Available online at www.sciencedirect.com
ScienceDirect
Fuzzy Sets and Systems 471 (2023) 108678
FUZZY sets and systems
www.elsevier.com/locate/fss

Simplicial depths for fuzzy random variables

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Received 7 June 2022; received in revised form 29 June 2023; accepted 7 August 2023
Available online 11 August 2023

Abstract

The recently defined concept of statistical depth function for fuzzy sets provides a theoretical framework for ordering fuzzy sets with respect to the distribution of a fuzzy random variable. One of the most used and studied statistical depth functions for multivariate data is simplicial depth, based on multivariate simplices. We introduce a notion of pseudosimplices generated by fuzzy sets and propose three generalizations of simplicial depth to fuzzy sets. Their theoretical properties are analyzed and the behavior of the proposals is illustrated through a study of both synthetic and real data.

Keywords: Fuzzy data; Fuzzy random variable; Nonparametric statistics; Statistical depth; Simplicial depth

1. Introduction

In the general framework of fuzzy data, data are classes of objects with continuous membership [35]. They are generally represented as functions from $\mathbb{R}^p$ to $[0, 1]$, whereas multivariate data are points in $\mathbb{R}^p$. Statistical depth functions are a quantification of the intuitive notion that the median is the point that is most 'in the middle', providing a center-outward ordering of the points in a space with respect to a probability distribution or dataset. Thus, statistical depth opens an avenue for extending rank-based and quantile-based statistical procedures from the real line to more complex spaces. While ordering data is trivial in the real line, in the sense that moving outward is just going towards $-\infty$ or ∞, it becomes harder for multivariate data (and even harder for more complex types of data) as no natural total order is present. To understand some of the challenges involved in ordering elements of a space with dimension higher than one, let us consider the first idea one might have to generalize the univariate median: applying the coordinate-wise median to obtain a multivariate median in $\mathbb{R}^p$. The coordinate-wise median may lie outside the convex hull of the data, which is against the idea that the median should be as much 'in the middle' of the data as possible. Moreover, by changing the coordinate system (which does not affect the data themselves, only how we represent them) the coordinate-wise median of the dataset can be modified. Even in simple cases, like the vertices of an equilateral triangle

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https://doi.org/10.1016/j.fss.2023.108678

and its center of mass, the coordinate-wise median fails to provide the intuitive solution that the innermost point is the center of mass [6].

Tukey [33] first introduced depth for multivariate data, computing the depth of a point x as the least fraction of data in some halfspace containing x. Some pre-existent notions in multivariate analysis can be expressed in the language of depth. For instance, Mahalanobis distance gives rise to Mahalanobis depth and the notion of convex hull of a set of points to convex hull peeling depth [2]. Liu [13] introduced simplicial depth, which is one of the best known and most popular depth functions, based on the notion of a simplex generated by a set of points. She proved a number of nice properties which then inspired Zuo and Serfling's abstract definition of statistical depth function, constituted by a list of desirable properties [37]. In intuitive terms, these are as follows:

(M1) *Affine invariance.* A change of coordinates should not affect the depth values.
(M2) *Maximality at the center of symmetry.* If a distribution is symmetric, the deepest point should be the center of symmetry.
(M3) *Monotonicity from the deepest point.* Depth values should decrease along any ray that departs from a deepest point.
(M4) *Vanishing at infinity.* The depth value of x should go to 0 as its norm goes to infinity.

It should be underlined that these are not clear-cut axioms. Failing to satisfy some property, or doing so only under some conditions, is not considered enough for a function to be excluded from being a depth function.

Today, the number of depth functions runs in the dozens and this is a broad and active topic in non-parametric statistics. With the rise of functional data analysis and the apparition of several adaptations of multivariate depth notions to the functional setting, Nieto-Reyes and Battey proposed a list of desirable properties for depth functions in functional (metric) spaces [21]. They also proposed an instance of depth satisfying all those properties in [22] which was later applied to a real data analysis in [23]. The connections between depth functions and fuzzy sets were noted by Terán [30,31], who showed that some depth functions can be rigorously interpreted as fuzzy sets and *vice versa*. In [8] we proposed two definitions of statistical depth for fuzzy data; although fuzzy sets are functions, these definitions list desirable properties tailored to fuzzy sets. We also generalized Tukey depth as a first example of depth for fuzzy data satisfying the proposed properties. Sinova [28] also considered depth for fuzzy data and defined depth-trimmed means. She applied depth functions for functional data to fuzzy sets via their support functions, functions that we also use. However, she did not propose a general definition of depth in the fuzzy setting. Thus, she did not check whether the properties constituting the definition were satisfied, which is a major objective of this paper.

It is important to show that most of the relevant examples of depth can be adapted to the fuzzy setting. Firstly, to justify the viability of the notions of depth for fuzzy data. Secondly, to create a library of depth functions with guaranteed good theoretical properties in order to apply them in practice. And thirdly, to test the abstract definitions proposed in [8] and understand whether they are fine as they stand or might need to be adjusted. In this paper, we study the problem of adapting Liu's simplicial depth to the fuzzy setting. As mentioned above, it is one of the best known and most used depth functions for multivariate data. For instance, Liu et al. [14] developed techniques to study multivariate distributional characteristics using simplicial depth, and other depth functions. The multivariate definition of simplicial depth assigns to each point $x \in \mathbb{R}^p$ the probability that x lies in the convex hull of $p+1$ independent observations. Provided the distribution is continuous, with probability 1, those observations define a p-dimensional simplex (a triangle when $p = 2$, a tetrahedron when $p = 3$, and so on). That p-dimensional simplex has a non-empty interior, which may contain x or not. If x is very outlying in the distribution, the probability that the simplex contains x is very small. Thus x is deeper insofar as, loosely speaking, it is more likely that the data points in a small sample 'capture' x among them.

When extending this notion to functional data, López-Pintado and Romo [16] already realized that using the convex hull to determine which functions are 'among' other functions is naive. We face similar problems in the fuzzy case. In the end, the convex hull of finitely many points is a finite-dimensional set, so in any infinite-dimensional space the vast majority of the elements in the space will be excluded from it. This creates a propensity to assign zero depth which will require an adaptation in line with that in [16]. Another obstacle is that some multivariate definitions do not transfer immediately to the fuzzy setting. For instance, Tukey depth is based on the notion of a halfspace but spaces of fuzzy sets, not being linear spaces, cannot be 'halved' by hyperplanes and a workaround is needed [8]. In this case, simplicial depth rests on the notion of simplex in $\mathbb{R}^p$, which, as will be discussed, also needs a workaround. That results in a plurality of ways to extend simplicial depth.

