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New results on convergence in distribution of fuzzy random variables

Miriam Alonso de la Fuente

Departamento de Estadística e I.O. y D.M., Universidad de Oviedo, E-33071 Oviedo, Spain

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ABSTRACT

We study some properties of convergence in distribution of fuzzy random variables in the metric d_p, more precisely, if this type of convergence is preserved when we apply some functions between the space of fuzzy sets and other metric spaces. In particular, we show that this convergence is preserved when we take the closed convex hull, the maximum, the minimum, the product and the quotient. Moreover, it is preserved when we apply some functionals, such as the value or the ambiguity. Finally, we show some results regarding convergence in distribution with respect to the metric d_∞.

1. Introduction

Convergence in distribution, defined as weak convergence of random elements in the space of fuzzy sets endowed with the metric d_p has been shown to behave well, since it is preserved under some operations, such as taking the sum of fuzzy random variables or the product of a fuzzy random variable and a real random variable (see [4]). It also verifies some usual convergence theorems, such as dominated convergence theorem (see [1] and [3]) and continuous mapping theorem (see [1]).

Functional of fuzzy sets, such as weighted averaging based on levels or value are used in the context of defuzzification (see [9]), hypothesis testing, ranking (see [19]) or approximation (see [11]).

In [4], we showed that convergence in distribution of k-tuples of fuzzy random variables, that is $(X_1, \dots, X_k)$ where X_i is a fuzzy random variable for every $i \in \{1, \dots, n\}$, is equivalent to convergence in distribution of their product in a space of fuzzy sets of higher dimension. Taking advantage of this result, we will show that if we build a fuzzy random variable as a result of an application of operations between two random variables, the result is also a fuzzy random variable and convergence in distribution is preserved. In particular, we will consider the lattice operation of taking the convex hull of the union of fuzzy sets, the maximum and minimum of fuzzy random variables and the product and quotient of fuzzy random variables taking positive values.

In [2], we proved the equivalence between convergence in distribution of a trapezoidal fuzzy random variable with respect to the metric d_p and convergence in distribution of the four points that identify it as a random vector. Now, we will improve this result showing that convergence of this random vector implies convergence in distribution of the fuzzy random vector in the metric d_∞.

In another result in [2], we showed the equivalence of convergence in distribution of a sequence of fuzzy random variables in d_p and convergence in distribution of their support functions as random elements of the Banach space where support functions are defined. In this paper we will prove similar results for the d_∞ metric, focusing on convergence by α-cuts, which are not verified when d_p is taken.

E-mail addresses: alonsofmiriam@uniovi.es, uo233280@uniovi.es.

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We will finish the article pointing out that convergence in distribution of fuzzy random variables in d_∞ implies convergence in distribution of every α-cut in the Hausdorff metric.

2. Preliminaries

Denote by $(\mathbb{E},\tau)$ a topological space and by $(\mathbb{E},d)$ a metric space. Denote by $(\Omega,\mathcal{A},P)$ a probability space and by $\mathcal{B}_{\mathbb{E}}$ the *Borel σ-algebra* of $(\mathbb{E},\tau)$, i.e., the σ-algebra generated by its open sets. A Borel measurable mapping with values in $\mathbb{E}$ will be called a *random element* of $\mathbb{E}$. Let A be a subset of $\mathbb{E}$. Then its convex hull (that is, the smallest convex set which contains A) will be denoted by $\operatorname{co} A$ and its closure (the smallest closed set that contains A) by $\operatorname{cl} A$. The Lebesgue measure in $[0,1]$ will be denoted by ℓ.

Let $(\mathbb{E},d)$ be a metric space and let X_n, X be random elements in $\mathbb{E}$. Then X_n *converges weakly* to X if $E[f(X_n)] \to E[f(X)]$ for every continuous bounded function $f:\mathbb{E}\to\mathbb{R}$, where $E[f(X)]$ and $E[f(X_n)]$ denote the expectation of the random variables $f(X)$ and $f(X_n)$, respectively.

The space $(\mathbb{E},\tau)$ is *Polish* if τ is generated by a complete separable metric. A *Lusin space* is the image of a Polish space under a continuous bijective mapping and a *Suslin space* is the image of a Polish space under a continuous surjective mapping.

Lemma 2.1. *[10, Theorem 1] Let τ_1, τ_2 be two Suslin topologies on a set $\mathbb{E}$. Then $\mathcal{B}_1=\mathcal{B}_2$, where $\mathcal{B}_i$ denotes the Borel σ-algebra generated by τ_i, if and only if the supremum topology of τ_1 and τ_2, that is, the topology generated by the union of τ_1 and τ_2, is a Suslin topology on $\mathbb{E}$.*

Denote by $\mathcal{K}(\mathbb{R}^d)$ the space of non-empty compact subsets of $\mathbb{R}^d$ and by $\mathcal{F}(\mathbb{R}^d)$ the space of all upper semicontinuous functions $U:\mathbb{R}^d\to[0,1]$ such that

$$U_\alpha=\{x\in\mathbb{R}^d : U(x)\ge\alpha\}$$

belong to $\mathcal{K}(\mathbb{R}^d)$ for every $\alpha\in(0,1]$ and

$$U_0=\operatorname{cl}\{x\in\mathbb{R}^d : U(x)>0\}$$

is compact. For each $\alpha\in[0,1]$, U_α will be called *α-cut* of U.

Denote by $\mathcal{F}_c(\mathbb{R}^d)$ the space of upper semicontinuous functions $U:\mathbb{R}^d\to[0,1]$ such that U_α belong to $\mathcal{K}_c(\mathbb{R}^d)$, i.e. the subset of $\mathcal{K}(\mathbb{R}^d)$ which contains the convex sets, for every $\alpha\in[0,1]$ and by $\widehat{\mathcal{F}}_{c,1}(\mathbb{R})$ the set of all upper semicontinuous functions $U:\mathbb{R}\to[0,1]$ with $U_\alpha\in\mathcal{K}_c(\mathbb{R})$ and

$$\int_{(0,1]} d_H(U_\alpha,\{0\})d\alpha<\infty.$$

The elements of the space $\mathcal{F}_c(\mathbb{R})$ will be called *fuzzy numbers*.

The *Hausdorff metric* between two compact convex sets K and L is

$$d_H(K,L)=\max\{\sup_{x\in K}\inf_{y\in L}\|x-y\|,\sup_{y\in L}\inf_{x\in K}\|x-y\|\}.$$

The *norm* of a compact convex set K is

$$\|K\|=d_H(K,\{0\}).$$

Next, let us introduce the following metrics between fuzzy sets, which will be used in this paper.

For any $U\in\mathcal{F}_c(\mathbb{R}^d)$, denote by $\operatorname{end} U$ its *endograph*, i.e.,

$$\operatorname{end} U=\{(x,\alpha)\in\mathbb{R}^d\times[0,1] : U(x)\ge\alpha\}.$$

Then the *endograph metric* between $U,V\in\mathcal{F}_c(\mathbb{R}^d)$ is defined by

$$d_{\text{end}}(U,V)=d_H(\operatorname{end} U,\operatorname{end} V).$$

It is known that d_{end} is weaker than d_p, defined as

$$d_p(U,V)=\left[\int_{(0,1]}\left(d_H(U_\alpha,V_\alpha)\right)^p d\alpha\right]^{1/p},$$

where $p\in[1,\infty)$, and the d_p-metrics are weaker than d_∞, given by

$$d_\infty(U,V)=\sup_{\alpha\in[0,1]} d_H(U_\alpha,V_\alpha).$$

For recent results related to the well-definedness of the d_p metrics in the space of fuzzy sets, the reader is referred to [14] and [15].

Notice that the mapping d_1 defines a metric in the space $\widehat{\mathcal{F}}_{c,1}(\mathbb{R})$.

Finally, the metric ρ_p is equivalent to d_p, which has the expression

$$\rho_p(U,V)=\left(\int_{[0,1]}\frac{|\sup U_\alpha-\sup V_\alpha|^p}{2}d\alpha+\int_{[0,1]}\frac{|\inf U_\alpha-\inf V_\alpha|^p}{2}d\alpha\right)^{1/p}$$

for $p\in[1,\infty)$.

Each $U\in\mathcal{F}_c(\mathbb{R}^d)$ is uniquely determined by its *support function*, given by

$$\begin{aligned}s_U:\mathbb{S}^{d-1}\times[0,1]&\to\mathbb{R}\\ U&\to s_U(r,\alpha)=\sup\langle r,\alpha\rangle.\end{aligned}$$

Let $\alpha\in[0,1]$. Denote by

$$\begin{aligned}L_\alpha:(\mathcal{F}_c(\mathbb{R}^d),d_p)&\to(\mathcal{K}_c(\mathbb{R}^d),d_H)\\ U&\mapsto U_\alpha.\end{aligned}$$

Then L_α is measurable in d_p and continuous in d_∞. Denote by σ_L the smallest σ-algebra which makes all mappings L_α continuous.

