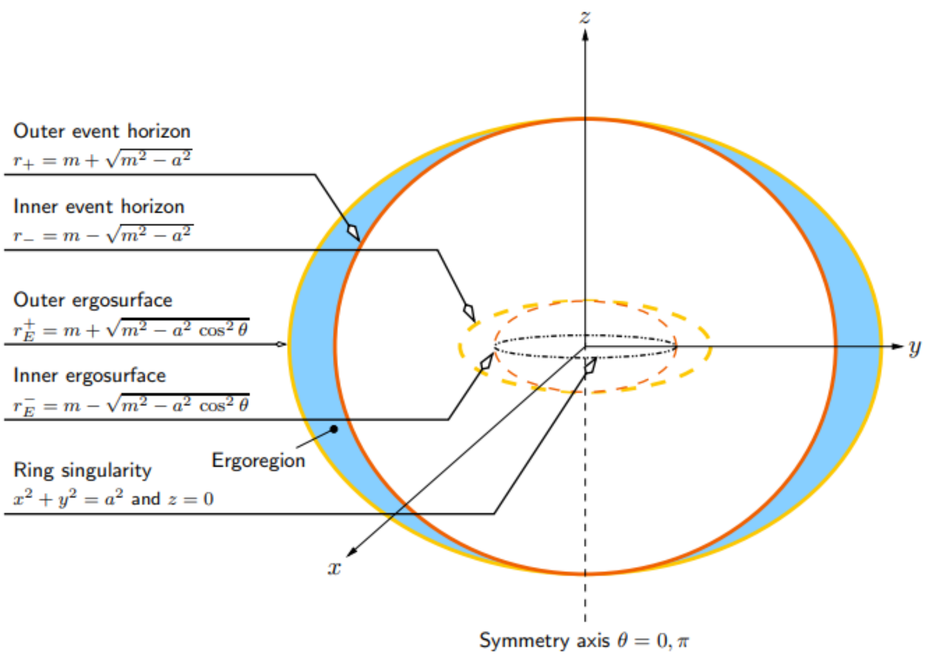

Figure 9: Schematic location of the horizons, ergosurfaces, and curvature singularity in the Kerr spacetime.

4 Supergravity

As we have seen, general relativity can be coupled to scalars φ, gauge fields A_μ and through the vielbein formalism also spinors ψ. Including also the metric, that means we have fields of the following spins

	φ	ψ	A_μ	?	$g_{\mu\nu}$
s	0	$\frac{1}{2}$	1	$\frac{3}{2}$	2

This begs the question: what about spin $\frac{3}{2}$ fields? These can indeed be included, which leads to supergravity, which enjoys an additional symmetry called "Supersymmetry".

4.1 The free Rarita–Schwinger field

As a first approximation to supergravity, we will study the free spin-3/2 field, the Rarita-Schwinger field Ψ_μ, referred to as a gravitini. The free limit means that the various fields don't interact, allowing us to consider them separately. Its structure is the one of a vector whose components are all spinors. As such, it transforms under both local Lorentz transformations and diffeomorphisms as

$$\Psi_\mu \to \Omega\Psi_\mu \quad \Longrightarrow \quad \delta_\Omega \Psi_\mu = \lambda^{ab}\gamma_{ab}\Psi_\mu \quad \text{and} \quad \Psi'_\mu \to \frac{\partial x^\nu}{\partial x'^\mu}\Psi_\nu \tag{216}$$

which also has the gauge symmetry

$$\Psi_\mu \to \Psi_\mu + \nabla_\mu \epsilon \quad \Longrightarrow \quad \delta_\epsilon \Psi_\mu = \nabla_\mu \epsilon \tag{217}$$

where $\epsilon = \epsilon(x)$ is a spinor.

The action of ψ_μ is given by

$$S_{\Psi_\mu} = i\int d^4x\sqrt{-g}\bar{\Psi}_\mu \gamma^{\mu\nu\rho}\nabla_\nu \Psi_\rho, \tag{218}$$

where $\bar{\Psi}_\mu = (\gamma^0\Psi_\mu)^\dagger$. The equation of motion reads

$$\gamma^{\mu\nu\rho}\nabla_\nu\Psi_\rho = 0, \tag{219}$$

which has some resemblance with its electromagnetic analogous $\nabla_\mu F^{\mu\nu} = 0$. Under an infinitesimal gauge transformation $\delta_\epsilon\Psi_\mu = \nabla_\mu\epsilon$ transforms as

$$\delta_\epsilon S_{\Psi_\mu} = \int d^4x\sqrt{-g}\left(-iG_{\mu\nu}\bar{\epsilon}\gamma^\mu\Psi^\nu\right) \tag{220}$$

up to boundary terms, where $G_{\mu\nu}$ is the Einstein tensor. Thus, in Ricci flat spaces $R_{\mu\nu} = 0$, the action is indeed gauge invariant.

The solution also suggest that gravity (encoded in the Einstein tensor) affects general ϵ_α-gauge invariance since

$$\delta_\epsilon S_\Psi \propto G_{\mu\nu}\bar{\epsilon}\gamma^\mu\Psi^\nu \neq 0 \tag{221}$$

in general. The construction of a theory that is fully gauge invariant leads to supergravity.

4.2 $\mathcal{N} = 1$ supergravity with $\Lambda = 0$

To construct a theory that is fully gauge invariant, we need to introduce a new type of symmetry, "supersymmetry". We will focus on the minimal $\mathcal{N} = 1$ $D = 4$ case, minimal supergravity in this section since it contains a part of the action common to all supergravity theories.

To see how this works, we can consider the combined action

$$S = S_g + S_{\psi_\mu} \tag{222}$$

$$S_g = \frac{1}{8\pi G}\int d^4x\sqrt{-g}g^{\mu\nu}R_{\mu\nu} = \frac{1}{8\pi G}\int d^4x\sqrt{-g}\left(e^\mu_a e^\nu_b R_{\mu\nu}{}^{ab}(e)\right) \tag{223}$$

$$S_\psi = -\frac{i}{8\pi G}\int d^4x\sqrt{-g}\bar{\Psi}_\mu\gamma^{\mu\nu\rho}\nabla_\nu\Psi_\rho \tag{224}$$

where we assume no torsion and that Ψ_μ is "Majorana". This means there exists an intertwiner B such that

$$B^{-1}\gamma_\mu B = \gamma^*_\mu \tag{225}$$

in terms of which $\Psi^c_\mu \equiv B\Psi^*_\mu = \Psi_\mu$.

The idea is to supplement $\delta_\epsilon \Psi_\mu = \nabla_\mu \epsilon$ by some transformation δe_μ^a such that $\delta_\epsilon S = 0$. Working up to quadratic order in Ψ_μ, it is possible to show

$$\delta_\epsilon e_\mu^a = \frac{i}{2}\bar{\epsilon}\gamma^a \Psi_\mu \quad , \quad \delta_\epsilon \Psi_\mu = \nabla_\mu \epsilon \tag{226}$$

indeed leaves the combined action invariant.

However, we are left with some $\bar{\epsilon}\Psi_\mu^3$ terms that do not cancel.

To resolve this issue, it is necessary to generalize the action of Ψ_μ to include quadratic terms. This leads to $S = S_g + S_{\Psi_\mu}$ where

$$S_\Psi = -\frac{i}{8\pi G}\int d^4x\sqrt{-g}\left(\bar{\Psi}_\mu \gamma^{\mu\nu\rho}\nabla_\nu \Psi_\rho + \mathcal{L}_{\text{Quartic}}\right) \tag{227}$$

where

$$\mathcal{L}_{\text{Quartic}} = -\frac{1}{16}[\left(\bar{\Psi}^\rho \gamma^\mu \Psi^\nu\right)\left(\bar{\Psi}_\rho \gamma_\mu \Psi_\nu + 2\bar{\Psi}_\rho \gamma^\nu \Psi_\mu\right) - 4\left(\bar{\Psi}_\mu \gamma^\rho \Psi_\rho\right)\left(\bar{\Psi}^\mu \gamma^\nu \Psi_\nu\right)] \tag{228}$$

and the transformation rules become

$$\delta_\epsilon e_\mu^a = \frac{i}{2}\bar{\epsilon}\gamma^a \Psi_\mu \tag{229}$$

$$\delta_\epsilon \Psi_\mu = \hat{\nabla}_\mu \epsilon = \nabla_\mu \epsilon + K_{\mu\nu\rho}\gamma^{\nu\rho}\epsilon \tag{230}$$

with

$$K_{\mu\nu\rho} = -\frac{i}{4}\left(\bar{\Psi}_\mu \gamma_\rho \Psi_\nu - \bar{\Psi}_\nu \gamma_\mu \Psi_\rho + \bar{\Psi}_\rho \gamma_\nu \Psi_\mu\right) \tag{231}$$

This makes the theory fully ϵ gauge invariant, and in the process, we have introduced local supersymmetry. The classical theory of its own is not very interesting, as we should fix Ψ_μ to zero, meaning that (230) reduces to

$$\nabla_\mu \epsilon = 0 \tag{232}$$

while we have restricted to vacuum solutions of GR ($R_{\mu\nu} = 0$).

It is possible to add matter to $\mathcal{N} = 1$ supergravity in such a way that supersymmetry is preserved, leading to more interesting theories, even classically.

Such generalizations obey an equation similar to (232) and in many cases, this can be shown to imply Einstein's equations, which is helpful because solving (232) or its generalizations is often much easier.

4.3 Anti-de Sitter supergravity

If we want to expand the supergravity theory described so far, one of the ways to do so is to derive it in another type of spacetime, let's say AdS_n rather than Minkowski. We need a modified covariant derivative $\hat{\nabla}_\mu$, which acts on spinors as

$$\hat{\nabla}_\mu \epsilon \equiv \left(\nabla_\mu - \frac{1}{2L}\gamma_\mu\right)\epsilon = \left(\partial_\mu + \frac{1}{4}\omega_{\mu ab}\gamma^{ab} - \frac{1}{2L}\gamma_\mu\right)\epsilon \tag{233}$$

where it's commutator

$$\left[\hat{\nabla}_\mu, \hat{\nabla}_\nu\right]\epsilon = \frac{1}{4}\left(R_{\mu\nu ab}(e) + \frac{1}{L^2}(e_{a\mu}e_{b\nu} - e_{b\mu}e_{a\nu})\right)\gamma^{ab}\epsilon \\ \equiv \frac{1}{4}\hat{R}_{\mu\nu ab}(e)\gamma^{ab}\epsilon \tag{234}$$

and

$$\hat{R}_{\mu a} \equiv \hat{R}_{\mu\nu ab}e^{b\nu} = R_{\mu a} + \frac{3}{L^2}e_{a\mu} \tag{235}$$

$$\hat{R} \equiv \hat{R}_{\mu a}e^{a\mu} = R + \frac{12}{L^2} \tag{236}$$

As a starting point, we consider the action (222) and replace $\nabla_\nu \to \hat{\nabla}_\nu$. Afterwards we add a cosmological term whose value is so $R_{\mu\nu} = 0$. We find

$$S = \frac{1}{8\pi G}\int d^4x\,\sqrt{-g}\left(R - i\bar{\Psi}_\mu\gamma^{\mu\nu\rho}\hat{\nabla}_\nu\Psi_\rho + \frac{6}{L^2}\right) \\ = \frac{1}{8\pi G}\int d^4x\,\sqrt{-g}\left(R - i\bar{\Psi}_\mu\gamma^{\mu\nu\rho}\nabla_\nu\Psi_\rho - \frac{i}{L}\bar{\Psi}_\mu\gamma^{\mu\nu}\Psi_\nu + \frac{6}{L^2}\right) \tag{237}$$

The term "anti" in anti-de Sitter comes from the constant negative potential term appearing in (237), which can be viewed as a negative cosmological constant[5]. We can be tempted to identify the term $\propto \Psi_\mu^2$ as a mass-like term, where we identify

$$m_\Psi = \frac{1}{L} \tag{238}$$

but this is a wrong interpretation, since the true nature of (237) is a description of a *massless* gravitino in an AdS_4 background geometry.

4.4 $\mathcal{N} = 2$ minimal gauge supergravity

Among all the black-hole solutions we've review earlier, the extreme Reissner-Nordström (RN) black hole occupies a special position because of its complete stability with respect to both classical and quantum processes. This case is special since it admits supersymmetry within the context of $\mathcal{N} = 2$ (ungauged) supergravity. Analogues of the RN solutions to Einstein-Maxwell theory with a cosmological constant Λ have been known for some time, that's why we're concerned with identifying cosmological analogues of the extreme RN black holes with respect to supersymmetry in section 5.

In the following, we will define what is necessary in the final section to address the RN solutions classification within the context of $\mathcal{N} = 2$ gauged supergravity, and then we will assume that the gravitini vanish in the background, with the field equations derived from (239) becoming the Einstein-Maxwell equations defined above, with $\Lambda = -3g^2$. This allows us to use the solutions of a cosmological RN black hole as background solutions to gauged $\mathcal{N} = 2$ supergravity.

The Lagrangian of the theory has four fields, namely graviton, gravitini (really two Majorana gravitini combined into a single complex gravitini $\Psi_\mu \equiv \Psi_\mu^1 + i\Psi_\mu^2$) and a Maxwell vector field A_μ minimally coupled to the gravitini with strength g ($g \neq 0$ is what we mean by gauge supergravity). The action is

[5] In de Sitter moels, used for actual cosmological descriptions of the universe, the sign is the opposite.

$$S = \int d^4x\sqrt{-g}\Big[R - 2i\bar{\Psi}_\mu\gamma^{\mu\nu\rho}\mathcal{D}_\nu\Psi_\rho - F_{\mu\nu}F^{\mu\nu} + \frac{1}{2}(F+\tilde{F})^{\mu\nu}\bar{\Psi}_\rho\gamma_{[\mu}\gamma^{\rho\sigma}\gamma_{\nu]}\Psi_\sigma + \\ +2ig\bar{\Psi}_\mu\gamma^{\mu\nu}\Psi_\nu + 6g^2\Big] \tag{239}$$

where supersymmetry fixes $\Lambda = -3g^2$. The Lorentz- and gauge-covariant derivative $\mathcal{D}_\mu$ acting on spinorial objects is defined by

$$\mathcal{D}_\mu = \nabla_\mu - igA_\mu \tag{240}$$

in terms of the Lorentz-covariant derivative $\nabla_\mu = \partial_\mu + \frac{1}{4}(\omega_\mu)_{ab}\gamma^{ab}$. The supercovariant field strength is

$$\hat{F}_{\mu\nu} = F_{\mu\nu} - \text{Im}\left(\bar{\Psi}_\mu\Psi_\nu\right) \tag{241}$$

which will become the well known electromagnetic tensor when we evaluate the theory in the background.