The paper is organized as follows. Section 2 contains the required notation and background on fuzzy sets and statistical depth for a comprehensive understanding of the next sections. An operative adaptation of simplices to spaces of sets and fuzzy sets is presented in Section 3. The definitions of the proposed variants of simplicial depth are in Section 4. Their status with respect to the desirable properties in the definitions of depth for fuzzy data [8] is studied in Section 5, assuming that the distribution is 'continuous' in certain sense. Examples with real and simulated data are worked out in Section 6, while a discussion is presented in Section 7. All proofs are deferred to Appendix A.

2. Notation and preliminaries

The following notation is used throughout. The symbol $=^{\mathcal{L}}$ denotes equality in distribution of random variables, and $\mathcal{M}_{p\times p}(\mathbb{R})$ is the set of all $p \times p$ real matrices.

Let $\mathcal{K}_c(\mathbb{R}^p)$ be the class of non-empty compact and convex subsets of $\mathbb{R}^p$. Any set $K \in \mathcal{K}_c(\mathbb{R}^p)$ can be identified with a fuzzy set, its *indicator function* $\mathrm{I}_K : \mathbb{R}^p \to \mathbb{R}$ where $\mathrm{I}_K(x) = 1$ if $x \in K$ and $\mathrm{I}_K(x) = 0$ otherwise. The support function of a set $K \in \mathcal{K}_c(\mathbb{R}^p)$ is the mapping $s_K : \mathbb{S}^{p-1} \to \mathbb{R}$ defined as

$$s_K(u) := \sup\{\langle u, k\rangle : k \in K\}$$

for every $u \in \mathbb{S}^{p-1}$. $\mathbb{S}^{p-1} := \{x \in \mathbb{R}^p : \|x\| \le 1\}$ is the unit sphere of $\mathbb{R}^p$, with $\|.\|$ denoting the Euclidean norm and $\langle\cdot,\cdot\rangle$ denotes the usual inner product in $\mathbb{R}^p$.

A function $A : \mathbb{R}^p \to [0, 1]$ is a *fuzzy set* on $\mathbb{R}^p$ (or a fuzzy subset of $\mathbb{R}^p$). Let $\mathcal{F}_c(\mathbb{R}^p)$ denote the class of all fuzzy sets A on $\mathbb{R}^p$ such that the α-level of A, given by

$$A_\alpha := \{x \in \mathbb{R}^p : A(x) \ge \alpha\}$$

if $\alpha \in (0, 1]$ and the closed support of A if $\alpha = 0$, is non-empty, compact and convex for every $\alpha \in [0, 1]$. To simplify our notation, we will refer to an element of $\mathcal{F}_c(\mathbb{R}^p)$ as a "fuzzy set" in the following discussion. The *support function* of $A \in \mathcal{F}_c(\mathbb{R}^p)$ is the mapping $s_A : \mathbb{S}^{p-1} \times [0, 1] \to \mathbb{R}$ such that $s_A(u, \alpha) := \sup_{v\in A_\alpha}\langle u, v\rangle$ for every $u \in \mathbb{S}^{p-1}$ and $\alpha \in [0, 1]$. In $\mathcal{F}_c(\mathbb{R})$, the subclass of *trapezoidal fuzzy sets* [10, Section 10.7] is used very often. Four values $a, b, c, d \in \mathbb{R}$ with $a \le b \le c \le d$ determine the trapezoidal fuzzy set

$$\mathrm{Tra}(a,b,c,d)(x) := \begin{cases} \dfrac{x-a}{b-a}, & \text{if} \quad a < x < b, \\ 1, & \text{if} \quad b \le x \le c, \\ \dfrac{d-x}{d-c}, & \text{if} \quad c < x < d, \\ 0, & \text{otherwise.} \end{cases}$$

2.1. Arithmetics and Zadeh's extension principle

Let $A, B \in \mathcal{F}_c(\mathbb{R}^p)$ and $\gamma \in \mathbb{R}$. The formulae

$$(A+B)(t) := \sup_{x,y\in\mathbb{R}^p : x+y=t} \min\{A(x), B(y)\}, \text{ and}$$

$$(\gamma\cdot A)(t) := \sup_{x\in\mathbb{R}^p : t=\gamma\cdot x} A(y) = \begin{cases} A\left(\frac{t}{\gamma}\right), & \text{if} \quad \gamma \ne 0 \\ I_{\{0\}}(t), & \text{if} \quad \gamma = 0 \end{cases}$$

valid for arbitrary $t \in \mathbb{R}^p$, define an addition and a product by scalars in $\mathcal{F}_c(\mathbb{R}^p)$.

Given $A, B \in \mathcal{F}_c(\mathbb{R}^p)$, $\gamma \in [0, \infty)$, $u \in \mathbb{S}^{p-1}$ and $\alpha \in [0, 1]$, a useful identity that makes use of these operations is

$$s_{A+\gamma \cdot B}(u, \alpha) = s_A(u, \alpha) + \gamma \cdot s_B(u, \alpha). \tag{2.1}$$

A consequence of this is

$$s_{K+\gamma \cdot L}(u) = s_K(u) + \gamma \cdot s_L(u) \tag{2.2}$$

for any $K, L \in \mathcal{K}_c(\mathbb{R}^p)$, $\gamma \in [0, \infty)$ and $u \in \mathbb{S}^{p-1}$.

Zadeh's extension principle [36] enables the application of a crisp (non-fuzzy) function to fuzzy sets in the following way. Given a (crisp) function $f : \mathbb{R}^p \to \mathbb{R}^p$ and a fuzzy set $A \in \mathcal{F}_c(\mathbb{R}^p)$, the image $f(A) \in \mathcal{F}_c(\mathbb{R}^p)$ is the fuzzy set defined by

$$f(A)(t) := \sup\{A(y) : y \in \mathbb{R}^p, f(y) = t\},$$

for all $t \in \mathbb{R}^p$.