Next, we will need some characterizations of convergence of sequences of fuzzy sets given, in a more general setting, in [13].

Lemma 2.2. *[13, Theorem 6.4, (i) ⇔ (ii)] Let $U_n,U\in\mathcal{F}_c(\mathbb{R}^d)$. Then $d_H(\operatorname{end}U,\operatorname{end}U_n)\to0$ if and only if $d_H(U_\alpha,(U_n)_\alpha)\to0$ for almost every $\alpha\in(0,1)$.*

Lemma 2.3. *[13, Theorem 6.5] Let $U_n,U\in\mathcal{F}_c(\mathbb{R}^d)$ such that $\bigcup_{n=1}^\infty(U_n)_0$ is bounded. If $d_H(\operatorname{end}U,\operatorname{end}U_n)\to0$, then $d_p(U_n,U)\to0$.*

Lemma 2.4. *[13, Theorem 6.6] Let $U_n,U\in\mathcal{F}_c(\mathbb{R}^d)$. If $d_p(U_n,U)\to0$, then $d_H(\operatorname{end}U_n,\operatorname{end}U)\to0$.*

Thank to these lemmas, we are allowed to establish the following result.

Lemma 2.5. *Let $U_n,U\in\mathcal{F}_c(\mathbb{R}^d)$ such that $\bigcup_{n=1}^\infty(U_n)_0$ is bounded. Then $d_H(\operatorname{end}U,\operatorname{end}U_n)\to0$ if and only if $d_p(U_n,U)\to0$.*

Let $U,V,U',V'\in\mathcal{F}_c(\mathbb{R}^d)$. Denote by

$$\begin{aligned}d_{\text{end}}^{\max}((U,V),(U',V'))&=\max\{d_{\text{end}}(U,U'),d_{\text{end}}(V,V')\},\\ d_p^{\max}((U,V),(U',V'))&=\max\{d_p(U,U'),d_p(V,V')\}\end{aligned}$$

for $p\in[1,\infty)$ and

$$d_\infty^{\max}((U,V),(U',V'))=\max\{d_\infty(U,U'),d_\infty(V,V')\}.$$

We will consider some operations between fuzzy numbers given in [7]. The first of them is the *maximum* of a pair of fuzzy numbers

$$\begin{aligned}\widetilde{\max}:(\mathcal{F}_c(\mathbb{R})\times\mathcal{F}_c(\mathbb{R}),d_p^{\max})&\to(\mathcal{F}_c(\mathbb{R}),d_{\text{end}})\\ (U,V)&\mapsto\widetilde{\max}(U,V)\end{aligned}$$

where

$$(\widetilde{\max}(U,V))_\alpha=\max\{U_\alpha,V_\alpha\}=[\max\{\inf U_\alpha,\inf V_\alpha\},\max\{\sup U_\alpha,\sup V_\alpha\}]$$

Analogously, the *minimum* of two fuzzy numbers is given by

$$\begin{aligned}\widetilde{\min}:(\mathcal{F}_c(\mathbb{R})\times\mathcal{F}_c(\mathbb{R}),d_p^{\max})&\to(\mathcal{F}_c(\mathbb{R}),d_{\text{end}})\\ (U,V)&\mapsto\widetilde{\min}(U,V)\end{aligned}$$

where

$$(\widetilde{\min}(U,V))_\alpha=\min\{U_\alpha,V_\alpha\}=[\min\{\inf U_\alpha,\inf V_\alpha\},\min\{\sup U_\alpha,\sup V_\alpha\}]$$

The *product* and *quotient* of fuzzy sets can be expressed via its α-cuts when the fuzzy sets involved take on only positive values.

$$\widetilde{\text{prod}}:(\mathcal{F}_c((0,\infty))\times\mathcal{F}_c((0,\infty)),d_p^{\max})\to(\mathcal{F}_c((0,\infty)),d_{\text{end}})$$

$$(U,V) \mapsto \widetilde{\mathrm{prod}}(U,V)$$

where

$$(\widetilde{\mathrm{prod}}(U,V))_\alpha = [\inf U_\alpha \cdot \inf V_\alpha, \sup U_\alpha \cdot \sup V_\alpha]$$

$$\widetilde{\mathrm{quot}} : (\mathcal{F}_c((0,\infty)) \times \mathcal{F}_c((0,\infty)), d_p^{\max}) \to (\mathcal{F}_c((0,\infty)), d_{\mathrm{end}})$$
$$(U,V) \mapsto \widetilde{\mathrm{quot}}(U,V)$$

where

$$(\widetilde{\mathrm{quot}}(U,V))_\alpha = [\inf U_\alpha / \sup V_\alpha, \sup U_\alpha / \inf V_\alpha]$$

An element of $\mathcal{K}_c(\mathbb{R})$, that is, a compact interval $K = [k_1, k_2]$, can be characterized via its infimum $\inf K = k_1$ and supremum $\sup K = k_2$, or by its midpoint $\mathrm{mid}\, K = \frac{k_1+k_2}{2}$ and spread $\mathrm{spr}\, K = \frac{k_2-k_1}{2}$.

We will consider four functionals on fuzzy numbers used in [6] and [12], the *value*

$$\mathrm{Val} : (\mathcal{F}_c(\mathbb{R}), d_p) \to \mathbb{R}$$
$$U \mapsto \mathrm{Val}(U) = \int_{[0,1]} \alpha(\sup U_\alpha + \inf U_\alpha) d\alpha = 2 \int_{[0,1]} \alpha\, \mathrm{mid}\, U_\alpha d\alpha,$$

the *ambiguity*

$$\mathrm{Amb} : (\mathcal{F}_c(\mathbb{R}), d_p) \to \mathbb{R}$$
$$U \mapsto \mathrm{Amb}(U) = \int_{[0,1]} \alpha(\sup U_\alpha - \inf U_\alpha) d\alpha = 2 \int_{[0,1]} \alpha\, \mathrm{spr}\, U_\alpha d\alpha,$$

the *weighted averaging based on levels*

$$\mathrm{wabl} : (\mathcal{F}_c(\mathbb{R}), d_p) \to \mathbb{R}$$
$$U \mapsto \mathrm{wabl}(U) = \int_{[0,1]} \mathrm{mid}\, U_\alpha d\alpha,$$

and the *width*

$$\mathrm{width} : (\mathcal{F}_c(\mathbb{R}), d_p) \to \mathbb{R}$$
$$U \mapsto \mathrm{width}(U) = \int_{[0,1]} (\sup U_\alpha - \inf U_\alpha) d\alpha = 2 \int_{[0,1]} \mathrm{spr}\, U_\alpha d\alpha.$$

A *random set* is a measurable mapping $X : (\Omega, \mathcal{A}, P) \to \mathcal{K}_c(\mathbb{R}^d)$ with respect to the Borel σ-algebra generated by the Hausdorff metric. If $d = 1$, then X will be called *random interval*. A *fuzzy random variable* is a mapping $X : (\Omega, \mathcal{A}, P) \to \mathcal{F}_c(\mathbb{R}^d)$ such that for every $\alpha \in [0,1]$, $X_\alpha : (\Omega, \mathcal{A}, P) \to \mathcal{K}_c(\mathbb{R}^d)$, given by $X_\alpha(\omega) = X(\omega)_\alpha$, is a random set. The following characterization in [16, Theorem 6.6.(1)] allows us to identify fuzzy random variables with measurable mappings with respect to the Borel σ-algebra generated by d_p, which will be denoted by $\mathcal{B}_{d_p}$.

Proposition 2.6. (Krätschmer) *Let $p \in [1,\infty)$. A mapping $X : \Omega \to \mathcal{F}_c(\mathbb{R}^d)$ is a fuzzy random variable if and only if it is a random element of the space $(\mathcal{F}_c(\mathbb{R}^d), d_p)$.*

A sequence $\{X_n\}_n$ of fuzzy random variables *converges in distribution in d_p* to a fuzzy random variable X if $X_n \to X$ weakly as random elements of the space $(\mathcal{F}_c(\mathbb{R}^d), d_p)$. Analogously, it *converges in distribution in d_∞* to X if $X_n \to X$ weakly as random elements of the space $(\mathcal{F}_c(\mathbb{R}^d), d_\infty)$.

Sequences of fuzzy random variables satisfy the continuous mapping theorem, as stated in [1, Theorem 3.5] and [4].

