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Evaluating uncertainty with Vertical Barrier Models

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A B S T R A C T

Vertical Barrier Models (VBM) are a family of imprecise probability models that generalise a number of well known distortion/neighbourhood models (such as the Pari-Mutuel Model, the Linear-Vacuous Model, and others) while still being relatively simple. Several of their properties were established in previous works; in this paper we explore, in a finite framework, further facets of these models: their interpretation as neighbourhood models, the structure of their credal set in terms of maximum number of its extreme points, the result of merging operations with VBMs, the properties of their mass function, the conditions for VBMs to be belief functions or maxitive measures and the approximation of other models by VBMs.

1. Introduction

In a context of imprecise information, a natural procedure for determining a robust model is to consider a neighbourhood around some precise probability measure P_0. This neighbourhood can be obtained by choosing a distance between probability measures and the degree of robustness we want to take into account; alternatively, we may directly transform P_0 by means of a suitable distorting function. Whichever the procedure, the imprecise probability model that is obtained is usually referred to as a *distortion model*. This type of model has the advantage of being generally computationally more tractable and easy to explain to non-experts.

Our focus in this paper is in one of such distortion models: the Vertical Barrier Model (VBM). It was introduced in [7] within the larger family of *Nearly-linear* (NL) models. NL models perform a linear affine transformation to P_0; given $a \in \mathbb{R}, b > 0$, they determine a lower probability given by $\underline{P}(A) = bP_0(A) + a$ whenever this value belongs to $[0,1]$, $\underline{P}(A) = 0$ if $bP_0(A) + a < 0$ and $\underline{P}(A) = 1$ if $bP_0(A) + a > 1$. Vertical Barrier Models add some constraints to parameters a, b. They are the most prominent subfamily of NL models, because *(a)* they are always (coherent and) 2-monotone, and *(b)* they include the Pari-Mutuel Model, the Linear-Vacuous Model and other remarkable distortion models as special cases [7]. As such, they provide a nice balance between generality and tractability.

In the past, a number of features of VBMs were investigated [7,25–27], in the frame of NL models: in particular, formulae for their natural extensions are given in [27], their being stable with conditioning (i.e. conditioning a VBM on an event B returns a VBM) is assessed in [25] and their dilation properties are established in [25,26].

Our objective in this paper is to complement this picture by investigating further features of VBMs. In doing this, we also relate to previous work discussing these aspects for partly overlapping models in [18–20], and determine the extent to which the properties of these particular cases are generalised.

After recalling essential preliminary notions in Section 2, we investigate in Section 3 how VBMs may be viewed as neighbourhood models. For this, a distorting function d_{VBM} is introduced in Section 3.1, demonstrating some of its properties and proving that the

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lower probability of a VBM can be obtained as the lower envelope of a neighbourhood of P_0, defined by means of d_{VBM}. Section 3.2 shows that some relevant neighbourhood models, such as the Constant Odds Ratio model, are not included into VBMs. The structure of the credal set of a VBM is investigated in Section 4. The main result of this section achieves a strict bound on the maximum number of extreme points of the credal set. Section 5 explores the behaviour of VBMs under the merging procedures of disjunction, conjunction and convex mixture. Section 6 investigates the properties of the Möbius inverse of a VBM. Section 7 discusses special VBMs. In Section 7.1, necessary and/or sufficient conditions are sought for a VBM to be (or more precisely not to be) a belief function, while Section 7.2 characterises those VBMs that are maxitive (or minitive). Section 8 deals with VBMs on product spaces. Finally, Section 9 investigates the role of VBMs in inner and outer approximations of imprecise probabilities. Our concluding comments are given in Section 10.

A preliminary version of this paper was presented at ISIPTA 2023 [17]. The present paper includes the discussion of the Möbius inverse of the lower probability of a VBM, the analysis of the multivariate case, the study of inner and outer approximations with VBMs, as well as additional results and examples in the other sections.

2. The Vertical Barrier Model: basic properties

Consider an arbitrary possibility space Ω. Let $\mathbb{P}(\Omega)$ denote the set of all finitely additive probability measures defined on the power set $\mathcal{P}(\Omega)$. We shall use $\subseteq$ to denote inclusion and $\subset$ to denote strict inclusion between events.

Vertical Barrier Models are defined as follows [7]:

Definition 1. Consider $P_0 \in \mathbb{P}(\Omega)$ and two parameters a, b with $a \leq 0, b > 0$ and $a + b \in [0, 1]$. The corresponding *Vertical Barrier Model* (VBM) is identified by (P_0, a, b); its lower probability $\underline{P}$ is given by

$$\underline{P}(A) = \max\{bP_0(A) + a, 0\}\ \forall A \subset \Omega \text{ and } \underline{P}(\Omega) = 1. \tag{1}$$

The above definition implies that $\underline{P}(\emptyset) = 0$. Note that $P_0(\{\omega\})$ is not required to be strictly positive for every ω; in this paper, the positivity assumption shall only occasionally be imposed.

As mentioned in the Introduction, VBMs are part of the family of Nearly-linear models; other members of this family are obtained by varying the constraints on the parameters a, b. VBMs are superior to alternative NL uncertainty measures in that they always satisfy 2-*monotonicity*:

$$\underline{P}(A \cup B) + \underline{P}(A \cap B) \geq \underline{P}(A) + \underline{P}(B)\ \forall A, B \subseteq \Omega;$$

as a consequence [32, Corollary 6.3], since also $\underline{P}(\emptyset) = 0, \underline{P}(\Omega) = 1$, VBMs are coherent. We will use $\mathcal{M}(\underline{P})$ to denote the *credal set* associated with $\underline{P}$, given by

$$\mathcal{M}(\underline{P}) := \{P \in \mathbb{P}(\Omega) \mid P(A) \geq \underline{P}(A)\ \forall A \subseteq \Omega\},$$

and $\overline{P}$ for the conjugate *upper probability*, given by $\overline{P}(A) = 1 - \underline{P}(A^c)$ for every $A \subseteq \Omega$. One way to define coherence of $\underline{P}$, $\overline{P}$ is to require that $\underline{P}(A) = \min_{P\in\mathcal{M}(\underline{P})} P(A)$ and $\overline{P}(A) = \max_{P\in\mathcal{M}(\underline{P})} P(A)$ for every $A \subseteq \Omega$ [34].

Using conjugacy, the upper probability of a VBM can be computed as

$$\overline{P}(A) = \min\{bP_0(A) + c, 1\}\ \forall A \neq \emptyset \text{ and } \overline{P}(\emptyset) = 0, \tag{2}$$

where

$$c = 1 - (a + b). \tag{3}$$

Observe that $bP_0(A) + c \geq 0$ for any event A.

As shown in [27], Vertical Barrier Models include as particular cases some of the most important distortion models in the literature:

- When $a < 0$ and $a + b = 1$, they correspond to the *Pari-Mutuel Model* (PMM) [34], whose lower probability is often written as $\underline{P}_{PMM}(A) = \max\{(1 + \delta)P_0(A) - \delta, 0\}$, with $\delta > 0$, so that $a = -\delta$, $b = 1 + \delta$.
- when $a = 0$ and $b < 1$, they boil down to *Linear-Vacuous mixtures* (LV) [13];
- finally, when $b = 1$ and $a \in (-1, 0)$ they correspond to the *Total Variation model* (TV) [12].

A VBM can formalise a larger variety of real-world situations than any of its just recalled special cases. One way to see this is to think of the behavioural interpretation of an upper probability $\overline{P}(A)$ as an infimum selling price for A. Recall that a probability $P_0(A)$ is instead a fair price for either buying or selling A [9]. Thus, from a seller's perspective, $\overline{P}(A)$ should ensure a *loading* $\overline{P}(A) - P_0(A) \geq 0$ over the fair price $P_0(A)$ for every A, to guarantee gain prospects in the long run.

With the PMM, the upper probability $\overline{P}_{PMM}(A) = \min\{(1 + \delta)P_0(A), 1\}$ returns a loading that tends to 0 as $P_0(A) \to 0$. This may conflict with the seller's need to cover some fixed costs independent of the value of P_0. In such a case, the seller may want $\overline{P}(A)$ not

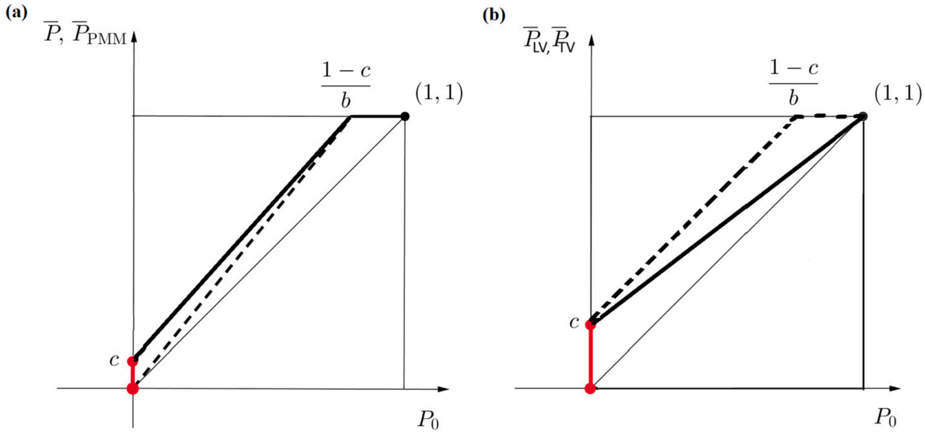


Fig. 1. (a) Upper probability of a VBM (bold line) and of a PMM (dotted line). (b) Upper probability of a LV (bold line) and of a TV (dotted line) model.

to be lower than the value $c > 0$. This can be achieved with a VBM with $c > 0$, and is represented in the $(P_0, \overline{P})$ plane by the *vertical barrier* formed by the segment with endpoints 0 and c in the $\overline{P}$ axis, as seen in Fig. 1(a).

A vertical barrier may be set up also with a LV model, where $\overline{P}_{LV}(A) = bP_0(A) + (1 - b)$. Here $c = 1 - b > 0$, and the loading $(1 - b)(1 - P_0(A))$ is *decreasing* as P_0 increases. Alternatively, the TV model ensures a *constant* loading of $c = -a > 0$ as long as $\overline{P}_{TV}(A) < 1$, as seen in Fig. 1 (b).

Note however that neither a LV nor a TV model is able to simultaneously guarantee both a vertical barrier and an *increasing* loading (as long as $\overline{P} < 1$), while a VBM can do so if $b > 1$, as in Fig. 1 (a).

A result established in [27, Section 3], relating a VBM with some of its special cases and that is relevant for the sequel, is that the lower (or upper) probability of any VBM can be expressed as a convex combination of a PMM and a vacuous probability. Specifically, denote by $\underline{P}$ (the lower probability of) a VBM (P_0, a, b). If $a + b > 0$, it holds that

$$\underline{P}(A) = (a + b)\underline{P}_{PMM}(A) + (1 - (a + b))\underline{P}_V(A)\ \forall A, \tag{4}$$

where $\underline{P}_{PMM}$ is used to denote the PMM determined by $P_0, \delta = -\frac{a}{a+b}$ and $\underline{P}_V$ denotes the *vacuous lower probability*, given by $\underline{P}_V(A) = 0$ for every $A \subset \Omega$ and $\underline{P}_V(\Omega) = 1$. On the other hand, when $a + b = 0$ the VBM $\underline{P}$ is equal to $\underline{P}_V$.

It is not difficult to establish that the converse also holds:

Lemma 1. *Let $\underline{P}_{PMM}$ be the PMM associated with a probability measure P_0 and $\delta > 0$, and consider $\alpha \in (0, 1)$. Then the convex mixture $\underline{P} := (1 - \alpha)\underline{P}_{PMM} + \alpha\underline{P}_V$ is a VBM.*

Proof. Clearly, $\underline{P}(\Omega) = 1$. For $A \neq \Omega$, we have

$$\underline{P}(A) = (1 - \alpha)\max\{(1 + \delta)P_0(A) - \delta, 0\} + \alpha\underline{P}_V(A) = \max\{(1 - \alpha)(1 + \delta)P_0(A) - (1 - \alpha)\delta, 0\},$$

which identifies a VBM (P_0, a, b) with $a = -(1 - \alpha)\delta < 0$, $b = (1 - \alpha)(1 + \delta) > 0$ and $a + b = 1 - \alpha \in (0, 1)$. □

While the convex mixture of a PMM and a vacuous lower probability originates a VBM, a model that is more general than both, the convex mixture VBM-vacuous is still a VBM. Our next result makes this explicit:

Lemma 2. *Let $\underline{P}$ be the lower probability of a VBM (P_0, a, b) and consider $\alpha \in (0, 1)$. Then the convex mixture $\underline{P}' = \alpha\underline{P} + (1 - \alpha)\underline{P}_V$ is the lower probability associated with the VBM $(P_0, \alpha a, \alpha b)$. Moreover, $\underline{P}'(A) \leq \underline{P}(A)$, and consequently $\overline{P}'(A) \geq \overline{P}(A)$ for every $A \in \mathcal{P}(\Omega)$.*

Proof. Trivially $\underline{P}'(\Omega) = 1$. For every $A \neq \Omega$ we have that

$$\underline{P}'(A) = \alpha\underline{P}(A) + (1 - \alpha)\underline{P}_V(A) = \max\{\alpha bP_0(A) + \alpha a, 0\}.$$

Thus, $\underline{P}'$ is a VBM with $a' := \alpha a < 0, b' := \alpha b > 0, a' + b' = \alpha(a + b) < 1$. Moreover,

$$\underline{P}'(A) = 0 \Leftrightarrow \underline{P}(A) = 0 \Leftrightarrow P_0(A) \leq -\frac{a'}{b'} = -\frac{a}{b};$$

when instead $\underline{P}'(A) \cdot \underline{P}(A) > 0$, it holds that $\underline{P}'(A) = \alpha bP_0(A) + \alpha a = \alpha\underline{P}(A) < \underline{P}(A)$. We conclude that $\underline{P}'(A) \leq \underline{P}(A)$ for every $A \in \mathcal{P}(\Omega)$. □

Lemma 2 implies that imprecision increases in the mixture VBM (P_0, a', b') compared to the initial VBM: $\overline{P}'(A) - \underline{P}'(A) \geq \overline{P}(A) - \underline{P}(A)\ \forall A \in \mathcal{P}(\Omega)$.

Lemmas 1 and 2 are relevant because, as we shall establish in Section 5 later on, a mixture of VBMs will not be a VBM in general; it is then interesting to know about the existence of some particular mixtures that remain within the VBM family.

While the features of VBMs discussed so far obtain whatever is the cardinality (also infinite) of the possibility space Ω, most of the properties of VBMs to be investigated next typically require a finite environment. Therefore, in the sequel, we shall assume that the possibility space Ω is *finite*.

In this paper, we shall pay special attention to a particular case of VBMs: those we shall call *non-null*. A VBM is non-null when $\underline{P}(A) > 0$ for every $A \neq \emptyset$. In that case,

$$\underline{P}(A) = bP_0(A) + a\ \forall A \neq \emptyset, \Omega, \quad \underline{P}(\emptyset) = 0, \quad \underline{P}(\Omega) = 1.$$

It also follows from the finitary assumption on Ω that non-null VBMs can be characterised by the condition

$$\min_{\omega \in \Omega} P_0(\{\omega\}) > -\frac{a}{b}.$$

3. VBMs as neighbourhood models

Distortion models appear in the literature in two forms: either as a transformation of a probability measure by means of some function [2–4] or as lower envelopes of neighbourhoods of a probability measure [12,13,31]. A unified procedure was presented in [19], showing that the first type of models can be embedded into the second by considering the neighbourhood determined by a suitable premetric. In this section, we show how this can be done for VBMs.

3.1. Expression in terms of a distorting function

Given a distorting function $d : \mathbb{P}(\Omega) \times \mathbb{P}(\Omega) \to [0, \infty)$, a probability measure $P_0 \in \mathbb{P}(\Omega)$ and $\delta > 0$, we define the set

$$B_d^{\delta}(P_0) = \{P \in \mathbb{P}(\Omega) \mid d(P, P_0) \leq \delta\},$$

and call it the *distortion model* on P_0 associated with d, δ.

In this section, we shall prove that the credal set $\mathcal{M}(\underline{P})$ of a VBM can be obtained as the distortion model associated with some distorting function. Since the LV and PMM were already dealt with in [19], here we shall consider parameters $a < 0 < b$ such that $a + b < 1$. Given such parameters, let us define the function $d_{VBM_{(a,b)}}$ on $\mathbb{P}(\Omega) \times \mathbb{P}(\Omega)$ by

$$d_{VBM_{(a,b)}}(P, P_0) = \max_{A \subseteq \Omega} \frac{P_0(A) - P(A)}{(1-b)P_0(A) - a}. \tag{5}$$

In order to alleviate the notation, we shall denote $d_{VBM_{(a,b)}}$ as d_{VBM} in the sequel, because the properties we shall establish will hold irrespective of the a, b that we fix.

Lemma 3. *d_{VBM} is well-defined and takes values in* $[0, +\infty)$.