Let us consider $M \in \mathcal{M}_{p\times p}(\mathbb{R})$ be a non-singular matrix and consider the mapping $f : \mathbb{R}^p \to \mathbb{R}^p$ defined by $f(x) = M \cdot x$. Then, if we apply the function f to a fuzzy set $A \in \mathcal{F}_c(\mathbb{R}^p)$ we obtain a new fuzzy set $M \cdot A$ defined as

$$(M \cdot A)(t) = \sup\{A(y) : y \in \mathbb{R}^p, M \cdot y = t\}.$$

By [8, Proposition 8.2],

$$s_{M\cdot A}(u, \alpha) = \|M^T \cdot u\| \cdot s_A\left(\frac{1}{\|M^T \cdot u\|} \cdot M^T \cdot u, \alpha\right) \tag{2.3}$$

for any $A \in \mathcal{F}_c(\mathbb{R}^p)$, non-singular $M \in \mathcal{M}_{p\times p}(\mathbb{R})$, $u \in \mathbb{S}^{p-1}$ and $\alpha \in [0, 1]$.

2.2. *Metrics*

We will use several metrics in $\mathcal{F}_c(\mathbb{R}^p)$. For any fuzzy sets $A, B \in \mathcal{F}_c(\mathbb{R}^p)$,

$$d_r(A, B) := \begin{cases} \left(\int_{[0,1]} \left(d_{\mathcal{H}}(A_\alpha, B_\alpha)\right)^r \mathrm{d}\nu(\alpha)\right)^{1/r} & \text{if} \quad r \in [1, \infty), \\ \sup_{\alpha\in[0,1]} d_{\mathcal{H}}(A_\alpha, B_\alpha) & \text{if} \quad r = \infty, \end{cases}$$

where

$$d_{\mathcal{H}}(S, T) := \max\left\{\sup_{s\in S} \inf_{t\in T} \| s - t \|, \sup_{t\in T} \inf_{s\in S} \| s - t \|\right\}$$

denotes the *Hausdorff metric* and ν the Lebesgue measure in $[0, 1]$. The metric space $(\mathcal{F}_c(\mathbb{R}^p), d_r)$ is non-complete and separable for any $r \in [1, \infty)$, while the metric space $(\mathcal{F}_c(\mathbb{R}^p), d_\infty)$ is non-separable and complete [5]. It is possible to consider L^r-type metrics [5], for instance, given any $A, B \in \mathcal{F}_c(\mathbb{R}^p)$,

$$\rho_r(A, B) := \left(\int_{\mathbb{S}^{p-1}} \int_{[0,1]} |s_A(u, \alpha) - s_B(u, \alpha)|^r \,\mathrm{d}\nu(\alpha)\,\mathrm{d}\mathcal{V}_p(u)\right)^{1/r}$$

where $\mathcal{V}_p$ denotes the normalized Haar measure (uniform probability distribution) in $\mathbb{S}^{p-1}$. The metrics d_r and ρ_r (for the same value of r) are equivalent.

2.3. Fuzzy random variables

There exist different definitions of fuzzy random variables in the literature. Here we consider the Puri's and Ralescu's approach (see [24]). Let $(\Omega, \mathcal{A}, \mathbb{P})$ be a probability space. A *random compact set* [20] is a function $\Gamma : \Omega \to \mathcal{K}_c(\mathbb{R}^p)$ such that $\{\omega \in \Omega : \Gamma(\omega) \cap K \neq \emptyset\} \in \mathcal{A}$ for each $K \in \mathcal{K}_c(\mathbb{R}^p)$. A *fuzzy random variable* [24] is a function $\mathcal{X} : \Omega \to \mathcal{F}_c(\mathbb{R}^p)$ such that $\mathcal{X}_\alpha(\omega)$ is a random compact set for all $\alpha \in [0, 1]$, where the α-level mapping $\mathcal{X}_\alpha : \Omega \to \mathcal{K}_c(\mathbb{R}^p)$ is defined by $\mathcal{X}_\alpha(\omega) := \{x \in \mathbb{R}^p : \mathcal{X}(\omega)(x) \geq \alpha\}$ for any $\omega \in \Omega$. Note that given a random variable $X : \Omega \to \mathbb{R}^p$, composing it with the indicator function gives a fuzzy random variable $I_X : \Omega \to \mathcal{F}_c(\mathbb{R}^p)$.

It is not straightforward from this definition that a fuzzy random variable is a measurable function. If we consider in $\mathcal{F}_c(\mathbb{R}^p)$ the σ-algebra generated by the mappings $L_\alpha : \mathcal{F}_c(\mathbb{R}^p) \to \mathcal{K}_c(\mathbb{R}^p)$, defined by $L_\alpha(A) = A_\alpha$ for any $\alpha \in [0, 1]$, we have that $\mathcal{X}$ is a fuzzy random variable if and only if it is measurable with respect to that measurable space. The σ-algebra considered above is the smallest σ-algebra which makes each L_α measurable. Krätschmer [11] proved that this σ-algebra is the Borel σ-algebra generated by the metrics d_r or ρ_r for any $r \in [1, \infty)$. Let $\mathcal{X} : \Omega \to \mathcal{F}_c(\mathbb{R}^p)$ be a fuzzy random variable, the support function of $\mathcal{X}$ is the function $s_{\mathcal{X}} : \mathbb{S}^{p-1} \times [0, 1] \times \Omega \to \mathbb{R}$ where $s_{\mathcal{X}}(u, \alpha, \omega) := s_{\mathcal{X}(\omega)}(u, \alpha)$, for all $u \in \mathbb{S}^{p-1}$, $\alpha \in [0, 1]$ and $\omega \in \Omega$. Throughout the paper, $(\Omega, \mathcal{A}, \mathbb{P})$ denotes the probabilistic space associated with a general fuzzy random variable $\mathcal{X}$. Let $L^0[\mathcal{F}_c(\mathbb{R}^p)]$ denote the class of all fuzzy random variables on the measurable space $(\Omega, \mathcal{A})$ and $C^0[\mathcal{F}_c(\mathbb{R}^p)] \subseteq L^0[\mathcal{F}_c(\mathbb{R}^p)]$ the class of all fuzzy random variables $\mathcal{X}$ such that $s_{\mathcal{X}}(u, \alpha)$ is a continuous real random variable for each $(u, \alpha) \in \mathbb{S}^{p-1} \times [0, 1]$.