Lemma 2.7. *Let $(\mathbb{E}, d)$ be a metric space. Let X_n and X be fuzzy random variables such that $X_n \to X$ in distribution in d_p. If $f : (\mathcal{F}_c(\mathbb{R}^d), d_p) \to \mathbb{E}$ is a P_X-almost surely continuous function, then $f(X_n) \to f(X)$ in distribution in d.*

Moreover, Dudley's version of Skorokhod representation theorem for separable spaces (see [8, Theorem 3]) holds for fuzzy random variables. Notice that it can be applied to the Cartesian product of fuzzy random variables.

Lemma 2.8. *Let $p \in [1, \infty)$. Let P_n, P be probability measures on σ_L, such that $P_n \to P$ in distribution. Then there exist fuzzy random variables $X_n, X : (\Omega', \mathcal{A}', P') \to (\mathcal{F}_c(\mathbb{R}^d), d_p)$, such that*

(a) *The distributions of X_n and X are P_n and P, respectively.*
(b) *$X_n(\omega') \to X(\omega')$ in d_p for every $\omega' \in \Omega'$.*

We will also need this lemma from Proposition 2 in [2], which is a consequence of continuous mapping theorem.

Lemma 2.9. *Let X_n, X be fuzzy random variables such that $X_n \to X$ in distribution in d_p. Then $aX_n \to aX$ in distribution in d_p for every $a \in \mathbb{R}$.*

Set

$$\begin{aligned} \varphi_\times : (\mathcal{F}_c(\mathbb{R}^d) \times \mathcal{F}_c(\mathbb{R}^{d'}), d_p^{\max}) &\to (\mathcal{F}_c(\mathbb{R}^{d+d'}), d_p) \\ (U, V) &\to U \times V, \end{aligned}$$

where

$$(U \times V)(x) = \min\{U(x_1), V(x_2)\} \in [0,1]$$

for every $x = (x_1, x_2) \in \mathbb{R}^{d+d'}$, with $x_1 \in \mathbb{R}^d$ and $x_2 \in \mathbb{R}^{d'}$. We will denote by $\phi_\times$ the inverse mapping of $\varphi_\times$, which is well defined (see Lemma 2.10 below).

The following lemmas from [4] are useful when dealing with Cartesian products of fuzzy random variables.

Lemma 2.10. *The mapping $\varphi_\times$ is an homeomorphism onto its image and $\varphi_\times(\mathcal{F}_c(\mathbb{R}^d) \times \mathcal{F}_c(\mathbb{R}^{d'}))$ is a measurable subset of $\mathcal{F}_c(\mathbb{R}^{d+d'})$.*

Lemma 2.11. *Let $\{X_n\}_n$ and $\{Y_n\}_n$ be sequences of fuzzy random variables and let X and Y be fuzzy random variables. Then $(X_n, Y_n) \to (X, Y)$ in distribution in $d_p^{\max}$ if and only if $X_n \times Y_n \to X \times Y$ in distribution in d_p.*

Lemma 2.12. *Let $\{X_n\}_n$ and $\{Y_n\}_n$ be sequences of fuzzy random variables, let X be a fuzzy random variable and let $U \in \mathcal{F}_c(\mathbb{R}^d)$. If $X_n \to X$ and $Y_n \to U$ in distribution in d_p, then $(X_n, Y_n) \to (X, U)$ as random elements in the space $\mathcal{F}_c(\mathbb{R}^d) \times \mathcal{F}_c(\mathbb{R}^d)$.*

Denote by $\phi_\cup$ the union between fuzzy sets.

$$\begin{aligned} \phi_\cup : (\mathcal{F}(\mathbb{R}^d) \times \mathcal{F}(\mathbb{R}^d), d_p^{\max}) &\to (\mathcal{F}(\mathbb{R}^d), d_p) \\ (U, V) &\mapsto U \cup V = \max\{U(x), V(x)\}, \end{aligned}$$

which is

$$(U \cup V)_\alpha = U_\alpha \cup V_\alpha$$

for every $\alpha \in [0, 1]$.

In [4, Lemma 5.3], we showed that $\phi_\cup$ is continuous in d_p.

Denote by $\phi_\vee$ the operation given by

$$\begin{aligned} \phi_\vee : (\mathcal{F}_c(\mathbb{R}^d) \times \mathcal{F}_c(\mathbb{R}^d), d_p^{\max}) &\to (\mathcal{F}(\mathbb{R}^d), d_p) \\ (U, V) &\mapsto \mathrm{co}(U \cup V), \end{aligned}$$

where $\mathrm{co}(U \cup V)$ is the *convex hull* of $U \cup V$, given by

$$(\mathrm{co}(U \cup V))_\alpha = \mathrm{co}((U \cup V)_\alpha) = \mathrm{co}(U_\alpha \cup V_\alpha),$$

for every $\alpha \in [0, 1]$.

We define a *trapezoidal random fuzzy set* $Tra(\xi_1, \xi_2, \xi_3, \xi_4)$ as

$$X(\omega)(x) = \begin{cases} 0 & \text{if } x < \xi_1(\omega) \\ \frac{x - \xi_1(\omega)}{\xi_2(\omega) - \xi_1(\omega)} & \text{if } \xi_1(\omega) \le x < \xi_2(\omega) \\ 1 & \text{if } \xi_2(\omega) \le x \le \xi_3(\omega) \\ \frac{\xi_4(\omega) - x}{\xi_4(\omega) - \xi_3(\omega)} & \text{if } \xi_3(\omega) < x \le \xi_4(\omega) \\ 0 & \text{if } x > \xi_4(\omega). \end{cases}$$

As shown in [2, Theorem 2], convergence in distribution of a trapezoidal random fuzzy set in the metric d_p for $p \in [1, \infty)$ is equivalent to convergence in distribution of $(\xi_1, \xi_2, \xi_3, \xi_4)$ as a random vector in $\mathbb{R}^4$.

Lemma 2.13. *Let $\{U_n\}_n$ be a sequence of trapezoidal fuzzy numbers converging to some $U \in \widehat{\mathcal{F}}_{c,1}(\mathbb{R})$ in d_1. Then $\{\|(U_n)_0\|\}_n$ is bounded.*

Corollary 2.14. *Let $\{U_n\}_n$ be a convergent sequence of trapezoidal fuzzy numbers. Denote by U its limit. Then $d_{\text{end}}(U_n, U) \to 0$ if and only if $d_p(U_n, U) \to 0$ for every $p \in [1, \infty)$.*

Proof. It is a consequence of Lemmas 2.5 and 2.13. $\square$

3. Operations between fuzzy random variables

The main objective of this section is to show that the operations between fuzzy sets defined before preserve convergence in distribution.

The first operation considered is $\phi_\vee$.

Lemma 3.1. *The function $\phi_\vee$ is continuous in d_p.*

Proof. Let $\{(U_n, V_n)\}_n$ be a convergent sequence to (U, V) in $d_p^{\max}$. Then, by [4], $d_p((U_n \cup V_n), (U \cup V)) \to 0$ and since $d_p(\text{co}(U_n \cup V_n), \text{co}(U \cup V)) \leq d_p((U_n \cup V_n), (U \cup V))$, the sequence $\{\text{co}(U_n \cup V_n)\}_n$ converges to $\text{co}(U \cup V)$ in d_p. $\square$

Proposition 3.2. *Let $\{X_n\}_n$ and $\{Y_n\}_n$ be sequences of random variables in $\mathbb{R}^d$ such that $\{(X_n, Y_n)\}_n$ converges in distribution to (X, Y) in $d_p^{\max}$. Then $\text{co}(X_n \cup Y_n)$ converges in distribution in d_p to $\text{co}(X \cup Y)$.*

Proof. First, recall that $\{X_n \times Y_n\}_n$ converges in distribution in d_p to $X \times Y$ (Lemma 2.11) and by definition of $\phi_\vee$ and $\phi_\times$,

$$\text{co}(U \cup V) = \phi_\vee(U, V) = \phi_\vee(\phi_\times(U \times V)) = (\phi_\vee \circ \phi_\times)(U \times V).$$

Since $\phi_\vee$ and $\phi_\times$ are continuous by Lemmas 3.1 and 2.10, their composition is continuous. Therefore, by the continuous mapping theorem (Lemma 2.7), $\{(\phi_\vee \circ \phi_\times)(X_n, Y_n)\}_n$ converges in distribution in d_p to $(\phi_\vee \circ \phi_\times)(X, Y)$. $\square$

To show that $\widetilde{\max}$, $\widetilde{\min}$, $\widetilde{\text{prod}}$ and $\widetilde{\text{quot}}$ are fuzzy random variables, we will need this lemma.