The action (239) is invariant under the $\mathcal{N}=2$ supersymmetry transformations

$$\begin{aligned}\delta e^a_\mu &= \mathbb{R}\text{e}(\bar{\epsilon}\gamma^a\Psi_\mu)\\ \delta\Psi_\mu &= \hat{\nabla}_\mu\epsilon\\ \delta A_\mu &= \mathbb{I}\text{m}(\bar{\epsilon}\Psi_\mu)\end{aligned}$$

where ϵ is a Dirac spinor. The supercovariant derivative is

$$\hat{\nabla}_\mu \equiv \mathcal{D}_\mu + \frac{1}{2}g\gamma_\mu + \frac{i}{4}\hat{F}_{ab}\gamma^{ab}\gamma_\mu. \tag{242}$$

Before moving on to the more interesting problem of supersymmetric black holes in section 5, let us consider some simple solutions on $mathcalN = 2$ gauged and ungauged supergravity.

4.4.1 AdS$_4$

Before any computation, it's worth defining some useful Clifford algebra identities

$$\gamma^{n_1\dots n_k}\gamma^m = (-1)^k\left(\gamma^{mn_1\dots n_k} - k\,g^{m[n_1}\gamma^{n_2\dots n_k]}\right) \tag{243}$$

$$\gamma^m\gamma^{n_1\dots n_k} = \gamma^{mn_1\dots n_k} + kg^{m[n_1}\gamma^{n_2\dots n_k]} \tag{244}$$

It comes as a natural question, what solutions live in $\mathcal{N}=2$ supergravity? To answer that, we are going to consider a couple of particular solutions in the classical version of supergravity. First we consider the case of AdS$_4$, since it is the unique maximally supersymmetric solution for gauged supergravity ($g \neq 0$).

The AdS$_4$ metric reads

$$ds^2\,(AdS_4) = -V(r)dt^2 + \frac{1}{V(r)}dr^2 + r^2ds^2(S^2) \tag{245}$$

with $V(r) = 1 + \frac{r^2}{\alpha^2}$, where $\frac{1}{\alpha^2} = -\frac{1}{3}\Lambda = g^2$.

Upon fixing the gravitino and the ,the Killing spinor equations read

$$\hat{\nabla}_t = \partial_t + \frac{1}{2}g^2 r\gamma_{01} + \frac{1}{2}gU\gamma_0 = \partial_t - T(r) \tag{246}$$

$$\hat{\nabla}_r = \partial r + \frac{1}{2}g^{-1}U^{-1}\gamma_1 \tag{247}$$

$$\hat{\nabla}_\theta = \partial_\theta - \frac{1}{2}U\gamma_{12} + \frac{1}{2}gr\gamma_2 \tag{248}$$

$$\hat{\nabla}_\phi = \partial_\phi - \frac{1}{2}U\sin\theta\gamma_{13} - \frac{1}{2}\cos\theta\gamma_{23} + \frac{1}{2}gr\sin\theta\gamma_3 \tag{249}$$

Assuming $\epsilon(r,t,\theta,\phi) = \mathcal{M}_1(r)\mathcal{M}_2(t)\mathcal{M}_3(\theta)\mathcal{M}_4(\phi)\epsilon_0$, we have that

$$\begin{gathered}\partial_r\epsilon = -\frac{1}{2}gU^{-1}\gamma_1\epsilon \Longrightarrow \ln\mathcal{M}_1(r) = -\int g\frac{1}{2}U^{-1}\gamma_1 dr = \left\{\begin{array}{l} gr = \sinh(\theta) \\ gdr = \cosh(\theta)d\theta \end{array}\right\} = \\ = -\frac{1}{2}g\gamma_1\int\frac{1}{\sqrt{1+\sinh^2\theta}}\cosh(\theta)\frac{d\theta}{g} = -\frac{1}{2}\gamma_1\theta = -\frac{1}{2}\gamma_1\operatorname{arcsinh}(gr) \Longrightarrow \\ \Longrightarrow \mathcal{M}_1(r) = e^{-\frac{1}{2}\gamma_1\operatorname{arcsinh} gr}\end{gathered} \tag{250}$$

Calling $\epsilon(r,t,\theta,\phi) = \mathcal{M}_1(r)\tilde{\epsilon}(t,\theta,\phi)$, the time component

$$\partial_t\epsilon = \mathcal{M}_1\partial_t\tilde{\epsilon}(t,\theta,\phi) = \left(-\frac{1}{2}g^2 r\gamma_0 - \frac{1}{2}gu\gamma_0\right)\epsilon = T(r)\epsilon \tag{251}$$

So we have that

$$\partial_t\tilde{\epsilon}(t,\theta,\phi) = \mathcal{M}_1(r)^{-1}T(r)M_1(r)\tilde{\epsilon}(t,\theta,\phi) \tag{252}$$

where

$$T(r) = -\frac{1}{2}g\left(gr\gamma_{01} + U\gamma_0\right) = -\frac{1}{2}g\gamma_0\left(gr\gamma_1 + U\right) \tag{253}$$

To solve this we can use the properties of an exponential of the type $e^{\alpha X}$, where α is a variable and X is a matrix obeying the property $X^2 = \mathbb{1}$. Then

$$X^2 = \mathbb{1} \quad ; \quad X^3 = X \tag{254}$$

so

$$\begin{gathered}e^{\alpha X} = \sum_{n=0}^{\infty}\frac{\alpha^n X^n}{n!} = \mathbb{1} + \alpha X + \frac{\alpha^2}{2!}X^2 + \frac{\alpha^3}{3!}X^3 + \frac{\alpha^4}{4!}X^4 + \frac{\alpha^5}{5!}X^5 + \cdots = \\ = \mathbb{1}\left(1 + \frac{\alpha^2}{2!} + \frac{\alpha^4}{4!} + \cdots\right) + X\left(\alpha + \frac{\alpha^3}{3!} + \frac{\alpha^5}{5!} + \cdots\right) = \sinh\alpha X + \cosh\alpha\mathbb{1}\end{gathered} \tag{255}$$

so calling $gr = \sinh\alpha$ and $U = \cosh\alpha$

$$gr\gamma_1 + U = e^{\alpha\gamma_1} = e^{\gamma_1\operatorname{arcsinh} gr} \tag{256}$$

Then

$$\begin{gathered}\partial_t\tilde{\epsilon}(t,\theta,\phi) = \mathcal{M}_1(r)^{-1}\left(-\frac{1}{2}g\gamma_0\right)e^{\gamma_1\operatorname{arcsinh} gr}M_1(r)\tilde{\epsilon}(t,\theta,\phi) = \\ = -\frac{1}{2}g\gamma_0\underbrace{\mathcal{M}_1(r)e^{\gamma_1\operatorname{arcsinh} gr}M_1(r)}_{\mathbb{1}}\tilde{\epsilon}(t,\theta,\phi)\end{gathered} \tag{257}$$

so

$$\partial_t\mathcal{M}_2(t) = -\frac{1}{2}g\gamma_0\mathcal{M}_2(t) \Longrightarrow \mathcal{M}_2(t) = e^{-\frac{1}{2}g\gamma_0 t} \tag{258}$$

For $\hat{\nabla}_\theta$, calling $\hat{\epsilon}(\theta,\phi) = \mathcal{M}_3(\theta)\mathcal{M}_4(\phi)\epsilon_0$, we have:

$$\mathcal{M}_1(r)\mathcal{M}_2(t)\partial_\theta\hat{\epsilon}(\theta,\phi) = \left(\frac{1}{2}U\gamma_{12} - \frac{1}{2}gr\gamma_2\right)\epsilon = N(r)\mathcal{M}_1(r)\mathcal{M}_2(t)\hat{\epsilon} \tag{259}$$

The form of $N(r)$, following the same procedure as before

$$N(r) = -\frac{1}{2}\gamma_2\left(U\gamma_1 + gr\right) = -\frac{1}{2}\gamma_2\gamma_1\left(U + gr\gamma_1\right) = \frac{1}{2}\gamma_{12}e^{\gamma_1 \operatorname{arcsinh} gr} \tag{260}$$

We want to pass $N(r)$ to the right side of $\mathcal{M}_2$, so we need to pay attention at how the gamma matrices products commute

$$\begin{cases} \mathcal{M}_1 \propto \gamma_1 \Longrightarrow \gamma_1\gamma_{12} = -\gamma_{12}\gamma_1 \\ \mathcal{M}_2 \propto \gamma_0 \Longrightarrow \gamma_0\gamma_{12} = \gamma_{12}\gamma_0 \end{cases} \tag{261}$$

therefore

$$\partial_\theta\hat{\epsilon} = \mathcal{M}_2^{-1}\mathcal{M}_1^{-1}N(r)\mathcal{M}_1\mathcal{M}_2\hat{\epsilon} = \mathcal{M}_2^{-1}\left(\frac{1}{2}\gamma_{12}\right)\mathcal{M}_2\hat{\epsilon} = \frac{1}{2}\gamma_{12}\hat{\epsilon} \tag{262}$$

so

$$\partial_\theta\mathcal{M}_3(\theta) = \frac{1}{2}\gamma_{12}\mathcal{M}_3(\theta) \Longrightarrow \mathcal{M}_3(\theta) = e^{\frac{1}{2}\gamma_{12}\theta} \tag{263}$$

In the last case, we have

$$\mathcal{M}_1\mathcal{M}_2\mathcal{M}_3\partial_\phi\mathcal{M}_4(\phi)\epsilon_0 = \left(\frac{1}{2}U\sin\theta\gamma_{13} + \frac{1}{2}\cos\theta\gamma_{23} - \frac{1}{2}gr\sin\theta\gamma_3\right)\epsilon = \Phi(r,\theta)\epsilon \tag{264}$$

Again

$$\begin{aligned} \Phi\left(r,\theta\right) &= \frac{1}{2}\cosh\alpha\sin\theta\gamma_{13} + \frac{1}{2}\cos\theta\gamma_{23} - \frac{1}{2}\sinh\alpha\sin\theta\gamma_3 = \\ &= \frac{1}{2}\cos\theta\gamma_{23} + \frac{1}{2}\sin\theta\gamma_{13}\left(\cosh\alpha + \sinh\alpha\gamma_1\right) = \\ &= \frac{1}{2}\cos\theta\gamma_{23} + \frac{1}{2}\sin\theta\gamma_{13}e^{\alpha\gamma_1} = \frac{1}{2}\gamma_{23}\left(\cos\theta - \sin\theta\gamma_{12}e^{\alpha\gamma_1}\right) \end{aligned} \tag{265}$$

and since

$$\mathcal{M}_1 \propto \gamma_1 \Longrightarrow \begin{cases} \gamma_1\gamma_{23} = \gamma_{23}\gamma_1 \\ \gamma_1\gamma_{12} = -\gamma_{12}\gamma_1 \end{cases} \tag{266}$$

then

$$\begin{aligned} \mathcal{M}_1^{-1}\Phi(r,\theta)\mathcal{M}_1 &= \frac{1}{2}\gamma_{23}\mathcal{M}_1^{-1}\left(\cos\theta - \sin\theta\gamma_{12}e^{\alpha\gamma_1}\right)\mathcal{M}_1 = \\ &= \frac{1}{2}\gamma_{23}\left(\cos\theta e^{\frac{\alpha}{2}\gamma_1} - \sin\theta\gamma_{12}e^{\frac{\alpha}{2}\gamma_1}\right)\mathcal{M}_1 = \\ &= \frac{1}{2}\gamma_{23}\left(\cos\theta - \sin\theta\gamma_{12}\right)e^{\frac{\alpha}{2}\gamma_1}e^{-\frac{\alpha}{2}\gamma_1} = \frac{1}{2}\gamma_{23}e^{-\gamma_{12}\theta} \end{aligned} \tag{267}$$

Now

$$\mathcal{M}_2 \propto \gamma_0 \Longrightarrow \begin{cases} \gamma_0\gamma_{23} = \gamma_{23}\gamma_0 \\ \gamma_0\gamma_{12} = \gamma_{12}\gamma_0 \end{cases} \tag{268}$$

so

$$\mathcal{M}_2^{-1}\frac{1}{2}\gamma_{23}e^{-\theta\gamma_{12}}\mathcal{M}_2 = \frac{1}{2}\gamma_{23}e^{-\theta\gamma_{12}} \tag{269}$$

and

$$\mathcal{M}_3 \propto \gamma_{12} \Longrightarrow \gamma_{12}\gamma_{23} = -\gamma_{23}\gamma_{12} \tag{270}$$

$$\frac{1}{2}\mathcal{M}_3^{-1}\gamma_{23}e^{-i\theta\gamma_{12}}\mathcal{M}_3 = \frac{1}{2}\gamma_{23}\mathcal{M}_3 e^{-i\theta\gamma_{12}}\mathcal{M}_3 = \frac{1}{2}\gamma_{23} \tag{271}$$

$$\partial_\phi \mathcal{M}_4 = \frac{1}{2}\gamma_{23}\mathcal{M}_4 \Longrightarrow \mathcal{M}_4(\phi) = e^{\frac{1}{2}\gamma_{23}\phi} \tag{272}$$

So the spinor reads:

$$\boxed{\epsilon(t,r,\theta,\phi) = \mathcal{M}_1(r)\mathcal{M}_2(t)\mathcal{M}_3(\theta)\mathcal{M}_4(\phi)\epsilon_0 = e^{-\frac{1}{2}\gamma_1 \operatorname{arcsinh}(gr)}e^{-\frac{1}{2}\gamma_0 gt}e^{\frac{1}{2}\gamma_{12}\theta}e^{\frac{1}{2}\gamma_{23}\phi}\epsilon_0} \tag{273}$$

which provides an explicit construction of the general Killing spinors in a AdS_4 spacetime. Note that the constant ϵ_0 is completely unconstrained so it depends on four complex or eight real constants which is the maximal possible for a Killing spinor in $\mathcal{N} = 2$ supergravity, meaning that the solution is maximally supersymmetric.

4.4.2 $\mathbf{AdS_2 \times S^2}$

Now let's consider a maximally supersymetric solution in the ungauged limit ($g = 0$), namely $AdS_2 \times S^2$, which appears as the near horizon limit of the RN BH with vanishing cosmological constant.