2.4. Fuzzy symmetry and depth: semilinear and geometric depth

Let $\mathcal{X} : \Omega \to \mathcal{F}_c(\mathbb{R}^p)$ be a fuzzy random variable and $A \in \mathcal{F}_c(\mathbb{R}^p)$ a fuzzy set. In [8], we proposed the *F-symmetry* notion for fuzzy random variables: $\mathcal{X}$ is *F-symmetric* with respect to A if, for all $(u, \alpha) \in \mathbb{S}^{p-1} \times [0, 1]$,

$$s_A(u, \alpha) - s_{\mathcal{X}}(u, \alpha) =^{\mathcal{L}} s_{\mathcal{X}}(u, \alpha) - s_A(u, \alpha).$$

It can be checked that the indicator function $I_{\{X\}}$ of a p-dimensional random vector X is F-symmetric if and only if X is a symmetrically distributed random vector.

Let Med be the (possibly multivalued) median operator on real random variables. It is also proved in [8] that, for all $u \in \mathbb{S}^{p-1}$ and $\alpha \in [0, 1]$,

$$s_A(u, \alpha) \in \mathrm{Med}(s_{\mathcal{X}}(u, \alpha)), \text{ if } \mathcal{X} \text{ is } F\text{-symmetric with respect to } A. \tag{2.4}$$

In the sequel, given a real sample $x_1, \ldots, x_n$, $\mathrm{Med}(x_1, \ldots, x_n)$ denotes its median.

Let $\mathcal{H} \subseteq L^0[\mathcal{F}_c(\mathbb{R}^p)]$, $\mathcal{J} \subseteq \mathcal{F}_c(\mathbb{R}^p)$, and let $d : \mathcal{F}_c(\mathbb{R}^p) \times \mathcal{F}_c(\mathbb{R}^p) \to [0, \infty)$ be a metric. The following properties are considered in [8]. In them, A denotes any element of $\mathcal{J}$ such that $D(A; \mathcal{X}) = \sup\{D(B; \mathcal{X}) : B \in \mathcal{J}\}$, i.e., a fuzzy set of maximal depth with respect to the distribution of $\mathcal{X}$.

P1. $D(M \cdot C + B; M \cdot \mathcal{X} + B) = D(C; \mathcal{X})$ for any non-singular matrix $M \in \mathcal{M}_{p\times p}(\mathbb{R})$, any $B, C \in \mathcal{J}$ and any $\mathcal{X} \in \mathcal{H}$.
P2. For (some notion of symmetry and) any symmetric fuzzy random variable $\mathcal{X} \in \mathcal{H}$, $D(U; \mathcal{X}) = \sup_{B\in\mathcal{F}_c(\mathbb{R}^p)} D(B; \mathcal{X})$, where $U \in \mathcal{J}$ is a center of symmetry of $\mathcal{X}$.
P3a. $D(A; \mathcal{X}) \geq D((1 - \lambda) \cdot A + \lambda \cdot B; \mathcal{X}) \geq D(B; \mathcal{X})$ for all $\lambda \in [0, 1]$ and all $B \in \mathcal{F}_c(\mathbb{R}^p)$.
P3b. $D(A; \mathcal{X}) \geq D(B; \mathcal{X}) \geq D(C; \mathcal{X})$ for all $B, C \in \mathcal{J}$ satisfying $d(A, C) = d(A, B) + d(B, C)$.
P4a. $\lim_{\lambda\to\infty} D(A + \lambda \cdot B; \mathcal{X}) = 0$ for all $B \in \mathcal{J} \setminus \{\mathrm{I}_{\{0\}}\}$.
P4b. $\lim_{n\to\infty} D(A_n; \mathcal{X}) = 0$ for every sequence of fuzzy sets $\{A_n\}_n$ such that the $\lim_{n\to\infty} d(A_n, A) = \infty$.

In Property **P2**, F-symmetry is considered in this work. Another notion of symmetry is also proposed in [8]. According to [8], a mapping $D(\cdot; \cdot) : \mathcal{J} \times \mathcal{H} \to [0, \infty)$ is a *semilinear depth function* if it satisfies P1, P2, P3a and P4a for each fuzzy random variable $\mathcal{X} \in \mathcal{H}$. It is a *geometric depth function* with respect to d if it satisfies P1, P2, P3b and P4b for each fuzzy random variable $\mathcal{X} \in \mathcal{H}$. Notice that semilinear depth only depends on the arithmetics of $\mathcal{F}_c(\mathbb{R}^p)$ while geometric depth depends on the choice of a specific metric.

3. Pseudosimplices in $\mathcal{F}_c(\mathbb{R}^p)$

One of the most well-known statistical depth functions for multivariate data is simplicial depth [13]. Simplicial depth is an instance of what Zuo and Serfling [37] called 'Type A depth', i.e., the depth of a point is the probability that it lies in a certain random set constructed from independent and identically distributed copies of the random vector. As such, it is the coverage function of a random set, and a connection to fuzzy sets is immediate [9]. Further examples of Type A depth functions are majority depth [27,14], convex hull peeling depth [2], spherical depth [7], and lens depth [15].

Let $(\Omega, \mathcal{A}, \mathbb{P})$ be the underlying probability space. The simplicial depth of $x \in \mathbb{R}^p$ with respect to a probability distribution P on $\mathbb{R}^p$ is defined to be

$$SD(x; P) := \mathbb{P}\left(\omega \in \Omega : x \in S[X_1(\omega), \dots, X_{p+1}(\omega)]\right), \tag{3.1}$$

where $X_1, \dots, X_{p+1}$ are independent and identically distributed random variables with distribution P and, for any $x_1, \dots, x_{p+1} \in \mathbb{R}^p$, $S[x_1, \dots, x_{p+1}]$ is the set

$$S[x_1, \dots, x_{p+1}] := \left\{\lambda_1 x_1 + \dots + \lambda_{p+1} x_{p+1} : \sum_{i=1}^{p+1} \lambda_i = 1, \lambda_i \geq 0\right\} \tag{3.2}$$

i.e., $S[x_1, \dots, x_{p+1}]$ is the convex hull of the points $x_1, \dots, x_{p+1}$. A characterization of simplices in $\mathbb{R}^p$ is provided in the next result.