Lemma 3.3. *Let $X : \Omega \to \mathcal{F}_c(\mathbb{R}^d)$. Then the following statements are equivalent:*

- *X is a fuzzy random variable*
- *X is measurable in the endograph topology.*

Proof. First, recall that by [1, Proposition 5.4], $(\mathcal{F}_c(\mathbb{R}^d), d_p)$ is a Lusin space for every $p \in [1, \infty)$, hence a Suslin space. Since $(\mathcal{F}_c(\mathbb{R}^d), d_{\text{end}})$ is the continuous image of a Suslin space (via the identity mapping $i : (\mathcal{F}_c(\mathbb{R}^d), d_p) \to (\mathcal{F}_c(\mathbb{R}^d), d_{\text{end}})$), it is also Suslin. And it is clear that the supremum of both topologies is τ_{d_p}, the topology generated by d_p. Then, by Lemma 2.1 and Proposition 2.6, $\mathcal{B}_{d_{\text{end}}} = \mathcal{B}_{d_p}$. $\square$

Lemma 3.4. *The function $\widetilde{\max}$ is continuous in d_{end}.*

Proof. Let $\{(U_n, V_n)\}_n \subseteq \mathcal{F}_c(\mathbb{R}) \times \mathcal{F}_c(\mathbb{R})$ be a sequence such that

$$d_{\text{end}}^{\max}((U_n, V_n), (U, V)) = \max\{d_{\text{end}}(U_n, U), d_{\text{end}}(V_n, V)\} \to 0.$$

Then, by Lemma 2.2, for almost every $\alpha \in [0, 1]$ it follows that $d_H((U_n)_\alpha, U_\alpha) \to 0$ and $d_H((V_n)_\alpha, V_\alpha) \to 0$. Denote by C the subset of $[0, 1]$ which contains every α fulfilling both conditions. Let $\alpha \in C$. Then

$$d_H(\widetilde{\max}(U_n, V_n)_\alpha, \widetilde{\max}(U, V)_\alpha) = d_H(\max((U_n)_\alpha, (V_n)_\alpha), \max(U_\alpha, V_\alpha))$$

$$= \max\{|\max\{\inf(U_n)_\alpha, \inf(V_n)_\alpha\} - \max\{\inf U_\alpha, \inf V_\alpha\}|,$$

$$|\max\{\sup(U_n)_\alpha, \sup(V_n)_\alpha\} - \max\{\sup U_\alpha, \sup V_\alpha\}|\} \to 0,$$

since

$$\max\{\inf(U_n)_\alpha, \inf(V_n)_\alpha\} \to \max\{\inf U_\alpha, \inf V_\alpha\}$$

and

$$\max\{\sup(U_n)_\alpha, \sup(V_n)_\alpha\} \to \max\{\sup U_\alpha, \sup V_\alpha\}$$

for every $\alpha \in C$. Therefore

$$d_H(\widetilde{\max}(U_n, V_n)_\alpha, \widetilde{\max}(U, V)_\alpha) \to 0$$

for almost every $\alpha \in [0, 1]$, yielding

$$d_{\text{end}}(\widetilde{\max}(U_n, V_n), \widetilde{\max}(U, V)) \to 0,$$

using Lemma 2.2 again. □

Lemma 3.5. *Let X, Y be fuzzy random variables. Then $\widetilde{\max}(X, Y)$ is also a fuzzy random variable.*

Proof. Since $\widetilde{\max}$ is continuous in d_{end} (Lemma 3.4), it is measurable in d_p. Then $\widetilde{\max}(X, Y)$ is a random element in $(\mathcal{F}_c(\mathbb{R}), d_p)$, which is equivalent to being a fuzzy random variable (Proposition 2.6). □

Theorem 3.6. *Let $(X_n, Y_n) \to (X, Y)$ in distribution in $d_p^{\max}$ and let $M > 0$ be such that $\max\{\|(X_n)_0\|, \|(Y_n)_0\|, \|X_0\|, \|Y_0\|\} \le M$. Then*

$$\widetilde{\max}(X_n, Y_n) \to \widetilde{\max}(X, Y)$$

in distribution in d_p.

Proof. By Dudley's version of Skorokhod representation theorem (Lemma 2.8), there exist pairs of fuzzy random variables $(X'_n, Y'_n), (X', Y') : (\Omega', \mathcal{A}', P') \to (\mathcal{F}_c(\mathbb{R}) \times \mathcal{F}_c(\mathbb{R}), d_p^{\max})$ such that $(X'_n, Y'_n)(\omega') \to (X', Y')(\omega')$ in $d_p^{\max}$ for every $\omega' \in \Omega'$, $\ell_{(X'_n, Y'_n)} = P_{(X_n, Y_n)}$ and $P'_{(X', Y')} = P_{(X, Y)}$.

First of all, we have to show that these variables given by Skorokhod's theorem are bounded. Set

$$\mathfrak{A} = \{U \in \mathcal{F}_c(\mathbb{R}) : \|U_0\| \le M\} = L_0^{-1}((-\infty, M]).$$

Then, since L_0 is a measurable function, $\mathfrak{A} \in \mathcal{B}_{d_p}$. Denote

$$\mathfrak{H} = \mathfrak{A} \times \mathfrak{A} \in \mathcal{B}_{d_p} \otimes \mathcal{B}_{d_p},$$

since the product of measurable sets is measurable. Next,

$$P'(\{\omega' \in \Omega' : \|(X'_0, Y'_0)(\omega')\| \le M\}) = P_{(X', Y')}(\mathfrak{H}) = P_{(X,Y)}(\mathfrak{H})$$
$$= P(\{\omega \in \Omega : \|(X_0, Y_0)(\omega)\| \le M\}) = 1.$$

Analogously, $(X'_n, Y'_n)(\omega')$ is bounded by the constant M for almost every ω'. Let $\mathfrak{D}$ be a measurable null set which contains

$$\{(X'_0, Y'_0) \in \mathfrak{H}\} \cup \left(\bigcup_{n \in \mathbb{N}} \{((X'_n)_0, (Y'_n)_0) \in \mathfrak{H}\} \right).$$

Then $\|(X'_0, Y'_0)(\omega')\| \le M$ and $\|((X'_n)_0, (Y'_n)_0)(\omega')\| \le M$ for each $\omega' \notin \mathfrak{D}$.

Moreover, $\widetilde{\max}(X'_n, Y'_n)$ and $\widetilde{\max}(X', Y')$ are fuzzy random variables by Lemma 3.5. Let us show that $\widetilde{\max}(X'_n, Y'_n)$ has the same distribution as $\widetilde{\max}(X_n, Y_n)$. Let $B \in \mathcal{B}_{d_p}$. Then

$$P'_{\widetilde{\max}(X'_n, Y'_n)}(B) = P'(\{\omega' \in \Omega' : \widetilde{\max}(X'_n, Y'_n)(\omega') \in B\})$$
$$= P'(\{\omega' \in \Omega' : (X'_n, Y'_n)(\omega') \in \widetilde{\max}^{-1}(B)\}) = P(\{\omega \in \Omega : (X_n, Y_n)(\omega) \in \widetilde{\max}^{-1}(B)\})$$
$$= P(\{\omega \in \Omega : \widetilde{\max}(X_n, Y_n)(\omega) \in B\}) = P_{\widetilde{\max}(X_n, Y_n)}(B),$$

where $\widetilde{\max}^{-1}(B)$ because of the measurability of $\widetilde{\max}$ (Lemma 3.5).

It remains to show that for every $\omega' \notin \mathfrak{D}$, $\widetilde{\max}(X'_n, Y'_n)(\omega') \to \widetilde{\max}(X', Y')(\omega')$ in d_p. By the continuity of $\widetilde{\max}$ in d_{end},

$$d_{\text{end}}(\widetilde{\max}(X'_n, Y'_n)(\omega'), \widetilde{\max}(X', Y')(\omega')) \to 0$$

for each $\omega \in \Omega'$. Moreover, if $\omega' \notin \mathfrak{D}$, $\|(X'_0, Y'_0)(\omega')\|, \|((X'_n)_0, (Y'_n)_0)(\omega')\| \le M$, yielding $\|\widetilde{\max}(X', Y')(\omega')_0\|, \|\widetilde{\max}(X'_n, Y'_n)(\omega')_0\| \le M$. Then by Lemma 2.5,

$$d_p(\widetilde{\max}(X'_n, Y'_n)(\omega'), \widetilde{\max}(X', Y')(\omega')) \to 0,$$

for every $\omega' \notin \mathfrak{D}$. Therefore, the sequence of pairs of fuzzy random variables $\{\widetilde{\max}(X'_n, Y'_n)\}_n$ converges almost sure in d_p to $\widetilde{\max}(X', Y')$. In conclusion, $\{\widetilde{\max}(X_n, Y_n)\}_n$ converges in distribution in d_p to $\widetilde{\max}(X, Y)$. $\square$

Corollary 3.7. *Let $(X_n, Y_n) \to (X, Y)$ in distribution in $d_p^{\max}$ and let $M > 0$ be such that $\max\{\|(X_n)_0\|, \|(Y_n)_0\|, \|X_0\|, \|Y_0\|\} \le M$. Then*

$$\widetilde{\min}(X_n, Y_n) \to \widetilde{\min}(X, Y)$$

in distribution in d_p.