We have that the AdS_2 metric is

$$ds^2\,(AdS_2) = -r^2 dt^2 + \frac{1}{r^2}dr^2 \tag{274}$$

but we can write it in a more convenient form

$$ds^2\,(AdS_2) = -\sinh^2(r)dt^2 + dr^2 \tag{275}$$

where the vielbeins are

$$e^a = L(\sinh(r)dt, dr, d\theta, \sin\theta d\phi)^a \tag{276}$$

To compute the spinors we need the spin connection, whose non-zero components are

$$\begin{aligned}\omega_{01} &= -\frac{\coth(r)}{L}e^0 \\ \omega_{23} &= -\frac{\cot\theta}{r}e^3\end{aligned}$$

where the antisymmetry of ω_{ab} is understood. The 4-potential is

$$A_t = -\cosh(r)Q \qquad , \qquad A_\phi = -H\cos\theta \tag{277}$$

which gives the field strenght

$$F = \frac{Q}{L^2}\cdot e^0 \wedge e^1 + \frac{H}{L^2}e^2 \wedge e^3 = Q\sinh(r)dt \wedge dr + H\sin\theta d\theta \wedge d\phi \tag{278}$$

The supercovariant derivatives decompose as

$$\begin{aligned}\hat{\nabla}_t &= \partial_t + \frac{1}{2}(-\cosh)\gamma^{01} + ig\cosh Q + \frac{1}{2}gL\sinh\gamma_0 + \frac{i}{2}\frac{1}{L^2}\left(Q\gamma^{01} + H\gamma^{23}\right)L\sinh\gamma_0 = \\ &= \partial_t + \frac{1}{2}\gamma^{01}\sinh + igQ\cosh + \frac{1}{2}gL\sinh\gamma_0 - \frac{i}{2L}\sinh\left(Q\gamma^1\gamma^{023}\right) \quad (279)\\ \hat{\nabla}_r &= \partial_r + \frac{1}{2}g\gamma_1 L + \frac{i}{2L}\left(Q\gamma^{01} + Hr^{23}\right)\gamma_1 \quad (280)\\ \hat{\nabla}_\theta &= \partial_\theta + \frac{1}{2}g\gamma_\theta + \frac{i}{2L^2}(\cdots)\gamma_\theta = \partial_\theta + \frac{L}{2}g\gamma_2 + \frac{i}{2L}(Q\gamma^{01} + H\gamma^{23})\gamma_2 \quad (281)\\ \hat{\nabla}_\phi &= \partial_\phi + \frac{1}{2}(-\cos\theta)\gamma^{23} + igH\cos\theta + \frac{1}{2}g\gamma_\phi + \frac{i}{2L^2}(\ldots)\gamma_\phi = \\ &= \partial_\phi - \frac{1}{2}\cos\theta\gamma^{23} + igH\cos\theta + \frac{1}{2}gL\sin\theta\gamma_3 + \frac{i}{2L}\sin\theta\left(Q\gamma^{01} + H\gamma^{23}\right)\gamma_3 \quad (282)\end{aligned}$$

We will assume $\epsilon(r,t,\theta,\phi) = \mathcal{M}_1(r)\mathcal{M}_2(t)\mathcal{M}_3(\theta)\mathcal{M}_4(\phi)\epsilon_0$. Starting from the $\hat{\nabla}_r$ equation

$$\begin{aligned}\partial_r\mathcal{M}_1 &= \left[-\frac{1}{2}g\gamma_1 L - \frac{i}{2L}\left(Q\gamma^0 + H\gamma^{123}\right)\right]\mathcal{M}_1 \Longrightarrow \\ \Longrightarrow \mathcal{M}_1(r) &= \exp\left[-\frac{1}{2}\left(g\gamma_1 L - \frac{i}{L}\left(Q\gamma_0 - H\gamma_{123}\right)\right)r\right] \overset{g=0}{=} \exp\left[\frac{i}{2L}\left(Q\gamma_0 - H\gamma_{123}\right)r\right]\end{aligned} \quad (283)$$

For convenience, we're going to define the operator

$$P = \frac{i}{L}\left(Q\gamma_0 - H\gamma_{123}\right) \quad (284)$$

which fulfils $P^2 = \mathbb{1}$, since we had $L^2 = M^2 = Z^2$ in this case.

For the t direction

$$\mathcal{M}_1 \propto \gamma_0, \gamma_{123} \Longrightarrow \begin{cases}\gamma_0\gamma^{01} = -\gamma^{01}\gamma_0 \\ \gamma_{123}\gamma^{01} = -\gamma^{01}\gamma_{123}\end{cases} \quad (285)$$

so

$$\begin{aligned}\partial_t\mathcal{M}_2(t) &= \mathcal{M}_1^{-1}\left[\frac{1}{2}\gamma^{01}\cosh + \frac{i}{2L}(\underbrace{Q\gamma^1 + H\gamma^{023}}_{-\gamma^{01}(Q\gamma_0 - H\gamma_{123})})\sinh\right]\mathcal{M}_1\mathcal{M}_2 = \\ &\overset{(288)}{=} \frac{1}{2}\mathcal{M}_1^{-1}\gamma^{01}\left[\cosh - \frac{i}{L}\left(Q\gamma_0 - H\gamma_{123}\right)\sinh\right]\mathcal{M}_1\mathcal{M}_2 = \frac{1}{2}\gamma^{01}\underbrace{\mathcal{M}_1 e^{-Pr}\mathcal{M}_1}_{\mathbb{1}}\mathcal{M}_2\end{aligned} \quad (286)$$

then

$$\mathcal{M}_2(t) = e^{\frac{1}{2}\gamma^{01}t} \quad (287)$$

Now for θ

$$\mathcal{M}_1 \propto \gamma_0, \gamma_{123} \Longrightarrow \begin{cases}\gamma_0\gamma^{12} = \gamma^{12}\gamma_0 \\ \gamma_{123}\gamma^{12} = \gamma^{12}\gamma_{123}\end{cases} \quad (288)$$

so

$$\begin{aligned}\partial_\theta\mathcal{M}_3 &= \mathcal{M}_2^{-1}\mathcal{M}_1^{-1}\left(-\frac{i}{2L}\underbrace{\left(Q\gamma^{01} + H\gamma^{23}\right)}_{\gamma^{12}(Q\gamma^0 + H\gamma^{123})}\gamma_2\right)\mathcal{M}_1\mathcal{M}_2\mathcal{M}_3 = \\ &= M_2^{-1}\frac{i}{2L}\gamma^{12}\left(Q\gamma_0 - H\gamma_{123}\right)\mathcal{M}_2\mathcal{M}_3 = \frac{i}{2L}\gamma^{12}\left(Q\gamma_0 - H\gamma_{123}\right)\mathcal{M}_3\end{aligned} \quad (289)$$

then

$$M_3(\theta) = \exp\left(\frac{i}{2L}\gamma^{12}\left(Q\gamma_0 - H\gamma_{123}\right)\theta\right) \quad (290)$$

and for ϕ

$$\partial_\phi \mathcal{M}_4 = \mathcal{M}_3^{-1}\mathcal{M}_2^{-1}\mathcal{M}_1^{-1}\overbrace{\left(\frac{1}{2}\cos\theta\gamma^{23} - \frac{i}{2L}\sin\theta\underbrace{(Q\gamma^{01}+H\gamma^{23})\gamma_3}_{-\gamma^{13}(Q\gamma_0 - H\gamma_{123})}\right)}^{C(\theta)}\mathcal{M}_1\mathcal{M}_2\mathcal{M}_3\mathcal{M}_4 \tag{291}$$

where

$$\begin{aligned} C(\theta) &= \frac{1}{2}\gamma^{23}(\cos\theta - \overbrace{\gamma^{23}\gamma^{13}}^{\gamma^{12}}\sin\theta\,(Q\gamma_0 - H\gamma_{123})) = \\ &= \frac{1}{2}\gamma^{23}\left(\cosh(i\theta) - \frac{\gamma^{12}}{L}(Q\gamma_0 - H\gamma_{123})\sinh(i\theta)\right) = e^{-\frac{i}{L}\gamma^{12}(Q\gamma_0 - H\gamma_{123})\theta} = (\mathcal{M}_3^{-1})^2 \end{aligned} \tag{292}$$

then

$$\partial_\phi \mathcal{M}_4 = \frac{1}{2}\gamma^{23}\mathcal{M}_4 \Longrightarrow \mathcal{M}_4(\phi) = e^{\frac{1}{2}\gamma^{23}\phi} \tag{293}$$

So finally

$$\boxed{\epsilon(r,t,\theta,\phi) = e^{\frac{1}{2}Pr}e^{\frac{1}{2}\gamma^{01}t}e^{\frac{1}{2}\gamma^{12}P\theta}e^{\frac{1}{2}\gamma^{23}\phi}\epsilon_0} \tag{294}$$

with P defined in (284), which provides an explicit construction of a general Killing spinors in the $\mathrm{AdS}_2\times S^2$ limit.

5 Supersymmetric black holes in cosmological Einstein-Maxwell theory

This section is based on the L. J. Romans paper [5]., whose computations I reproduce.

Our final goal is going to be build the Killing spinors in all supersymmetric RN solutions. Something special about the "cosmic monopoles" supersymmetric RN BH class is that they don't have flat-space analogue, since the magnetic charge blows up in the formal limit to flat space, as we will see.

5.1 Cosmological Einstein-Maxwell theory

If we take the Lagrangian for Einstein-Maxwell theory with cosmological constant Λ to be

$$S = \int d^4x(R - F_{\mu\nu}F^{\mu\nu} - 2\Lambda) \tag{295}$$

the field equations are

$$R_{\mu\nu} = 2F_{\mu\rho}F_\nu{}^\rho - \frac{1}{2}g_{\mu\nu}F_{\rho\sigma}F^{\rho\sigma} + \Lambda g_{\mu\nu} \tag{296}$$

and

$$\nabla_\mu F^{\mu\nu} = 0 \quad , \quad dF = 0 \tag{297}$$

The previous BH we considered where $\Lambda = 0$. Now we're considering BHs with non-vanishing cosmological constant within Einstein-Maxwell theory. There is the following solution for arbitrary Λ.

As we've seen, the metric of this BHs has the stationary, spherically symmetric form

$$ds^2 = -Vdt^2 + \frac{dr^2}{V} + r^2\left(d\theta^2 + \sin^2\theta d\phi^2\right) \tag{298}$$

with vielbeins

$$e^a = \left(Udt, U^{-1}dr, rd\theta, r\sin\theta d\phi\right)^a \tag{299}$$

where $U = \sqrt{V}$. The vector potential is taken to be

$$A_t = \frac{Q}{r} \qquad \text{and} \qquad A_\phi = -H\cos\theta \tag{300}$$

so F has non-vanishing F_{ab} components

$$F_{01} = \frac{Q}{r^2} \qquad \text{and} \qquad F_{23} = \frac{H}{r^2} \tag{301}$$

where Q and H are the electric and magnetic charge respectively. If we define a “total charge” Z as $Z^2 = Q^2 + H^2$, $V(r)$ takes the form

$$V(r) = 1 - \frac{2M}{r} + \frac{Z^2}{r^2} - \frac{1}{3}\Lambda r^2 \tag{302}$$

5.2 Supersymmetric Reissner-Nordström solutions

If we fix the spinors of $\mathcal{N} = 2$ gauged supergravity to zero and fix $\Lambda = -3g^2$, we observe that its action coincides with that of cosmological Einstein-Maxwell theory.

A solution is supersymmetric if it obey the Killing spinor equation

$$\hat{\nabla}_\mu \epsilon = 0 \tag{303}$$

What form does each of this equations take in general? We have for $\Psi_\mu = 0$ that

$$\hat{\nabla}_\mu \epsilon = \partial_\mu \epsilon + \frac{1}{4}\omega_\mu{}^{ab}\gamma_{ab}\epsilon - igA_\mu\epsilon + \frac{1}{2}g\gamma_\mu\epsilon + \frac{i}{4}F_{ab}\gamma^{ab}\gamma_\mu\epsilon \tag{304}$$

where the non-zero components of ω_{ab} are

$$\omega_{01} = -U'e^0$$
$$\omega_{a1} = \frac{U}{r}e^a \quad for \quad a = 2,3$$
$$\omega_{23} = -\frac{\cot\theta}{r}e^3$$

where the anti-symmetry of ω_{ab} is understood. To obtain them we've used **Cartan's First Structure Equation** (138).

What form does each one of the Killing spinor equations take? We have

$$\hat{\nabla}_t\epsilon = \left[\partial_t + \frac{1}{4}\overbrace{\left(\omega_{t01}\gamma^{01} + \omega_{t10}\gamma^{10}\right)}^{2\omega_{t01}\gamma^{01}} - igA_t + \frac{1}{2}g\gamma_t + \frac{i}{2}\left(F_{01}\gamma^{01} + F_{23}\gamma^{23}\right)\gamma_t\right]\epsilon = \tag{305}$$
$$= \left[\partial_t - \frac{1}{2}U'U\gamma^{01} - ig\frac{Q}{r} + \frac{1}{2}gU\gamma_0 + \frac{i}{2}\left(\frac{Q}{r^2}\gamma^{01} + \frac{H}{r^2}\gamma^{23}\right)U\gamma_0\right]\epsilon$$

$$\hat{\nabla}_r\epsilon = \left[\partial_r + \frac{1}{2}gU^{-1}\gamma_1 + \frac{i}{4}\not{F}U^{-1}\gamma_1\right]\epsilon \tag{306}$$

$$\hat{\nabla}_\theta\epsilon = \left[\partial_\theta - \frac{1}{2}U\gamma^{12} + \frac{1}{2}gr\gamma_2 + \frac{i}{4}\not{F}r\gamma_2\right]\epsilon \tag{307}$$

$$\hat{\nabla}_\phi\epsilon = \left[\partial_\phi - \frac{1}{2}\left(U\sin\theta\gamma^{13} + \cos\theta\gamma^{23}\right) + igH\cos\theta + \frac{1}{2}gr\sin\theta\gamma_3 + \frac{i}{4}\not{F}r\sin\theta\gamma_3\right]\epsilon \tag{308}$$

where we've defined

$$\not{F} = F_{ab}\gamma^{ab} = \frac{2}{r^2}(Q\gamma^{01} + H\gamma^{23}) \tag{309}$$

using the property

$$F_{ab} = -F_{ba} \quad ; \quad \gamma^{ab} = -\gamma^{ba} \tag{310}$$

5.2.1 Algebraic constraints

It is possible to obtain some purely algebraic constraints on ϵ from (303) via

$$\Omega_{\mu\nu}\epsilon = \left[\hat{\nabla}_\mu, \hat{\nabla}_\nu\right]\epsilon = 0 \tag{311}$$

In the following, we might be sloppy at the time of explicitly putting the spinor ϵ, yet it is always there.