Proposition 3.1. *For any $x_1, \dots, x_{p+1} \in \mathbb{R}^p$,*

$$S[x_1, \dots, x_{p+1}] = \left\{x \in \mathbb{R}^p : \langle u, x\rangle \in [m(u), M(u)] \text{ for all } u \in \mathbb{S}^{p-1}\right\},$$

with $m(u) := \min\{\langle u, x_1\rangle, \dots, \langle u, x_{p+1}\rangle\}$ and $M(u) := \max\{\langle u, x_1\rangle, \dots, \langle u, x_{p+1}\rangle\}$.

If the X_i's are affinely independent, $S[X_1, \dots, X_{p+1}]$ is by definition a (random) p-dimensional simplex, which explains the name 'simplicial depth'. Indeed, the X_i's are affinely independent, almost surely, provided that $\mathbb{P}$ assigns zero probability to any lower-dimensional subspace of $\mathbb{R}^p$; which is the case for continuous distributions. In the statistical depth literature, the name 'simplex' reflects the fact that exactly $p + 1$ points are taken for the convex hull, although it can fail to be p-dimensional for an arbitrary distribution $\mathbb{P}$. With this in mind, we will freely call $S[X_1, \dots, X_{p+1}]$ a simplex in the sequel.

Before proposing examples of depth functions inspired by simplicial depth, we study how to adapt simplices to our context. To the best of our knowledge, the literature contains no notion of a simplex in $\mathcal{F}_c(\mathbb{R}^p)$. In [3], however, a *band* generated by compact and convex sets is defined, which coincides with our definition of a pseudosimplex in $\mathcal{K}_c(\mathbb{R}^p)$ (Definition 3.2 below). We analyze it first in order to use it in our proposed definition of a pseudosimplex in $\mathcal{F}_c(\mathbb{R}^p)$.

Definition 3.2. The *pseudosimplex* generated by $A_1, \dots, A_{p+1} \in \mathcal{K}_c(\mathbb{R}^p)$ is

$$S_c[A_1, \dots, A_{p+1}] := \left\{A \in \mathcal{K}_c(\mathbb{R}^p) : s_A(u) \in [m(u), M(u)] \text{ for all } u \in \mathbb{S}^{p-1}\right\},$$

where $m(u) := \min\{s_{A_1}(u), \dots, s_{A_{p+1}}(u)\}$ and $M(u) := \max\{s_{A_1}(u), \dots, s_{A_{p+1}}(u)\}$.

Our justification for Definition 3.2 is that, according to Proposition 3.1, the simplex generated by $p + 1$ points, $x_1, \dots, x_{p+1}$, coincides with the set of points whose projections in every direction $u \in \mathbb{S}^{p-1}$ are in the closed interval generated by the minimum and the maximum of $\{\langle u, x_1\rangle, \dots, \langle u, x_{p+1}\rangle\}$. Thus, replacing in this characterization the inner products by the support function of the elements in $\mathcal{K}_c(\mathbb{R}^p)$ yields Definition 3.2.

As simplices are defined to be subsets of linear spaces, and $\mathcal{K}_c(\mathbb{R}^p)$ and $\mathcal{F}_c(\mathbb{R}^p)$ are not linear but they embed into appropriate linear spaces (e.g., by identifying their elements with support functions), there arises the question whether, after such an embedding, $S_c[A_1, \dots, A_{p+1}]$ becomes an infinite-dimensional simplex [34, Section 1.5, pp. 46–53]. The name 'pseudosimplex' avoids prejudicing the question.

Since the operations of sum and product by a scalar are available in $\mathcal{K}_c(\mathbb{R}^p)$ (Section 2.1), an alternative would be to define the simplex generated by $A_1, \ldots, A_{p+1} \in \mathcal{K}_c(\mathbb{R}^p)$ to be the set of all convex combinations of these generating elements, that is

$$\left\{ A \in \mathcal{K}_c(\mathbb{R}^p) : A = \sum_{i=1}^{p+1} \lambda_i \cdot A_i, \text{ with } \sum_{i=1}^{p+1} \lambda_i = 1 \text{ and } \lambda_i \geq 0 \right\}. \tag{3.3}$$

That corresponds to the convex hull of the set $\{A_1, \ldots, A_{p+1}\}$ when $\mathcal{K}_c(\mathbb{R}^p)$ is regarded as a convex combination space [32]. The next result proves that the convex hull is contained in the corresponding pseudosimplex. Example 3.4 shows that both sets are not necessarily equal.

Proposition 3.3. *For any $A_1, \ldots, A_{p+1} \in \mathcal{K}_c(\mathbb{R}^p)$,*

$$\left\{ A \in \mathcal{K}_c(\mathbb{R}^p) : A = \sum_{i=1}^{p+1} \lambda_i \cdot A_i, \sum_{i=1}^{p+1} \lambda_i = 1, \lambda_i \geq 0 \right\} \subseteq S_c[A_1, \ldots, A_{p+1}].$$

Example 3.4. Let $p = 1$, $A = [0, 1]$ and $B = [3, 4]$. Then

$$S_c[A, B] = \big\{[x, y] : x \in [0, 3], y \in [1, 4]\big\}$$

while the simplex in the sense of Equation (3.3) is

$$S := \{[3\lambda, 1 + 3\lambda] : \lambda \in [0, 1]\}.$$

For instance, $\{2\} \in S_c[A, B]$ but $\{2\} \notin S$.

Our choice of the pseudosimplex, instead of the convex hull simplex in (3.3), is based on cases like the last example. Intuitively, it is hard to deny that $\{2\}$ is between A and B in a definite sense, but it cannot be written as a convex combination of them. In this connection, see Proposition 3.8 below concerning the role of 'betweenness' in the definition of pseudosimplices in the fuzzy case.

We will extend now the notion of a pseudosimplex to the fuzzy case by working α-level by α-level.

Definition 3.5. The *pseudosimplex* generated by $A_1, \ldots, A_{p+1} \in \mathcal{F}_c(\mathbb{R}^p)$ is

$$S_F[A_1, \ldots, A_{p+1}] := \big\{A \in \mathcal{F}_c(\mathbb{R}^p) : A_\alpha \in S_c[(A_1)_\alpha, \ldots, (A_{p+1})_\alpha] \text{ for all } \alpha \in [0, 1]\big\},$$

where $(A_i)_\alpha$ denotes the α-level of A_i.