Proof. Let us check that $\widetilde{\min}(U, V) = (-1)\widetilde{\max}(-U, -V)$. Since

$$\begin{aligned}
&\widetilde{\max}(-U, -V)_\alpha = \max(-U_\alpha, -V_\alpha)\\
&= [\max\{-\sup U_\alpha, -\sup V_\alpha\}, \max\{-\inf U_\alpha, -\inf V_\alpha\}]\\
&= [-\min\{\sup U_\alpha, \sup V_\alpha\}, -\min\{\inf U_\alpha, \inf V_\alpha\}],
\end{aligned}$$

then

$$(-1)\widetilde{\max}(-U, -V)_\alpha = [\min\{\inf U_\alpha, \inf V_\alpha\}, \min\{\sup U_\alpha, \sup V_\alpha\}] = \widetilde{\min}(U, V)_\alpha.$$

If $(X_n, Y_n) \to (X, Y)$ in distribution in $d_p^{\max}$, by Lemma 2.11, $X_n \times Y_n \to X \times Y$ in distribution in d_p. Then by the continuous mapping theorem $-(X_n \times Y_n) \to -(X \times Y)$ in distribution in d_p, that is, $(-X_n) \times (-Y_n) \to (-X) \times (-Y)$. Therefore, by Lemma 2.11, $(-X_n, -Y_n) \to (-X, -Y)$. Next, by Theorem 3.6, $\widetilde{\max}(-X_n, -Y_n) \to \widetilde{\max}(-X, -Y)$ in distribution in d_p. And by Lemma 2.9, $-\widetilde{\max}(-X_n, -Y_n) \to -\widetilde{\max}(-X, -Y)$, that is, $\widetilde{\min}(X_n, Y_n) \to \widetilde{\min}(X, Y)$ in distribution in d_p. $\square$

Lemma 3.8. *The function $\widetilde{\text{prod}}$ is continuous in $d_{\text{end}}^{\max}$.*

Proof. Let $\{(U_n, V_n)\}_n \subseteq \mathcal{F}_c((0, \infty)) \times \mathcal{F}_c((0, \infty))$ be a convergent sequence to (U, V) in $d_{\max}$. Let $C \subseteq [0, 1]$ be the set of α's such that $d_H((U_n)_\alpha, U_\alpha) \to 0$ and $d_H((V_n)_\alpha, V_\alpha) \to 0$. Then

$$\begin{aligned}
&d_H(\widetilde{\text{prod}}(U_n, V_n)_\alpha, \widetilde{\text{prod}}(U, V)_\alpha)\\
&= d_H([\inf(U_n)_\alpha \cdot \inf(V_n)_\alpha, \sup(U_n)_\alpha \cdot \sup(V_n)_\alpha], [\inf U_\alpha \cdot \inf V_\alpha, \sup U_\alpha \cdot \sup V_\alpha])\\
&= \max\{|\inf(U_n)_\alpha \cdot \inf(V_n)_\alpha - \inf U_\alpha \cdot \inf V_\alpha|, |\sup(U_n)_\alpha \cdot \sup(V_n)_\alpha - \sup U_\alpha \cdot \sup V_\alpha|\} \to 0,
\end{aligned}$$

since

$$|\inf(U_n)_\alpha \cdot \inf(V_n)_\alpha - \inf U_\alpha \cdot \inf V_\alpha| \to 0$$

and

$$|\sup(U_n)_\alpha \cdot \sup(V_n)_\alpha - \sup U_\alpha \cdot \sup V_\alpha| \to 0$$

for each $\alpha \in C$. Then by Lemma 2.2,

$$d_{\text{end}}(\widetilde{\text{prod}}(U_n, V_n), \widetilde{\text{prod}}(U, V)) \to 0,$$

that is, $\widetilde{\text{prod}}$ is a continuous function in $d_{\text{end}}^{\max}$. $\square$

Lemma 3.9. *Let X and Y be fuzzy random variables. Then $\widetilde{\text{prod}}(X, Y)$ is a fuzzy random variable.*

Theorem 3.10. *Let $(X_n, Y_n) \to (X, Y)$ in distribution in $d_p^{\max}$ and let $M, m > 0$ be such that $\sup(X_n)_0, \sup X_0, \sup(Y_n)_0, \sup Y_0 < M$ and $\inf(X_n)_0, \inf X_0, \inf(Y_n)_0, \inf Y_0 > m$ for every $n \in \mathbb{N}$. Then*

$$\widetilde{\text{prod}}(X_n, Y_n) \to \widetilde{\text{prod}}(X, Y)$$

in distribution in d_p.

Proof. Like in the proof of Theorem 3.6, denote by (X'_n, Y'_n) and (X', Y') the pairs of fuzzy random variables given by Lemma 2.8. Notice that the set

$$\mathfrak{B} = \{U \in \mathcal{F}_c(\mathbb{R}) : \inf U_0 > m, \sup U_0 < M\}$$

is measurable, since sup and inf are measurable functions in d_p. Denote

$$\mathfrak{J} = \mathfrak{B} \times \mathfrak{B} \in \mathcal{B}_{d_p} \otimes \mathcal{B}_{d_p}.$$

Then

$$P'(\{\omega' \in \Omega' : \inf X'_0(\omega') > m, \sup X'_0(\omega') < M, \inf Y'_0(\omega') > m, \sup Y'_0(\omega') < M\})$$
$$= P(\{\omega \in \Omega : \inf X_0(\omega) > m, \sup X_0(\omega) < M, \inf Y_0(\omega) > m, \sup Y_0(\omega) < M\}) = 1.$$

Analogously, for every $n \in \mathbb{N}$,

$$P'(\{\omega' \in \Omega' : \inf(X'_n)_0(\omega') > m, \sup(X'_n)_0(\omega') < M, \inf(Y'_n)_0(\omega') > m, \sup(Y'_n)_0(\omega') < M\}) = 1.$$

Denote by $\mathfrak{G}$ a null measurable set of Ω' which contains

$$\{(X'_0, Y'_0) \in \mathfrak{J}\} \cup \left(\bigcup_{n\in\mathbb{N}} \{((X'_n)_0, (Y'_n)_0) \in \mathfrak{J}\} \right).$$

As in Theorem 3.6, for every $\omega' \notin \mathfrak{G}$ and $n \in \mathbb{N}$, it holds that $\sup(X'_n)_0(\omega'), \sup X'_0(\omega'), \sup(Y'_n)_0(\omega'), \sup Y'_0(\omega') < M$ and $\inf(X'_n)_0(\omega'), \inf X'_0(\omega'), \inf(Y'_n)_0(\omega'), \inf Y'_0(\omega') > m$. Next, we have to show that $\widetilde{\mathrm{prod}}(X'_n, Y'_n)$ and $\widetilde{\mathrm{prod}}(X_n, Y_n)$ have the same distribution. For $B \in \mathcal{B}_{d_p}$,

$$P'_{\widetilde{\mathrm{prod}}(X'_n,Y'_n)}(B) = P'(\{\omega' \in \Omega' : \widetilde{\mathrm{prod}}(X'_n, Y'_n)(\omega') \in B\})$$
$$= P'(\{\omega' \in \Omega' : (X'_n, Y'_n)(\omega') \in \widetilde{\mathrm{prod}}^{-1}(B)\}) = P(\{\omega \in \Omega : (X_n, Y_n)(\omega) \in \widetilde{\mathrm{prod}}^{-1}(B)\})$$
$$= P(\{\omega \in \Omega : \widetilde{\mathrm{prod}}(X_n, Y_n)(\omega) \in B\}) = P_{\widetilde{\mathrm{prod}}(X_n,Y_n)}(B),$$

by the measurability of $\widetilde{\mathrm{prod}}$.