If we define

$$\hat{\nabla}_\mu = \nabla_\mu + \Delta_\mu \tag{312}$$

where

$$\Delta_\mu = -igA_\mu + \frac{1}{2}g\gamma_\mu + \frac{i}{4}F_{ab}\gamma^{ab}\gamma_\mu \tag{313}$$

on expands (311)

$$\begin{aligned}(\hat{\nabla}_\mu\hat{\nabla}_\nu - \hat{\nabla}_\nu\hat{\nabla}_\mu)\epsilon &= (\nabla_\mu + \Delta_\mu)(\nabla_\nu + \Delta_\nu) - (\mu \longleftrightarrow \nu)\\ &= \nabla_\mu\nabla_\nu + \nabla_\mu\Delta_\nu + \Delta_\mu\nabla_\nu + \Delta_\mu\Delta_\nu - (\mu \longleftrightarrow \nu)\\ &= \underbrace{[\nabla_\mu, \nabla_\nu]\epsilon}_{\frac{1}{4}R_{\mu\nu ab}\gamma^{ab}\epsilon} + (\nabla_\mu\Delta_\nu - \nabla_\nu\Delta_\mu)\epsilon + [\Delta_\mu, \Delta_\nu]\epsilon = 0\end{aligned} \tag{314}$$

where we've been careful in the intermediate steps, since we have extra terms from the fact that the ∇_μ is acting both on Δ_μ and ϵ. We now need to substitute (313) into each expresssion. We find

$$\begin{aligned}&\nabla_\mu\Delta_\nu = \nabla_\mu\left[-igA_\nu + \left(\frac{1}{2}g + \frac{i}{4}\not{F}\right)\gamma_\nu\right] = -ig\nabla_\mu A_\nu + \frac{1}{2}\overbrace{(\nabla_\mu e^a_\nu)}^{0}g\gamma_a +\\ &+\frac{i}{4}F_{ab}\overbrace{(\nabla_\mu e^a_\alpha e^b_\beta)}^{\propto\nabla_\mu e^a_\alpha + \nabla_\mu e^b_\beta = 0}\gamma^{\alpha\beta} + \frac{i}{4}\not{F}\overbrace{(\nabla_\mu e^a_\nu)}^{0}\gamma_a + \frac{i}{4}(\nabla_\mu F_{ab})\gamma^{ab}\gamma_\nu = -ig\nabla_\mu A_\nu + \frac{i}{4}(\nabla_\mu F_{ab})\gamma^{ab}\gamma_\nu\end{aligned} \tag{315}$$

so

$$\nabla_\mu \Delta_\nu - \nabla_\nu \Delta_\mu = -ig(\overbrace{\partial_\mu A_\nu - \partial_{nu} A_\mu}^{F_{\mu\nu}} \overbrace{-\Gamma^\lambda_{\mu\nu} A_\lambda + \Gamma^\lambda_{\nu\mu} A_\lambda}^{=0 \text{ assuming no torsion}})+ \\ + \frac{i}{4}\left[(\nabla_\mu F_{ab})\,\gamma^{ab}\gamma_\nu - (\nabla_\nu F_{ab})\,\gamma^{ab}\gamma_\mu\right] = -igF_{\mu\nu} + \frac{i}{2}\gamma^{ab}\gamma_{[\nu}\left(\nabla_{\mu]} F_{ab}\right) \tag{316}$$

While for the $[\Delta_\mu, \Delta_\nu]$ term we have

$$\begin{aligned} \Delta_\mu \Delta_\nu - \Delta_\nu \Delta_\mu &= \left[-igA_\mu + \left(\frac{1}{2}g + \frac{i}{4}\not{F}\right)\gamma_\mu\right]\left[-igA_\nu + \left(\frac{1}{2}g + \frac{i}{4}\not{F}\right)\gamma_\nu\right] - (\mu \longleftrightarrow \nu) = \\ = -g^2 \cancel{A_\mu A_\nu} - \cancel{ig(\dots)A_\mu\gamma_\nu} &- \cancel{ig(\dots)A_\nu\gamma_\mu} + \left(\frac{1}{2}g + \frac{i}{4}\not{F}\right)\gamma_\mu\left(\frac{1}{2}g + \frac{i}{4}\not{F}\right)\gamma_\nu - (\mu \longleftrightarrow \nu) = \\ &= \left(\frac{1}{2}g + \frac{i}{4}\not{F}\right)\gamma_\mu\left(\frac{1}{2}g + \frac{i}{4}\not{F}\right)\gamma_\nu - \left(\frac{1}{2}g + \frac{i}{4}\not{F}\right)\gamma_\nu\left(\frac{1}{2}g + \frac{i}{4}\not{F}\right)\gamma_\mu = \\ &= \frac{1}{4}g^2\gamma_\mu\gamma_\nu + \frac{1}{2}g\gamma_\mu\frac{i}{4}\not{F}\gamma_\nu + \frac{i}{4}\not{F}\gamma_\nu\frac{1}{2}g\gamma_\mu + \frac{i^2}{4^2}\not{F}\gamma_\mu\not{F}\gamma_\nu - (\mu \longleftrightarrow \nu) = \\ = \frac{1}{2}g^2\gamma_{\mu\nu} + \frac{i}{8}g\left(\gamma_\mu\not{F}\gamma_\nu - \gamma_\nu\not{F}\gamma_\mu\right) &+ \frac{i}{8}g\not{F}\left(\gamma_\mu\gamma_\nu - \gamma_\nu\gamma_\mu\right) - \frac{1}{16}\not{F}\left(\gamma_\mu\not{F}\gamma_\nu - \gamma_\nu\not{F}\gamma_\mu\right) = \\ &= \left(\frac{1}{2}g^2 + \frac{i}{4}g\not{F}\right)\gamma_{\mu\nu} + \left(\frac{i}{8}g - \frac{1}{16}\not{F}\right)\left(\gamma_\mu\not{F}\gamma_\nu - \gamma_\nu\not{F}\gamma_\mu\right) \end{aligned} \tag{317}$$

where the red crossed terms cancel with the ones inside $(\mu \longleftrightarrow \nu)$.

The result can then be expressed as

$$\Omega_{\mu\nu} = \overbrace{\frac{1}{4}R_{\mu\nu ab}\gamma^{ab}}^{A} - igF_{\mu\nu} + \overbrace{\frac{i}{2}\gamma^{ab}\gamma_{[\nu}\left(\nabla_{\mu]}F_{ab}\right)}^{B} + \tag{318}$$

$$+\left(\frac{1}{2}g^2 + \frac{i}{4}g\not{F}\right)\gamma_{\mu\nu} + \left(\frac{i}{8}g - \frac{1}{16}\not{F}\right)\underbrace{\left(\gamma_\mu\not{F}\gamma_\nu - \gamma_\nu\not{F}\gamma_\mu\right)}_{C} \tag{319}$$

but we can do better, it is possible to factorise $\Omega_{\mu\nu}$ as

$$\Omega_{\mu\nu} = X_{\mu\nu}\Theta \tag{320}$$

where the $X_{\mu\nu}$ components are in general non-singular, making (311) only able to be satisfied iif ϵ is an "eigenspinor" of Θ with zero eigenvalue. Then form of Θ is

$$\Theta \equiv U + gr\gamma_1 + \left\{\frac{1}{r} - \frac{M}{Z^2}\right\}(i\gamma_0 Q - i\gamma_{123}H) \tag{321}$$

with (320) reducing to

$$\Theta\epsilon = 0 \tag{322}$$

Before continuing, let's compute a couple of $X_{\mu\nu}$ terms to show that (320) is indeed true. Starting from $\Omega_{t\phi}$, the A term is composed of the Riemann curvature tensor (presented as a curvature 2-form) which can be obtained using the spin connection through **Cartan's Second Structure Equation** (139).

If we expand it, we get the curvature tensor

$$R_{\mu\nu ab} = \partial_\mu(\omega_\nu)_{ab} - \partial_\nu(\omega_\mu)_{ab} + (\omega_\mu)_{ac}(\omega_\nu)^c{}_b - (\omega_\nu)_{ac}(\omega_\mu)^c{}_b \tag{323}$$

so the A term reads

$$\frac{1}{4}R_{t\phi ab}\gamma^{ab} = \frac{1}{4}\left(\overset{0\ \forall a,b}{\cancel{\partial_t\omega_{\phi ab}}} - \overset{0\ \forall a,b}{\cancel{\partial_\phi\omega_{tab}}} + \omega_{tac}\omega_\phi{}^c{}_b - \omega_{\phi ac}\omega_t{}^c{}_b\right)\gamma^{ab} \tag{324}$$

For what values of a and b this isn't zero?

$$\omega_{tac} \neq 0 \quad iif \quad \begin{cases} a=0, c=1 \Longrightarrow \omega_\phi{}^1{}_b \propto \omega_{\phi 1b} \neq 0 \quad iif \quad b=3 \\ a=1, c=0 \Longrightarrow \omega_\phi{}^0{}_b \propto \omega_{\phi 0b} = 0 \quad \forall b \end{cases}$$
$$\omega_{\phi ac} \neq 0 \quad iif \quad \begin{cases} a=1, c=3 \Longrightarrow \omega_t{}^3{}_b = 0 \quad \forall b \\ a=3, c=1 \Longrightarrow \omega_t{}^1{}_b \neq 0 \quad iif \quad b=3 \\ a=2, c=3 \Longrightarrow \omega_t{}^3{}_b = 0 \quad \forall b \\ a=3, c=2 \Longrightarrow \omega_t{}^2{}_b \neq 0 \quad iif \quad b=3 \end{cases} \tag{325}$$

therefore

$$\frac{1}{4}R_{t\phi ab}\gamma^{ab} = \frac{1}{4}(\omega_{t01}\omega_\phi{}^1{}_3\gamma^{03} - \omega_{\phi 31}\omega_t{}^1{}_0\gamma^{30} - \omega_{\phi 32}\omega_t{}^2{}_3\overset{0}{\cancel{\gamma^{33}}}) = \frac{1}{2}\omega_{t01}\omega_{\phi 13}\gamma^{03} = \frac{1}{2}U'U^2\sin\theta\gamma^{03} \tag{326}$$

The B term reads

$$B = \frac{i}{2}\gamma_{ab}\gamma_{[\phi}\left(\nabla_{t]}F^{ab}\right) \tag{327}$$

where we have that

$$\nabla_\mu F^{ab} = \partial_\mu F^{ab} + (\omega_\mu)^a{}_c F^{cb} + (\omega_\mu)^b{}_c F^{ac} \tag{328}$$

and since none of the F^{ab} components depends on t or ϕ, we only have to care about the spin connection terms. The contribution to B is

$$\nabla_t F^{ab} = (\omega_t)^a{}_c F^{cb} + (\omega_t)^b{}_c F^{ac} \tag{329}$$
$$\nabla_\phi F^{ab} = (\omega_\phi)^a{}_c F^{cb} + (\omega_\phi)^b{}_c F^{ac} \tag{330}$$

For the t direction, we have that all the a,b terms give zero, since the unique non zero $(\omega_t)_{ab}$ components are the ones where $a = 0, 1$ and $b = 1, 0$ respectively. This translates into

$$\nabla_t F^{01} = (\omega_t)^0{}_c F^{c1} + (\omega_t)^1{}_c F^{0c} \tag{331}$$

and vice versa. So, if we choose the c's which make F^{c1} or F^{0c} not zero, we end up with a spin connection of the form $(\omega_t)^0{}_0$ and $(\omega_t)^1{}_1$, which using the antisymmetry property

$$\omega^a{}_b = \eta^{ac}\omega_{cb} = -\eta^{ac}\omega_{bc} \tag{332}$$

is always zero, since the flat-space metric is diagonal. Now for the ϕ direction, we have to be more careful. We have two components which at first sight we could think they don't contribute, but we have to account for the spin connection. They are

$$\nabla_\phi F^{03} = (\omega_\phi)^0{}_c F^{c3} + (\omega_\phi)^3{}_c F^{0c} = (\omega_\phi)^3{}_1 F^{01} = -U\sin\theta\frac{Q}{r^2} \tag{333}$$

since $F^{c3} \neq 0$ iif $c = 2$, leading to $(\omega_\phi)^0{}_2 = 0$. We also have

$$\nabla_\phi F^{12} = (\omega_\phi)^1{}_c F^{c2} + (\omega_\phi)^2{}_c F^{1c} = (\omega_\phi)^1{}_3 F^{32} + \cancelto{0}{(\omega_\phi)^2{}_0} F^{10} = U \sin\theta \frac{H}{r^2} \tag{334}$$

Then the term of B that contributes is

$$\begin{aligned}
-\frac{i}{4}\gamma_{ab}\gamma_t\nabla_\phi F^{ab} &= -\frac{i}{2}(\gamma_{03}\gamma_t\nabla_\phi F^{03} + \gamma_{12}\gamma_t\nabla_\phi F^{12}) = \\
&= \frac{i}{2}\left(\gamma_{03}U\gamma_0 U\sin\theta\frac{Q}{r^2} - \gamma_{12}U\gamma_t U\sin\theta\frac{H}{r^2}\right) = \\
&= \frac{i}{2}\frac{U^2\sin\theta}{r^2}(Q\gamma^3 + H\gamma^{012}) = -\frac{i}{2}\frac{U^2\sin\theta\gamma^{03}}{r^2}(-Q\gamma^0 - H\underbrace{\gamma^0\gamma^3\gamma^{012}}_{\gamma^3\gamma^{12}=\gamma^{312}=\gamma^{123}}) = \\
&= -\frac{i}{2}\frac{U^2\sin\theta\gamma^{03}}{r^2}(Q\gamma_0 - H\gamma_{123})
\end{aligned} \tag{335}$$

For now, we have

$$\begin{aligned}
\Omega_{t\phi} = \frac{1}{2}U'U^2\sin\theta\gamma^{03} - ig\overbrace{F_{t\phi}}^{\propto F_{03}=0} - \frac{i}{2}\frac{U^2\sin\theta\gamma^{03}}{r^2}(Q\gamma_0 - H\gamma_{123}) + \\
+\left(\frac{1}{2}g^2 + \frac{i}{4}g\not{F}\right)\gamma_{t\phi} + \left(\frac{i}{8}g - \frac{1}{16}\not{F}\right)\underbrace{(\gamma_t\not{F}\gamma_\phi - \gamma_\phi\not{F}\gamma_t)}_{C}
\end{aligned} \tag{336}$$

The C term is composed of

$$\gamma_t\not{F}\gamma_\phi = 2e_t^0 e_\phi^3\gamma_0\not{F}\gamma_3 = -2Ur\sin\theta\gamma^0\not{F}\gamma^3 \tag{337}$$

where

$$\gamma^0\not{F}\gamma^3 = 2\left(\frac{Q}{r^2}\underbrace{\gamma^0\gamma^{01}\gamma^3}_{-\gamma^{13}} + \frac{H}{r^2}\underbrace{\gamma^0\gamma^{23}\gamma^3}_{\gamma^{02}}\right) = 2\left(-\frac{Q}{r^2}\gamma^{13} + \frac{H}{r^2}\gamma^{02}\right) \tag{338}$$

and

$$\gamma_\phi\not{F}\gamma_t = 2e_\phi^3 e_t^0\gamma_0\not{F}\gamma_3 = -2Ur\sin\theta\gamma^3\not{F}\gamma^0 \tag{339}$$

with

$$\gamma^3\not{F}\gamma^0 = 2\left(\frac{Q}{r^2}\gamma^3\gamma^{01}\gamma^0 + \frac{H}{r^2}\gamma^3\gamma^{23}\gamma^0\right) = 2\left(-\frac{Q}{r^2}\gamma^{13} + \frac{H}{r^2}\gamma^{02}\right) \tag{340}$$

therefore

$$C = \gamma_t\not{F}\gamma_\phi - \gamma_\phi\not{F}\gamma_t = 0 \tag{341}$$

So we end up with

$$\begin{aligned}\Omega_{t\phi} &= \frac{1}{2}U'U^2\sin\theta\gamma^{03} - \frac{i}{2}\frac{U^2\sin\theta\gamma^{03}}{r^2}(Q\gamma_0 - H\gamma_{123}) - \frac{1}{2}g^2Ur\sin\theta\gamma^{03} + \\ &+ \frac{ig}{4}\times 2\left(\frac{Q}{r^2}\gamma^{01} + \frac{H}{r^2}\gamma^{23}\right)\underbrace{e_t^0 e_\phi^3\gamma_{03}}_{-Ur\sin\theta\gamma^{03}} = \\ &= \frac{1}{2}U'U^2\sin\theta\gamma^{03} - \frac{i}{2}\frac{U^2\sin\theta\gamma^{03}}{r^2}(Q\gamma_0 - H\gamma_{123}) - \\ &- \frac{1}{2}g^2Ur\sin\theta\gamma^{03} - \frac{ig}{2}Ur\sin\theta\overbrace{\left(\frac{Q}{r^2}\underbrace{\gamma^{01}\gamma^{03}}_{\gamma^{13}} + \frac{H}{r^2}\underbrace{\gamma^{23}\gamma^{03}}_{\gamma^{02}}\right)}^{L}\end{aligned} \tag{342}$$