As fuzzy sets are a generalization of ordinary sets in $\mathbb{R}^p$, it is interesting to underline that the notion of a pseudosimplex generated by crisp sets relates to that of a simplex in the multivariate case. For that, we consider the class of fuzzy sets

$$\mathcal{R}^p := \big\{\mathrm{I}_{\{x\}} \in \mathcal{F}_c(\mathbb{R}^p) : x \in \mathbb{R}^p\big\},$$

which can be identified with $\mathbb{R}^p$.

Proposition 3.6. *For any $x_1, \ldots, x_{p+1} \in \mathbb{R}^p$,*

$$S_F\left[I_{\{x_1\}}, \ldots, I_{\{x_{p+1}\}}\right] \cap \mathcal{R}^p = \big\{I_{\{x\}} : x \in S[x_1, \ldots, x_{p+1}]\big\}.$$

The proof is trivial. A direct implication of the proposition is

$$\big\{\mathrm{I}_{\{x\}} : x \in S[x_1, \ldots, x_{p+1}]\big\} \subsetneq S_F\left[\mathrm{I}_{\{x_1\}}, \ldots, \mathrm{I}_{\{x_{p+1}\}}\right] \tag{3.4}$$

provided there exist $i, j \in \{1, \ldots, p+1\}$ such that $x_i \neq x_j$. Denoting the segment joining x_i and x_j by $\overline{x_i x_j}$, by Definition 3.5, we have

$$\mathrm{I}_{\overline{x_i x_j}} \in S_F\left[\mathrm{I}_{\{x_1\}}, \ldots, \mathrm{I}_{\{x_{p+1}\}}\right].$$

However,

$$\mathrm{I}_{\overline{x_i x_j}} \notin \left\{\mathrm{I}_{\{x\}} : x \in S[x_1, \ldots, x_{p+1}]\right\}$$

because $\overline{x_i x_j}$ is not a single point.

Another implication is that the result in Proposition 3.6 is true for $\mathcal{K}_c(\mathbb{R}^p)$. Let us denote the set of all singletons by $\mathcal{R}_c^p := \{\{x\} \in \mathcal{K}_c(\mathbb{R}^p) : x \in \mathbb{R}^p\}$.

Corollary 3.7. *For any $x_1, \ldots, x_{p+1} \in \mathbb{R}^p$,*

$$S_c[\{x_1\}, \ldots, \{x_{p+1}\}] \cap \mathcal{R}_c^p = \{\{x\} : x \in S[x_1, \ldots, x_{p+1}]\}.$$

One also has for this case the inclusion in (3.4). An example is that of the pseudosimplex generated by $\{0\}$ and $\{3\}$, which contains not only singletons but also sets like the interval $[1, 2]$ which lies entirely in the gap between 0 and 3.

The Ramík–Řimanék partial order in $\mathcal{F}_c(\mathbb{R})$ [25, Definition 3] is given by

$$A_1 \preceq A_2 \Leftrightarrow \inf(A_1)_\alpha \leq \inf(A_2)_\alpha,\ \sup(A_1)_\alpha \leq \sup(A_2)_\alpha \ \ \forall \alpha \in (0, 1].$$

This provides a natural (partial) ordering in $\mathcal{F}_c(\mathbb{R})$, which ranking methods for fuzzy numbers should be consistent with.

Proposition 3.8. *Let $A_1, A_2 \in \mathcal{F}_c(\mathbb{R})$. If $A_1 \preceq A_2$ then $S_F[A_1, A_2]$ is the set of all $A \in \mathcal{F}_c(\mathbb{R})$ such that $A_1 \preceq A \preceq A_2$.*

Propositions 3.6 and 3.8 confirm that pseudosimplices are consistent with a natural notion of 'being between' for fuzzy numbers; as opposed to what would have happened with convex hull simplices.

4. Simplicial depth functions for fuzzy sets

Our analogs to simplicial depth are not the direct result of plugging the fuzzy pseudosimplex into the simplicial depth formula. To understand why, we first propose and discuss a straightforward adaptation.

The *naive simplicial depth*, based on $\mathcal{J} \subseteq \mathcal{F}_c(\mathbb{R}^p)$ and $\mathcal{H} \subseteq L^0[\mathcal{F}_c(\mathbb{R}^p)]$, of a fuzzy set $A \in \mathcal{J}$ with respect to a fuzzy random variable $\mathcal{X} \in \mathcal{H}$ is

$$D_{nS}(A; \mathcal{X}) := \mathbb{P}\left(A \in S_F[\mathcal{X}_1, \ldots, \mathcal{X}_{p+1}]\right), \tag{4.1}$$

where $\mathcal{X}_1, \ldots, \mathcal{X}_{p+1}$ are $p+1$ independent and identically distributed random variables with distribution $\mathbb{P}_\mathcal{X}$. Setting

$$m_\mathcal{X}(u, \alpha) := \min\{s_{\mathcal{X}_1}(u, \alpha), \ldots, s_{\mathcal{X}_{p+1}}(u, \alpha)\}, \tag{4.2}$$

$$M_\mathcal{X}(u, \alpha) := \max\{s_{\mathcal{X}_1}(u, \alpha), \ldots, s_{\mathcal{X}_{p+1}}(u, \alpha)\} \tag{4.3}$$

for any $(u, \alpha) \in \mathbb{S}^{p-1} \times [0, 1]$, one can also express this function as

$$D_{nS}(A; \mathcal{X}) = \mathbb{P}\left(s_A(u, \alpha) \in [m_\mathcal{X}(u, \alpha), M_\mathcal{X}(u, \alpha)] \text{ for all } (u, \alpha) \in \mathbb{S}^{p-1} \times [0, 1]\right). \tag{4.4}$$

It is not self-evident that D_{nS} is well defined:

(i) In (4.1), it is not clear whether $S_F[\mathcal{X}_1, \ldots, \mathcal{X}_{p+1}]$ is a random set in $\mathcal{F}_c(\mathbb{R}^p)$, which would ensure that the probability makes sense.
(ii) In (4.4), the event depends on uncountably many (u, α), making it an uncountable intersection which might fail to be measurable.

Thus it becomes necessary to establish the measurability of those events.