Proof. To see that d_{VBM} is well-defined, observe that $(1-b)P_0(A) - a > 0$ for every $A \subseteq \Omega$:

- $P_0(A) = 0 \Rightarrow (1-b)P_0(A) - a = -a > 0$;
- $P_0(A) = 1 \Rightarrow (1-b)P_0(A) - a = 1 - b - a > 0$;
- $P_0(A) \in (0,1) \Rightarrow 0 < a(P_0(A) - 1) = aP_0(A) - a < (1-b)P_0(A) - a$;

on the other hand, $d_{VBM}(P, P_0) \geq \frac{P_0(\emptyset) - P(\emptyset)}{(1-b)P_0(\emptyset) - a} = 0$ and moreover $d_{VBM}(P, P_0)$ is bounded using that Ω is finite. □

Let us state some properties of d_{VBM}:

Proposition 4. *Let $a < 0$, $b > 0$ with $a + b < 1$ and let d_{VBM} be given by Equation (5).*

(a) $d_{VBM}(P, P_0) = 0 \Leftrightarrow P = P_0$. *[definiteness]*
(b) $d_{VBM}(\alpha P_1 + (1-\alpha)P_2, P_0) \leq \max\{d_{VBM}(P_1, P_0), d_{VBM}(P_2, P_0)\}$ *for every* $\alpha \in [0,1]$. *[quasiconvexity]*
(c) $\forall P_0, P_1, P_2 \in \mathbb{P}(\Omega), \forall \epsilon > 0, \exists \delta > 0$ *such that if* $||P_1 - P_2|| < \delta$ *then* $|d_{VBM}(P_1, P_0) - d_{VBM}(P_2, P_0)| < \epsilon$*, where* $|| \cdot ||$ *is the supremum norm, given by* $||P_1 - P_2|| = \max_{A \subseteq \Omega} |P_1(A) - P_2(A)|$. *[continuity]*

Proof. (a) Trivially, $P = P_0$ implies that $d_{VBM}(P, P_0) = 0$. Conversely, the equality $d_{VBM}(P, P_0) = 0$ implies that $P_0(A) \leq P(A)\ \forall A \subseteq \Omega$, which, taking into account that Ω is finite, is equivalent to $P_0(A) = P(A)\ \forall A \subseteq \Omega$.

(b) When $\alpha \in \{0,1\}$ the thesis is trivial. Let $\alpha \in (0,1)$. Observe that for any event A

$$\frac{P_0(A) - (\alpha P_1(A) + (1-\alpha)P_2(A))}{(1-b)P_0(A) - a} = \alpha \frac{P_0(A) - P_1(A)}{(1-b)P_0(A) - a} + (1-\alpha)\frac{P_0(A) - P_2(A)}{(1-b)P_0(A) - a},$$

and this sum is bounded by the maximum of $\left\{ \frac{P_0(A)-P_1(A)}{(1-b)P_0(A)-a}, \frac{P_0(A)-P_2(A)}{(1-b)P_0(A)-a} \right\}$.

(c) Define $m = \min_{A \subseteq \Omega}\{(1-b)P_0(A) - a\}$. It is shown in the proof of Lemma 3 that $(1-b)P_0(A) - a > 0\ \forall A$. Therefore, it is $m > 0$ by the finiteness of Ω. Fix now $\epsilon > 0$ and take $\delta := m\epsilon$. Then

$$\begin{aligned}
&|d_{VBM}(P_1, P_0) - d_{VBM}(P_2, P_0)| \\
&= \left| \max_{A \subseteq \Omega} \frac{P_0(A) - P_1(A)}{(1-b)P_0(A) - a} - \max_{A \subseteq \Omega} \frac{P_0(A) - P_2(A)}{(1-b)P_0(A) - a} \right| \\
&= \left| \max_{A \subseteq \Omega} \frac{P_0(A) - P_1(A)}{(1-b)P_0(A) - a} + \min_{A \subseteq \Omega} \frac{P_2(A) - P_0(A)}{(1-b)P_0(A) - a} \right| \\
&= \left| \max_{A \subseteq \Omega} \left(\frac{P_0(A) - P_1(A)}{(1-b)P_0(A) - a} + \min_{A \subseteq \Omega} \frac{P_2(A) - P_0(A)}{(1-b)P_0(A) - a} \right) \right| \\
&\le \left| \max_{A \subseteq \Omega} \left(\frac{P_0(A) - P_1(A)}{(1-b)P_0(A) - a} + \frac{P_2(A) - P_0(A)}{(1-b)P_0(A) - a} \right) \right| \\
&= \left| \max_{A \subseteq \Omega} \frac{P_2(A) - P_1(A)}{(1-b)P_0(A) - a} \right| \\
&\le \frac{||P_1 - P_2||}{m} < \frac{m\epsilon}{m} = \epsilon,
\end{aligned}$$

from which the thesis follows. □

Next, we prove that d_{VBM} is indeed the distorting function associated with the VBMs. An alternative, more complex proof could be made using arguments similar to those in [19,20] together with Proposition 4.

Theorem 5. *Let $\underline{P}$ be a VBM associated with a probability measure P_0 and with parameters $a < 0, b > 0$ such that $a + b < 1$. Then $\mathcal{M}(\underline{P}) = B^1_{d_{VBM}}(P_0)$.*

Proof. Consider $P \in \mathbb{P}(\Omega)$. Then $P \in B^1_{d_{VBM}}(P_0)$ iff

$$d_{VBM}(P, P_0) \le 1 \Leftrightarrow \max_{A \subset \Omega} \frac{P_0(A) - P(A)}{(1-b)P_0(A) - a} \le 1 \Leftrightarrow P_0(A) - P(A) \le (1-b)P_0(A) - a\ \forall A \subset \Omega \Leftrightarrow P(A) \ge bP_0(A) + a\ \forall A \subset \Omega.$$

Since $P(A) \ge 0\ \forall A \subseteq \Omega$ and $P(\Omega) = \underline{P}(\Omega) = 1$, this is equivalent to $P(A) \ge \underline{P}(A)\ \forall A \subseteq \Omega$, which in turn is equivalent to $P \in \mathcal{M}(\underline{P})$. □

Remark 1. Observe that the radius δ of the neighbourhood above is always equal to 1, and seemingly does not depend on the parameters a, b; this is because a, b are incorporated in the definition of d_{VBM}.

If we consider a different value of $\delta > 0$, following the same reasoning as in the proof of Theorem 5 we obtain that

$$d_{VBM}(P, P_0) \le \delta \Leftrightarrow P(A) \ge (1 - \delta(1-b))P_0(A) + a\delta\ \forall A \subset \Omega.$$

Note however, that this does not imply that the associated ball agrees with the credal set of the VBM with parameters $(a\delta, 1-\delta(1-b))$, because we need in addition that these parameters satisfy the restrictions associated with a VBM; namely, it should be $a\delta \le 0 < 1 - \delta(1-b)$ and $a\delta + 1 - \delta(1-b) \in [0,1]$. It turns out that, if $a < 0 < b$ and $a + b < 1$, as we are assuming in the statement of Theorem 5, these restrictions hold if and only if $0 < \delta \le \frac{1}{1-(a+b)}$. ♦

As we have said, VBMs include as particular cases the PMM, LV and TV distortion models. The distorting functions associated with these models were investigated in [19,20] for the particular case where $P_0(\{\omega\}) > 0$ for every ω. Let us study the connection between those functions and the function d_{VBM} given in Equation (5) for arbitrary P_0 and for $a < 0 < b$ with $a + b < 1$. For this, extend Equation (5) to the case where $a = 0$ or $a + b = 1$ and also by taking the maximum on those events A where the denominator is different from zero, i.e., with

$$d_{VBM}(P, P_0) = \max_{A:(1-b)P_0(A)-a \neq 0} \frac{P_0(A) - P(A)}{(1-b)P_0(A) - a}. \tag{6}$$

- If $a + b = 1$ and $P_0(\{\omega\}) > 0\ \forall\omega$, then (6) becomes

$$d_{VBM}(P, P_0) = \max_{A\subset\Omega} \frac{P_0(A) - P(A)}{-a(1 - P_0(A))} = \max_{A\subset\Omega} \frac{P_0(A) - P(A)}{\delta(1 - P_0(A))},$$

which corresponds to $\frac{d_{PMM}}{\delta}$ for $\delta = -a$ and d_{PMM} the distorting function of a PMM in [19, Section 4.1].
- If $a = 0, b < 1$ and $P_0(\{\omega\}) > 0\ \forall\omega$, then

$$d_{VBM}(P, P_0) = \max_{\emptyset\neq A\subseteq\Omega} \frac{P_0(A) - P(A)}{(1 - b)P_0(A)}$$

which corresponds to $\frac{d_{LV}}{\delta}$ for $\delta = 1 - b$ and d_{LV} the distorting function of a LV in [19, Section 4.2].
- Finally, if $a < 0$ and $b = 1$, we obtain

$$d_{VBM}(P, P_0) = \max_{A\subset\Omega} \frac{P_0(A) - P(A)}{-a},$$

that is a scalar transformation of the total variation distance d_{TV}.

Taking into account these connections, it is not difficult to establish that d_{VBM} is not symmetric nor it satisfies the triangle inequality in general: we simply need to refer to the counterexamples for d_{PMM} in [19, Proposition 4.1 (b)].

3.2. Relationship with other distortion models

We have already mentioned that VBMs include as particular cases some of the most prominent distortion models considered in the literature [27]. In spite of this great generality, it is not hard to show that they do not include other important models, such as the Constant Odds Ratio (COR), Kolmogorov and L_1-models discussed in [19,20].[1] In the case of the L_1-models, it suffices to observe that they do not satisfy the property of 2-monotonicity [20, Example 4.3] while VBMs do [7, Proposition 4.1]. To see that they do not include the COR or Kolmogorov models either,[2] note that for every $A, B \subset \Omega$ such that $A \cap B = \emptyset, A \cup B \neq \Omega$, if the lower probability $\underline{P}$ associated with the VBM determined by (P_0, a, b) satisfies $\min\{\underline{P}(A), \underline{P}(B)\} > 0$, then by Equation (1)

$$\underline{P}(A \cup B) - \underline{P}(A) - \underline{P}(B) = bP_0(A \cup B) + a - bP_0(A) - a - bP_0(B) - a = -a. \tag{7}$$

This is the main idea behind the following counterexample:

Example 1. Consider $\Omega = \{\omega_1, \omega_2, \omega_3\}$, P_0 the probability measure associated with the mass function $(0.5, 0.3, 0.2)$ and $\delta = 0.1$. From [19, Example 6.1], the COR model it induces satisfies

A	$\{\omega_1\}$	$\{\omega_2\}$	$\{\omega_3\}$	$\{\omega_1,\omega_2\}$	$\{\omega_1,\omega_3\}$	$\{\omega_2,\omega_3\}$
$\underline{P}_{COR}(A)$	0.4737	0.2784	0.1837	0.7826	0.6774	0.4737

Since

$$\underline{P}_{COR}(\{\omega_1,\omega_2\}) - \underline{P}_{COR}(\{\omega_1\}) - \underline{P}_{COR}(\{\omega_2\}) = 0.0305 \neq 0.02 = \underline{P}_{COR}(\{\omega_1,\omega_3\}) - \underline{P}_{COR}(\{\omega_1\}) - \underline{P}_{COR}(\{\omega_3\}),$$

we deduce from Equation (7) that $\underline{P}_{COR}$ is not a VBM.

On the other hand, the Kolmogorov model induced by (P_0, δ) is given by

A	$\{\omega_1\}$	$\{\omega_2\}$	$\{\omega_3\}$	$\{\omega_1,\omega_2\}$	$\{\omega_1,\omega_3\}$	$\{\omega_2,\omega_3\}$
$\underline{P}_K(A)$	0.4	0.1	0.1	0.7	0.5	0.4

Since

$$\underline{P}_K(\{\omega_1,\omega_2\}) - \underline{P}_K(\{\omega_1\}) - \underline{P}_K(\{\omega_2\}) = 0.2 \neq 0 = \underline{P}_K(\{\omega_1,\omega_3\}) - \underline{P}_K(\{\omega_1\}) - \underline{P}_K(\{\omega_3\}),$$

we deduce from Equation (7) that $\underline{P}_K$ is not a VBM either. ♦

[1] We refer to [19,20] for the expression of the lower probabilities of these models and a deeper account of their properties.

[2] As remarked by a reviewer, it may be argued that the non-inclusion of COR models in the VBM family is immediate, since they are not determined in general by their restriction to events; what we show in our next example is that the lower probability they determine is not a VBM either.

4. Structure of the credal set

One important feature of an imprecise probability model is the complexity of its associated credal set, in terms of the maximal number of extreme points. These not only determine the model by taking lower envelopes, but can also be used to ease the computations in some contexts, for instance when conditioning. For this reason, in this section we investigate the number of extreme points for VBMs.

Recall that given a credal set $\mathcal{M}$, an element $P \in \mathcal{M}$ is an *extreme point* when $P = \alpha P_1 + (1-\alpha)P_2$ for $\alpha \in (0,1), P_1, P_2 \in \mathcal{M}$ implies that $P = P_1 = P_2$. Let $ext(\mathcal{M}(\underline{P}))$ be the set of extreme points of $\mathcal{M}(\underline{P})$.

Recalling again that any VBM is 2-monotone, the extreme points of $\mathcal{M}(\underline{P})$ are determined by the permutations of Ω [11, Section 3.3]. Denote by S_n the permutations of $\{1,\dots,n\}$. Then, given $\sigma \in S_n$, we can define the probability measure P_σ by means of the equalities

$$P_\sigma(\{\omega_{\sigma(1)}\}) = \overline{P}(\{\omega_{\sigma(1)}\}),$$
$$P_\sigma(\{\omega_{\sigma(j)}\}) = \overline{P}(\{\omega_{\sigma(1)},\dots,\omega_{\sigma(j)}\}) - \overline{P}(\{\omega_{\sigma(1)},\dots,\omega_{\sigma(j-1)}\}) \ \forall j = 2,\dots,n.$$

It holds that $ext(\mathcal{M}(\underline{P})) = \{P_\sigma : \sigma \in S_n\}$. This allows us to bound above the number of extreme points of $\mathcal{M}(\underline{P})$ by $n!$. As we shall show next, this bound can be tightened. For this, let us express $\overline{P}$ as a convex combination of $\overline{P}_{PMM}$ and the vacuous upper probability $\overline{P}_V$, that is given by

$$\overline{P}_V(A) = 1 \ \forall A \neq \emptyset \text{ and } \overline{P}_V(\emptyset) = 0;$$

from Equation (4) and conjugacy, it is

$$\overline{P} = (1-\alpha)\overline{P}_{PMM} + \alpha\overline{P}_V, \text{ with } \alpha = 1-(a+b). \tag{8}$$

Using Equation (8), we can get more insight on the structure of P_σ, as follows: let $j_\sigma = \min\{i : \overline{P}(\{\omega_{\sigma(1)},\dots,\omega_{\sigma(i)}\}) = 1\}$. Note that $i < j_\sigma$ implies that $\overline{P}_{PMM}(\{\omega_{\sigma(1)},\dots,\omega_{\sigma(i)}\}) < 1$. If $j_\sigma \geq 2$, this produces

$$P_\sigma(\{\omega_{\sigma(1)}\}) = (1-\alpha)\overline{P}_{PMM}(\{\omega_{\sigma(1)}\}) + \alpha \tag{9}$$
$$\begin{aligned} P_\sigma(\{\omega_{\sigma(i)}\}) &= \overline{P}(\{\omega_{\sigma(1)},\dots,\omega_{\sigma(i)}\}) - \overline{P}(\{\omega_{\sigma(1)},\dots,\omega_{\sigma(i-1)}\}) \\ &= (1-\alpha)\overline{P}_{PMM}(\{\omega_{\sigma(1)},\dots,\omega_{\sigma(i)}\}) + \alpha - (1-\alpha)\overline{P}_{PMM}(\{\omega_{\sigma(1)},\dots,\omega_{\sigma(i-1)}\}) - \alpha \\ &= (1-\alpha)\overline{P}_{PMM}(\{\omega_{\sigma(i)}\}) \text{ (for } i = 2,\dots,j_\sigma - 1) \end{aligned} \tag{10}$$
$$\begin{aligned} P_\sigma(\{\omega_{\sigma(j_\sigma)}\}) &= 1 - ((1-\alpha)\overline{P}_{PMM}(\{\omega_{\sigma(1)},\dots,\omega_{\sigma(j_\sigma-1)}\}) + \alpha) \\ &= (1-\alpha)\underline{P}_{PMM}(\{\omega_{\sigma(j_\sigma)},\dots,\omega_{\sigma(n)}\}) \end{aligned} \tag{11}$$
$$P_\sigma(\{\omega_{\sigma(j_\sigma+1)}\}) = \dots = P_\sigma(\{\omega_{\sigma(n)}\}) = 0. \tag{12}$$

This means that, when $j_\sigma \geq 2$, the same extreme point shall be induced by $(j_\sigma - 2)!(n-j_\sigma)!$ permutations: the ones with the same elements in the positions $\{2,\dots,j_\sigma - 1\}$ and in positions $\{j_\sigma + 1,\dots,n\}$. This shall help us in bounding the number of different extreme points, without having to perform their computation.

Denote now, for a real positive x, $\lfloor x \rfloor = \max\{n \in \mathbb{N} : n \leq x\}$, $\lceil x \rceil = \min\{n \in \mathbb{N} : n \geq x\}$.