We can rewrite the L term as

$$L = (-Q\gamma^3 + H\gamma^{02}\gamma^1)\gamma^1 = \gamma^{03}(-Q\overbrace{\gamma^{03}\gamma^3}^{\gamma^0} + H\underbrace{\gamma^{03}\gamma^{02}\gamma^1}_{\gamma^3\gamma^2\gamma^1=\gamma^{32}\gamma^1=\gamma^{132}=-\gamma^{123}})\gamma^1 = \gamma^{03}(Q\gamma_0 - H\gamma_{123})\gamma^1 \tag{343}$$

having

$$\Omega_{t\phi} = \frac{1}{2}U\sin\theta\left[(U'U - g^2r)\gamma^{03} - \frac{ig}{r}\gamma^{03}(Q\gamma_0 - H\gamma_{123})\gamma^1 - \frac{iU}{r^2}\gamma^{03}(Q\gamma_0 - H\gamma_{123})\right] \tag{344}$$

The derivative of U is

$$U' = \frac{1}{2}U^{-1}\left(\frac{2M}{r^2} - \frac{2Z^2}{r^3} + 2g^2r\right) \implies U'U = -\frac{Z^2}{r^2}\left(\frac{1}{r} - \frac{M}{Z^2}\right) + g^2r \tag{345}$$

and

$$\boxed{(Q\gamma_0 - H\gamma_{123})(Q\gamma_0 - H\gamma_{123}) = -Q^2 - QH\underbrace{\gamma_0\gamma_{123}}_{\gamma_{0123}} - HQ\underbrace{\gamma_{123}\gamma_0}_{-\gamma_{0123}} + H^2\underbrace{\gamma_{123}\gamma_{123}}_{-\mathbb{I}} = -Z^2} \tag{346}$$

substituting both (345) and (346) in (344)

$$\begin{aligned}\Omega_{t\phi} &= \frac{U\sin\theta}{2r^2}\gamma^{03}(Q\gamma_0 - H\gamma_{123})\left[\left(\frac{1}{r} - \frac{M}{Z^2}\right)(Q\gamma_0 - H\gamma_{123}) - iU - igr\gamma^1\right]\times\frac{i}{i} = \\ &= \underbrace{\frac{U\sin\theta}{2ir^2}\gamma^{03}(Q\gamma_0 - H\gamma_{123})}_{X_{t\phi}}\underbrace{\left[\left(\frac{1}{r} - \frac{M}{Z^2}\right)(iQ\gamma_0 - iH\gamma_{123}) + U + gr\gamma_1\right]}_{\Theta}\end{aligned} \tag{347}$$

Now for $\Omega_{t\theta}$, the A part

$$A = \frac{1}{4}R_{t\theta ab}\gamma^{ab} = \frac{1}{4}(\cancelto{0}{\partial_t\omega_{\theta ab}} - \cancelto{0}{\partial_\theta\omega_{tab}} + \omega_{tac}\omega_\theta{}^c{}_b - \omega_{\theta ac}\omega_t{}^c{}_b)\gamma^{ab} = \frac{1}{2}\omega_{t01}\omega_{\theta 12}\gamma^{02} = \frac{1}{2}U'U^2\gamma^{02} \tag{348}$$

where

$$\begin{aligned}&\omega_{tac}\neq 0 \quad iif \quad \begin{cases} a=0, c=1 \implies \omega_\theta{}^1{}_b \neq 0 \quad iif \quad b=2 \\ a=1, c=0 \implies \omega_\theta{}^0{}_b = 0\ \forall b\end{cases} \\ &\omega_{\theta ac}\neq 0 \quad iif \quad \begin{cases} a=1, c=2 \implies \omega_t{}^3{}_b = 0 \quad \forall b \\ a=1, c=3 \implies \omega_t{}^1{}_b = 0 \quad \forall b \\ a=2, c=1 \implies \omega_t{}^3{}_b \neq 0 \quad iif \quad b=0 \\ a=3, c=1 \implies \omega_t{}^2{}_b \neq 0 \quad iif \quad b=0\end{cases}\end{aligned} \tag{349}$$

Then

$$B=\frac{i}{2}\gamma_{ab}\gamma_{[\theta}\left(\nabla_{t]}F^{ab}\right)=-\frac{i}{4}\gamma_{ab}\gamma_t\nabla_\theta F^{ab}=-\frac{i}{2}\left(-\gamma_{02}U^2\frac{Q}{r^2}\gamma_0-\gamma_{13}\gamma_0U^2\frac{H}{r^2}\right)= \\ =-\frac{iU^2}{2r^2}\gamma^{02}(Q\gamma_0+H\underbrace{\gamma_{02}\gamma_{013}}_{\gamma_2\gamma_{13}=-\gamma_{123}})=-\frac{iU^2}{2r^2}\gamma^{02}(Q\gamma_0-H\gamma_{123}) \tag{350}$$

where the non-zero terms are

$$\nabla_\theta F^{02}=\cancelto{0}{\partial_\theta F^{02}}+\cancelto{0\ \forall c}{(\omega_\theta)^0{}_c}F^{c1}+(\omega_\theta)^2{}_cF^{0c}=(\omega_\theta)^2{}_1F^{01}=-\frac{U}{r^2}Q \tag{351}$$

$$\nabla_\theta F^{13}=(\omega_\theta)^1{}_cF^{c3}+(\omega_\theta)^3{}_cF^{1c}=(\omega_\theta)_{12}F^{23}=-\frac{U}{r^2}H \tag{352}$$

and

$$\gamma_t\not{F}\gamma_\theta=-2Ur\gamma^0\not{F}\gamma^2=-\frac{4U}{r}(Q\underbrace{\gamma^0\gamma^{01}\gamma^2}_{-\gamma^{12}}+H\underbrace{\gamma^0\gamma^{23}\gamma^2}_{-\gamma^{03}})=\frac{4U}{r}(Q\gamma^{12}+H\gamma^{03}) \tag{353}$$

$$\gamma_\theta\not{F}\gamma_t=-2Ur\gamma^2\not{F}\gamma^0=-\frac{4U}{r}(Q\underbrace{\gamma^2\gamma^{01}\gamma^0}_{-\gamma^{12}}+H\underbrace{\gamma^2\gamma^{23}\gamma^0}_{-\gamma^{03}})=\frac{4U}{r}(Q\gamma^{12}+H\gamma^{03}) \tag{354}$$

$$\gamma_t\not{F}\gamma_\theta-\gamma_\theta\not{F}\gamma_t=0 \tag{355}$$

So

$$\Omega_{t\theta}=\frac{1}{2}U'U^2\gamma^{02}-\cancelto{0}{igF_{t\theta}}-\frac{iU^2}{2r^2}\gamma^{02}(Q\gamma_0-H\gamma_{123})+\frac{1}{2}g^2\gamma_{t\theta}+\frac{ig}{4}\times 2\left(\frac{Q}{r^2}\gamma^{01}+\frac{H}{r^2}\gamma^{23}\right)\gamma_{t\theta}= \\ =\frac{1}{2}U'U^2\gamma^{02}-\frac{iU^2}{2r^2}\gamma^{02}(Q\gamma_0-H\gamma_{123})-\frac{1}{2}Urg^2\gamma^{02}-\frac{ig}{2}\left(\frac{Q}{r^2}\underbrace{\gamma^{01}\gamma^{02}}_{-\gamma^{02}\gamma^{01}}+\frac{H}{r^2}\underbrace{\gamma^{23}\gamma^{02}}_{-\gamma^{02}\gamma^{23}}\right)Ur= \\ =\frac{1}{2}U\gamma^{02}\left[\underbrace{U'U^2-g^2r}_{-\frac{Z^2}{r^2}\left(\frac{1}{r}-\frac{M}{r^2}\right)}+\frac{ig}{r}\underbrace{(Q\gamma^{01}+H\gamma^{23})}_{(Q\gamma_0-H\gamma_{123})\gamma_1}-\frac{iU}{r^2}(Q\gamma_0-H\gamma_{123})\right]= \\ \overset{(346)}{=}\underbrace{\frac{U}{2ir^2}\gamma^{02}(Q\gamma_0-H\gamma_{123})}_{X_{t\theta}}\left[\left(\frac{1}{r}-\frac{M}{r^2}\right)(iQ\gamma_0-iH\gamma_{123})+gr\gamma_1+U\right] \tag{356}$$

Finally, a more complex term $\Omega_{\theta\phi}$

$$A=\frac{1}{4}R_{\theta\phi ab}\gamma^{ab}=\frac{1}{4}\left(\partial_\theta\omega_{\phi ab}-\cancelto{0}{\partial_\phi\omega_{\theta ab}}+\omega_{\theta ac}\omega_\phi{}^c{}_b-\omega_{\phi ac}\omega_\phi{}^c{}_b\right)\gamma^{ab}= \\ =\frac{1}{2}(\cancel{-U\cos\theta\gamma^{13}}+\sin\theta\gamma^{23}+\cancel{\omega_{\theta12}\omega_{\phi23}\gamma^{13}}-\omega_{\theta12}\omega_{\phi13}\gamma^{23})= \\ =\frac{1}{2}\sin\theta(1-U^2)\gamma^{23} \tag{357}$$

where

$$\partial_\theta\omega_{\phi ab}\neq 0\quad iif\quad \begin{cases} a=1,b=3\to\partial_\theta(-U\sin\theta)=-U\cos\theta \\ a=2,b=3\to\partial_\theta(-\cos\theta)=\sin\theta \end{cases} \tag{358}$$

and

$$\omega_{\theta ac} \neq 0 \quad iif \quad \begin{cases} a=1, c=2 \Longrightarrow \omega_\phi{}^2{}_b \neq 0 \quad iif \quad b=3 \\ a=2, c=1 \Longrightarrow \omega_\phi{}^1{}_b \neq 0 \quad iif \quad b=3 \end{cases} \tag{359}$$

$$\omega_{\phi ac} \neq 0 \quad iif \quad \begin{cases} a=1, c=3 \Longrightarrow \omega_\theta{}^3{}_b = 0 \quad \forall b \\ a=2, c=3 \Longrightarrow \omega_\theta{}^3{}_b = 0 \quad \forall b \\ a=3, c=1 \Longrightarrow \omega_\theta{}^1{}_b \neq 0 \quad iif \quad b=2 \\ a=3, c=2 \Longrightarrow \omega_\theta{}^2{}_b \neq 0 \quad iif \quad b=1 \end{cases} \tag{360}$$

$$\begin{aligned} B &= \frac{i}{2}\gamma_{ab}\gamma_{[\phi}\left(\nabla_{\theta]}F^{ab}\right) = \frac{i}{4}\gamma_{ab}(\gamma_\phi\nabla_\theta F^{ab} - \gamma_\theta\nabla_\phi F^{ab}) \overset{\substack{(333),(334)\\(351),(352)}}{=} \\ &= -\frac{iU}{2r^2}\Big[-\underbrace{\gamma_{03}\gamma_2}_{\gamma_{203}=-\gamma_{230}} r\sin\theta Q + \underbrace{\gamma_{12}\gamma_2}_{\gamma_1} r\sin\theta H + \underbrace{\gamma_{02}\gamma_3}_{\gamma_{302}=\gamma_{230}} r\sin\theta Q + \underbrace{\gamma_{13}\gamma_3}_{\gamma_1} r\sin\theta H\Big] = \\ &= -\frac{iU}{2r}\sin\theta(2Q\gamma_{23}\gamma_0 + 2H\gamma_1) = -\frac{iU}{r}\sin\theta\gamma^{23}(Q\gamma_0 - H\gamma_{123}) \end{aligned} \tag{361}$$

also

$$\gamma_\theta \not{F}\gamma_\phi = r^2\sin\theta\Bigg(\underbrace{\gamma^2\gamma^{01}\gamma^3}_{\gamma^{2301}=\gamma^{0123}}\frac{Q}{r^2} + \underbrace{\gamma^2\gamma^{23}\gamma^3}_{\not{\mathbb{1}}}\frac{H}{r^2}\Bigg)\times 2 = 2\sin\theta(Q\gamma^{0123} + H) \tag{362}$$

$$\gamma_\phi \not{F}\gamma_\theta = -2\sin\theta(Q\gamma^{0123} + H) \tag{363}$$

then

$$\gamma_\theta \not{F}\gamma_\phi - \gamma_\phi \not{F}\gamma_\theta = 4\sin\theta(Q\gamma^{0123} + H) \tag{364}$$

Therefore

$$\begin{aligned} \Omega_{\theta\phi} = &-\frac{iU}{r}\sin\theta\gamma^{23}\overbrace{(Q\gamma_0 - H\gamma_{123})}^{P} + \frac{1}{2}\sin\theta(1-U^2)\gamma^{23} - igF_{\theta\phi} + \left(\frac{1}{2}g^2 + \frac{ig}{4}\not{F}\right)\gamma_{\theta\phi} + \\ &+ \left(\frac{ig}{8} - \frac{1}{16}\not{F}\right)4\sin\theta(Q\gamma^{0123} + H) \overset{\substack{(366),(367)\\(369),(368)}}{=} -\frac{iU}{r}\sin\theta\gamma^{23}P + \frac{1}{2}\sin\theta\gamma^{23}\left(\frac{2M}{r} - \frac{Z^2}{r^2} - \cancel{g^2r^2}\right) \\ &\underbrace{-igH\sin\theta}_{A} + \cancel{\frac{1}{2}g^2r^2\sin\theta\gamma^{23}} + \underbrace{\frac{ig}{2}\sin\theta\gamma^{23}(Q\gamma^0 + H\gamma^{123})\gamma^1}_{B} + \underbrace{\frac{ig}{2}\sin\theta\gamma^{23}(Q\gamma^0 - H\gamma^{123})\gamma^1}_{C} - \\ &-\frac{1}{2r^2}\sin\theta Z^2\gamma^{23} \overset{(346),(370)}{=} \frac{\sin\theta}{r}\gamma^{23}P\left[-iU + \left(\frac{1}{r} - \frac{M}{r^2}\right)P - igr\gamma_1\right]\times\frac{i}{i} = \\ = &\underbrace{\frac{1}{ir}\sin\theta\gamma^{23}(Q\gamma_0 - H\gamma_{123})}_{X_{\theta\phi}}\Theta \end{aligned} \tag{365}$$

where we've used

$$1 - U^2 = \frac{2M}{r} - \frac{Z^2}{r^2} - g^2r^2 \tag{366}$$

$$(Q\gamma^{0123} + H) = \gamma^{23}(Q\gamma^{01} - H\gamma^{23}) = \gamma^{23}(Q\gamma^0 - H\gamma^{123})\gamma^1 \tag{367}$$

$$\not{F}\gamma^{23} \propto (Q\gamma^{01} + H\gamma^{23})\gamma^{23} = \gamma^{23}(Q\gamma^0 + H\gamma^{123})\gamma^1 \tag{368}$$

$$\not{F}(Q\gamma^{0123}+H)\propto(Q\gamma^{01}+H\gamma^{23})(Q\gamma^{0123}+H)= \\ =Q^2\underbrace{\gamma^{01}\gamma^{0123}}_{\gamma^{23}}+\cancel{QH\gamma^{01}}+HQ\underbrace{\cancel{\gamma^{23}\gamma^{0123}}}_{-\gamma^{01}}+H^2\gamma^{23}=Z^2\gamma^{23} \tag{369}$$

$$C+B+A=\frac{ig}{2}\sin\theta\gamma^{23}[-Q\gamma_0-H\gamma^{123}-Q\gamma_0+H\gamma^{123}+2H\gamma^{23}\gamma^1]\gamma^1=-ig\sin\theta\gamma^{23}P\gamma_1 \tag{370}$$

For (311) to be true, it must be fulfilled that $\det\Theta=0$ for all r. We have

$$\det\Theta=\left[1-2gH-\frac{\left(M^2-2gHMr\right)}{Z^2}\right]\times\left[1+2gH-\frac{\left(M^2+2gHMr\right)}{Z^2}\right] \tag{371}$$

so

$$1\pm2gH-\frac{M^2}{Z^2}=0\quad\text{and}\quad gHM=0 \tag{372}$$

This leads to 3 cases that can be compatible with supersymmetry.

$$g=0\quad,\quad Z^2=M^2 \tag{373}$$

$$H=0\quad,\quad Q^2=M^2 \tag{374}$$

$$M=0\quad,\quad H=\pm\frac{1}{2g} \tag{375}$$

One thing to note is that, for the solutions where $g\neq0$, the metric function $V(r)$ is always positive (as we will see in the following sections), meaning that the singularity at the origin is not surrounded by an horizon, contradicting Penrose's principle of cosmic censorship.