Let $\omega' \notin \mathfrak{G}$. By the continuity of $\widetilde{\mathrm{prod}}$ with respect to $d^{\max}_{\mathrm{end}}$, it holds that

$$d_{\mathrm{end}}(\widetilde{\mathrm{prod}}(X'_n, Y'_n)(\omega'), \widetilde{\mathrm{prod}}(X', Y')(\omega')) \to 0$$

for every $\omega' \in \Omega'$. Furthermore, for every $\omega' \notin \mathfrak{G}$, $\|(X'_0, Y'_0)(\omega')_0\| \le M$, yielding $|\sup X_0(\omega') \cdot \sup Y_0(\omega')| \le M^2$, that is, $\|\widetilde{\mathrm{prod}}(X', Y')(\omega')_0\| \le M^2$. Analogously, $\|\widetilde{\mathrm{prod}}(X'_n, Y'_n)(\omega')_0\| \le M^2$ for every $n \in \mathbb{N}$. Consequently,

$$d_p(\widetilde{\mathrm{prod}}(X'_n, Y'_n)(\omega'), \widetilde{\mathrm{prod}}(X', Y')(\omega')) \to 0$$

for every $\omega' \notin \mathfrak{G}$, that is, $\widetilde{\mathrm{prod}}(X'_n, Y'_n)$ converges almost surely in d_p to $\widetilde{\mathrm{prod}}(X', Y')$. In conclusion, $\widetilde{\mathrm{prod}}(X_n, Y_n)$ converges in distribution in d_p to $\widetilde{\mathrm{prod}}(X, Y)$. □

Lemma 3.11. *The function* $\widetilde{\mathrm{quot}}$ *is continuous in* $d^{\max}_{\mathrm{end}}$.

Proof. Let $\{(U_n, V_n)\}_n \subseteq \mathcal{F}_c((0, \infty)) \times \mathcal{F}_c((0, \infty))$ be a convergent sequence, denote by (U, V) its limit. Set

$$C = \{\alpha \in [0, 1] : \max\{d_H((U_n)_\alpha, U_\alpha), d_H((V_n)_\alpha, V_\alpha)\} \to 0\}.$$

Then

$$d_H(\widetilde{\mathrm{quot}}(U_n, V_n)_\alpha, \widetilde{\mathrm{quot}}(U, V)_\alpha)$$
$$= d_H([\inf(U_n)_\alpha / \sup(V_n)_\alpha, \sup(U_n)_\alpha / \inf(V_n)_\alpha], [\inf U_\alpha / \sup V_\alpha, \sup U_\alpha / \inf V_\alpha])$$
$$= \max\{|\inf(U_n)_\alpha / \sup(V_n)_\alpha - \inf U_\alpha / \sup V_\alpha|, |\sup(U_n)_\alpha / \inf(V_n)_\alpha - \sup U_\alpha / \inf V_\alpha|\} \to 0,$$

since

$$|\inf(U_n)_\alpha / \sup(V_n)_\alpha - \inf U_\alpha / \sup V_\alpha| \to 0$$

and

$$|\sup(U_n)_\alpha / \inf(V_n)_\alpha - \sup U_\alpha / \inf V_\alpha| \to 0$$

for every $\alpha \in C$. Then

$$d_{\mathrm{end}}(\widetilde{\mathrm{quot}}(U_n, V_n), \widetilde{\mathrm{quot}}(U, V)) \to 0.$$

In conclusion, $\widetilde{\mathrm{quot}}$ is a continuous function in $d^{\max}_{\mathrm{end}}$. □

The proof of the following result is analogous to that of Theorem 3.10.

Proposition 3.12. *In the conditions of Theorem 3.10,* $\widetilde{\mathrm{quot}}(X_n, Y_n) \to \widetilde{\mathrm{quot}}(X, Y)$ *in distribution in* d_p.

We finally obtain a Slutski type theorem for fuzzy random variables under the d_p metrics.

Corollary 3.13. *Let* $\{X_n\}_n$ *and* $\{Y_n\}_n$ *be sequences of fuzzy random variables which converge in distribution in* d_p *to a fuzzy random variable* X *and a fuzzy set* U*, respectively.*

(a) $\mathrm{co}(X_n \cup Y_n)$ *converges in distribution in* d_p *to* $\mathrm{co}(X_n \cup U)$.
(b) *If* $\|(X_n)_0\|$, $\|(Y_n)_0\|$ *and* $\|X_0\|$ *are bounded by a constant* M *for each* $n \in \mathbb{N}$*, then* $\widetilde{\max}(X_n, Y_n)$ *converges in distribution in* d_p *to* $\widetilde{\max}(X, U)$ *and* $\widetilde{\min}(X_n, Y_n)$ *converges in distribution in* d_p *to* $\widetilde{\min}(X, U)$.
(c) *If there exist* $m > 0$ *and* $M > 0$ *such that* $\sup(X_n)_0, \sup X_0, \sup(Y_n)_0 < M$ *and* $\inf(X_n)_0, \inf X_0, \inf(Y_n)_0 > m$ *for every* $n \in \mathbb{N}$*, then* $\widetilde{\mathrm{prod}}(X_n, Y_n)$ *converges in distribution in* d_p *to* $\widetilde{\mathrm{prod}}(X, U)$ *and* $\widetilde{\mathrm{quot}}(X_n, Y_n)$ *converges in distribution in* d_p *to* $\widetilde{\mathrm{quot}}(X, U)$.

Proof. For part (a), we have to apply Lemma 2.12 and Proposition 3.2 to obtain $\mathrm{co}(X_n \cup Y_n) \to \mathrm{co}(X_n \cup U)$ in distribution in d_p.

Part (b) is a combination of Lemma 2.12 and Propositions 3.6 and 3.7 and part (c) is analogous. □

4. Functionals of fuzzy random variables

We start this section showing that the functionals of fuzzy sets considered in page 5 are Lipschitz (and hence continuous) with respect to the metric ρ_1.

Lemma 4.1. *The mappings* Val, Amb, wabl *and* width *are Lipschitz when* $\mathcal{F}_c(\mathbb{R}^d)$ *is endowed with the metric* ρ_1.

Proof. Let $U, V \in \mathcal{F}_c(\mathbb{R})$. Then

$$
\begin{aligned}
&|\,\mathrm{wabl}(U) - \mathrm{wabl}(V)| = |\int_{[0,1]} \mathrm{mid}\, U_\alpha d\alpha - \int_{[0,1]} \mathrm{mid}\, V_\alpha d\alpha| \\
&= |\int_{[0,1]} \frac{\sup U_\alpha + \inf U_\alpha}{2} d\alpha - \int_{[0,1]} \frac{\sup V_\alpha + \inf V_\alpha}{2} d\alpha| \\
&= |\int_{[0,1]} \frac{\sup U_\alpha - \sup V_\alpha}{2} d\alpha + \int_{[0,1]} \frac{\inf U_\alpha - \inf V_\alpha}{2} d\alpha| \\
&\leq |\int_{[0,1]} \frac{\sup U_\alpha - \sup V_\alpha}{2} d\alpha| + |\int_{[0,1]} \frac{\inf U_\alpha - \inf V_\alpha}{2} d\alpha| \\
&\leq \int_{[0,1]} \frac{|\sup U_\alpha - \sup V_\alpha|}{2} d\alpha + \int_{[0,1]} \frac{|\inf U_\alpha - \inf V_\alpha|}{2} d\alpha = \rho_1(U, V).
\end{aligned}
$$

Next,

$$
\begin{aligned}
&|\,\mathrm{width}(U) - \mathrm{width}(V)| = |2\int_{[0,1]} \mathrm{spr}\, U_\alpha d\alpha - 2\int_{[0,1]} \mathrm{spr}\, V_\alpha d\alpha| \\
&= |\int_{[0,1]} (\sup U_\alpha - \inf U_\alpha) d\alpha - \int_{[0,1]} (\sup V_\alpha - \inf V_\alpha) d\alpha| \\
&= |\int_{[0,1]} (\sup U_\alpha - \sup V_\alpha) d\alpha + \int_{[0,1]} (\inf V_\alpha - \inf U_\alpha) d\alpha| \\
&\leq |\int_{[0,1]} (\sup U_\alpha - \sup V_\alpha) d\alpha| + |\int_{[0,1]} (\inf V_\alpha - \inf U_\alpha) d\alpha| \\
&\leq \int_{[0,1]} |\sup U_\alpha - \sup V_\alpha| d\alpha + \int_{[0,1]} |\inf V_\alpha - \inf U_\alpha| d\alpha = 2\rho_1(U, V).
\end{aligned}
$$