Lemma 6. *Given $n, k \in \mathbb{N}$ with $n \geq k$,*

$$k!(n-k)! \geq \left\lfloor \frac{n}{2} \right\rfloor ! \left\lceil \frac{n}{2} \right\rceil ! \tag{13}$$

As a consequence, for any $j \geq 2$, it holds that

$$(j-2)!(n-j)! \geq \left\lfloor \frac{n}{2} - 1 \right\rfloor ! \left\lceil \frac{n}{2} - 1 \right\rceil !$$

Proof. Consider $\binom{n}{k}$ for a fixed n. It is well-known that $\binom{n}{k} \leq \binom{n}{\frac{n}{2}}$ if n is even and $\binom{n}{k} \leq \binom{n}{\frac{n-1}{2}} = \binom{n}{\frac{n+1}{2}}$ if n is odd. In the first case, we obtain

$$\frac{n!}{k!(n-k)!} \leq \frac{n!}{\left(\frac{n}{2}\right)!\left(\frac{n}{2}\right)!},$$

from which Equation (13) follows. In the second, we obtain

$$\frac{n!}{k!(n-k)!} \leq \frac{n!}{\left(\frac{n-1}{2}\right)!\left(n-\frac{n-1}{2}\right)!} = \frac{n!}{\left(\frac{n-1}{2}\right)!\left(\frac{n+1}{2}\right)!},$$

giving again (13).

Applying this equation to $k := j-2$ and $n := n-2$, it follows that

$$(j-2)!(n-j)! = (j-2)!(n-2-(j-2))! \geq \left\lfloor \frac{n-2}{2} \right\rfloor ! \left\lceil \frac{n-2}{2} \right\rceil ! = \left\lfloor \frac{n}{2} - 1 \right\rfloor ! \left\lceil \frac{n}{2} - 1 \right\rceil !. \quad \square$$

Proposition 7. *Given a VBM $\underline{P}$ on a space Ω of cardinality n, any vertex of the credal set $\mathcal{M}(\underline{P})$ is obtained from at least $\lfloor \frac{n}{2} - 1 \rfloor ! \lceil \frac{n}{2} - 1 \rceil !$ different permutations of Ω.*

Proof. Consider a permutation σ. When $j_\sigma = \min\{i : \overline{P}(\{\omega_{\sigma(1)}, \dots, \omega_{\sigma(i)}\}) = 1\} \geq 2$, there are $(j_\sigma - 2)!(n-j_\sigma)!$ different permutations that originate the same vertex, and by Lemma 6 this value is not smaller than $\lfloor \frac{n}{2} - 1 \rfloor ! \lceil \frac{n}{2} - 1 \rceil !$.

On the other hand, if $j_\sigma = 1$ and consequently $\overline{P}(\{\omega_{\sigma(1)}\}) = 1$, then all the permutations σ' satisfying $\sigma'(1) = \sigma(1)$ produce the same extreme point, and there are

$$(n-1)! \geq \left\lfloor \frac{n}{2} - 1 \right\rfloor ! \left\lceil \frac{n}{2} - 1 \right\rceil !$$

such permutations. $\square$

This allows us to establish the main result in this section:

Theorem 8. *Given a VBM $\underline{P}$ on a space Ω of cardinality n, the maximum number of extreme points of $\mathcal{M}(\underline{P})$ is*

$$\frac{n!}{\lfloor \frac{n}{2} - 1 \rfloor ! \lceil \frac{n}{2} - 1 \rceil !}. \tag{14}$$

Proof. That $\frac{n!}{\lfloor \frac{n}{2} - 1 \rfloor ! \lceil \frac{n}{2} - 1 \rceil !}$ is an upper bound of the number of extreme points is a consequence of Proposition 7.

But since TV models are a particular case of VBM, the maximal number of extreme points must be at least as large as the maximal number of extreme points for TV models, that was established in [20, Proposition A.1] to be

$$\frac{n!}{\lfloor \frac{n}{2} - 1 \rfloor !(n - \lfloor \frac{n}{2} \rfloor - 1)!} = \frac{n!}{\lfloor \frac{n}{2} - 1 \rfloor ! \lceil \frac{n}{2} - 1 \rceil !}.$$

The double inequality gives the result. $\square$

We observe that the maximal number of extreme points in Equation (14) is strictly larger than the one for the other particular cases discussed in [19], namely:

- n in the case of LV models;
- $\frac{n!}{\lfloor \frac{n}{2} \rfloor \lfloor \frac{n}{2} - 1 \rfloor ! \lceil \frac{n}{2} + 1 \rceil !}$ in the case of the PMM.

In the case of non-null VBMs the upper bound (14) is notably lowered:

Proposition 9. *Given a non-null VBM $\underline{P}$ on a space Ω of cardinality n, the maximum number of extreme points of $\mathcal{M}(\underline{P})$ is $n(n-1)$.*

Proof. By monotonicity of $\underline{P}$, we have that, $\forall A \in \mathcal{P}(\Omega), A \neq \emptyset, A \neq \Omega$, $\underline{P}(A) > 0$ and therefore $\overline{P}(A) < 1$. Hence, $j_\sigma = n$ for any permutation in the proof of Proposition 7, meaning that any vertex is originated by $(n-2)!$ different permutations. Thus, the bound (14) reduces to

$$\frac{n!}{(n-2)!} = n(n-1). \quad \square$$

We recall that the same bound $n(n-1)$ applies to TV models too, when $\underline{P}$ is strictly positive, as shown in [20, Proposition 2.5]. It is also interesting to remark that not only the tight bound on the number of extreme points is useful, but also the identities established in Equations (9)÷(12). To see an application in finance where this comes into play for the particular case of LV mixtures, we refer to [28].

5. Processing Vertical Barrier Models

Next we investigate the behaviour of the family of VBMs under a number of merging operations. By *merging*, we refer to the procedure where we aggregate belief models, defined on the same domain Ω, into a unique one. These models may arise as the

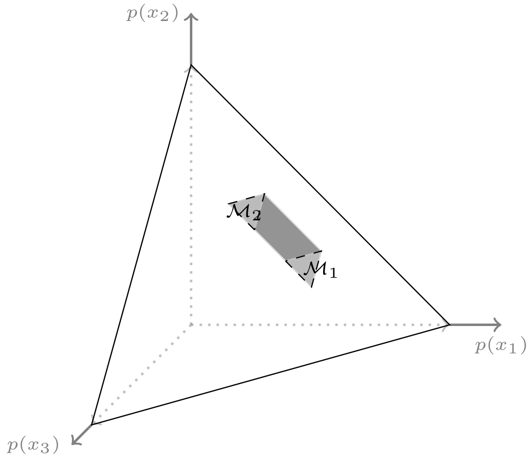


Fig. 2. Credal sets of the lower probabilities in Example 2 and their disjunction.

opinion of different experts or from several data sources, for instance. The problem of aggregating imprecise beliefs has been analysed from the axiomatic point of view by Walley in [33]. Other relevant works on this topic are [23,24].

In this paper, we shall focus on the three most fundamental merging procedures: those of disjunction, conjunction and convex mixture.

Definition 2. Given two credal sets $\mathcal{M}_1, \mathcal{M}_2$ on Ω, their *disjunction* is given by $\mathcal{M}_1 \cup \mathcal{M}_2$.

If we interpret $\mathcal{M}_1, \mathcal{M}_2$ as the sets of probability measures that are considered acceptable by two different experts, the disjunction $\mathcal{M}_1 \cup \mathcal{M}_2$ considers those probability measures that are acceptable for at least one of them. This is illustrated in Fig. 2 on a space of cardinality three.

If we denote by $\underline{P}_1, \underline{P}_2, \underline{P}^{\cup}$ the lower probabilities obtained as lower envelopes of $\mathcal{M}_1, \mathcal{M}_2$ and $\mathcal{M}_1 \cup \mathcal{M}_2$, respectively, it holds that

$$\underline{P}^{\cup}(A) = \min\{\underline{P}_1(A), \underline{P}_2(A)\} \, \forall A \subseteq \Omega.$$

The disjunction $\mathcal{M}_1 \cup \mathcal{M}_2$ is not a convex set of probability measures in general; it is not difficult to show that $\underline{P}^{\cup}$ is also the lower envelope of the convex hull $ch(\mathcal{M}_1 \cup \mathcal{M}_2)$.

It was shown in [10, Example 2] that the disjunction of two PMMs does not produce a PMM in general. The same example can be used to establish that the family of VBMs is not closed under disjunction. For this, we shall use that, given a VBM $\underline{P}$ associated with (P_0, a, b), it is

$$\underline{P}(A) + \underline{P}(A^c) = b + 2a \, \forall A \subset \Omega \text{ such that } \underline{P}(A) > 0, \underline{P}(A^c) > 0. \tag{15}$$

Example 2. Let $\Omega = \{\omega_1, \omega_2, \omega_3\}$ and consider the probability measures P_0^1, P_0^2 on $\mathcal{P}(\Omega)$ associated with $(0.5, 0.3, 0.2)$ and $(0.3, 0.5, 0.2)$, respectively. Let $a_1 = a_2 = -0.1$, $b_1 = b_2 = 1.1$ and denote by $\underline{P}_1, \underline{P}_2$ the VBMs determined by (P_0^1, a_1, b_1) and (P_0^2, a_2, b_2), respectively. Then $\underline{P}_1, \underline{P}_2$ and their disjunction $\underline{P}^{\cup}$ are given in the following table:

A	$\{\omega_1\}$	$\{\omega_2\}$	$\{\omega_3\}$	$\{\omega_1, \omega_2\}$	$\{\omega_1, \omega_3\}$	$\{\omega_2, \omega_3\}$
$\underline{P}_1(A)$	0.45	0.23	0.12	0.78	0.67	0.45
$\underline{P}_2(A)$	0.23	0.45	0.12	0.78	0.45	0.67
$\underline{P}^{\cup}(A)$	0.23	0.23	0.12	0.78	0.45	0.45

These sets are also graphically depicted in Fig. 2.

If $\underline{P}^{\cup}$ were the lower probability associated with a VBM, and observing that $\underline{P}^{\cup}(A)$ is strictly positive for every non-impossible event A, there should be a probability measure P_0 on $\mathcal{P}(\Omega)$ and two parameters $b > 0, a \leq 0$ such that $\underline{P}^{\cup}(A) = bP_0(A) + a$ for every $A \neq \emptyset, \Omega$. But then applying Equation (15) it should be

$$0.68 = \underline{P}^{\cup}(\{\omega_2\}) + \underline{P}^{\cup}(\{\omega_1, \omega_3\}) = b + 2a = \underline{P}^{\cup}(\{\omega_3\}) + \underline{P}^{\cup}(\{\omega_1, \omega_2\}) = 0.9,$$

a contradiction. Thus, $\underline{P}^{\cup}$ does not belong to the family of VBMs. ♦

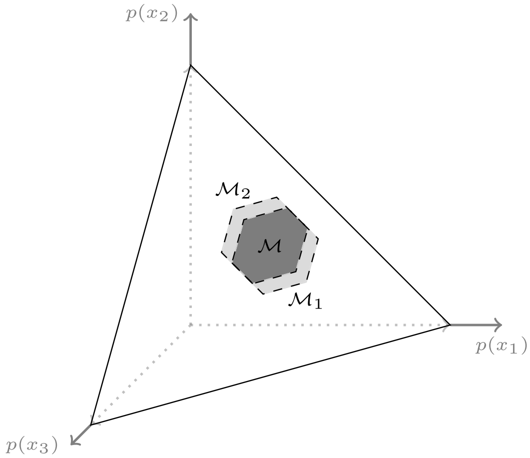


Fig. 3. Credal sets of the lower probabilities in Example 3 and their conjunction.

The second merging operation we analyse in this paper is that of conjunction:

Definition 3. Given two credal sets $\mathcal{M}_1, \mathcal{M}_2$ on Ω, their *conjunction* is given by $\mathcal{M}_1 \cap \mathcal{M}_2$.

If we interpret $\mathcal{M}_1, \mathcal{M}_2$ as the sets of precise probabilities that are considered acceptable by two different experts, the conjunction $\mathcal{M}_1 \cap \mathcal{M}_2$ only considers those probability measures that are acceptable for both of them. The process is illustrated in Fig. 3.

Unlike disjunction, the process of conjunction always produces a convex (though possibly empty) credal set. Note that, if we denote by $\underline{P}_1, \underline{P}_2, \underline{P}^{\cap}$ the lower envelopes of $\mathcal{M}_1, \mathcal{M}_2$ and $\mathcal{M}_1 \cap \mathcal{M}_2$, it will hold that

$$\underline{P}^{\cap}(A) \geq \max\{\underline{P}_1(A), \underline{P}_2(A)\} \; \forall A \subseteq \Omega,$$

with the inequality being possibly strict on some events. It also holds that $\mathcal{M}(\underline{P}^{\cap}) = \mathcal{M}(\underline{P}_1) \cap \mathcal{M}(\underline{P}_2)$; $\underline{P}^{\cap}$ corresponds to the *natural extension* of $\max\{\underline{P}_1, \underline{P}_2\}$ in the terminology of Walley [34]. This means that $\underline{P}^{\cap} = \max\{\underline{P}_1, \underline{P}_2\}$ when the latter is a coherent lower probability.

It was shown in [10, Example 5] that the family of TV models is not closed under conjunction. Using that example, we can easily establish that the family of VBMs is not closed under conjunction either:

Example 3. Let $\Omega = \{\omega_1, \omega_2, \omega_3\}$ and consider the probability measures P_0^1, P_0^2 on $\mathcal{P}(\Omega)$ associated with $(0.41, 0.37, 0.22)$ and $(0.37, 0.41, 0.22)$, respectively. Let $b_1 = b_2 = 1$, $a_1 = a_2 = -0.12$, and denote by $\underline{P}_1, \underline{P}_2$ the VBM models determined by (P_0^1, a_1, b_1) and (P_0^2, a_2, b_2), respectively. To see that in this case the conjunction $\underline{P}^{\cap}$ of $\underline{P}_1, \underline{P}_2$ coincides with $\max\{\underline{P}_1, \underline{P}_2\}$, it suffices to take into account that the latter is a coherent lower probability, since it is the lower envelope of the set of probability measures

$$\{(0.29, 0.37, 0.34), (0.37, 0.29, 0.34), (0.49, 0.41, 0.1), (0.41, 0.49, 0.1)\}.$$

Then $\underline{P}_1, \underline{P}_2$ and their conjunction $\underline{P}^{\cap}$ are given in the following table:

A	$\{\omega_1\}$	$\{\omega_2\}$	$\{\omega_3\}$	$\{\omega_1, \omega_2\}$	$\{\omega_1, \omega_3\}$	$\{\omega_2, \omega_3\}$
$\underline{P}_1(A)$	0.29	0.25	0.1	0.66	0.51	0.47
$\underline{P}_2(A)$	0.25	0.29	0.1	0.66	0.47	0.51
$\underline{P}^{\cap}(A)$	0.29	0.29	0.1	0.66	0.51	0.51

The associated credal sets are represented in Fig. 3.

Again, for $\underline{P}^{\cap}$ to be the lower probability associated with a VBM, there should be a probability measure P_0 on $\mathcal{P}(\Omega)$ and two parameters $b > 0, a \leq 0$ such that $\underline{P}^{\cap}(A) = bP_0(A) + a$ for every $A \neq \emptyset, \Omega$. But this is impossible since we should have

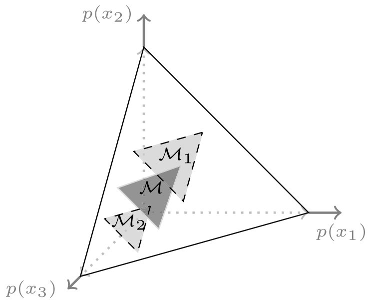
$$0.8 = \underline{P}^{\cap}(\{\omega_2\}) + \underline{P}^{\cap}(\{\omega_1, \omega_3\}) = b + 2a = \underline{P}^{\cap}(\{\omega_3\}) + \underline{P}^{\cap}(\{\omega_1, \omega_2\}) = 0.76,$$

where the second and third equalities follow from Equation (15). Thus, $\underline{P}^{\cap}$ does not belong to the family of VBMs. ♦

The third and last merging operation we consider in this paper is that of mixture:

Definition 4. Let $\mathcal{M}_1, \mathcal{M}_2$ be two credal sets on Ω and consider $\alpha \in (0,1)$. Their *mixture* corresponds to the credal set $\alpha\mathcal{M}_1 + (1-\alpha)\mathcal{M}_2 := \{P : \exists P_1 \in \mathcal{M}_1, P_2 \in \mathcal{M}_2 \text{ such that } P = \alpha P_1 + (1-\alpha)P_2\}$.

The process of mixture is illustrated in the following figure:



Mixture of credal sets

The mixture aggregation procedure is an intermediate solution between the disjunction and conjunction operations, and can be seen as giving a weight to the opinion of each expert. It is easy to see that the mixture $\alpha\mathcal{M}_1 + (1-\alpha)\mathcal{M}_2$ produces a convex credal set. If we denote by $\underline{P}_1, \underline{P}_2$ and $\underline{P}_\alpha$ the lower envelopes of the credal sets $\mathcal{M}_1, \mathcal{M}_2$ and of their mixture $\alpha\mathcal{M}_1 + (1-\alpha)\mathcal{M}_2$, it holds that

$$\underline{P}_\alpha = \alpha\underline{P}_1 + (1-\alpha)\underline{P}_2.$$

That VBMs are not closed under conjunction or disjunction in general is somewhat intuitive, if we take into account that the larger family of 2-monotone models they are included in is not closed either; on the other hand, it is easy to see that the property of 2-monotonicity is preserved under convex combinations. All of this leads us towards the intuition that VBMs may be closed under mixtures, or at least that they will be so much more often than they are with respect to conjunction and disjunction.