Proposition 4.1. *The function D_{nS} is well defined.*

The proposed naive simplicial depth generalizes the multivariate, simplicial depth, as observed below by taking $\mathcal{J} = \mathcal{R}^p$.

Proposition 4.2. *For any random variable X on $\mathbb{R}^p$ and $x \in \mathbb{R}^p$,*

$$D_{nS}\left(I_{\{x\}}; I_X\right) = SD(x; \mathbb{P}_X).$$

The proof follows directly. Although we replaced convex hull simplices by pseudosimplices, which are generally larger, this naive depth function may still result in a high number of ties at zero, which is inappropriate for certain applications such as classification. That is a consequence of the fuzzy set having to be completely contained in the pseudosimplex. An analogous problem was observed by López-Pintado and Romo when adapting simplicial depth to functional data in [16]. Their definition of *band depth* aims at ordering functional data and stems from simplicial depth in the same way as our naive simplicial fuzzy depth. To overcome this shortcoming, in [16] a *modified band depth* is introduced which inspires our next definition. A similar reasoning is also found in [17,18], both in the functional setting.

Definition 4.3. The *modified simplicial depth*, based on $\mathcal{J} \subseteq \mathcal{F}_c(\mathbb{R}^p)$ and $\mathcal{H} \subseteq L^0[\mathcal{F}_c(\mathbb{R}^p)]$, of a fuzzy set $A \in \mathcal{J}$ with respect to a random variable $\mathcal{X} \in \mathcal{H}$ is

$$D_{mS}(A; \mathcal{X}) := \mathrm{E}\left(\mathcal{V}_p \otimes \nu\{(u,\alpha) \in \mathbb{S}^{p-1} \times [0,1] : s_A(u,\alpha) \in [m_{\mathcal{X}}(u,\alpha), M_{\mathcal{X}}(u,\alpha)]\}\right),$$

where $m_{\mathcal{X}}(u,\alpha)$ and $M_{\mathcal{X}}(u,\alpha)$ are defined in (4.2) and (4.3) and $\mathcal{X}_1, \dots, \mathcal{X}_{p+1}$ are independent and identically distributed random variables with distribution $\mathbb{P}_{\mathcal{X}}$.

By Fubini's theorem,

$$D_{mS}(A; \mathcal{X}) = \int_{\mathbb{S}^{p-1}} \int_{[0,1]} \mathbb{P}\left(s_A(u,\alpha) \in [m_{\mathcal{X}}(u,\alpha), M_{\mathcal{X}}(u,\alpha)]\right) \mathrm{d}\nu(\alpha)\, \mathrm{d}\mathcal{V}_p(u), \tag{4.5}$$

as justified by the next proposition.

Proposition 4.4. *The function D_{mS} is well defined.*

The formulation in (4.5) has inspired us to introduce the following definition of simplicial, fuzzy, depth, which is also motivated by the Tukey depth in [33], defined as an infimum over $\mathbb{S}^{p-1}$.

Definition 4.5. The *simplicial depth* based on $\mathcal{J} \subseteq \mathcal{F}_c(\mathbb{R}^p)$ and $\mathcal{H} \subseteq L^0[\mathcal{F}_c(\mathbb{R}^p)]$ of a fuzzy set $A \in \mathcal{J}$ with respect to a random variable $\mathcal{X} \in \mathcal{H}$ is

$$D_{FS}(A; \mathcal{X}) := \inf_{u \in \mathbb{S}^{p-1}} \mathrm{E}\left(\nu\{\alpha \in [0,1] : s_A(u,\alpha) \in [m_{\mathcal{X}}(u,\alpha), M_{\mathcal{X}}(u,\alpha)]\}\right),$$

where $m_{\mathcal{X}}(u,\alpha)$ and $M_{\mathcal{X}}(u,\alpha)$ are defined in (4.2) and (4.3) and $\mathcal{X}_1, \dots, \mathcal{X}_{p+1}$ are independent and identically distributed random variables with distribution $\mathbb{P}_{\mathcal{X}}$.

Again by Fubini's theorem,

$$D_{FS}(A; \mathcal{X}) = \inf_{u \in \mathbb{S}^{p-1}} \int_{[0,1]} \mathbb{P}(s_A(u,\alpha) \in [m_{\mathcal{X}}(u,\alpha), M_{\mathcal{X}}(u,\alpha)])\, \mathrm{d}\nu(\alpha). \tag{4.6}$$

Proposition 4.6. *The function D_{FS} is well defined.*

The difference between Definitions 4.3 and 4.5 can be understood as follows. In (4.5), we take the average over $\mathbb{S}^{p-1}$ of the integral over $[0,1]$, while in (4.6) we take the infimum over $\mathbb{S}^{p-1}$ of the integral over $[0,1]$, that is, we consider the direction $u \in \mathbb{S}^{p-1}$ for which the integral is smallest. The next example examines the difference between

D_{mS} and D_{FS}, and their suitability under different scenarios. The example is in $\mathcal{F}_c(\mathbb{R})$, for which Definitions 4.3 and 4.5 reduce to

$$D_{mS}(A;\mathcal{X})=\frac{1}{2}\sum_{u\in\{-1,1\}}\mathrm{E}\left(\nu\{\alpha\in[0,1]:s_A(u)\in[m_{\mathcal{X}}(u,\alpha),M_{\mathcal{X}}(u,\alpha)]\}\right) \tag{4.7}$$

and

$$D_{FS}(A;\mathcal{X})=\min_{u\in\{-1,1\}}\mathrm{E}\left(\nu\{\alpha\in[0,1]:s_A(u)\in[m_{\mathcal{X}}(u,\alpha),M_{\mathcal{X}}(u,\alpha)]\}\right), \tag{4.8}$$

respectively.