We are not guaranteed that solving the first integrability condition, namely (311), guarantee a solution for the Killing spinor equation (303). To address this problem, we must use the original first-order equation, but taking advantage of the conditions that Θ gives us. Let's compute the spinor for each one of the conditions.

5.3 Flat-space extreme Reissner-Nordström solutions

Starting from the case $g=0$, $Z^2=M^2$, we are dealing with the ungauged $\mathcal{N}=2$ theory in flat space. We have

$$U=\sqrt{V}=\sqrt{1-\frac{2M}{r}+\frac{Z^2}{r^2}+g^2r^2}\Bigg|_{\substack{g=0\\M^2=Z^2}}=\sqrt{1-\frac{2M}{r}+\frac{M^2}{r^2}}=1-\frac{M}{r} \tag{376}$$

then

$$\Theta\epsilon=\left[U+\left(\frac{1}{r}-\frac{1}{M}\right)(i\gamma_0Q-i\gamma_{123}H)\right]\epsilon=U\left[1-\frac{1}{M}(i\gamma_0Q-i\gamma_{123}H)\right]\epsilon=0\Longrightarrow \\ \Longrightarrow\frac{i}{M}(\gamma_0Q-\gamma_{123}H)\epsilon=\epsilon\Longrightarrow\boxed{H\gamma_{123}\epsilon=(\gamma_0Q+iM)\epsilon} \tag{377}$$

The Killing spinor equations are

$$\begin{aligned}\hat{\nabla}_t\epsilon &= \left[\partial_t - \frac{1}{2}U'U\gamma^{01} + \frac{i}{2}\left(\frac{Q}{r^2}\gamma^{01} + \frac{H}{r^2}\gamma^{23}\right)\gamma_t\right]\epsilon = 0 \Longrightarrow \\ \Longrightarrow \partial_t\epsilon &= \left[\frac{1}{2}U'U\gamma^{01} - \frac{i}{2r^2}\underbrace{\left(Q\gamma^{01} + H\gamma^{23}\right)}_{\chi}U\gamma_0\right]\epsilon\end{aligned} \tag{378}$$

$$\hat{\nabla}_r\epsilon = \partial_r\epsilon + \underbrace{\frac{iU^{-1}}{2r^2}\chi\gamma_1}_{-A(r)}\epsilon = 0 \Rightarrow \partial_r\epsilon = A(r)\epsilon \tag{379}$$

$$\hat{\nabla}_\theta\epsilon = 0 \Rightarrow \partial_\theta\epsilon = \left[\frac{1}{2}U\gamma^{12} - \frac{i}{2r}\chi\gamma_2\right]\epsilon = B(r)\epsilon = 0 \tag{380}$$

$$\begin{aligned}\hat{\nabla}_\phi\epsilon &= \partial_\phi\epsilon - \frac{1}{2}\left(U\sin\theta\gamma^{13} + \cos\theta\gamma^{23}\right)\epsilon + \frac{i}{2r}\sin\theta\chi\gamma_3\epsilon = 0 \Longrightarrow \\ \Longrightarrow \partial_\phi\epsilon &= \left[\frac{1}{2}\left(U\sin\theta\gamma^{13} + \cos\theta\gamma^{23}\right) - \frac{i}{2r}\sin\theta\chi\gamma_3\right]\epsilon = H(r,\theta)\epsilon\end{aligned} \tag{381}$$

If we use the relation (377), from the time direction we get a trivial equation

$$\partial_t\epsilon = \frac{1}{2}U\left[\frac{M}{r^2}\gamma^{01} - \frac{i}{r^2}\underbrace{\chi\gamma_0}_{\gamma^{01}\chi = \gamma^{01}\frac{M}{i}}\right]\epsilon = 0 \tag{382}$$

so our spinor doesn't depend on time.

Assuming that we can decompose our spinor as

$$\epsilon(r,\theta,\phi) = f(r)\mathcal{M}_1(\theta)\mathcal{M}_2(\phi)\epsilon_0 \tag{383}$$

where the $\mathcal{M}$'s are exponential matrices and ϵ_0 is a constant spinor. It's convenient to define $\tilde{\epsilon} = \mathcal{M}_1(\theta)\mathcal{M}_2(\phi)\epsilon_0$. For the r direction, we have that

$$\begin{aligned}\partial_r\epsilon = \partial_r f(r)\tilde{\epsilon} = A(r)f(r)\tilde{\epsilon} \quad &\Longrightarrow \quad \frac{\partial_r f(r)}{f(r)} = \partial_r(\ln f(r)) = A(r) \Longrightarrow \\ \Longrightarrow \ln f = \int A(r)dr = -\frac{i\chi}{2}\gamma_1\int\frac{1}{r(r-M)} &= \frac{-i\chi\gamma_1}{2}\cdot\frac{\ln\left(1-\frac{M}{r}\right)}{M} \Longrightarrow \\ \Longrightarrow f(r) = \exp&\left(-\frac{i\chi\gamma_1}{2M}\ln\left(1-\frac{M}{r}\right)\right)\end{aligned} \tag{384}$$

but

$$\chi\gamma_1 = (Q\underbrace{\gamma^{01}\gamma_1}_{\gamma^0} + H\underbrace{\gamma^{23}\gamma_1}_{\gamma^{123}}) = Q\gamma^0 + H\gamma^{123} \overset{(377)}{=} Q\gamma^0 - Q\gamma^0 + iM = iM \tag{385}$$

$$f(r) = \exp\left[\frac{1}{2}\ln\left(1-\frac{M}{r}\right)\right] = \sqrt{1-\frac{M}{r}} = \sqrt{U} = V(r) \tag{386}$$

For the θ direction

$$\partial_\theta\epsilon = f(r)\partial_\theta\tilde{\epsilon} = B(r)f(r)\tilde{\epsilon} \Rightarrow \partial_\theta\tilde{\epsilon} = B(r)\tilde{\epsilon} \Rightarrow \tilde{\epsilon} \propto \exp(B(r)\theta) \tag{387}$$

which has to be the $\mathcal{M}_1(\theta)$ part. Using the projector

$$\gamma^{21}\times(377) \Longrightarrow \gamma^{21}\times\left[H\gamma^{123}\epsilon = \left(-Q\gamma^0 + iM\right)\epsilon\right] \Longrightarrow H\gamma^3\epsilon = (-Q\underbrace{\gamma^{21}\gamma^0}_{\gamma^{021} = -\gamma^{201}} + iM\gamma^{21})\epsilon \tag{388}$$

$$\chi\gamma_2 = (Q\underbrace{\gamma^{01}\gamma_2}_{\gamma^{201}} + H\underbrace{\gamma^{23}\gamma_2}_{-\gamma^3}) = Q\gamma^{201} - H\gamma^3 \overset{(388)}{=} -iM\gamma^{21} = iM\gamma^{12} \tag{389}$$

and $B(r)$ reduces to

$$B(r) = \left(\frac{1}{2}U - \frac{i}{2r}iM\right)\gamma^{12} = \left(\frac{1}{2} - \frac{M}{2r} + \frac{M}{2r}\right)\gamma^{12} = \frac{1}{2}\gamma^{12} \tag{390}$$

so

$$\mathcal{M}_1(\theta) = \exp\left(\frac{1}{2}\gamma^{12}\theta\right) \tag{391}$$

Now, for the ϕ part

$$\partial_\phi \epsilon = f(r)\mathcal{M}_1(\theta)\partial_\phi\hat{\epsilon} = H(r,\theta)f(r)\mathcal{M}_1(\theta)\hat{\epsilon} \tag{392}$$

where $\hat{\epsilon} = \mathcal{M}_2(\phi)\epsilon_0$. Therefore

$$\partial_\phi\hat{\epsilon} = \mathcal{M}_1^{-1}(\theta)H(r,\theta)\mathcal{M}_1(\theta)\hat{\epsilon} \tag{393}$$

Working a little bit on $H(r,\theta)$

$$\gamma^{31} \times (377) \Longrightarrow H\underbrace{\gamma^{31}\gamma^{123}}_{-\gamma^2}\epsilon = \gamma^{31}\left(-Q\gamma^0 + iM\right)\epsilon \Longrightarrow H\gamma^2\epsilon = (Q\gamma^{031} - iM\gamma^{31})\epsilon \tag{394}$$

$$\chi\gamma_3 = (Q\underbrace{\gamma^{01}\gamma_3}_{\gamma^{301}=-\gamma^{031}} + H\underbrace{\gamma^{23}\gamma_3}_{\gamma^2}) = -Q\gamma^{031} + H\gamma^2 \overset{(394)}{=} -iM\gamma^{31} = iM\gamma^{13} \tag{395}$$

so

$$\begin{aligned} H(r,\theta) &= \left[\frac{1}{2}\left(U\sin\theta\gamma^{13} + \cos\theta\gamma^{23}\right) - \frac{i}{2r}\sin\theta\cdot\left(iM\gamma^{13}\right)\right] = \\ &= \frac{1}{2}\sin\theta\gamma^{13}\left(U + \frac{M}{r}\right) + \frac{1}{2}\cos\theta\gamma^{23} = \frac{1}{2}\left(\sin\theta\gamma^{13} + \cos\theta\gamma^{23}\right) = H(\theta) \end{aligned} \tag{396}$$

It seems that $H(\theta)$ can not be further simplified, supposing a problem to our computations since it disables us to factorise the spinor into variable independent parts.

To overcome this, we can use the properties of an exponential of the type $e^{\alpha X}$, where α is a variable and X is a matrix obeying the property $X^2 = -\mathbb{1}$. Then

$$X^2 = -\mathbb{1} \quad ; \quad X^3 = -X \quad ; \quad X^4 = \mathbb{1} \quad ; \quad X^5 = X \tag{397}$$

so

$$\begin{aligned} e^{\alpha X} &= \sum_{n=0}^{\infty}\frac{\alpha^n X^n}{n!} = \mathbb{1} + \alpha X + \frac{\alpha^2}{2!}X^2 + \frac{\alpha^3}{3!}X^3 + \frac{\alpha^4}{4!}X^4 + \frac{\alpha^5}{5!}X^5 + \cdots = \\ &= \mathbb{1}\left(1 - \frac{\alpha^2}{2!} + \frac{\alpha^4}{4!} - \cdots\right) + X\left(\alpha - \frac{\alpha^3}{3!} + \frac{\alpha^5}{5!} - \cdots\right) = \mathbb{1}\cos\alpha + X\sin\alpha \end{aligned} \tag{398}$$

obtaining

$$H(\theta) = \frac{1}{2}\gamma^{23}\left(\cos\theta - \sin\theta\gamma^{23}\gamma^{13}\right) = \frac{1}{2}\gamma^{23}\left(\cos\theta - \sin\theta\gamma^{12}\right) = \frac{1}{2}\gamma^{23}e^{-\gamma^{12}\theta} \tag{399}$$

and

$$\partial_\phi\hat{\epsilon} = \mathcal{M}_1^{-1}(\theta)H(\theta)\mathcal{M}_1(\theta)\hat{\epsilon} = e^{-\frac{1}{2}\gamma^{12}\theta}\frac{1}{2}\gamma^{23}e^{-\gamma^{12}\theta}e^{\frac{1}{2}\gamma^{12}\theta}\hat{\epsilon} \tag{400}$$

Let's try to move $\mathcal{M}_1^{-1}(\theta)$ to the right of $H(\theta)$. We have

$$\mathcal{M}_1^{-1}(\theta)\gamma^{23} = \sum_{n=0}^{\infty}\frac{(-1)^n\theta^n}{2^n n!}(\gamma^{12})^n\gamma^{23} \overset{(402)}{=} \gamma^{23}\sum_{n=0}^{\infty}\frac{(-1)^{2n}\theta^n}{2^n n!}(\gamma^{12})^n = \gamma^{23}e^{\frac{1}{2}\gamma^{12}\theta} = \gamma^{23}\mathcal{M}_1(\theta) \tag{401}$$

where we've used the property

$$(\gamma^{12})^n\gamma^{23} = (\gamma^{12})^{n-1}\underbrace{\gamma^{12}\gamma^{23}}_{\gamma^{23}(-\gamma^{23}\gamma^{12}\gamma^{23})=-\gamma^{23}\gamma^{12}} = -(\gamma^{12})^{n-1}\gamma^{23}\gamma^{12} = \text{n-1 more times} = (-1)^n\gamma^{23}(\gamma^{12})^n \tag{402}$$

Then

$$\partial_\phi\hat{\epsilon} = \frac{1}{2}\gamma^{23}e^{\frac{1}{2}\gamma^{12}\theta}e^{-\gamma^{12}\theta}e^{\frac{1}{2}\gamma^{12}\theta}\hat{\epsilon} = \frac{1}{2}\gamma^{23}\hat{\epsilon} \quad \Longrightarrow \quad \hat{\epsilon} \propto \exp\left(\frac{1}{2}\gamma^{23}\phi\right) = \mathcal{M}_2(\phi) \tag{403}$$

To finish, let's see what form ϵ_0 has (maybe it's not trivial). The spinor we've found so far must satisfy the (322) projector condition

$$\Theta\epsilon = 0 \Longrightarrow \overbrace{\frac{i}{M}\left(\gamma_0 Q - \gamma_{123}H\right)}^{C} f(r)\mathcal{M}_1\mathcal{M}_2\epsilon_0 = f(r)\mathcal{M}_1\mathcal{M}_2\epsilon_0 \tag{404}$$

Does C conmute with $\mathcal{M}_1$ and $\mathcal{M}_2$?

$$C \propto \gamma^0 Q + \gamma^{123}H \Longrightarrow \begin{cases} C\mathcal{M}_1 \propto \gamma^0\gamma^{12} + \gamma^{123}\gamma^{12} = \gamma^{012} + \gamma^{312}\gamma^{12} = \gamma^{12}\gamma^0 + \gamma^3\underbrace{\gamma^{12}\gamma^{12}}_{-\mathbb{1}} = \gamma^{12}(\gamma^0 + \gamma^{123}) \\ C\mathcal{M}_2 \propto \gamma^0\gamma^{23} + \gamma^{123}\gamma^{23} = \gamma^{23}\gamma^0 - \gamma^1 = \gamma^{23}(\gamma^0 + \underbrace{\gamma^{23}\gamma^1}_{\gamma^{123}}) \end{cases} \tag{405}$$

so we end up with

$$C\epsilon = f\mathcal{M}_1\mathcal{M}_2C\epsilon_0 = f\mathcal{M}_1\mathcal{M}_2\epsilon_0 \quad \Longrightarrow \quad \boxed{C\epsilon_0 = \epsilon_0} \tag{406}$$

The final spinor in full generality is

$$\boxed{\epsilon(r,\theta,\phi) = f(r)\mathcal{M}_1(\theta)\mathcal{M}_2(\phi)C\epsilon_0 = \sqrt{U(r)}\exp\left(\frac{1}{2}\gamma^{12}\theta\right)\exp\left(\frac{1}{2}\gamma^{23}\phi\right)C\epsilon_0} \tag{407}$$

which provides an explicit construction of the Killing spinors for a general (flat-space) extreme RN black hole, of where because $C\epsilon_0 = \epsilon_0$ imply that half of the components of ϵ_0 are projected out so supersymmetry is halved.