Next,

$$
\begin{aligned}
|\operatorname{Val}(U)-\operatorname{Val}(V)| &= |2\int_{[0,1]} \alpha \operatorname{mid} U_\alpha d\alpha - 2\int_{[0,1]} \alpha \operatorname{mid} V_\alpha d\alpha| \\
&= |\int_{[0,1]} \alpha(\sup U_\alpha + \inf U_\alpha) d\alpha - \int_{[0,1]} \alpha(\sup V_\alpha + \inf V_\alpha) d\alpha| \\
&= |\int_{[0,1]} \alpha(\sup U_\alpha - \sup V_\alpha) d\alpha + \int_{[0,1]} \alpha(\inf U_\alpha - \inf V_\alpha) d\alpha| \\
&\leq |\int_{[0,1]} \alpha(\sup U_\alpha - \sup V_\alpha) d\alpha| + |\int_{[0,1]} \alpha(\inf U_\alpha - \inf V_\alpha) d\alpha| \\
&\leq \int_{[0,1]} \alpha|\sup U_\alpha - \sup V_\alpha| d\alpha + \int_{[0,1]} \alpha|\inf U_\alpha - \inf V_\alpha| d\alpha \\
&\leq \int_{[0,1]} |\sup U_\alpha - \sup V_\alpha| d\alpha + \int_{[0,1]} |\inf U_\alpha - \inf V_\alpha| d\alpha = 2\rho_1(U,V).
\end{aligned}
$$

Finally,

$$
\begin{aligned}
|\operatorname{Amb}(U)-\operatorname{Amb}(V)| &= |2\int_{[0,1]} \alpha \operatorname{spr} U_\alpha d\alpha - 2\int_{[0,1]} \alpha \operatorname{spr} V_\alpha d\alpha| \\
&= |\int_{[0,1]} \alpha(\sup U_\alpha - \inf U_\alpha) d\alpha - \int_{[0,1]} \alpha(\sup V_\alpha - \inf V_\alpha) d\alpha| \\
&= |\int_{[0,1]} \alpha(\sup U_\alpha - \sup V_\alpha) d\alpha + \int_{[0,1]} \alpha(\inf V_\alpha - \inf U_\alpha) d\alpha| \\
&\leq |\int_{[0,1]} \alpha(\sup U_\alpha - \sup V_\alpha) d\alpha| + |\int_{[0,1]} \alpha(\inf V_\alpha - \inf U_\alpha) d\alpha| \\
&\leq \int_{[0,1]} \alpha|\sup U_\alpha - \sup V_\alpha| d\alpha + \int_{[0,1]} \alpha|\inf V_\alpha - \inf U_\alpha| d\alpha \\
&\leq \int_{[0,1]} |\sup U_\alpha - \sup V_\alpha| d\alpha + \int_{[0,1]} |\inf U_\alpha - \inf V_\alpha| d\alpha = 2\rho_1(U,V). \quad \square
\end{aligned}
$$

Lemma 4.2. *Let X be a fuzzy random variable. Then* $\operatorname{Val}(X), \operatorname{Amb}(X), \operatorname{wabl}(X)$ *and* $\operatorname{width}(X)$ *are random variables.*

Proof. Since these mappings are Lipschitz by Lemma 4.1, they are measurable in ρ_1 and d_1, hence the composition with a fuzzy random variable is measurable too. □

Proposition 4.3. *Let $\{X_n\}_n$ be a sequence of fuzzy random variables which converges in distribution in d_1 to X. Then*

- $\operatorname{Val}(X_n) \to \operatorname{Val}(X)$,
- $\operatorname{Amb}(X_n) \to \operatorname{Amb}(X)$,
- $\operatorname{wabl}(X_n) \to \operatorname{wabl}(X)$,
- $\operatorname{width}(X_n) \to \operatorname{width}(X)$

in distribution.

Proof. By Lemma 4.1, these mappings are continuous and the continuous mapping theorem (Lemma 2.7) can be applied. □

5. Convergence in d_∞

Our proof of the equivalence between convergence in distribution of a trapezoidal fuzzy random variable and convergence in distribution of the random vectors that characterize it in [2, Theorem 2] relied on the boundedness of the 0-cuts of a convergent sequence of trapezoidal fuzzy numbers (see [4, Lemma 3.6] for the detailed proof of this fact). Now, we will strengthen this result via the continuity of the mapping which identifies a 4-dimensional vector with a trapezoidal fuzzy set with respect to the metric d_∞.

Lemma 5.1. *Let* $A = \{(u_1, u_2, u_3, u_4) \in \mathbb{R}^4 : u_1 \le u_2 \le u_3 \le u_4\}$*. Then the mapping*

$$\varphi : (A, d_{\max}) \to (\mathcal{F}_c(\mathbb{R}), d_\infty)$$
$$(u_1, u_2, u_3, u_4) \mapsto Tra(u_1, u_2, u_3, u_4),$$

where

$$d_{\max}((u_1, u_2, u_3, u_4), (v_1, v_2, v_3, v_4)) = \max\{|u_1 - v_1|, |u_2 - v_2|, |u_3 - v_3|, |u_4 - v_4|\},$$

is injective and Lipschitz.

Proof. Clearly, φ is injective. Let $(u_1, u_2, u_3, u_4), (v_1, v_2, v_3, v_4) \in \mathbb{R}^4$. Denote by U the trapezoidal fuzzy number $Tra(u_1, u_2, u_3, u_4)$ and by V the trapezoidal fuzzy number $Tra(v_1, v_2, v_3, v_4)$. Let $\alpha \in [0, 1]$,

$$\begin{aligned}
d_H(U_\alpha, V_\alpha) &= \max\{|\inf U_\alpha - \inf V_\alpha|, |\sup U_\alpha - \sup V_\alpha|\} \\
&= \max\{|(1-\alpha)\inf U_0 + \alpha \inf U_1 - (1-\alpha)\inf V_0 - \alpha \inf V_1|, \\
&\quad |\alpha \sup U_1 + (1-\alpha)\sup U_0 - \alpha \sup V_1 - (1-\alpha)\sup V_0|\} \\
&= \max\{|(1-\alpha)(u_1 - v_1) + \alpha(u_2 - v_2)|, |\alpha(u_3 - v_3) + (1-\alpha)(u_4 - v_4)|\} \\
&\le \max\{|u_1 - v_1|, |u_2 - v_2|, |u_3 - v_3|, |u_4 - v_4|\} = d_{\max}((u_1, u_2, u_3, u_4), (v_1, v_2, v_3, v_4)).
\end{aligned}$$

Therefore

$$d_\infty(U, V) = \sup_{\alpha \in [0,1]} d_H(U_\alpha, V_\alpha) \le d_{\max}((u_1, u_2, u_3, u_4), (v_1, v_2, v_3, v_4)). \quad \square$$

Since $d_p \le d_\infty$ for $p \in [1, \infty)$, we get the following result.

Corollary 5.2. *The mapping* φ *is Lipschitz when* $\mathcal{F}_c(\mathbb{R})$ *is endowed with the metric* d_p*.*

Proposition 5.3. *Let* $p \in [1, \infty)$ *and let* $X_n \sim Tra(\xi_{n_1}, \xi_{n_2}, \xi_{n_3}, \xi_{n_4})$ *and* $X \sim Tra(\xi_1, \xi_2, \xi_3, \xi_4)$ *be random trapezoidal fuzzy sets. Then*

1. *If* $X_n \to X$ *in distribution in* d_p*, then* $(\xi_{n_1}, \xi_{n_2}, \xi_{n_3}, \xi_{n_4}) \to (\xi_1, \xi_2, \xi_3, \xi_4)$ *in distribution.*
2. *If* $(\xi_{n_1}, \xi_{n_2}, \xi_{n_3}, \xi_{n_4}) \to (\xi_1, \xi_2, \xi_3, \xi_4)$ *in distribution, then* $X_n \to X$ *in distribution in* d_∞*.*

Proof. 1. It is the first implication in [2, Theorem 2].