Consider thus two VBMs

$$\underline{P}_i(A) = \max\{b_i P_0^i(A) + a_i, 0\},\ \forall A \subset \Omega,\ \underline{P}_i(\Omega) = 1\ (i = 1,2),$$

and their convex combination

$$\underline{P}_\alpha(A) = \alpha\underline{P}_1(A) + (1-\alpha)\underline{P}_2(A),\ \forall A \subseteq \Omega \text{ and for some fixed } \alpha \in (0,1).$$

As we shall see, $\underline{P}_\alpha$ is not a VBM in general. In order to prove this, we give first the following supporting result:

Lemma 10. *Assume that $\underline{P}_\alpha$ is the VBM associated with $(P_0^\alpha, a_\alpha, b_\alpha)$.*

(a) *If there are $A, B \subset \Omega$ such that $A \cap B = \emptyset$, $A \cup B \subset \Omega$, $\underline{P}_i(A) > 0$, $\underline{P}_i(B) > 0$ $(i = 1,2)$, then*

$$a_\alpha = \alpha a_1 + (1-\alpha)a_2. \tag{16}$$

(b) *If there exists $F \subset \Omega$ such that $\underline{P}_i(F) > 0, \underline{P}_i(F^c) > 0$ $(i = 1,2)$, then*

$$b_\alpha + 2a_\alpha = \alpha(b_1 + 2a_1) + (1-\alpha)(b_2 + 2a_2). \tag{17}$$

Proof. (a) Firstly, note that given A, B as in the statement, $\underline{P}_\alpha(A) > 0$, $\underline{P}_\alpha(B) > 0$. Hence, by applying Equation (7) to $\underline{P}_\alpha$, $\underline{P}_1$, $\underline{P}_2$, we get

$$\begin{aligned}-a_\alpha &= \underline{P}_\alpha(A \cup B) - \underline{P}_\alpha(A) - \underline{P}_\alpha(B)\\ &= \alpha(\underline{P}_1(A \cup B) - \underline{P}_1(A) - \underline{P}_1(B)) + (1-\alpha)(\underline{P}_2(A \cup B) - \underline{P}_2(A) - \underline{P}_2(B)) = -\alpha a_1 - (1-\alpha)a_2,\end{aligned}$$

from which the thesis follows.

(b) Note that $\underline{P}_\alpha(F) > 0, \underline{P}_\alpha(F^c) > 0$. Hence, by applying Equation (15) to $\underline{P}_\alpha$, $\underline{P}_1$, $\underline{P}_2$, we get

$$b_\alpha + 2a_\alpha = \underline{P}_\alpha(F) + \underline{P}_\alpha(F^c) = \alpha(\underline{P}_1(F) + \underline{P}_1(F^c)) + (1-\alpha)(\underline{P}_2(F) + \underline{P}_2(F^c)) = \alpha(b_1 + 2a_1) + (1-\alpha)(b_2 + 2a_2). \qquad \square$$

Proposition 11. *Under any of the following conditions, the mixture $\underline{P}_\alpha$ is not a VBM no matter the value of $\alpha \in (0,1)$:*

(a) *If the hypotheses of Lemma 10(a) hold and there are $C, D \subset \Omega$, such that $C \cap D = \emptyset$, $C \cup D \subset \Omega$, $\underline{P}_i(C) > 0$ $(i = 1, 2)$, $\underline{P}_1(D) > 0$, $b_2 P_0^2(D) + a_2 < 0$.*
(b) *If the hypotheses of Lemma 10(b) hold and there exists $F \subset \Omega$ such that $\underline{P}_1(F) > 0$, $\underline{P}_1(F^c) > 0$, $\underline{P}_2(F^c) > 0$ and $b_2 P_0^2(F) + a_2 < 0$.*

Proof. We proceed by contradiction, assuming ex-absurdo that $\underline{P}_\alpha$ is a VBM.

(a) Note that

$$\underline{P}_2(C) = b_2 P_0^2(C) + a_2, \quad \underline{P}_2(D) = 0, \quad \underline{P}_2(C \cup D) = b_2 P_0^2(C \cup D) + a_2,$$
$$\underline{P}_\alpha(C) > 0, \qquad \underline{P}_\alpha(D) = \alpha \underline{P}_1(D) > 0.$$

Applying Equation (7) to $\underline{P}_\alpha$, $\underline{P}_1$, we get

$$\begin{aligned} -a_\alpha &= \underline{P}_\alpha(C \cup D) - \underline{P}_\alpha(C) - \underline{P}_\alpha(D) \\ &= \alpha(\underline{P}_1(C \cup D) - \underline{P}_1(C) - \underline{P}_1(D)) + (1-\alpha)(\underline{P}_2(C \cup D) - \underline{P}_2(C)) \\ &= -\alpha a_1 + (1-\alpha)(b_2 P_0^2(C \cup D) + a_2 - b_2 P_0^2(C) - a_2) \\ &= -\alpha a_1 + (1-\alpha) b_2 P_0^2(D). \end{aligned}$$

Since (16) holds by Lemma 10(a), $-(1-\alpha)a_2 = -a_\alpha + \alpha a_1 = (1-\alpha) b_2 P_0^2(D)$, whence $b_2 P_0^2(D) + a_2 = 0$, a contradiction.

(b) Note that

$$\underline{P}_2(F) = 0, \qquad \underline{P}_2(F^c) = b_2 P_0^2(F^c) + a_2,$$
$$\underline{P}_\alpha(F) = \alpha \underline{P}_1(F) > 0, \qquad \underline{P}_\alpha(F^c) > 0.$$

Applying Equation (15) to $\underline{P}_\alpha$, $\underline{P}_1$, we get

$$\begin{aligned} b_\alpha + 2a_\alpha &= \underline{P}_\alpha(F) + \underline{P}_\alpha(F^c) \\ &= \alpha(\underline{P}_1(F) + \underline{P}_1(F^c)) + (1-\alpha)\underline{P}_2(F^c) \\ &= \alpha(b_1 + 2a_1) + (1-\alpha)(b_2 P_0^2(F^c) + a_2). \end{aligned}$$

Since (17) holds by Lemma 10(b), we get

$$(1-\alpha)(b_2 + 2a_2) = (1-\alpha)(b_2 P_0^2(F^c) + a_2),$$

whence $b_2 + 2a_2 = b_2 - b_2 P_0^2(F) + a_2$, which is equivalent to $b_2 P_0^2(F) + a_2 = 0$, a contradiction. □

In spite of this result, part (a) of the next proposition establishes relatively general sufficient conditions for the mixture of two VBMs to be again a VBM:

Proposition 12. *Let $\underline{P}_1, \underline{P}_2$ be the lower probabilities associated with two VBMs, and let $\mathcal{N}_i := \{A \in \mathcal{P}(\Omega) : \underline{P}_i(A) = 0\}$.*

(a) *If either $\underline{P}_1$ or $\underline{P}_2$ is vacuous or if $\mathcal{N}_1 = \mathcal{N}_2$, then for any $\alpha \in (0, 1)$ the mixture*

$$\underline{P}_\alpha := \alpha \underline{P}_1 + (1-\alpha)\underline{P}_2 \tag{18}$$

is the lower probability of a VBM.

(b) *Conversely, let $\mathcal{N}_1 \neq \mathcal{N}_2$. If there exist three pairwise disjoint events A_1, A_2, B such that $\underline{P}_i(A_1) > 0$, $\underline{P}_i(A_2) > 0$ for $i = 1, 2$ and $\underline{P}_1(B) > 0$, $b_2 P_0^2(B) + a_2 < 0$, then $\underline{P}_\alpha$ is not a VBM.*

Proof. (a) When either $\underline{P}_1$ or $\underline{P}_2$ is vacuous, the result is a consequence of Lemma 2. Otherwise, let us denote $\mathcal{N} := \mathcal{N}_1 = \mathcal{N}_2$. It follows from Definition 1 and Equation (18) that $\underline{P}_\alpha(\Omega) = 1$. For any $A \notin \mathcal{N}$, $A \neq \Omega$, it holds that $\underline{P}_i(A) = b_i P_0^i(A) + a_i$ for $i = 1, 2$, whence

$$\begin{aligned} \underline{P}_\alpha(A) &= \alpha[b_1 P_0^1(A) + a_1] + (1-\alpha)[b_2 P_0^2(A) + a_2] \\ &= (\alpha b_1 P_0^1 + (1-\alpha) b_2 P_0^2)(A) + [\alpha a_1 + (1-\alpha)a_2] \\ &= (\alpha b_1 + (1-\alpha) b_2) \cdot \frac{\alpha b_1 P_0^1(A) + (1-\alpha) b_2 P_0^2(A)}{\alpha b_1 + (1-\alpha) b_2} + [\alpha a_1 + (1-\alpha)a_2] \\ &:= b^\alpha P_0^\alpha(A) + a^\alpha, \end{aligned}$$

where $b^\alpha = \alpha b_1 + (1-\alpha)b_2$, $a^\alpha = \alpha a_1 + (1-\alpha)a_2$ and

$$P_0^\alpha(A) = \frac{\alpha b_1 P_0^1(A) + (1-\alpha)b_2 P_0^2(A)}{\alpha b_1 + (1-\alpha)b_2}.$$

On the other hand, if $A \in \mathcal{N}$, then $b_i P_0^i(A) + a_i \leq 0$ for $i = 1,2$ and from the above reasoning also $b^\alpha P_0^\alpha(A) + a^\alpha \leq 0$.
This implies that $\underline{P}_\alpha$ is the VBM $(P_0^\alpha, a^\alpha, b^\alpha)$, noting also that by construction $b^\alpha > 0, a^\alpha \leq 0$, $a^\alpha + b^\alpha \leq 1$ and P_0^α is a probability measure.

(b) Apply Proposition 11(a) with $C = A_1, D = B$ (while A_1, A_2 have the role of A, B in Lemma 10(a)). □

A relevant special case of Proposition 12(a) occurs when $\mathcal{N}_1 = \mathcal{N}_2 = \emptyset$, i.e., for non-null VBMs. This corresponds to the case where both experts believe that all non-impossible events have positive lower probability. Note that the subfamily of non-null VBMs is not closed under disjunction or conjunction, as Examples 2 and 3 show.

As for Proposition 12(b), it obtains in particular if we only know that there exists B such that $\underline{P}_1(B) > 0 > b_2 P_0^2(B) + a_2$ and there exist ω_1, ω_2 such that $\underline{P}_i(\{\omega_1\}) > 0, \underline{P}_i(\{\omega_2\}) > 0, i = 1,2$. In fact, it holds that $\omega_1, \omega_2 \notin B$ (or else it would be $\underline{P}_i(B) > 0$ for $i = 1,2$). Therefore, ω_1, ω_2 can take the role of A_1, A_2 in Proposition 12(b).

Proposition 12(b) lets us establish that $\underline{P}_\alpha$ is not a VBM in a number of common situations when $\mathcal{N}_1 \neq \mathcal{N}_2$. Yet, there remain some essentially peculiar cases where Proposition 12 does not allow to conclude whether $\underline{P}_\alpha$ is a VBM or not, as the following example shows:

Example 4. Let $\Omega = \{\omega_1, \omega_2, \omega_3\}$ and consider the probability measures P_0^1, P_0^2 on $\mathcal{P}(\Omega)$ associated with $(0.7, 0.1, 0.2)$ and $(0.1, 0.7, 0.2)$, respectively. Let $\underline{P}_1, \underline{P}_2$ be the VBM models determined by $(P_0^1, -0.7, 1)$ and $(P_0^2, -0.7, 1)$, respectively. Then $\underline{P}_1, \underline{P}_2$ and their mixture $\underline{P}_\alpha$ are given in the following table:

A	$\{\omega_1\}$	$\{\omega_2\}$	$\{\omega_3\}$	$\{\omega_1,\omega_2\}$	$\{\omega_1,\omega_3\}$	$\{\omega_2,\omega_3\}$
$\underline{P}_1(A)$	0	0	0	0.1	0.2	0
$\underline{P}_2(A)$	0	0	0	0.1	0	0.2
$\underline{P}_\alpha(A)$	0	0	0	0.1	0.2α	$0.2(1-\alpha)$

We observe that none of $\underline{P}_1, \underline{P}_2$ is vacuous and also that

$$\mathcal{N}_1 = \{\{\omega_1\}, \{\omega_2\}, \{\omega_3\}, \{\omega_2,\omega_3\}\} \neq \mathcal{N}_2 = \{\{\omega_1\}, \{\omega_2\}, \{\omega_3\}, \{\omega_1,\omega_3\}\}.$$

Note also that the hypotheses of Proposition 12(b) do not apply either.

On the other hand, $\underline{P}_\alpha$ corresponds to a TV model determined by the probability measure $\left(\frac{0.7}{3} + 0.2\alpha, \frac{0.7}{3} + 0.2(1-\alpha), \frac{1}{3}\right)$ and $a = -\frac{1.7}{3}, b = 1$, and as a consequence it is also a VBM. ♦

6. Properties of the mass function of a VBM

In this section, we shall investigate the properties of the mass function of a VBM, also called the *Möbius inverse* of $\underline{P}$. It is given by [29]

$$m_{\underline{P}}(A) = \sum_{B \subseteq A} (-1)^{|A \setminus B|} \underline{P}(B), \tag{19}$$

and we shall simply denote the Möbius inverse by m when there is no ambiguity about its associated lower probability.

We shall show that, in several common instances, more direct formulae are available for $m(A)$. Since by Equation (19) $m(\{\omega\}) = \underline{P}(\{\omega\})$ for every $\omega \in \Omega$, our focus will be on those events A with $|A| > 1$.

Let us define the function $m^* : \mathcal{P}(\Omega) \to \mathbb{R}$ by

$$m^*(A) = \sum_{\emptyset \neq B \subseteq A} (-1)^{|A \setminus B|} (bP_0(B) + a),\ m^*(\emptyset) = 0. \tag{20}$$

Lemma 13. *For every $A \subset \Omega$ with $|A| > 1$, it holds that $m^*(A) = (-1)^{|A|+1} a$.*

Proof. By (20),

$$m^*(A) = b \sum_{\emptyset \neq B \subseteq A} (-1)^{|A \setminus B|} P_0(B) + a \sum_{\emptyset \neq B \subseteq A} (-1)^{|A \setminus B|}.$$

Concerning the first term, it is

$$\sum_{\emptyset\neq B\subseteq A}(-1)^{|A\setminus B|}P_0(B)=\sum_{B\subseteq A}(-1)^{|A\setminus B|}P_0(B)=m_{P_0}(A)=0,$$

since $|A|>1$ and the mass function m_{P_0} of a probability measure (here P_0) may be non-zero on singletons only [29]. With respect to the second term,

$$a\sum_{\emptyset\neq B\subseteq A}(-1)^{|A\setminus B|}=a\left(\sum_{B\subseteq A}(-1)^{|A\setminus B|}-(-1)^{|A|}\right)=(-1)^{|A|+1}a,$$

using the equality $\sum_{B\subseteq A}(-1)^{|A\setminus B|}=0$. □

From this lemma we deduce the expression of the Möbius inverse for non-null VBMs, since in that case $m(A)=m^*(A)$ for every A:

Proposition 14. *If $\underline{P}$ is a non-null VBM, then $m(A)=(-1)^{|A|+1}a$ for every $A\neq\Omega$ with $|A|>1$, and $m(A)=bP_0(A)+a$ if $|A|=1$.*

While this gives us the mass function of a relevant subfamily of VBMs, if $\underline{P}(A)=0$ for some $A\neq\emptyset$ it may be useful to consider the following notion:

Definition 5. An event $N\subset\Omega$ is called a *maximal null event* with respect to $\underline{P}$ if $\underline{P}(N)=0$ and $\underline{P}(C)>0$ for every $C\supset N$.

Given an event A with $|A|>1$, there are three possible scenarios: (i) A intersects no maximal null event, (ii) it intersects only one maximal null event or (iii) it intersects more than one maximal null event.

If A intersects no maximal null event, then necessarily $\underline{P}(B)>0\ \forall B\subseteq A, B\neq\emptyset$. Hence,

$$m(A)=\sum_{B\subseteq A}(-1)^{|A\setminus B|}\underline{P}(B)=\sum_{\emptyset\neq B\subseteq A}(-1)^{|A\setminus B|}(bP_0(B)+a)=m^*(A)=(-1)^{|A|+1}a,$$

using Equation (20), and applying Lemma 13 for the last equality.