5.4 Electric AdS extreme Reissner-Nordström solutions

Second case, $H = 0$, $Q^2 = M^2$. Then $Z^2 = Q^2 = M^2$, and

$$U(r) = \sqrt{1 - \frac{2M}{r} + \frac{M^2}{r^2} + g^2r^2} = \sqrt{\left(1 - \frac{M}{r}\right)^2 + g^2r^2} \tag{408}$$

Our Θ operator now becomes

$$\Theta\epsilon = \left[U + gr\gamma_1 + \overbrace{\left(\frac{1}{r} - \frac{M}{M^2}\right)}^{\frac{1}{r}-\frac{1}{M}=-\frac{1}{M}\left(1-\frac{M}{r}\right)} i\gamma_0 Q\right]\epsilon = 0 \quad \Longrightarrow \quad \boxed{\left(U + gr\gamma_1\right)\epsilon = i\left(1 - \frac{M}{r}\right)\gamma_0\epsilon} \tag{409}$$

where we've assumed $Q = M$ since the $Q = -M$ case can be recast into the actual one by a field redefinition. We can use the projector to simplify the Killing spinor equations. Starting from the t direction

$$\hat{\nabla}_t = \partial_t + \frac{1}{2}\omega_{t01}\gamma^{01} - igA_t + \frac{1}{2}g\gamma_t + \frac{i}{2}F_{01}\gamma^{01}\gamma_t = \partial_t - \frac{U'U}{2}\gamma^{01} - ig\frac{Q}{r} + \frac{1}{2}g\gamma_t + \frac{i}{2}\frac{Q}{r^2}\gamma^{01}U\gamma_0 =$$
$$= \partial_t - \frac{U}{2}\left[U'\gamma^{01} + g\gamma^0 + \frac{iQ}{r^2}\gamma^1 + 2ig\frac{Q}{r}U^{-1}\right] = \partial_t - T(r) \tag{410}$$

We have

$$\begin{aligned} U' &= \frac{1}{2}\cdot\frac{1}{\sqrt{\cdots}}\cdot\left[2\left(1-\frac{M}{r}\right)\frac{M}{r^2} + 2g^2 r\right] = U^{-1}\left[\left(1-\frac{M}{r}\right)\frac{M}{r^2} + g^2 r\right] = \\ &= U^{-1}\left[\left(1-\frac{M}{r}\right)\frac{M}{r^2} + \frac{1}{r}\left(U^2 - \left(1-\frac{M}{r}\right)^2\right)\right] = \\ &\overset{(412)}{=} U^{-1}\left[\left(1-\frac{M}{r}\right)\underbrace{\left(\frac{M}{r^2} - \frac{1}{r}\left(1-\frac{M}{r}\right)\right)}_{-\frac{1}{r}\left(1-\frac{2M}{r}\right)} + \frac{U^2}{r}\right] = -\frac{U^{-1}}{r}\left(1-\frac{M}{r}\right)\left(1-\frac{2M}{r}\right) + \frac{U}{r} \end{aligned} \tag{411}$$

since

$$g^2 r = \left[U^2 - \left(1-\frac{M}{r}\right)^2\right]\frac{1}{r} \tag{412}$$

and from (409) we know

$$\gamma^{01}\times\left(g\gamma^1\epsilon = -\frac{1}{r}\left[i\left(1-\frac{M}{r}\right)\gamma_0 + U\right]\epsilon\right) \Longrightarrow g\gamma^0\epsilon = -\frac{1}{r}\left[i\left(1-\frac{M}{r}\right)\gamma^1 + U\gamma^{01}\right]\epsilon \tag{413}$$

$$\gamma^{1}\times\left(g\gamma^1\epsilon = -\frac{1}{r}\left[i\left(1-\frac{M}{r}\right)\gamma_0 + U\right]\epsilon\right) \Longrightarrow U\gamma^1\epsilon = \left[i\left(1-\frac{M}{r}\right)\gamma^{01} - gr\right]\epsilon \tag{414}$$

so T(r) becomes

$$\begin{aligned} T(r) &\overset{(411)(413)}{=} \frac{U}{2}\left[\left(-\frac{U^{-1}}{r}\left(1-\frac{M}{r}\right)\left(1-\frac{2M}{r}\right) + \cancel{\frac{U}{r}}\right)\gamma^{01} + \right. \\ &\quad \left. + \frac{iM}{r^2}\gamma^1 + \frac{2iMgU^{-1}}{r} - \frac{i}{r}\left(1-\frac{M}{r}\right)\gamma^1 - \cancel{\frac{U}{r}}\gamma^{01}\right] \\ &= -\frac{1}{2r}\left(1-\frac{M}{r}\right)\left(1-\frac{2M}{r}\right)\gamma^{01} + \frac{iM}{r^2}U\gamma^1 + \frac{iMg}{r} - \frac{iU}{2r}\gamma^1 \\ &= -\frac{1}{2r}\left(1-\frac{M}{r}\right)\left(1-\frac{2M}{r}\right)\gamma^{01} + \frac{iU}{2r}\left(\frac{2M}{r} - 1\right)\gamma^1 + \frac{iMg}{r} \\ &= -\frac{1}{2r}\left(1-\frac{2M}{r}\right)\left[\left(1-\frac{M}{r}\right)\gamma^{01} + iU\gamma^1\right] + \frac{iMg}{r} \\ &\overset{(414)}{=} -\frac{1}{2r}\left(1-\frac{2M}{r}\right)\left[\cancel{\left(1-\frac{M}{r}\right)\gamma^{01}} - \cancel{\left(1-\frac{M}{r}\right)\gamma^{01}} - irg\right] + \frac{iMg}{r} \\ &= \frac{ig}{2} - \frac{igM}{2} + \frac{iMg}{2} = \frac{ig}{2} \end{aligned} \tag{415}$$

so we ended up with

$$\partial_t\epsilon = \frac{ig}{2}\epsilon \tag{416}$$

Now, for the radial part

$$\hat{\nabla}_r = \partial_r + \frac{1}{2}gu^{-1}\gamma_1 + \frac{i}{2}U^{-1}\frac{Q}{r^2}\underbrace{\gamma^{01}\gamma_1}_{\gamma^0} = \partial_r + \frac{1}{2}U^{-1}\left(g\gamma^1 + \frac{iQ}{r^2}\gamma^0\right) = \partial_r - R(r) \tag{417}$$

We have that

$$(409) \Longrightarrow \frac{iM\gamma^0}{r^2}\epsilon = \left(g\gamma^1 + \frac{U}{r} + \frac{i\gamma^0}{r}\right)\epsilon \tag{418}$$

so

$$R(r) = -\frac{1}{2}U^{-1}\left(g\gamma^1 + \frac{iM\gamma^0}{r^2}\right) \overset{(418)}{=} -\frac{1}{2}U^{-1}\left(2g\gamma^1 + \frac{U}{r} + \frac{i\gamma^0}{r}\right) = -\frac{1}{2}U^{-1}\left(2g\gamma^1 + \frac{i\gamma^0}{r}\right) - \frac{1}{2r} \tag{419}$$

For the θ direction

$$\hat{\nabla}_\theta = \partial_\theta - \frac{1}{2}U\gamma^{12} + \frac{1}{2}gr\gamma_2 + \frac{i}{2}\frac{Q}{r}\underbrace{\gamma^{01}\gamma^2}_{\gamma^{201}} = \partial_\theta - H(\theta) \tag{420}$$

and because

$$\gamma^2 \times (414) \Longrightarrow U\gamma^{12} = -i\left(1 - \frac{M}{r}\right)\gamma^{201} + gr\gamma^2 \tag{421}$$

we have

$$\begin{aligned} H(\theta) &= \left[\frac{1}{2}U\gamma^{12} - \frac{1}{2}gr\gamma^2 - \frac{i}{2}\frac{M}{r}\gamma^{201}\right] = \\ &\overset{(421)}{=} \left[-\frac{i}{2}\left(1 - \frac{M}{r}\right)\gamma^{201} + \frac{1}{2}gr\gamma^2 - \frac{1}{2}gr\gamma^2 - \frac{i}{2}\frac{M}{2}\gamma^{201}\right] = -\frac{i}{2}\gamma^{201} \end{aligned} \tag{422}$$

And for the ϕ component

$$\begin{aligned} \hat{\nabla}_\phi &= \partial\phi - \frac{1}{2}\left(U\sin\theta\gamma^{13} + \cos\theta\gamma^{23}\right) + \frac{1}{2}gr\sin\theta\gamma_3 + \frac{i}{2}\frac{Q}{r}\sin\theta\gamma^{01}\gamma_3 = \\ &= \partial_\phi - \frac{1}{2}\left(U\sin\theta\gamma^{13} + \cos\theta\gamma^{23}\right) + \frac{1}{2}\sin\theta\left(gr + \frac{iQ\gamma^{01}}{r}\right)\gamma_3 = \partial_\phi - F(r,\theta) \end{aligned} \tag{423}$$

$$\gamma^3 \times (414) \Longrightarrow U\gamma^{13} = -i\left(1 - \frac{M}{r}\right)\gamma^{301} + gr\gamma^3 \tag{424}$$

so

$$\begin{aligned} F(r,\theta) &= \left[\frac{1}{2}\left(U\sin\theta\gamma^{13} + \cos\theta\gamma^{23}\right) - \frac{1}{2}\sin\theta\left(gr + \frac{iM\gamma^{01}}{r}\right)\gamma_3\right] = \\ &\overset{(424)}{=} \left[\frac{1}{2}\sin\theta\left(\cancel{gr\gamma^3} - i\left(1 - \cancel{\frac{M}{r}}\right)\gamma^{301}\right) + \frac{1}{2}\cos\theta\gamma^{23} - \frac{1}{2}\sin\theta\left(\cancel{gr} + \cancel{\frac{iM\gamma^{01}}{r}}\right)\gamma^3\right] = \\ &= \left(\frac{1}{2}\cos\theta\gamma^{23} - \frac{i}{2}\sin\theta\gamma^{301}\right) = \frac{1}{2}\gamma^{23}(\cos\theta + i\sin\theta\underbrace{\gamma^{23}\gamma^{301}}_{\gamma^{201}}) = \frac{1}{2}\gamma^{23}e^{i\gamma^{201}\theta} = F(\theta) \end{aligned} \tag{425}$$

Then, assuming a spinor of the form

$$\epsilon(t,\theta,\phi,r) = \tau(t)\mathcal{M}_1(\theta)\mathcal{M}_2(\phi)\varrho(r) \tag{426}$$

we had, for the t direction

$$\partial_t \epsilon = \partial_t \tau(t)\tilde{\epsilon} = \frac{ig}{2}\tau(t)\tilde{\epsilon} \quad \Longrightarrow \quad \tau(t) = \exp\left(\frac{ig}{2}t\right) \tag{427}$$

for θ

$$\partial_\theta \epsilon = \tau(t)\partial_\theta \mathcal{M}_1(\theta)\hat{\epsilon} = -\frac{i}{2}\gamma^{201}\tau(t)\mathcal{M}_1(\theta)\hat{\epsilon} \quad \Longrightarrow \quad \mathcal{M}_1(\theta) = \exp\left(-\frac{i}{2}\gamma^{012}\theta\right) \tag{428}$$

and for ϕ

$$\begin{gathered}\partial_\phi \epsilon = \tau(t)\mathcal{M}_1(\theta)\partial_\phi \mathcal{M}_2(\phi)\varrho(r) = \tau(t)F(\theta)\mathcal{M}_1(\theta)\mathcal{M}_2(\phi)\varrho(r) \Longrightarrow \\ \Longrightarrow \partial_\phi \mathcal{M}_2(\phi) = \mathcal{M}_1^{-1}(\theta)F(\theta)\mathcal{M}_1(\theta)\mathcal{M}_2(\phi) = \frac{1}{2}\gamma^{23}\mathcal{M}_1(\theta)e^{i\gamma^{201}\theta}\mathcal{M}_1(\theta)\mathcal{M}_2(\phi) = \frac{1}{2}\gamma^{23}\mathcal{M}_2(\phi) \Longrightarrow \\ \Longrightarrow \mathcal{M}_2(\phi) = \exp\left(\frac{1}{2}\gamma^{23}\phi\right)\end{gathered} \tag{429}$$

You may have noticed that we've left the radial component for last, as opposite with the previous case. That's not random, since now it is not as simple as before due to the r-dependence projection constraint. Thus we are going to need a little bit more machinery. We have that

$$R(r) \propto \gamma^1 \quad \text{and} \quad \gamma^0 \tag{430}$$

so

$$R\mathcal{M}_1 \Longrightarrow \begin{cases}\gamma^1\gamma^{012} = \gamma^{012}\gamma^1 \\ \gamma^0\gamma^{012} = \gamma^{012}\gamma^0\end{cases} \tag{431}$$

$$R\mathcal{M}_2 \Longrightarrow \begin{cases}\gamma^1\gamma^{23} = \gamma^{23}\gamma^1 \\ \gamma^0\gamma^{23} = \gamma^{23}\gamma^0\end{cases} \tag{432}$$

so $R(r)$ commutes with both $\mathcal{M}_1$ and $\mathcal{M}_2$. Therefore

$$\begin{gathered}\partial_r \epsilon = \tau(t)\mathcal{M}_1\mathcal{M}_2\partial_r\varrho(r) = \tau(t)R(r)\mathcal{M}_1\mathcal{M}_2\varrho(r) = \tau(t)\mathcal{M}_1\mathcal{M}_2 R(r)\varrho(r) \Longrightarrow \\ \Longrightarrow \partial_r\varrho(r) = R(r)\varrho(r)\end{gathered} \tag{433}$$