2. By Skorokhod representation theorem [17], there exist random vectors $(\eta_{n,1}, \eta_{n,2}, \eta_{n,3}, \eta_{n,4}), (\eta_1, \eta_2, \eta_3, \eta_4)$ such that $\ell_{(\eta_{n,1},\eta_{n,2},\eta_{n,3},\eta_{n,4})} = P_{(\xi_{n,1},\xi_{n,2},\xi_{n,3},\xi_{n,4})}$, $\ell_{(\eta_1,\eta_2,\eta_3,\eta_4)} = P_{(\xi_1,\xi_2,\xi_3,\xi_4)}$ and $(\eta_{n,1}, \eta_{n,2}, \eta_{n,3}, \eta_{n,4})$ converges pointwise to $(\eta_1, \eta_2, \eta_3, \eta_4)$. Let $Y_n = Tra(\eta_{n_1}, \eta_{n_2}, \eta_{n_3}, \eta_{n_4})$ and $Y = Tra(\eta_1, \eta_2, \eta_3, \eta_4)$. We have to show that $Y_n(t) \to Y(t)$ in d_∞ for each $t \in [0, 1]$. Then, using Lemma 5.1,

$$d_\infty((Y_n)(t), Y(t)) \le d_{\max}((\eta_{n,1}(t), \eta_{n,2}(t), \eta_{n,3}(t), \eta_{n,4}(t)), (\eta_1(t), \eta_2(t), \eta_3(t), \eta_4(t)))$$

for every $t \in [0, 1]$. Finally, let $B \in \mathcal{B}_{(\mathcal{F}_c(\mathbb{R}), d_\infty)}$.

$$\begin{aligned}
\ell_{Y_n}(B) &= \ell(\{t \in [0,1] : Y_n(t) \in B\}) = \ell(\{t \in [0,1] : (\eta_{n,1}, \ldots, \eta_{n,4})(t) \in \varphi^{-1}(B)\}) \\
&= P(\{\omega \in \Omega : (\xi_{n,1}, \ldots, \xi_{n,4})(\omega) \in \varphi^{-1}(B)\}) = P(\{\omega \in \Omega : X_n(\omega) \in B\}) = P_{X_n}(B),
\end{aligned}$$

that is, Y_n and X_n have the same distribution. Analogously, $\ell_Y(B) = P_X(B)$ for every Borel set $B \subseteq \mathcal{F}_c(\mathbb{R})$. In conclusion, $X_n \to X$ in distribution in d_∞. $\square$

As a result, for any sequence of random trapezoidal fuzzy sets, we obtain the equivalence between convergence in distribution in the metrics d_p and d_∞. In particular, any convergent sequence of trapezoidal fuzzy numbers with respect to the metric d_1 is convergent in d_∞, which implies that the sequence of 0-cuts is bounded.

Proposition 5.4. *Let* X_n, X *be fuzzy random variables such that* $X_n \to X$ *weakly in* d_∞*. Then* $s_{X_n}(r, \alpha) \to s_X(r, \alpha)$ *weakly for every* $r \in \mathbb{R}^d$ *and for every* $\alpha \in [0, 1]$*.*

Proof. By Skorokhod's representation theorem [18, Proposition 10], there exist fuzzy random variables Y_n, Y such that $Y_n(t) \to Y(t)$ for each $t \in [0, 1]$. Then $(Y_n)_\alpha \to Y_\alpha$ in d_H for each $\alpha \in [0, 1]$.

Then, since

$$d_H((Y_n)_\alpha, Y_\alpha) = \sup_{r \in \mathbb{S}^{d-1}} \|s_{Y_n}(r, \alpha) - s_Y(r, \alpha)\|,$$

it follows that for every $\varepsilon > 0$ there exists $n_0 \in N$ such that

$$\sup_{r \in \mathbb{S}^{d-1}} \|s_{Y_n}(r,\alpha) - s_Y(r,\alpha)\| \le \varepsilon.$$

Therefore, for each $r \in \mathbb{S}^{d-1}$, $\alpha \in [0,1]$ and $t \in [0,1]$, $\|s_{Y_n}(r,\alpha) - s_Y(r,\alpha)\| \le \varepsilon$.

Thus, for fixed $\alpha \in [0,1]$ and $r \in \mathbb{S}^{d-1}$, $s_{Y_n(t)}(r,\alpha) \to s_{Y(t)}(r,\alpha)$ for each $t \in [0,1]$.

Now, let us show that $P_{s_{X(\cdot)}(r,\alpha)} = \ell_{s_{Y(\cdot)}(r,\alpha)}$. Let $a \in \mathbb{R}$ and $s = \max_{x \in U_\alpha} \langle r, x\rangle$ be a function.

$$P_{s_{X(\cdot)}(r,\alpha)}((-\infty,a]) = P(\{\omega \in \Omega : s_{X(\cdot)}(r,\alpha) \le a\})$$
$$= P(\{\omega \in \Omega : \max_{x \in X_\alpha(\omega)} \langle r, x\rangle \le a\}) = P(\{\omega \in \Omega : X_\alpha(\omega) \in s^{-1}((-\infty,a])\})$$
$$P(\{t \in [0,1] : Y_\alpha(t) \in s^{-1}((-\infty,a])\}) = P_{s_{Y(\cdot)}(r,\alpha)}((-\infty,a]).$$

Analogously, $P_{s_{X_n(\cdot)}(r,\alpha)} = \ell_{s_{Y_n(\cdot)}(r,\alpha)}$. In conclusion, $s_{X_n}(r,\alpha) \to s_X(r,\alpha)$ weakly. □

Remark 5.1. Notice that if we try to generalize the previous result to convergence in d_p, we will obtain that for every $t \in [0,1]$ there is a subset of α-cuts converging, but it may be different for each t.

Corollary 5.5. *Let X_n, X be fuzzy random variables such that $X_n \to X$ in probability in d_∞. Then $s_{X_n}(r,\alpha) \to s_X(r,\alpha)$ in probability for every $r \in \mathbb{R}^d$ and $\alpha \in [0,1]$.*

Proof. Let $\{X_{n'}\}_n$ be any subsequence of $\{X_n\}_n$ and let $\{X_{n''}\}_n$ be a subsequence of $\{X_{n'}\}_n$ which converges almost sure to X. Reasoning as in the proof of Proposition 5.4, for fixed $\alpha \in [0,1]$ and $r \in \mathbb{S}^{d-1}$, $s_{X_{n''}(t)}(r,\alpha) \to s_{X(\omega)}(r,\alpha)$ for each $\omega \in \Omega$. Therefore every subsequence $\{s_{X_{n'}(t)}(r,\alpha)\}_n$ of $\{s_{X_n(t)}(r,\alpha)\}_n$ has a further subsequence $\{s_{X_{n''}(t)}(r,\alpha)\}_n$ which converges almost surely to $s_{X(t)}(r,\alpha)$. In conclusion, $s_{X_n}(r,\alpha) \to s_X(r,\alpha)$ in probability for every $r \in \mathbb{R}^d$ and for every $\alpha \in [0,1]$. □

Proposition 5.6. *Let X_n, X be d_∞-Borel measurable fuzzy random variables such that $X_n \to X$ in d_∞. Then for every $\alpha \in [0,1]$, $(X_n)_\alpha \to X_\alpha$ in distribution in d_H.*

Proof. Let us show that $E[f((X_n)_\alpha)] \to E[f(X_\alpha)]$ for every continuous and bounded mapping $f : \mathcal{K}_c(\mathbb{R}^d) \to \mathbb{R}$. First, recall that for every $\alpha \in [0,1]$, the mapping L_α is continuous. Next, let $f : \mathcal{K}_c(\mathbb{R}^d) \to \mathbb{R}$ be any continuous and bounded mapping. Then $f(X_\alpha) = f(L_\alpha(X)) = (f \circ L_\alpha)(X)$, where $f \circ L_\alpha : \mathcal{F}_c(\mathbb{R}^d) \to \mathbb{R}$ is a continuous and bounded function. Then, by hypothesis, $E[f \circ L_\alpha(X_n)] \to E[f \circ L_\alpha(X)]$, that is, $E[f((X_n)_\alpha)] \to E[f(X_\alpha)]$, yielding $(X_n)_\alpha \to X_\alpha$ in distribution in d_H. □

Corollary 5.7. *Let X_n, X be d_∞-Borel measurable fuzzy random variables in $\mathbb{R}$ such that $X_n \to X$ in d_∞. Then for every $\alpha \in [0,1]$*

- $\inf(X_n)_\alpha \to \inf X_\alpha$,
- $\sup(X_n)_\alpha \to \sup X_\alpha$,
- $\operatorname{mid}(X_n)_\alpha \to \operatorname{mid} X_\alpha$,
- $\operatorname{spr}(X_n)_\alpha \to \operatorname{spr} X_\alpha$

in distribution.

Proof. Fix any $\alpha \in [0,1]$. Then, by Proposition 5.6, $(X_n)_\alpha \to X_\alpha$ in distribution in d_H. Since inf, sup, mid and spr are continuous in d_H, convergence in distribution is preserved. □

6. Concluding remarks

Throughout this paper, we have studied the compatibility of the definition of convergence in distribution of fuzzy random variables with respect to the metric d_p with some common operations between fuzzy sets. Moreover, we have shown how the relationship between convergence in L^p metrics and the endograph metric can be used to obtain new results related to convergence of fuzzy random variables.

As a future line of research, it would be interesting to study if convergence in distribution of fuzzy random variables with respect to L^∞ metrics behaves well with the structure of spaces of fuzzy sets. For the compatibility of convergence in distribution with functionals of fuzzy sets, we may consider showing whether those in [5], [6], [11] or [20] preserve convergence in distribution.

Declaration of competing interest

The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

Data availability

No data was used for the research described in the article.

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