In the second scenario A intersects one maximal null event only. In that case, the Möbius inverse is given by the following proposition:

Proposition 15. *For a given VBM $\underline{P}$, let $A\neq\Omega$ be an event with $|A|>1$ such that A intersects only one maximal null event N.*

(a) *If $|A\cap N|\geq 2$ then $m(A)=0$.*
(b) *If $|A\cap N|=1$ then $m(A)=(-1)^{|A|}bP_0(A\cap N)$.*

Proof. From Equation (19)

$$\begin{aligned}
m(A)&=\sum_{\emptyset\neq B\subseteq A}(-1)^{|A\setminus B|}\underline{P}(B)\\
&=\sum_{\emptyset\neq B\subseteq A, B\nsubseteq A\cap N}(-1)^{|A\setminus B|}\underline{P}(B)+\sum_{\emptyset\neq B\subseteq A\cap N}(-1)^{|A\setminus B|}\underline{P}(B)\\
&=\sum_{\emptyset\neq B\subseteq A, B\nsubseteq A\cap N}(-1)^{|A\setminus B|}\underline{P}(B)\\
&=\sum_{\emptyset\neq B\subseteq A, B\nsubseteq A\cap N}(-1)^{|A\setminus B|}(bP_0(B)+a)\\
&=\sum_{\emptyset\neq B\subseteq A}(-1)^{|A\setminus B|}(bP_0(B)+a)-\sum_{\emptyset\neq B\subseteq A\cap N}(-1)^{|A\setminus B|}(bP_0(B)+a)\\
&=(-1)^{|A|+1}a-\sum_{\emptyset\neq B\subseteq A\cap N}(-1)^{|A\setminus B|}(bP_0(B)+a);
\end{aligned}\tag{21}$$

here, the fourth equality holds because if it were $\underline{P}(B)=0$ then B would be included in a maximal null event, that would thus intersect A; but this means that the maximal null event must necessarily be N, and then $B\subseteq A\cap N$, a contradiction. On the other hand, we are using Equation (20) and applying Lemma 13 for the last equality.

(a) Assume that $|A\cap N|\geq 2$. Given $B\subseteq A\cap N$, it is $|A\setminus B|=|A\setminus N|+|(A\cap N)\setminus B|$. Then in the last term of Equation (21) we obtain

$$\sum_{\emptyset\neq B\subseteq A\cap N}(-1)^{|A\setminus B|}(bP_0(B)+a)=(-1)^{|A\setminus N|}\sum_{\emptyset\neq B\subseteq A\cap N}(-1)^{|(A\cap N)\setminus B|}(bP_0(B)+a)$$

$$= (-1)^{|A\setminus N|}(-1)^{|A\cap N|+1}a = (-1)^{|A|+1}a,$$

using Equation (20) and applying Lemma 13 for the one but last equality. From Equation (21), we conclude that $m(A) = 0$.

(b) In this second case, it is

$$\sum_{\emptyset \neq B \subseteq A\cap N} (-1)^{|A\setminus B|}(bP_0(B) + a) = (-1)^{|A|-1}(bP_0(A\cap N) + a),$$

whence

$$\begin{aligned} m(A) &= (-1)^{|A|+1}a - (-1)^{|A|-1}bP_0(A\cap N) - (-1)^{|A|-1}a \\ &= ((-1)^{|A|+1} - (-1)^{|A|-1})a + (-1)^{|A|}bP_0(A\cap N) \\ &= (-1)^{|A|}bP_0(A\cap N). \quad \square \end{aligned}$$

In the third scenario A intersects more than one maximal null event. To study this case we start with a preliminary lemma:

Lemma 16. *If $\{N_1,\dots,N_m\}$ is the set of maximal null events with respect to $\underline{P}$, it is $\underline{P}(A) > 0$ for every A satisfying $A\cap N_i^c \neq \emptyset$ for every $i = 1,\dots,m$.*

Proof. To see this, assume ex-absurdo that $\underline{P}(A) = 0$ for one such A. Then the class $\mathcal{N} = \{B : A \subseteq B, \underline{P}(B) = 0\}$ is non-empty, and must have therefore a maximal element B^* with respect to set inclusion. But this maximal element is by definition a maximal null-event, a contradiction with the assumption $A\cap (B^*)^c \neq \emptyset$. $\square$

Next we give the expression of $m(A)$ in the case the intersections of A with the maximal null events are pairwise disjoint:

Proposition 17. *Let $N_1,\dots,N_m$ be the maximal null events that intersect $A \subset \Omega$. Let us express*

$$A \cap \left(\bigcup_{i=1}^m N_i\right) = \bigcup_{i=1}^r (A\cap N_{j_i}) \cup \bigcup_{i=1}^s (A\cap N_{k_i}),$$

with $|A\cap N_{j_i}| = 1$ for $i = 1,\dots,r$ and $|A\cap N_{k_i}| \geq 2$ for $i = 1,\dots,s$. Suppose that any two distinct events $A\cap N_i$ are disjoint. Then

$$m(A) = (-1)^{|A|}\left[(r+s-1)a + bP_0\left(A\cap \bigcup_{i=1}^r N_{j_i}\right)\right]. \tag{22}$$

Proof. From Equation (19),

$$\begin{aligned} m(A) &= \sum_{\substack{\emptyset\neq B\subseteq A \\ B\nsubseteq A\cap N_{j_i}, i=1,\dots,r \\ B\nsubseteq A\cap N_{k_i}, i=1,\dots,s}} (-1)^{|A\setminus B|}\underline{P}(B) + \sum_{i=1}^r \sum_{B\subseteq A\cap N_{j_i}} (-1)^{|A\setminus B|}\underline{P}(B) + \sum_{i=1}^s \sum_{B\subseteq A\cap N_{k_i}} (-1)^{|A\setminus B|}\underline{P}(B) \\ &= \sum_{\substack{\emptyset\neq B\subseteq A \\ B\nsubseteq A\cap N_{j_i}, i=1,\dots,r \\ B\nsubseteq A\cap N_{k_i}, i=1,\dots,s}} (-1)^{|A\setminus B|}\underline{P}(B) \\ &= \sum_{\emptyset\neq B\subseteq A} (-1)^{|A\setminus B|}(bP_0(B)+a) - \sum_{i=1}^r \sum_{\emptyset\neq B\subseteq A\cap N_{j_i}} (-1)^{|A\setminus B|}(bP_0(B)+a) - \sum_{i=1}^s \sum_{\emptyset\neq B\subseteq A\cap N_{k_i}} (-1)^{|A\setminus B|}(bP_0(B)+a), \end{aligned} \tag{23}$$

using Lemma 16 for the last equality.

Observe that $r + s = m \geq 2$. Assume next that $r \geq 1$; if $r = 0$ then $s \geq 1$ and a similar reasoning can be made. Using Equation (20) and Lemma 13 for the first term, and recalling the derivation of (b) and (a) in the proof of Proposition 15 for the second and third terms, respectively, the three terms in Equation (23) are equal to

$$\begin{aligned} &(-1)^{|A|+1}a - [(-1)^{|A|-1}\sum_{i=1}^r (bP_0(A\cap N_{j_i}) + a) + (-1)^{|A|+1}a\cdot s] \\ &= (-1)^{|A|+1}a - (-1)^{|A|-1}b\sum_{i=1}^r P_0(A\cap N_{j_i}) - (-1)^{|A|-1}a\cdot r - (-1)^{|A|+1}a\cdot s \\ &= -(-1)^{|A|-1}(r-1)a + (-1)^{|A|}b\sum_{i=1}^r P_0(A\cap N_{j_i}) + (-1)^{|A|}a\cdot s \end{aligned}$$

$$= (-1)^{|A|}\left[(r+s-1)a + bP_0(A \cap \left(\bigcup_{i=1}^{r} N_{j_i}\right))\right].$$

This completes the proof. □

Proposition 17 applies in particular when the maximal null events are pairwise disjoint. In that case, it allows us to completely determine the mass function of $\underline{P}$. To see an example, suppose that for a given VBM $\underline{P}$ is zero on all and only on all atomic events (whence $a < 0$). Then Proposition 17 implies that we have $r = n, s = 0$, and Equation (22) gives $\forall A \neq \Omega, |A| \geq 2$,

$$m(A) = (-1)^{|A|}[(n-1)a + bP_0(A)] = (-1)^{|A|}[(n-2)a + (a + bP_0(A))] = (-1)^{|A|}[(n-2)a + \underline{P}(A)].$$

Therefore, for $m(A)$ to be non-negative, it should hold that $(n-2)a \geq -\underline{P}(A)$ if $|A|$ is even and $(n-2)a \leq -\underline{P}(A)$ if $|A|$ is odd, and this for every $A \neq \Omega$.

It is also possible to relax the assumptions of Proposition 17 to derive further formulae for $m(A)$. However, these become increasingly complex and the advantage over the general formula (19) tends to fade. We give an idea of this through an example.

Example 5. Let $m = 3, r = 1, s = 2$ in Proposition 17, and further $|A \cap N_1| = 1$ (hence $|A \cap N_i| \geq 2$ for $i = 2, 3$). Then Equation (22) gives $m(A) = (-1)^{|A|}(2a + bP_0(A \cap N_1))$.

Now suppose instead that $A \cap N_2 \cap N_3 \neq \emptyset$. Proposition 17 no longer obtains, but we may modify Equation (23) in its proof adding the new term

$$\sum_{\emptyset \neq B \subseteq A \cap N_2 \cap N_3} (-1)^{|A \setminus B|}(bP_0(B) + a),$$

to take account of the terms subtracted twice when $A \cap N_2 \cap N_3 \neq \emptyset$. We can continue and obtain a variant of Equation (22), whose exact expression depends on whether $|A \cap N_2 \cap N_3| = 1$ or $|A \cap N_2 \cap N_3| \geq 2$. In the former case, the added term is equal to $(-1)^{|A|-1}(bP_0(A \cap N_2 \cap N_3) + a)$, making the structure of $m(A)$ more complex. And the complexity may further increase in the general situations: with more than three maximal null events we may for instance have to take account of the cases where $(A \cap N_i \cap N_j) \cap (A \cap N_k \cap N_h) \neq \emptyset$ (i, j, h, k all distinct). The complete solution would then require a sort of inclusion-exclusion mechanism, not simpler than Equation (19). ♦

7. Connection with other families of imprecise probability models

Next, we investigate the connection between VBMs and other families of imprecise probability models.

As established in [7, Proposition 4.1], the lower probability $\underline{P}$ of a VBM is always 2-monotone. However, not every 2-monotone lower probability is representable by a VBM; to see an example, note that, as established in [19, Section 6.2.1], the COR model on events determines a 2-monotone lower probability, and, as we have shown in Example 1, it is not a VBM in general. In this respect, note that any VBM should satisfy conditions (7) and (15), and these are not always satisfied by 2-monotone lower probabilities. In fact, in itself the property of 2-monotonicity is not dependent on the functional form of the transformation of P_0 giving $\underline{P}$ in a VBM. The simplest way to see this is to recall (see [32, p. 58] or [30, Proposition 6.9]) that any coherent lower probability on a three element space is 2-monotone, while not all of them are representable by a VBM.

Furthermore, there exist even lower probabilities obtained from a probability measure P_0 by the same rule as VBMs, but with different constraints on the parameters a and b, that may be (coherent and) 2-monotone. They are those Horizontal Barrier Models that satisfy some additional requirements, see [7, Section 5.2].

On the other hand, it was also characterised in [7, Proposition 7.3] under which conditions a VBM corresponds to a probability interval. In this section, we shall analyse under which conditions it corresponds to other particular cases of 2-monotone models: belief functions and minitive measures.

Recall that a lower probability $\underline{P}$ is a *belief function* if and only if for every $p \in \mathbb{N}$ and every $A_1, \ldots, A_p \in \mathcal{P}(\Omega)$,

$$\underline{P}\left(\bigcup_{i=1}^{p} A_i\right) \geq \sum_{J \subseteq \{1,\ldots,p\}} (-1)^{|J|-1}\underline{P}\left(\bigcap_{i \in J} A_i\right).$$

Let an agent assign a VBM (P_0, a, b), with P_0 defined on Ω.

7.1. Belief functions

We begin by giving a number of conditions that guarantee that the lower probability of a VBM is a belief function. A necessary and sufficient condition for this is that its mass function m is non-negative. Hence, using the results in Section 6 we can immediately see that VBMs will not determine a belief function in general: indeed, given a non-null VBM $\underline{P}$ on a possibility space of cardinality $|\Omega| \geq 4$, it follows from Proposition 14 that $m(A) = a$ for every A with $|A| = 3$; this means that the *only* non-null VBMs that determine a belief function are the LV models associated with a strictly positive probability P_0, for which $a = 0$. On the other hand, if $|\Omega| = 3$

then we deduce from Proposition 14 that $m(A) \geq 0$ for $|A| = 1, 2$ and $m(\Omega) = 1 - b$, so a non-null VBM induces a belief function if and only if $b \leq 1$.[3]

In fact, any LV model determines a belief function, even when zero lower probabilities are involved. On the other hand, when $a + b = 0$, it follows that $\underline{P}$ is the vacuous lower probability, that is a belief function. Taking all this into account, in the remainder of this section we shall assume that $a + b > 0 > a$.

Recall from Equation (4) that any VBM $\underline{P}$ satisfies $\underline{P}(A) = (a+b)\underline{P}_{PMM}(A)$ for every $A \neq \Omega$, where $\underline{P}_{PMM}$ is the PMM associated with the probability measure P_0 and the parameter $\delta = -\frac{a}{a+b}$. We shall refer to this $\underline{P}_{PMM}$ as the PMM *associated with* the VBM $\underline{P}$. From the correspondence between VBM and PMM, it is immediate to establish that:

Lemma 18.

(a) *For every $A \subseteq \Omega$, $\underline{P}(A) > 0 \Leftrightarrow \underline{P}_{PMM}(A) > 0$, and for $A \neq \Omega$, $m(A) = (a+b)m_{PMM}(A) \lesseqqgtr 0 \Leftrightarrow m_{PMM}(A) \lesseqqgtr 0$.*
(b) *$m(\Omega) = 1 - (a+b)(1 - m_{PMM}(\Omega))$.*
(c) *$m_{PMM}(\Omega) \geq 0 \Leftrightarrow m(\Omega) \geq 1 - (a+b)$.*

Proof. (a) Trivial.
(b) This is a consequence of the first statement together with the equality $1 = \sum_{B\subseteq\Omega} m(B) = m(\Omega) + \sum_{B\subset\Omega} m(B) = m(\Omega) + (a+b)(1 - m_{PMM}(\Omega))$.
(c) The second statement together with $a + b > 0$ implies that $m_{PMM}(\Omega) = \frac{m(\Omega)-(1-(a+b))}{a+b}$, from which the result follows. ☐

In the next theorem (only) we suppose that $P_0(\{\omega_i\}) > 0\ \forall i$. Further, and following [18], we define the *non-vacuity* index of $\underline{P}$ by

$$k = \min\{|A| : \underline{P}(A) > 0\}.$$

We shall denote $\mathcal{N}^c = \{A \subseteq \Omega : \underline{P}(A) > 0\}$ and $\mathcal{N}^c_{PMM} = \{A \subseteq \Omega : \underline{P}_{PMM}(A) > 0\}$, where $\underline{P}_{PMM}$ is the PMM associated with $\underline{P}$. Using this notion and the connection between VBM and PMM, we can give sufficient and necessary conditions for $\underline{P}$ to be a belief function:

Theorem 19. *Let $P_0(\{\omega_i\}) > 0\ \forall i$.*

(a) *The following are sufficient conditions for the lower probability $\underline{P}$ of a VBM to be a belief function:*
   (i) *$k = n - 1$ and $\sum_{\omega\in\Omega} \underline{P}(\{\omega\}^c) \leq 1$.*
   (ii) *There exists a unique event B such that $|B| = k < n - 1$ and $\mathcal{N}^c = \{A \subseteq \Omega : A \supseteq B\}$.*
   (iii) *There exists a unique event B such that $|B| = k - 1 < n - 2, bP_0(B) + a = 0$ and $\mathcal{N}^c = \{A \subseteq \Omega : A \supset B\}$.*
(b) *If $m(\Omega) \geq 1 - (a+b)$ and $\underline{P}$ is not vacuous, then it is necessary that one of (i)–(iii) holds for $\underline{P}$ to be a belief function.*

Proof. (a) Let us show that each of the conditions (i)–(iii) is sufficient for $\underline{P}$ to be a belief function.
   (i) In this case it is $m(A) = 0$ for every A with $|A| < n - 1$, $m(A) = \underline{P}(A)$ for every A with $|A| = n - 1$ and $m(\Omega) = 1 - \sum_{\omega\in\Omega} m(\{\omega\}^c) = 1 - \sum_{\omega\in\Omega} \underline{P}(\{\omega\}^c) \geq 0$, so $\underline{P}$ is a belief function.
   (ii) Let $\underline{P}_{PMM}$ be the PMM associated with $\underline{P}$. It follows that $\underline{P}_{PMM}$ satisfies the conditions of [18, Proposition 8], whence $m_{PMM}(A) \geq 0$ for every A and $m_{PMM}(\Omega) = 0$. Applying Lemma 18, we deduce that $m(A) \geq 0$ for every $A \neq \Omega$ and $m(\Omega) = 1 - (a+b) \geq 0$.
   (iii) The condition $bP_0(B) + a = 0$ is equivalent to the following condition in the associated PMM:

$$\delta = -\frac{a}{a+b} = \frac{bP_0(B)}{-bP_0(B) + b} = \frac{P_0(B)}{1 - P_0(B)}.$$