Following the appendix of [5] on how to solve the radial equation, if we put Θ as

$$\Theta = U + gr\gamma_1 - i\left(1 - \frac{M}{r}\right)\gamma_0 = 2U\frac{1}{2}\left(1 + \frac{gr}{U}\gamma_1 - \frac{i}{U}\left(1 - \frac{M}{r}\right)\gamma_0\right) = 2U\Pi(r) \tag{434}$$

which satisfies $\Pi\epsilon = 0$. Identifying

$$x(r) = \frac{gr}{U} \qquad ; \qquad y(r) = \frac{1}{U}\left(1 - \frac{M}{r}\right) \tag{435}$$

which satisfy the condition

$$x^2 + y^2 = \left(\frac{gr}{U}\right)^2 + \frac{\left(1 - \frac{M}{r}\right)^2}{U^2} = \frac{1}{U^2}\overbrace{\left(gr^2 + \left(1 - \frac{M}{r}\right)^2\right)}^{U^2} = 1 \tag{436}$$

and

$$\Gamma_1 = \gamma_1 \qquad ; \qquad \Gamma_2 = -i\gamma_0 \tag{437}$$

which satisfy the other condition

$$\Gamma_1\Gamma_2 = -i\gamma_1\gamma_0 = +i\gamma_0\gamma_1 = -\Gamma_2\Gamma_1 \tag{438}$$

If we combine $\Pi\eta = 0$ and the equation

$$\eta' = \left(a(r) + b(r)\Gamma_1 + c(r)\Gamma_2\right)\eta(r) \tag{439}$$

we get that

$$\Pi\eta = 0 \Longrightarrow \Gamma_2\eta = -\frac{1}{y(r)}\left(1 + x(r)\Gamma_1\right)\eta \tag{440}$$

so

$$\eta' = \Bigg[\underbrace{a(r) - \frac{c(r)}{y(r)}}_{\tilde{a}(r)} + \underbrace{\left(b(r) - \frac{x(r)}{y(r)}c(r)\right)}_{\tilde{b}(r)}\Gamma_1\Bigg]\eta(r) = \left(\tilde{a}(r) + \tilde{b}(r)\Gamma_1\right)\eta(r) \tag{441}$$

assuming $y(r) \neq 0$. So we can take $c(r) = 0$ and don't lose any generality.

A general solution is

$$\eta(r) = \left(u(r) + v(r)\Gamma_2\right)P\left(-\Gamma_1\right)\eta_0 \tag{442}$$

where

$$u = \sqrt{\frac{1+x}{y}}\, e^{\omega} \qquad ; \qquad v = -\sqrt{\frac{1-x}{y}}\, e^{\omega} \tag{443}$$

$$\omega = \int^r a\left(r'\right)dr' \tag{444}$$

and the projector P is defined as

$$P(\Gamma) = \frac{1}{2}(1+\Gamma) \tag{445}$$

so $\Gamma^2 = 1$ and $P^2 = P$. P acting on some spinor is zero if it gives -1 eigenvalue and is equal to the spinor if it gives the $+1$ eigenvalue.

Comparing with (439), we identify

$$\partial_r\epsilon = R(r)\epsilon = \Bigg(\underbrace{-U^{-1}g}_{b(r)}\gamma^1 \underbrace{-\frac{U^{-1}}{2r}}_{c(r)}(-i\gamma_0) \underbrace{-\frac{1}{2r}}_{a(r)}\Bigg)\epsilon \tag{446}$$

and using $\tilde{a}(r)$

$$\tilde{a}(r) = -\frac{1}{2r} - \frac{U^{-1}}{2r}\frac{1}{U^{-1}\left(1-\frac{M}{r}\right)} = \frac{M}{2}\frac{1}{r(r-M)} \tag{447}$$

to compute $\omega(r)$

$$\omega(r) = \int^r \tilde{a}(r')dr' = \frac{M}{2}\cdot\frac{\ln\left(1-\frac{M}{r}\right)}{M} = \ln\sqrt{1-\frac{M}{r}} = \ln\sqrt{Uy} \tag{448}$$

we have

$$u = \sqrt{\frac{1+x}{y}}\overbrace{e^{w}}^{\sqrt{Uy}} = \sqrt{(1+x)U} = \sqrt{U+gr} \tag{449}$$

$$v = -\sqrt{\frac{1-x}{y}}\sqrt{Uy} = -\sqrt{U-gr} \tag{450}$$

and finally, using (442)

$$\varrho(r) = \left(\sqrt{U+gr} + \sqrt{U-gr}\; i\gamma_0\right) P(-\gamma_1)\epsilon_0 \tag{451}$$

where all the constants we've omitted along the way can be introduced into ϵ_0.

One thing to note is that, because we have a projector in the spinor definition, supersymmetry is again half maximal.

The final spinor form if

$$\boxed{\epsilon(t,\theta,\phi,r) = \exp\left(\frac{ig}{2}t\right)\exp\left(-\frac{i}{2}\gamma^{012}\theta\right)\exp\left(\frac{1}{2}\gamma^{23}\phi\right)\left(\sqrt{U+gr} + \sqrt{U-gr}\; i\gamma_0\right) P(-\gamma_1)\epsilon_0} \tag{452}$$

which provides an explicit construction of the Killing spinors for a general electric AdS extreme Reissner-Nordström BH.

5.5 Exotic AdS solutions ("cosmic monopoles")

Last case where $M = 0, H = \frac{1}{2g}$ ($H = -\frac{1}{2g}$ it's equivalent to sending $g \to -g$). (322) can be cast into the form

$$\Theta\epsilon = \left[2U\Pi - \frac{1}{gr}\gamma_1 P\left(i\gamma_{23}\right)\right]\epsilon = 0 \tag{453}$$

where both

$$\Pi\epsilon = 0 \qquad \text{and} \qquad P(i\gamma_{23})\epsilon = 0. \tag{454}$$

with

$$\Pi\epsilon = \frac{1}{2}\left\{1 + \frac{1}{U}\left[\left(gr + \frac{1}{2gr}\right)\gamma^1 - i\gamma^0\frac{Q}{r}\right]\right\}\epsilon = 0 \tag{455}$$

obtaining two conditions this time

$$\begin{cases} \Pi\epsilon = 0 \Longrightarrow \left(gr + \frac{1}{2gr}\right)\gamma^1\epsilon = \left(i\gamma^0\frac{Q}{r} - U\right)\epsilon & (a) \\ P(i\gamma_{23})\epsilon = 0 \Longrightarrow i\gamma_{23}\;\epsilon = -\epsilon & (b) \end{cases} \tag{456}$$

Then, (456b) is telling us that the projector $P(i\gamma_{23})$ is getting the eigenvalue -1 when acting on ϵ.

The potential becomes

$$U(r) = \sqrt{1 + \frac{Z^2}{r^2} + g^2r^2} = \sqrt{\left(gr + \frac{1}{2gr}\right)^2 + \frac{Q^2}{r^2}} \tag{457}$$

where we clearly see that it doesn't have a flat-space limit, since H and $U(r)$ blow up when $g \to 0$. The overall AdS cosmological distance, defined as $\sim \frac{1}{g}$ gives us the characteristic scales for size and magnetic charge. For is usually referred to as a "cosmic monopole".

using $Z^2 = Q^2 + \frac{1}{4g^2}$. It's derivative reads

$$U' = \frac{1}{2}U^{-1}\left[2\left(gr + \frac{1}{2gr}\right)\left(g - \frac{1}{2gr^2}\right) - \frac{2Q^2}{r^3}\right] = \xi(r)U^{-1} \tag{458}$$

The time equation is

$$\hat{\nabla}_t = \partial_t - \frac{1}{2}UU'\gamma^{01} - ig\frac{Q}{r} - \frac{1}{2}gU\gamma^0 + \frac{i}{2}\left(\frac{Q}{r^2}\gamma^{01} + \frac{H}{r^2}\gamma^{23}\right)U\gamma_0 \tag{459}$$

where

$$UU'\gamma^{01} = \xi(r)\gamma^{01} = \gamma^0\xi(r)\gamma^0 = \gamma^0\left[\left(gr + \frac{1}{2gr}\right)\cdot\left(g - \frac{1}{2gr^2}\right) - \frac{Q^2}{r^3}\right]\gamma^1 = \stackrel{(456a)}{=} \gamma^0\left(g - \frac{1}{2gr^2}\right)\left(i\gamma^0\frac{Q}{r} - U\right) - \frac{Q^2}{r^3}\gamma^{01} \tag{460}$$

and

$$\gamma_{23}\gamma_0\epsilon = \gamma_0\gamma_{23}\epsilon = i\gamma_0\epsilon \tag{461}$$

so

$$\hat{\nabla}_t = \partial_t + \frac{igQ}{2r} + \frac{1}{2}gU\gamma^0 - \frac{iQ}{4gr^3} - \frac{U\gamma^0}{4gr^2} + \frac{Q^2}{2r^3}\gamma^{01} - ig\frac{Q}{r} - \frac{1}{2}gH\gamma^0 - \frac{iQ}{2r^2}U\gamma^1 + \frac{U}{4gr^2}\gamma^0 = \stackrel{(464)}{=} \partial_t - \frac{iQ}{4gr^3} + \frac{iQ^2\gamma^{01}}{2r^3} - \frac{iQ}{2r^2}\gamma^1\left[\frac{iQ}{r}\gamma^0 - \left(gr + \frac{1}{2gr}\right)\gamma^1\right] - \frac{igQ}{2r} = \partial_t \tag{462}$$

$$\partial_t\epsilon = 0 \Longrightarrow \text{time independent} \tag{463}$$

We've used

$$(456a) \Longrightarrow U\gamma^1\epsilon = \left[\frac{-i\gamma^{01}Q}{r} - \left(gr + \frac{1}{2gr}\right)\right]\epsilon \tag{464}$$

For the θ component we will need

$$(456a) \Longrightarrow U\gamma^{12}\epsilon = \left[-\frac{i\gamma^{012}Q}{r} + \left(gr + \frac{1}{2gr}\right)\gamma^2\right]\epsilon \tag{465}$$

$$\hat{\nabla}_\theta = \partial_\theta - \frac{1}{2}U\gamma^{12} + \frac{1}{2}gr\gamma^2 + \frac{i}{2}\left(\frac{Q}{r^2}\gamma^{01} + \frac{H}{r^2}\gamma^{23}\right)r\gamma^2 = \stackrel{(465)}{=} \partial_\theta - \frac{1}{2}\left[-\frac{i\gamma^{012}Q}{r} + \left(gr + \frac{1}{2gr}\right)\gamma^2\right] + \frac{1}{2}gr\gamma^2 + \frac{iQ}{2r}\gamma^{012} + \frac{1}{4gr}\gamma^2 = \partial_\theta \Longrightarrow \theta \text{ independent} \tag{466}$$

and for the ϕ component

$$\hat{\nabla}_\phi = \partial\phi - \frac{1}{2}\left(U\sin\theta\gamma^{13} + \cos\theta\gamma^{23}\right) + \frac{i}{2}\cos\theta + \frac{1}{2}gr\sin\theta\gamma^3 + \frac{i}{2}\left(\frac{Q}{r^2}\gamma^{01} + \frac{H}{r^2}\gamma^{23}\right)r\sin\theta\gamma^3 = \stackrel{(468)}{=} \partial_\phi - \frac{1}{2}\cos\theta\left(\gamma^{23} - i\mathbb{1}\right) + \frac{1}{2}gr\sin\theta\gamma^3 + \frac{iQ}{2r}\sin\theta\gamma^{013} + \frac{1}{4gr}\sin\theta\gamma^3 - -\frac{1}{2}\sin\theta\left[-\frac{iQ}{r}\gamma^{013} + \left(gr + \frac{1}{2gr}\right)\gamma^3\right] \stackrel{(456b)}{=} \partial_\phi \quad \Longrightarrow \quad \phi \text{ independent} \tag{467}$$

we've used

$$(456a) \Longrightarrow U\gamma^{13}\epsilon = \left[-\frac{i\gamma^{013}Q}{r} + \left(gr + \frac{1}{2gr}\right)\gamma^3\right]\epsilon \tag{468}$$

So our spinor only depends on r. To simplify the radial equation we will need

$$(456a) \Longrightarrow \frac{i\gamma^0 Q}{2r^2}\epsilon = \frac{1}{2r}\left[\left(gr + \frac{1}{2gr}\right)\gamma^1 + U\right]\epsilon \tag{469}$$

so

$$\begin{aligned}\hat{\nabla}_r &= \partial_r + \frac{1}{2}gU^{-1}\gamma^1 + \frac{i}{2}U^{-1}\left(\frac{Q}{r^2}\gamma^{01} + \frac{Q}{r^2}\gamma^{23}\right)\gamma^1 = \\ &= \partial_r + U^{-1}\left(\frac{1}{2}g\gamma^1 + \frac{iQ}{2r^2}\gamma^0 - \frac{1}{4gr^2}\gamma^1\right) = \\ &\overset{(469)}{=} \partial_r + U^{-1}\left(g\gamma^1 + \frac{U}{2r}\right) = \partial_r + \frac{g}{U}\gamma^1 + \frac{1}{2r}\end{aligned} \tag{470}$$

This has solution

$$\boxed{\epsilon(r) = \left(\sqrt{U + gr + \frac{1}{2gr}} + \sqrt{U - gr - \frac{1}{2gr}}\, i\gamma^0\right) P\left(-\gamma_1\right) P\left(-i\gamma_{23}\right)\epsilon_0} \tag{471}$$

where we've absorbed the $\frac{1}{\sqrt{Q}}$ factor inside ϵ_0. This time as we assumed (485) since in the derivation we assumed two projections, in the solution we have two projectors action on the constant spinor meaning that only one complex supercharge remains.

This is the explicit construction of the Killing spinors for a general exotic AdS Reissner-Nordström BH.

6 Discussion

Among the electrically and magnetically charged Reissner-Nordström solutions of Einstein-Maxwell theory with a cosmological constant Λ, we have identified a new type, supersymmetric black holes (for $\Lambda \leq 0$).

An important feature of supersymmetric RN solutions in AdS space is the presence of naked singularities, violating Penrose's cosmic censorship hypothesis. In flat space, the naked singularity of overextreme ($Z^2 > M^2$) RN solutions tends to be unstable, losing charge until reaching the extreme state with an apparent horizon.

To rigorously address the stability of the supersymmetric solutions, one might adapt existing stability proofs for black holes, supergravity theories, and anti-de Sitter backgrounds to the current context. The critical challenge lies in establishing proper boundary conditions at the singularity.

This work have several potential extensions worth exploring. Examining charged configurations might influence early inflationary models with de Sitter phases, and also cosmological analogues of black-hole solutions in field theories with scalar fields could be investigated. Finally, the discussed aspects of Reissner-Nordström solutions might reflect in the structure of certain conformal field theories that can be interpreted in terms of charged black holes.

7 References

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