   It follows that $\underline{P}_{PMM}$ satisfies the conditions in [18, Proposition 9], whence $m_{PMM}(A) \geq 0$ for every A and $m_{PMM}(\Omega) = 0$. Applying Lemma 18, we deduce again that $m(A) \geq 0$ for every $A \neq \Omega$ and $m(\Omega) = 1 - (a+b) > 0$.
(b) Assume next that $\underline{P}$ is a non-vacuous belief function. This means that its non-vacuity index k is strictly smaller than n. There are a number of possibilities:
   - If $k = n - 1$, then $\sum_{\omega\in\Omega} \underline{P}(\{\omega\}^c) = \sum_{\omega\in\Omega} m(\{\omega\}^c) \leq 1$, taking into account that $\sum_{A\subseteq\Omega} m(A) = 1$ and that $m(A) \geq 0$ for every A. Thus, condition (i) holds.
   - Assume next that $k < n - 1$. Since $\underline{P}$ is a belief function, we have that $m_{PMM}(A) \geq 0, \forall A \neq \Omega$, by Lemma 18 (a). Further, the assumption $m(\Omega) \geq 1 - (a+b)$ implies that $m_{PMM}(\Omega) \geq 0$ by Lemma 18 (c). Therefore, the following equivalence holds:

$$\underline{P} \text{ is a belief function } \Leftrightarrow \underline{P}_{PMM} \text{ is.}$$

   But since the non-vacuity index k is the same for the VBM $\underline{P}$ and its associated PMM $\underline{P}_{PMM}$, it follows from [18, Theorem 2] that, if $\underline{P}_{PMM}$ is a belief function and $k < n - 1$, there are two alternatives. Either there exists a unique B with $|B| = k$ and

[3] Observe that when $|\Omega| = 2$ any coherent lower probability, and in particular any VBM, non-null or not, is a belief function, and is a particular case of a LV model.

such that $\mathcal{N}^c_{PMM} = \{A \subseteq \Omega : A \supseteq B\} = \mathcal{N}^c$, and we are in case (ii); or there is a unique B with $|B| = k-1$ and such that $\mathcal{N}^c_{PMM} = \{A \subseteq \Omega : A \supset B\} = \mathcal{N}^c$. In this latter case, [18, Theorem 2] implies that $\delta = -\frac{a}{a+b} = \frac{P_0(B)}{1-P_0(B)}$, which is equivalent to $bP_0(B) + a = 0$. Therefore, we are in case (iii). □

Note that the necessity part in this proposition depends on the assumption $m(\Omega) \geq 1-(a+b)$, which in turn is equivalent to $m_{PMM}(\Omega) \geq 0$. It is not difficult to find VBMs that are belief functions for which the associated PMM is not:

Example 6. Let $\Omega = \{\omega_1, \omega_2, \omega_3, \omega_4\}, b = 0.7, a = -0.3$ and P_0 the probability measure determined by the mass function $(0.24, 0.26, 0.25, 0.25)$. These assignments correspond to the VBM given by:

$$\underline{P}(\{\omega_i\}) = 0 \ \forall i = 1,\dots,4, \quad \underline{P}(\{\omega_1,\omega_3\}) = \underline{P}(\{\omega_1,\omega_4\}) = 0.043$$
$$\underline{P}(\{\omega_1,\omega_2\}) = \underline{P}(\{\omega_3,\omega_4\}) = 0.05, \quad \underline{P}(\{\omega_2,\omega_3\}) = \underline{P}(\{\omega_2,\omega_4\}) = 0.057$$
$$\underline{P}(\{\omega_1,\omega_2,\omega_3\}) = \underline{P}(\{\omega_1,\omega_2,\omega_4\}) = 0.225$$
$$\underline{P}(\{\omega_1,\omega_3,\omega_4\}) = 0.218, \quad \underline{P}(\{\omega_2,\omega_3,\omega_4\}) = 0.232.$$

From these values we deduce that the mass function m is given by:

$$m(\{\omega_i\}) = 0 \ \forall i = 1,\dots,4, \quad m(\{\omega_i,\omega_j\}) = \underline{P}(\{\omega_i,\omega_j\}) \geq 0 \forall i \neq j$$
$$m(\{\omega_1,\omega_2,\omega_3\}) = m(\{\omega_1,\omega_2,\omega_4\}) = 0.075$$
$$m(\{\omega_1,\omega_3,\omega_4\}) = 0.082, \quad m(\{\omega_2,\omega_3,\omega_4\}) = 0.068, \quad m(\Omega) = 0.4.$$

Thus, $\underline{P}$ is a belief function. However, if we consider the associated PMM it is $m_{PMM}(\Omega) = -0.5$, meaning that the latter is not a belief function; and indeed we observe that none of the conditions (i)–(iii) holds. ♦

In the case of cardinality three, it is not difficult to find a sufficient condition for a VBM to induce a belief function:

Proposition 20. *Let $\Omega = \{\omega_1, \omega_2, \omega_3\}$ and consider a VBM on $\mathcal{P}(\Omega)$ associated with a probability measure P_0 and two parameters $a < 0 < b$ such that $a + b > 0$. Then $\underline{P}$ is a belief function if $b \leq 1$.*

Proof. We shall proceed by determining the mass function of $\underline{P}$ in the different alternatives:

Case 1: $\underline{P}(\{\omega_i\}) > 0$ for $i = 1,2,3$. In that case, we get:
- $m(\{\omega_i\}) = bP_0(\{\omega_i\}) + a > 0$ $(i = 1,2,3)$.
- $m(\{\omega_i,\omega_j\}) = -a \geq 0 \ \forall i \neq j$.
- $m(\Omega) = 1 - b \geq 0 \Leftrightarrow b \leq 1$.

Case 2: $\exists!\, \omega_i : \underline{P}(\{\omega_i\}) = 0$. Assume w.l.o.g. $i = 1$. Then the mass function of $\underline{P}$ is:
- $m(\{\omega_1\}) = 0, m(\{\omega_j\}) = bP_0(\{\omega_j\}) + a > 0$ $(j = 2,3)$.
- $m(\{\omega_1,\omega_2\}) = m(\{\omega_1,\omega_3\}) = bP_0(\{\omega_1\}), m(\{\omega_2,\omega_3\}) = -a \geq 0$.
- $m(\Omega) = 1 - (b + bP_0(\{\omega_1\}) + a) \geq 1 - b \geq 0$ if $b \leq 1$.

Case 3: $\exists!\, \omega_i : \underline{P}(\{\omega_i\}) > 0$. Assume w.l.o.g. $i = 3$. Then the mass function of $\underline{P}$ is:
- $m(\{\omega_1\}) = 0 = m(\{\omega_2\}), m(\{\omega_3\}) = bP_0(\{\omega_3\}) + a > 0$.
- $m(\{\omega_1,\omega_3\}) = bP_0(\{\omega_1\}), m(\{\omega_2,\omega_3\}) = bP_0(\{\omega_2\}), m(\{\omega_1,\omega_2\}) = \underline{P}(\{\omega_1,\omega_2\}) \geq 0$.
- $m(\Omega) = 1 - (b + a + \underline{P}(\{\omega_1,\omega_2\}) \geq 1 - b \geq 0$ if $b \leq 1$. The first inequality is immediate if $\underline{P}(\{\omega_1,\omega_2\}) = 0$, otherwise $1 - b - a - \underline{P}(\{\omega_1,\omega_2\}) = 1 - b - a - bP_0(\{\omega_1\}) - bP_0(\{\omega_2\}) - a \geq 1 - b$, because $-a - bP_0(\{\omega_i\}) \geq 0$ for $i = 1,2$.

Case 4: Finally, if $\underline{P}(\{\omega_i\}) = 0$ for $i = 1,2,3$, we get:
- $m(\{\omega_i\}) = 0$ $(i = 1,2,3)$.
- $m(\{\omega_i,\omega_j\}) = \underline{P}(\{\omega_i,\omega_j\}) \geq 0 \ \forall i \neq j$.
- $m(\Omega) = 1 - \sum_{i \neq j} \underline{P}(\{\omega_i,\omega_j\}) \geq 1 - b \geq 0$ if $b \leq 1$. Here, the first inequality can be verified by considering all the possible cases, and reasoning analogously to case 3. □

To see that this sufficient condition is not necessary, observe that there are cases with cardinality 3 where the PMM, that is a particular case of VBM with $a + b = 1$, induces a belief function (see the comments after [10, Proposition 5]).

To investigate further the role of belief functions within VBMs, we give next a condition assuring that $\underline{P}$ is *not* a belief function. The result can be intuitively regarded as a consequence of Proposition 14. Note that in the following proposition we allow P_0 to take zero values.

Proposition 21. *Let $\underline{P}$ be a VBM on $\mathcal{P}(\Omega)$ such that P_0 is an arbitrary probability (not necessarily strictly positive). If there exist $A_1, A_2, A_3 \in \mathcal{P}(\Omega)$ such that $A_i \cap A_j = \emptyset$, $\forall i \neq j$, $\underline{P}(A_i) > 0$, $i = 1,2,3$, $A = A_1 \cup A_2 \cup A_3 \neq \Omega$, then $\underline{P}$ is not a belief function.*

Proof. Consider a partition Ω_c coarser than Ω, such that A_1, A_2, A_3 are atomic events of Ω_c, and let $\underline{P}_c$ be the restriction of $\underline{P}$ on $\mathcal{P}(\Omega_c)$ and m_c its mass function.

Clearly, if $\underline{P}$ is a belief function on $\mathcal{P}(\Omega)$ also $\underline{P}_c$ is on $\mathcal{P}(\Omega_c)$. Thus, we shall show that $\underline{P}_c$ is not a belief function to prove the thesis. Compute then

$$\begin{aligned} m_c(A_1 \cup A_2 \cup A_3) &= bP_0(A_1 \cup A_2 \cup A_3) + a \\ &\quad - (bP_0(A_1 \cup A_2) + a + bP_0(A_1 \cup A_3) + a + bP_0(A_2 \cup A_3) + a) \\ &\quad + bP_0(A_1) + a + bP_0(A_2) + a + bP_0(A_3) + a \\ &= a < 0 \end{aligned}$$

Hence, $\underline{P}_c$ is not a belief function. □

In terms of (P_0, a, b), the condition in this proposition implies that $P_0(A_i) > -\frac{a}{b}$ for $i = 1, 2, 3$, from which it follows that it can only hold when $-\frac{3a}{b} < 1$. Proposition 21 also shows us that:

(a) the sufficient condition for belief functions $b \leq 1$ in Proposition 20 does not extend to cardinalities higher than 3.
(b) Belief functions are characterized when $m(\Omega) \geq 1 - (a + b)$ in Theorem 19. When $k < n - 1$, the set $\mathcal{N}^c = \{A \in \mathcal{P}(\Omega) : \underline{P}(A) > 0\}$ must be either (Theorem 19 (ii)) a filter generated by B or (Theorem 19 (iii)) 'nearly', in the sense that B does not belong to the filter.
When $m(\Omega) < 1 - (a + b)$, these constraints do not necessarily apply, as Example 6 shows. Yet, the path to obtain a belief function remains narrow: Proposition 21 requires (implicitly) that $n \geq 4$, but its other hypotheses are rather mild: it suffices that three disjoint, non-exhaustive events are given positive lower probability.

7.2. Maxitive measures

The lower probability $\underline{P}$ of a VBM is minitive when

$$\underline{P}(A_1 \cap A_2) = \min\{\underline{P}(A_1), \underline{P}(A_2)\} \ \forall A_1, A_2 \subseteq \Omega.$$

This is equivalent to determining when the conjugate upper probability $\overline{P}$ of $\underline{P}$ is maxitive, i.e., whether

$$\overline{P}(A_1 \cup A_2) = \max\{\overline{P}(A_1), \overline{P}(A_2)\} \ \forall A_1, A_2 \subseteq \Omega;$$

we shall focus on this second condition in this section. Let us then consider the upper probability of the VBM determined by a probability measure P_0 and two parameters $b > 0$ and $c \in [0, 1]$ by means of Equations (2) and (3).

Proposition 22. *Let $\overline{P}$ be the upper probability of a VBM.*

$$\overline{P} \text{ maxitive} \Leftrightarrow |\{\omega : P_0(\{\omega\}) > 0, \overline{P}(\{\omega\}) < 1\}| \leq 1.$$

Proof. Assume first of all that $P_0(\{\omega\}) > 0$ for every $\omega \in \Omega$, and let us establish that in that case

$$\overline{P} \text{ maxitive} \Leftrightarrow |\{\omega : \overline{P}(\{\omega\}) < 1\}| \leq 1. \tag{24}$$

(⇒) Let $\overline{P}$ be maxitive and assume ex-absurdo the existence of different ω_1, ω_2 in Ω satisfying $\overline{P}(\{\omega_1\}), \overline{P}(\{\omega_2\}) < 1$. Then

$$\max\{\overline{P}(\{\omega_1\}), \overline{P}(\{\omega_2\})\} = \max\{bP_0(\{\omega_1\}) + c, bP_0(\{\omega_2\}) + c\} < bP_0(\{\omega_1, \omega_2\}) + c,$$

taking into account that $P_0(\{\omega_1\}), P_0(\{\omega_2\}) > 0$ by assumption. Since on the other hand we also have that $\max\{\overline{P}(\{\omega_1\}, \overline{P}(\{\omega_2\})\} < 1$, it follows that

$$\max\{\overline{P}(\{\omega_1\}), \overline{P}(\{\omega_2\})\} < \min\{bP_0(\{\omega_1, \omega_2\}) + c, 1\} = \overline{P}(\{\omega_1, \omega_2\}),$$

a contradiction with the maxitivity of $\overline{P}$.

(⇐) If $\overline{P}(\omega) = 1$ for every $\omega \in \Omega$, then $\overline{P}$ is the vacuous upper probability, that is maxitive. On the other hand, if there is a unique ω' with $\overline{P}(\{\omega'\}) < 1$, then it follows by monotonicity of $\overline{P}$ that

$$\overline{P}(A) = \begin{cases} \overline{P}(\{\omega\}) & \text{if } A = \{\omega\} \\ 1 & \text{otherwise,} \end{cases}$$

which is also maxitive.

Consider next the general case. Let $\Omega^* = \{\omega \in \Omega : P_0(\{\omega\}) > 0\}$ denote the support of P_0. Then for any $A \subset \Omega$, it holds that

$$\overline{P}(A) = \min\{bP_0(A) + c, 1\} = \min\{bP_0(A \cap \Omega^*) + c, 1\} = \overline{P}(A \cap \Omega^*).$$

Let us define the upper probability $\overline{P}^* : \mathcal{P}(\Omega^*) \to [0,1]$ by $\overline{P}^*(A) = \overline{P}(A)$. Note that $\overline{P}^*(\Omega^*) = \overline{P}(\Omega^*) = \min\{b+c, 1\} = \min\{1-a, 1\} = 1$, and therefore $\overline{P}^*$ can be seen as a VBM on Ω^*, determined by the restriction of P_0 and the parameters b, c. Now,

$$\begin{aligned}
\overline{P} \text{ maxitive } &\Leftrightarrow \overline{P}(A_1 \cup A_2) = \max\{\overline{P}(A_1), \overline{P}(A_2)\} \forall A_1, A_2 \subseteq \Omega \\
&\Leftrightarrow \overline{P}(A_1 \cup A_2) = \max\{\overline{P}(A_1), \overline{P}(A_2)\} \forall A_1, A_2 \subset \Omega \\
&\Leftrightarrow \overline{P}((A_1 \cup A_2) \cap \Omega^*) = \max\{\overline{P}(A_1 \cap \Omega^*), \overline{P}(A_2 \cap \Omega^*)\} \forall A_1, A_2 \subset \Omega \\
&\Leftrightarrow \overline{P}^*(B_1 \cup B_2) = \max\{\overline{P}(B_1), \overline{P}(B_2)\} \forall B_1, B_2 \subseteq \Omega^* \\
&\Leftrightarrow \overline{P}^* \text{ maxitive} \\
&\Leftrightarrow |\{\omega \in \Omega^* : \overline{P}^*(\{\omega\}) < 1\}| \leq 1 \\
&\Leftrightarrow |\{\omega \in \Omega^* : \overline{P}(\{\omega\}) < 1\}| \leq 1,
\end{aligned}$$

where the one-but-last equivalence follows from Equation (24). □

Note that, by definition of the upper probability of a VBM, it holds that

$$\overline{P}(\{\omega\}) \in (0,1) \Leftrightarrow \overline{P}(\{\omega\}) \in (c,1) \Leftrightarrow bP_0(\{\omega\}) + c < 1 \Leftrightarrow P_0(\omega) < \frac{1-c}{b}.$$

The limited relevance of maxitive measures within VBMs is patent from Proposition 22: when P_0 is strictly positive on all non-trivial events, a VBM is maxitive iff it is vacuous or nearly, i.e. with non-vacuous imprecise probability assessment on at most one event.

On the other hand, note that the condition in Proposition 22 is *not* equivalent to $|\{\omega \in \Omega : \overline{P}(\{\omega\}) < 1\}| \leq 1$, as we can see in our next example:

Example 7. Let $\Omega = \{\omega_1, \omega_2, \omega_3\}$ and consider the VBM determined by the mass function $P_0 = (0.7, 0.3, 0)$ and the parameters $b = 1, a = -0.4$. Applying Equation (2), its upper probability is given by:

A	$\{\omega_1\}$	$\{\omega_2\}$	$\{\omega_3\}$	$\{\omega_1,\omega_2\}$	$\{\omega_1,\omega_3\}$	$\{\omega_2,\omega_3\}$
$\overline{P}(A)$	1	0.7	0.4	1	1	0.7

Then $\overline{P}$ is maxitive. However, $\{\omega : \overline{P}(\{\omega\}) < 1\} = \{\omega_2, \omega_3\}$. ⧫

8. Marginalisation and independence

In this section, we investigate the behaviour of VBMs in a multivariate setting. A similar study for the particular cases of PMM or LV models was carried out in [10].

Our first analysis is on the procedure of *marginalisation*, by which we move from a coherent lower probability $\underline{P}$ on a product space $\Omega^{\mathcal{X}} \times \Omega^{\mathcal{Y}}$ to its associated marginal models $\underline{P}^{\mathcal{X}}, \underline{P}^{\mathcal{Y}}$, given by

$$\underline{P}^{\mathcal{X}}(A) := \underline{P}(A \times \Omega^{\mathcal{Y}}) \ \forall A \subseteq \Omega^{\mathcal{X}}, \underline{P}^{\mathcal{Y}}(B) := \underline{P}(\Omega^{\mathcal{X}} \times B) \ \forall B \subseteq \Omega^{\mathcal{Y}}.$$

It is not difficult to establish that the marginal models are still within the VBM family. Indeed, if $\underline{P}$ is the VBM associated with (P_0, a, b), we obtain that

$$\underline{P}^{\mathcal{X}}(A) := \underline{P}(A \times \Omega^{\mathcal{Y}}) = \max\{bP_0(A \times \Omega^{\mathcal{Y}}) + a, 0\} = \max\{bP_0^{\mathcal{X}}(A) + a, 0\}$$

where $P_0^{\mathcal{X}}$ denotes the $\Omega^{\mathcal{X}}$-marginal of P_0. In other words, $\underline{P}^{\mathcal{X}}$ corresponds to the VBM determined by $(P_0^{\mathcal{X}}, a, b)$. A similar result can be established for $\underline{P}^{\mathcal{Y}}$.

Concerning independent products, in [10] two avenues were considered: the first consists in distorting two marginal probability measures $P_0^{\mathcal{X}}, P_0^{\mathcal{Y}}$ and then in considering the strong product $\underline{P}^{\mathcal{X}} \boxtimes \underline{P}^{\mathcal{Y}}$ of the resulting lower probabilities $\underline{P}^{\mathcal{X}}, \underline{P}^{\mathcal{Y}}$; the second instead contemplates first the independent product $P_0^{\mathcal{X}} \times P_0^{\mathcal{Y}}$ as a precise model on $\Omega^{\mathcal{X}} \times \Omega^{\mathcal{Y}}$ and then distorts it suitably. It was already established in [10] that neither of these approaches is more precise than the other. What we shall establish next is that the strong product of two VBMs is not a VBM in general:

Example 8. Consider the spaces $\Omega^{\mathcal{X}} = \{\omega_1, \omega_2\}$ and $\Omega^{\mathcal{Y}} = \{\omega'_1, \omega'_2\}$ and the probability measures $P_0^{\mathcal{X}}, P_0^{\mathcal{Y}}$ given by $P_0^{\mathcal{X}}(\{\omega_1\}) = 0.3, P_0^{\mathcal{X}}(\{\omega_2\}) = 0.7$, $P_0^{\mathcal{Y}}(\{\omega'_1\}) = 0.5 = P_0^{\mathcal{Y}}(\{\omega'_2\})$. Let $\underline{P}^{\mathcal{X}}, \underline{P}^{\mathcal{Y}}$ the LV models determined by $P_0^{\mathcal{X}}, P_0^{\mathcal{Y}}$ and $\delta = 0.1$ (in the context of VBMs, by $b = 0.9$ and $a = 0$). It was shown in [10, Example 7] that

$$\underline{P}^{\mathcal{X}} \boxtimes \underline{P}^{\mathcal{Y}}(\{\omega_1\} \times \{\omega'_1\}) = \underline{P}^{\mathcal{X}} \boxtimes \underline{P}^{\mathcal{Y}}(\{\omega_1\} \times \{\omega'_2\}) = 0.1215$$

and $\underline{P}^{\mathcal{X}}(\{\omega_1\}) = 0.27$. If $\underline{P}^{\mathcal{X}} \boxtimes \underline{P}^{\mathcal{Y}}$ were a VBM, it would follow applying Equation (7) with $A = \{\omega_1\} \times \{\omega'_1\}$ and $B = \{\omega_1\} \times \{\omega'_2\}$ that $-a = 0.27 - 0.1215 - 0.1215 = 0.027$. On the other hand,

$$\underline{P}^{\mathcal{X}} \boxtimes \underline{P}^{\mathcal{Y}}(\{\omega_2\} \times \{\omega'_1\}) = \underline{P}^{\mathcal{X}} \boxtimes \underline{P}^{\mathcal{Y}}(\{\omega_2\} \times \{\omega'_2\}) = 0.2835$$

and $\underline{P}^{\mathcal{X}}(\{\omega_2\}) = 0.63$. If $\underline{P}^{\mathcal{X}} \boxtimes \underline{P}^{\mathcal{Y}}$ were a VBM, then applying Equation (7) with $A = \{\omega_2\} \times \{\omega'_1\}$ and $B = \{\omega_2\} \times \{\omega'_2\}$ we should deduce that $-a = 0.63 - 0.567 = 0.063$. These two equations are incompatible, and as a consequence the strong product $\underline{P}^{\mathcal{X}} \boxtimes \underline{P}^{\mathcal{Y}}$ is not a VBM. ♦

Note that, even if in the example above we have focused on the strong product $\underline{P}^{\mathcal{X}} \boxtimes \underline{P}^{\mathcal{Y}}$ of two VBMs $\underline{P}^{\mathcal{X}}, \underline{P}^{\mathcal{Y}}$, the key in the example lies in the factorisation properties of this product, that are also satisfied by other products such as the independent or Kuznetsov; thus, the family of VBMs would not be closed under those procedures either. We refer to [6] for a more detailed analysis of independence within imprecise probabilities.

9. Inner and outer approximations with VBMs

Finally, we investigate how VBMs can be involved in the problem of approximating a coherent lower probability with another one.

9.1. Inner and outer approximations by VBMs

In this section we look at the problem of transforming a coherent lower probability $\underline{P}$ into a VBM $\underline{Q}$ that is close to it with a minimal variation of imprecision with respect to the original model. We shall consider two possibilities: *inner* approximations, that correspond to the case where $\underline{Q} \geq \underline{P}$, and *outer* approximations, where $\underline{Q} \leq \underline{P}$.

A first approach would be to consider those approximations that are *Pareto optimal* in terms of dominance, in the sense that there is no other approximation that can be placed between $\underline{Q}$ and $\underline{P}$. In the case of inner approximations, this means that there is no other VBM $\underline{Q}'$ satisfying $\underline{Q} \gneq \underline{Q}' \geq \underline{P}$, while for outer approximations there should be no other VBM $\underline{Q}'$ satisfying $\underline{Q} \lneq \underline{Q}' \leq \underline{P}$. However, this criterion alone does not single out one approximation, as we shall see later on in this section. It does not even entail a minimum variation of imprecision for its admissible approximations: in fact, it does not specify what variation of imprecision should mean.

For this reason, and following the work in [16,21,22], we shall consider the approximation that is *closest* to the original model, and for this we need to fix a distance between lower probabilities. In this paper, we consider the distance employed by Baroni and Vicig in [1]:

$$d_{BV}(\underline{P}, \underline{Q}) = \sum_{A \subseteq \Omega} |\underline{P}(A) - \underline{Q}(A)|.$$

This distance aggregates the discrepancy between the lower probabilities of all non-trivial events. It can be viewed as the overall variation of imprecision when replacing $\underline{P}$ with $\underline{Q}$. When $\underline{Q}$ is an inner approximation, it can be interpreted as a measure of the imprecision in $\underline{P}$ that is removed when considering $\underline{Q}$, while if $\underline{Q}$ is an outer approximation of $\underline{P}$ the value $d_{BV}(\underline{P}, \underline{Q})$ measures the extra imprecision that is incorporated in our model by using $\underline{Q}$. While this is not the only possible way of comparing two lower probabilities, it has the sensible property in our view of being a linear function of the discrepancies across all possible events.

The problem of outer approximating a coherent lower probability by a more tractable model was investigated in [16,21,22]. On the other hand, inner approximations were considered in [5,14,15]. We refer to those papers for motivation and additional discussion.

Concerning inner approximations, it is not difficult to characterise the optimal inner approximations in terms of the parameters a, b under some conditions:

Proposition 23. *Let $\underline{P}$ be a coherent lower probability on $\mathcal{P}(\Omega)$ satisfying $\underline{P}(A) > 0$ for every $A \neq \emptyset$. Then a VBM $\underline{Q}$ determined by (P_0, a, b) that inner approximates $\underline{P}$ minimises $d_{BV}(\underline{P}, \underline{Q})$ if and only if it minimises $b + 2a$.*

Proof. It suffices to observe that, if $\underline{Q}$ is an inner approximation of $\underline{P}$,

$$\begin{aligned} d_{BV}(\underline{P}, \underline{Q}) &= \sum_{A \subseteq \Omega} |\underline{P}(A) - \underline{Q}(A)| = \sum_{A \subseteq \Omega} (\underline{Q}(A) - \underline{P}(A)) = -\sum_{\emptyset \neq A \subset \Omega} \underline{P}(A) + \sum_{\emptyset \neq A \subset \Omega} (bP_0(A) + a) \\ &= -\sum_{\emptyset \neq A \subset \Omega} \underline{P}(A) + b(2^{n-1} - 1) + a(2^n - 2) = -\sum_{\emptyset \neq A \subset \Omega} \underline{P}(A) + (2^{n-1} - 1)(b + 2a). \end{aligned}$$

Thus, $d_{BV}(\underline{P}, \underline{Q})$ is minimised if and only if $b + 2a$ is minimised. □

This proposition allows us to establish that the optimal inner approximation is not unique in general:

Example 9. Let $\Omega = \{\omega_1, \omega_2, \omega_3\}$ and let $\underline{P}$ be the coherent lower probability given by:

A	$\{\omega_1\}$	$\{\omega_2\}$	$\{\omega_3\}$	$\{\omega_1,\omega_2\}$	$\{\omega_1,\omega_3\}$	$\{\omega_2,\omega_3\}$	Ω
$\underline{P}(A)$	0.15	0.2	0.2	0.5	0.5	0.45	1
$m_{\underline{P}}(A)$	0.15	0.2	0.2	0.15	0.15	0.05	0.1

To see that it is coherent, it suffices to observe that its mass function, given in the same table, is non-negative, and therefore that $\underline{P}$ is a belief function.

Let $\underline{Q}$ be a VBM that is an inner approximation of $\underline{P}$. Since $\underline{P}(A) > 0$, it must be $\underline{Q}(A) = bP_0(A) + a$ for some probability measure P_0 and parameters $a \leq 0, b > 0, a + b \in [0, 1]$. Applying Equation (15), it must be

$$\underline{Q}(\{\omega_2\}) + \underline{Q}(\{\omega_1, \omega_3\}) = b + 2a \geq 0.7 = \underline{P}(\{\omega_2\}) + \underline{P}(\{\omega_1, \omega_3\}).$$

To see that this bound can be attained by more than one inner approximation, let $\underline{Q}_1$ be the VBM determined by the probability measure $(0.3, 0.35, 0.35)$ and the parameters $a = -0.15, b = 1$, and let $\underline{Q}_2$ be the VBM determined by $(\frac{1}{3}, \frac{1}{3}, \frac{1}{3})$ and $a = -0.1, b = 0.9$. They are given by

A	$\{\omega_1\}$	$\{\omega_2\}$	$\{\omega_3\}$	$\{\omega_1,\omega_2\}$	$\{\omega_1,\omega_3\}$	$\{\omega_2,\omega_3\}$
$\underline{Q}_1(A)$	0.15	0.2	0.2	0.5	0.5	0.55
$\underline{Q}_2(A)$	0.2	0.2	0.2	0.5	0.5	0.5

Applying Proposition 23, we conclude that both $\underline{Q}_1, \underline{Q}_2$ are optimal inner approximations. ♦

Remark 2. Observe that any inner approximation that minimises $d_{BV}(\underline{P}, \underline{Q})$ is also Pareto optimal, in the sense that there is no other VBM $\underline{Q}'$ such that $\underline{P} \leq \underline{Q}' \lneq \underline{Q}$: if that were the case, it would ensue that $d_{BV}(\underline{P}, \underline{Q}') < d_{BV}(\underline{P}, \underline{Q})$, a contradiction. Then Example 9 allows us to conclude also that there is no unique inner approximation with respect to Pareto dominance.

We can also use the example to show that there may be an infinite number of optimal inner approximations: for any $0 < \alpha < 1$ the coherent lower probability $\underline{Q}_\alpha = \alpha \underline{Q}_1 + (1 - \alpha)\underline{Q}_2$ is a VBM by Proposition 12, it satisfies $\underline{Q}_\alpha \geq \underline{P}$ and $\sum_{A \neq \emptyset, \Omega} \underline{Q}_\alpha(A) = \sum_{A \neq \emptyset, \Omega} \underline{Q}_i(A) = 2.1$ for $i = 1, 2$, so that $d_{BV}(\underline{P}, \underline{Q}_\alpha) = d_{BV}(\underline{P}, \underline{Q}_i) = 0.1$. ♦

We shift our attention now to outer approximations. It was established in [21, Propositions 7 and 8] that any coherent lower probability $\underline{P}$ has a unique outer approximation in the family of PMM or LV models. As we shall see, this is not necessarily the case for VBMs.

With a similar reasoning to that of Proposition 23, we can characterise the outer approximations by non-null VBMs:

Proposition 24. *Let $\underline{P}$ be a coherent lower probability on $\mathcal{P}(\Omega)$ satisfying $\underline{P}(A) > 0$ for every $A \neq \emptyset$. Then a non-null VBM $\underline{Q}$ determined by (P_0, a, b) that outer approximates $\underline{P}$ minimises $d_{BV}(\underline{P}, \underline{Q})$ if and only if it maximises $b + 2a$.*

Proof. The reasoning is similar to that of Proposition 23, observing that now we should minimise $-\sum_{A \subset \Omega} \underline{Q}(A) = -(2^{n-1} - 1)(b + 2a)$, and that this is then equivalent to maximising $b + 2a$. □

Let us now establish that there may be more than one optimal outer approximation:

Example 10. Consider again the coherent lower probability $\underline{P}$ of Example 9. Any non-null VBM $\underline{Q}$ that outer approximates $\underline{P}$ will satisfy

$$b + 2a = \underline{Q}(\{\omega_1\}) + \underline{Q}(\{\omega_2, \omega_3\}) \leq \underline{P}(\{\omega_1\}) + \underline{P}(\{\omega_2, \omega_3\}) = 0.6$$

from which

$$\begin{aligned} d_{BV}(\underline{P}, \underline{Q}) &\geq \underline{P}(\{\omega_2\}) + \underline{P}(\{\omega_1, \omega_3\}) + \underline{P}(\{\omega_3\}) + \underline{P}(\{\omega_1, \omega_2\}) - \underline{Q}(\{\omega_2\}) - \underline{Q}(\{\omega_1, \omega_3\}) - \underline{Q}(\{\omega_3\}) - \underline{Q}(\{\omega_1, \omega_2\}) \\ &\geq 0.7 + 0.7 - 0.6 - 0.6 = 0.2. \end{aligned}$$

This distance is attained by more than one VBM: in particular, by $\underline{Q}_1$, the TV model determined by the probability measure $(0.35, 0.35, 0.3)$, $a = -0.2$ and $b = 1$, and by $\underline{Q}_2$, the TV model determined by the probability measure $(0.35, 0.3, 0.35)$, $a = -0.2, b = 1$. These are given by

A	$\{\omega_1\}$	$\{\omega_2\}$	$\{\omega_3\}$	$\{\omega_1,\omega_2\}$	$\{\omega_1,\omega_3\}$	$\{\omega_2,\omega_3\}$
$\underline{Q}_1(A)$	0.15	0.15	0.1	0.5	0.45	0.45
$\underline{Q}_2(A)$	0.15	0.1	0.15	0.45	0.5	0.45

Therefore, in this case there is no unique optimal outer approximation of $\underline{P}$ in the subfamily of non-null VBMs. Let us establish that in this case we cannot find any closer outer approximation in the family of VBMs, i.e., that $\min\{d_{BV}(\underline{P},\underline{Q}) : \underline{Q} \text{ is a VBM}\} = 0.2$. To see that this is the case, let $\underline{Q}$ be a VBM that outer approximates $\underline{P}$. If $\underline{Q}(A) = 0$ for some $A \neq \{\omega_1\}$, then

$$d_{BV}(\underline{P},\underline{Q}) \geq \underline{P}(A) - \underline{Q}(A) = \underline{P}(A) \geq 0.2.$$

On the other hand, if $\underline{Q}(\{\omega_1\}) = 0$ and $d_{BV}(\underline{P},\underline{Q}) \leq 0.2$, then necessarily it must be $\sum_{B\neq\{\omega_1\}}(\underline{P}(B) - \underline{Q}(B)) \leq 0.05$. This implies

$$\underline{Q}(\{\omega_1,\omega_2\}) + \underline{Q}(\{\omega_1,\omega_3\}) + \underline{Q}(\{\omega_2,\omega_3\}) = 2b + 3a \leq \underline{P}(\{\omega_1,\omega_2\}) + \underline{P}(\{\omega_1,\omega_3\}) + \underline{P}(\{\omega_2,\omega_3\}) = 1.45$$

Moreover, since

$$\underline{P}(\{\omega_1,\omega_2\}) + \underline{P}(\{\omega_1,\omega_3\}) + \underline{P}(\{\omega_2,\omega_3\}) - \underline{Q}(\{\omega_1,\omega_2\}) - \underline{Q}(\{\omega_1,\omega_3\}) - \underline{Q}(\{\omega_2,\omega_3\}) \leq \sum_{B\neq\{\omega_1\}} (\underline{P}(B) - \underline{Q}(B)) \leq 0.05,$$

it follows that

$$\underline{Q}(\{\omega_1,\omega_2\}) + \underline{Q}(\{\omega_1,\omega_3\}) + \underline{Q}(\{\omega_2,\omega_3\}) \geq \underline{P}(\{\omega_1,\omega_2\}) + \underline{P}(\{\omega_1,\omega_3\}) + \underline{P}(\{\omega_2,\omega_3\}) - 0.05 = 1.4.$$

Therefore, $2b + 3a \in [1.4, 1.45]$.

With a similar reasoning, we can conclude that

$$\underline{Q}(\{\omega_2\}) + \underline{Q}(\{\omega_1,\omega_3\}) = b + 2a \in [0.65, 0.7];$$

From $2b + 3a \in [1.4, 1.45]$ and $b + 2a \in [0.65, 0.7]$ it follows that $b + a \in [0.7, 0.8]$, or equivalently, that $2b + 2a \in [1.4, 1.6]$; this, again together with $2b + 3a \in [1.4, 1.45]$, implies that $a \in [-0.15, 0]$. Since

$$\begin{aligned}\underline{Q}(\{\omega_1\}) = 0 &\Rightarrow b \cdot P_0(\{\omega_1\}) \leq -a \leq 0.15\\ 0 < \underline{Q}(\{\omega_2\}) \leq \underline{P}(\{\omega_2\}) &\Rightarrow b \cdot P_0(\{\omega_2\}) \leq 0.2 - a \leq 0.35\\ 0 < \underline{Q}(\{\omega_3\}) \leq \underline{P}(\{\omega_3\}) &\Rightarrow b \cdot P_0(\{\omega_3\}) \leq 0.2 - a \leq 0.35\end{aligned}$$

it follows that $b \leq 0.85$, and since $b + 2a \in [0.65, 0.7]$ it ensues that $a \in [-0.1, 0]$. With a similar reasoning,

$$\begin{aligned}&b \cdot P_0(\{\omega_1\}) \leq -a \leq 0.1\\ &b \cdot P_0(\{\omega_2\}) \leq 0.2 - a \leq 0.3\\ &b \cdot P_0(\{\omega_3\}) \leq 0.2 - a \leq 0.3,\end{aligned}$$

whence $b \leq 0.7$. But $b + a \in [0.7, 0.8]$ means that it should be $a = 0, b = 0.7$, whence

$$\underline{Q}(\{\omega_1\}) = \min\{bP_0(\{\omega_1\}), 0\} = 0 \Rightarrow P_0(\{\omega_1\}) = 0,$$

and then

$$0.7 = b = bP_0(\{\omega_2\}) + bP_0(\{\omega_3\} = \underline{Q}(\{\omega_2\}) + \underline{Q}(\{\omega_3\}) \leq \underline{P}(\{\omega_2\}) + \underline{P}(\{\omega_3\}) = 0.4,$$

a contradiction. We conclude that $\min\{d_{BV}(\underline{P},\underline{Q}) : \underline{Q} \text{ is a VBM}\} = 0.2$ and as a consequence that $\underline{Q}_1, \underline{Q}_2$ are two different optimal inner approximations. This allows to prove that there are infinitely (actually uncountably) many optimal VBM outer approximations: it suffices to observe that for any $0 < \alpha < 1$ $\underline{Q}_\alpha = \alpha\underline{Q}_1 + (1-\alpha)\underline{Q}_2$ is a VBM by Proposition 12, that $\underline{Q}_\alpha \leq \underline{P}$ and that $\sum_{A\neq\emptyset,\Omega} \underline{Q}_\alpha(A) = \sum_{A\neq\emptyset,\Omega} \underline{Q}_i(A) = 1.8$ for $i = 1, 2$, so that $d_{BV}(\underline{P},\underline{Q}_\alpha) = d_{BV}(\underline{P},\underline{Q}_i) = 0.2$. ♦

Remark 3. The comments at the end of the example above, as well as those in Remark 2, point out towards a more general result: if we consider two non-null VBMs $\underline{Q}_1, \underline{Q}_2$ such that $d_{BV}(\underline{P},\underline{Q}_1) = d_{BV}(\underline{P},\underline{Q}_2) = \mu$, then for any $\alpha \in (0, 1)$ the coherent lower probability $\underline{Q}_\alpha = \alpha\underline{Q}_1 + (1-\alpha)\underline{Q}_2$ shall also be a VBM by Proposition 12. Moreover, it shall satisfy $d_{BV}(\underline{P},\underline{Q}_\alpha) = \mu$, being therefore an optimal inner (alternatively, outer) approximation of $\underline{P}$ when both $\underline{Q}_1, \underline{Q}_2$ are. ♦

The above example may lead us to believe that the optimal outer approximations of a non-null coherent lower probability in the family of VBMs are always non-null VBMs. However, it is possible to show that this is not the case in general. It does not seem easy to characterise in which cases the optimal outer approximations are non-null. Another pending issue would be looking for characterisations of the optimal inner and outer approximations by VBMs in terms of the parameters a, b even when zero lower probabilities are involved.

On the other hand, and reasoning similarly to Remark 2, it follows that any outer approximation $\underline{Q}$ that minimises the distance with respect to $\underline{P}$ will also be Pareto optimal, in that there will be no other lower probability $\underline{Q}'$ such that $\underline{Q} \lneq \underline{Q}' \leq \underline{P}$. As a consequence, Example 10 shows that the set of Pareto optimal outer approximations of a coherent lower probability in the family of VBMs may have more than one element.

These results may then lead us to look for procedures to single out a unique inner or outer approximation; following the ideas in [16], one option would be to consider those that minimise the quadratic distance, possibly after first focusing on those that minimise d_{BV}. Other strictly convex distances may also be of interest in this regard.

9.2. Outer approximations of VBMs

Unlike the preceding section, a VBM plays here the opposite role of being the imprecise probability model to be outer approximated. In this section, we shall investigate its outer approximations by means of either a PMM or a LV model. In both cases, we specialise to VBMs general results for coherent lower probabilities, stated in [21] and giving explicit formulae for these approximations, that are also unique.

It has to be observed that performing such approximations does not operationally produce notable simplifications: the VBM is already a rather simple model itself. We are interested towards the PMM and LV approximations mainly because they let us explore from a new perspective how much a VBM differs from its special cases. This complements our previous investigations on this issue, starting from the behavioural interpretations discussed in Section 2.

Let $\underline{P}$ be a VBM (P_0, a, b) on a space Ω with $|\Omega| = n$ and let $\overline{P}$ be its conjugate. Let us consider first the outer approximation of $\overline{P}$ by a PMM upper probability $\overline{P}_{PMM}$. Let us assume that for this approximation $\overline{P}(\{\omega\}) < 1\ \forall \omega \in \Omega$. Applying the results in [21, Section 4.2] and recalling that by Equation (3) $c = 1 - (a + b)$ we obtain

$$(1+\delta) = \sum_{\omega\in\Omega} \overline{P}(\{\omega\}) = b + nc,\ \ P_0^{PMM}(\{\omega\}) = \frac{\overline{P}(\{\omega\})}{\sum_{\omega\in\Omega}\overline{P}(\{\omega\})},$$

so that, for every $A \subset \Omega$,

$$\overline{P}_{PMM}(A) = \min\{(1+\delta)P_0^{PMM}(A), 1\} = \min\left\{\sum_{\omega\in A}\overline{P}(\{\omega\}), 1\right\} = \min\{bP_0(A) + |A|c, 1\} \tag{25}$$

and $\overline{P}_{PMM}(\Omega) = 1$.

To evaluate the quality of this approximation, we observe that the more the parameter c differs from 0, the more the original VBM deviates from a PMM and the more its PMM outer approximation is 'vague', in the sense that it gives $\overline{P}_{PMM}(A) = 1$ to a larger set of events A. This conclusion is supported by the following:

Proposition 25. *Given the VBM (P_0, a, b), if $a + b \leq \frac{1}{2}$ then $\overline{P}_{PMM}(A) = 1$ for every A with $|A| \geq 2$.*

Proof. From (25) we have that for every $A \neq \Omega$

$$\overline{P}_{PMM}(A) = 1 \Leftrightarrow |A| \geq \frac{1 - bP_0(A)}{c} = \frac{1 - bP_0(A)}{1-(a+b)}.$$

Suppose $|A| \geq 2$. Then the inequality above applies when it holds that $2 \geq \frac{1-bP_0(A)}{1-(a+b)}$, or, equivalently, when $P_0(A) \geq \frac{2(a+b)-1}{b}$. This holds automatically when $a + b \leq \frac{1}{2}$ since then $\frac{2(a+b)-1}{b} \leq 0$. □

Note that the condition $a + b \leq \frac{1}{2}$ is not particularly restrictive. On the other hand, since $\overline{P}_{PMM}(\{\omega\}) = \overline{P}(\{\omega\})$ in any optimal PMM outer approximation [21, Section 4.2], the PMM in the hypotheses of Proposition 25 is the vaguest possible such optimal approximation.

Let us turn now to the LV approximation. As shown in [21, Section 4.3], this type of approximation is not feasible[4] if $\sum_{\omega\in\Omega}\underline{P}(\{\omega\}) = 0$, while being otherwise unique.

Let us suppose then that $\sum_{\omega\in\Omega}\underline{P}(\{\omega\}) > 0$. Applying [21, Proposition 8], and defining, for any $A \subseteq \Omega$

$$A^+ = \{\omega \in A : \underline{P}(\{\omega\}) > 0\},$$

[4] Of course, there is always a feasible outer approximation if we consider the vacuous lower probability as a LV model; however, in [21] the focus is on non-trivial LV mixtures, i.e., in those where the weights of the linear and the vacuous model are both non-zero.

the optimal outer approximation is given by

$$\underline{P}_{LV}(A) = \sum_{\omega\in A} \underline{P}(\{\omega\}) = \sum_{\omega\in A^*} \underline{P}(\{\omega\}) = \sum_{\omega\in A^+} (bP_0(\{\omega\} + a) = bP_0(A^+) + a|A^+|, \tag{26}$$

and $\underline{P}_{LV}(\Omega) = 1$.

Let us evaluate this kind of outer approximation of $\overline{P}$. We observe what follows:

- Unlike PMM outer approximations, it is not possible that the LV approximation is 'too vague'. In fact, it may be that $\underline{P}(A) > 0$ and $\underline{P}_{LV}(A) = 0$: by Equation (26) it is $\underline{P}_{LV}(A) = 0$ iff $\underline{P}(\{\omega\}) = 0$ for every $\omega \in A$, which in turn is equivalent to $A^+ = \emptyset$. Yet, and in contradistinction with Proposition 25, it cannot be that $\underline{P}_{LV}(A) = 0$ for every A with $|A| \geq 2$, because this would imply that $\underline{P}(\{\omega\}) = 0$ for every ω and therefore that the LV approximation does not exist.
- A VBM is LV iff $a = 0$, and the more a deviates from 0, the less similar the VBM is from a LV model. What are the implications for the LV outer approximation? Take A with $\underline{P}(A) > 0$. The *approximation error* $\underline{P}(A) - \underline{P}_{LV}(A)$ is bounded below as follows:

$$\begin{aligned}\underline{P}(A) - \underline{P}_{LV}(A) &= bP_0(A) + a - bP_0(A^+) - a|A^+| \\ &\geq bP_0(A^+) + a - bP_0(A^+) - a|A^+| \\ &= -a(|A^+| - 1) > 0.\end{aligned}$$

  Thus, the greater imprecision of the approximating $\underline{P}_{LV}$ compared to $\underline{P}$ increases with $-a > 0$ (and with $|A^+| - 1$). Note however that a larger $-a$ increases also, for given b and P_0, the number of $\underline{P}$-null events, which are obviously also $\underline{P}_{LV}$-null and therefore subject to no approximation error.

Summarising, we see that both outer approximations tend to perform better when the VBM is 'close' to a PMM or a LV model, respectively. In the opposite case, the (non-trivial) LV approximation may simply not exist while the PMM does if $\overline{P}(\{\omega\}) < 1 \ \forall\omega \in \Omega$, but may be very vague.

10. Conclusions

Vertical Barrier Models have been originally introduced as distortion models, i.e. as functions of a given probability P_0. We have seen in this paper how they can be interpreted as neighbourhood models, originated from a neighbourhood of P_0 by means of a suitable distorting function. The complexity of the credal set of a VBM has been discussed in terms of the maximum number of its extreme points. Perhaps surprisingly, the bound we found is the same as for TV models, a proper subfamily of VBMs. Mixtures of VBMs are, under some conditions, still VBMs, while disjunction and conjunction operations do not retain this closure property. In the case of mixtures, a sufficient condition is the equality $\mathcal{N}_1 = \mathcal{N}_2$. We make this assumption in Proposition 12(a) and it holds in particular if the two VBMs to be combined are non-null. More generally, the interpretation of the assumption is that the opinions of two experts may differ on all events but those given null lower probability, in order for the mixture to be a VBM again.

Several features of VBMs that are (or are not) belief functions have been detected. Although VBMs that are belief functions are certainly not the rule, there is anyway some more flexibility with respect to the already well known subcase of Pari-Mutuel Models. Nevertheless, a complete characterisation of belief functions within VBMs is still to be determined.

Tables 1 and 2 summarise some of our results and establish a comparison with the properties of the PMM, LV and TV models:

Table 1
Summary of some properties of VBMs.

	Conj.	Disj.	Mixtures	Belief?
PMM	YES	NO	If non-null	[18, Thm. 2] (⇔)
LV	YES	NO	YES	YES
TV	NO	NO	If non-null	NO
VBM	NO	NO	Proposition 12(a) (sufficient)	Theorem 19 (sufficient)
Non-null VBM	NO	NO	YES	Only if LV or $\|\Omega\| = 3, b \leq 1$

Concerning the maximum number of extreme points of $\mathcal{M}(\underline{P})$ in terms of $n = |\Omega|$, the bounds are as in Table 2.

We provide a characterisation of the Möbius inverse of non-null VBMs in Proposition 14, as well as its expression in other cases of VBMs. In general, the technique we followed is advantageous over the standard formulae for computing the Möbius inverse as long as the structure of zero lower probability events remains simple.

With respect to the inner and outer approximations of coherent lower probabilities by means of VBMs, we can characterise the optimal inner and outer approximations in the subfamily of non-null VBMs, but not in general. Nevertheless, even in that case there is no unique solution, and we may have to consider the results discussed in [16] to choose between the different solutions.

Table 2
Complexity of the credal set of a VBM.

	Maximum number of extreme points of $\mathcal{M}(\underline{P})$
PMM	$\frac{n!}{\lfloor \frac{n}{2} \rfloor \lfloor \frac{n}{2}-1 \rfloor ! \lceil \frac{n}{2}+1 \rceil !}$
LV	n
TV	$\frac{n!}{\lfloor \frac{n}{2}-1 \rfloor ! \lceil \frac{n}{2}-1 \rceil !}$
VBM	$\frac{n!}{\lfloor \frac{n}{2}-1 \rfloor ! \lceil \frac{n}{2}-1 \rceil !}$
Non-null VBM	$n(n-1)$

It is also interesting to remark that, as we have established in Section 8, the family of VBMs is closed under marginalisation but not under independence. Similar results for some particular subfamilies were already established in [10].

Finally, explicit formulae for outer approximating a VBM by means of a PMM or a LV model are supplied. The related discussion highlights that VBMs cannot always be adequately approximated by these submodels of theirs. This fact, together with the reasoning on the behavioural interpretation of VBMs in Section 2, confirms that VBMs cannot be always replaced by their (more well known) submodels without losing something in terms of uncertainty representations. On the other hand, VBMs are not really much more complex than these submodels. This appears already from their expression and is confirmed by our results on their representation as neighbourhood models and on the cardinality of the extreme points of their credal sets.

CRediT authorship contribution statement

Enrique Miranda: Conceptualization, Formal analysis, Investigation, Methodology, Writing – original draft, Writing – review & editing. **Renato Pelessoni:** Conceptualization, Formal analysis, Investigation, Methodology, Writing – original draft, Writing – review & editing. **Paolo Vicig:** Conceptualization, Formal analysis, Investigation, Methodology, Writing – original draft, Writing – review & editing.

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Data availability

No data was used for the research described in the article.

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