Example 4.7. Let us consider the fuzzy random variable $\mathcal{X}:\Omega\to\mathcal{F}_c(\mathbb{R})$ such that

$$\mathbb{P}(\omega\in\Omega:\mathcal{X}(\omega)=\mathrm{I}_{[1,2]})=\mathbb{P}(\omega\in\Omega:\mathcal{X}(\omega)=\mathrm{I}_{[4,5]})=1/2.$$

Let $\mathcal{X}_1=\mathrm{I}_{[1,2]}$ and $\mathcal{X}_2=\mathrm{I}_{[4,5]}$ be two independent observations of $\mathcal{X}$. With this, for each $\alpha\in[0,1]$,

$$s_{\mathcal{X}_1}(-1,\alpha)=-1,\ s_{\mathcal{X}_1}(1,\alpha)=2,\ s_{\mathcal{X}_2}(-1,\alpha)=-4 \text{ and } s_{\mathcal{X}_2}(1,\alpha)=5.$$

Then (4.7) and (4.8) for a general $A\in\mathcal{F}_c(\mathbb{R})$ yield

$$D_{mS}(A;\mathcal{X})=\frac{1}{2}\Big[\nu\left\{\alpha\in[0,1]:s_A(1,\alpha)\in[2,5]\right\}+\nu\left\{\alpha\in[0,1]:s_A(-1,\alpha)\in[-4,-1]\right\}\Big] \tag{4.9}$$

and

$$D_{FS}(A;\mathcal{X})=\min\Big\{\nu\{\alpha\in[0,1]:s_A(1,\alpha)\in[2,5]\},\ \nu\{\alpha\in[0,1]:s_A(-1,\alpha)\in[-4,-1]\}\Big\}. \tag{4.10}$$

Let us present two cases:

(i) $R,G\in\mathcal{F}_c(\mathbb{R}^p)$ such that $D_{mS}(R;\mathcal{X})=D_{mS}(G;\mathcal{X})$ and $D_{FS}(R;\mathcal{X})\neq D_{FS}(G;\mathcal{X})$,
(ii) $R,G\in\mathcal{F}_c(\mathbb{R}^p)$ such that $D_{mS}(R;\mathcal{X})\neq D_{mS}(G;\mathcal{X})$ and $D_{FS}(R;\mathcal{X})=D_{FS}(G;\mathcal{X})$,

illustrated in Fig. 1.

(i) Let $R,G\in\mathcal{F}_c(\mathbb{R}^p)$ be defined, for every $t\in\mathbb{R}$, by

$$R(t):=(t-1/2)I_{[1/2,3/2]}(t)+(-t/2+7/4)I_{[3/2,7/2]}(t),$$
$$G(t):=(3t/2-23/4)I_{[23/6,9/2]}(t).$$

Consequently, $R_\alpha=[\alpha+1/2,7/2-2\cdot\alpha]$ and $G_\alpha=[(2/3)\cdot\alpha+23/6,9/2]$ for each $\alpha\in[0,1]$. Additionally, also for each $\alpha\in[0,1]$,

$$s_R(-1,\alpha)=-\alpha-1/2,\ s_R(1,\alpha)=7/2-2\cdot\alpha,$$
$$s_G(-1,\alpha)=-(2/3)\cdot\alpha-23/6 \text{ and } s_G(1,\alpha)=9/2$$

are their support functions.

To obtain the depth values, we first compute the Lebesgue measure of the α's for which these support functions are in the intervals established in (4.9) and (4.10). We illustrate the computation making use of the top row of Fig. 1. In the left plot, the thick red line is the part of the fuzzy set R for which $s_R(-1,\alpha)\in[-4,-1]$. This corresponds to $\alpha\in[.5,1]$ which has Lebesgue measure .5. In the right plot, the thick red line is the part of R such that $s_R(1,\alpha)\in[2,5]$, which corresponds to $\alpha\in[0,.75]$, with Lebesgue measure .75. These measures add up to 5/4 and their minimum is 1/2.

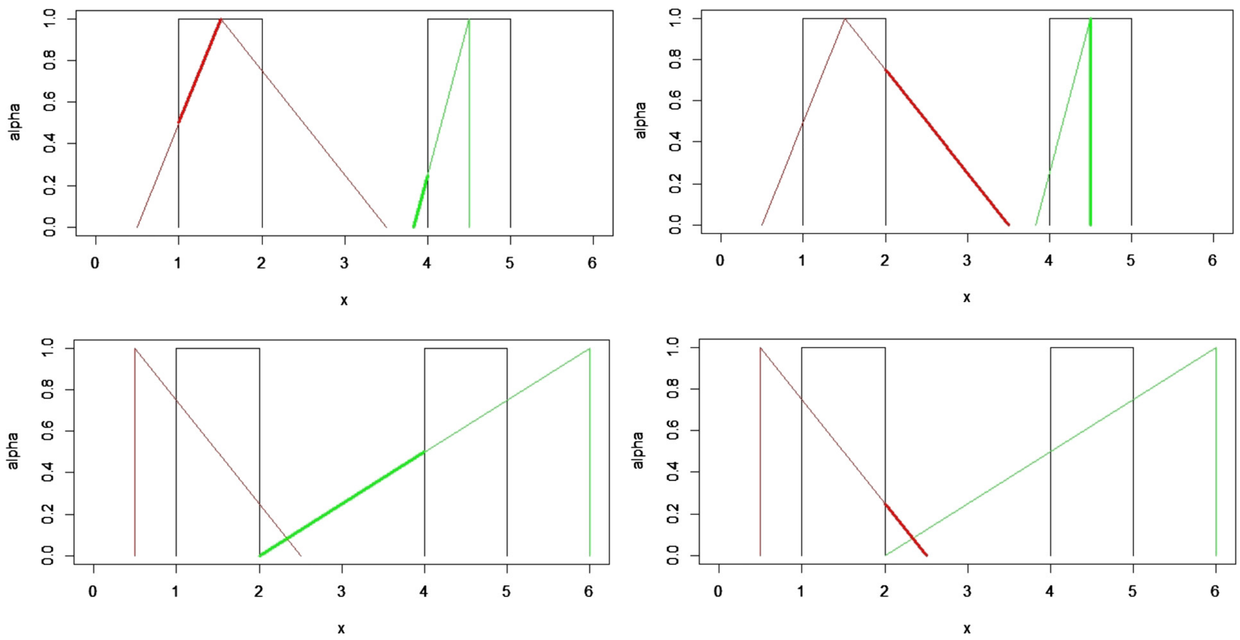

Fig. 1. Representation of Example 4.7, with part (i) in the top row and part (ii) in the bottom row. In each plot, the fuzzy sets $\mathcal{X}_i$ $(i = 1, 2)$ are represented in black, R in red and G in green. Thick lines indicate the parts of R and G for which the corresponding support function is in the interval $[m_{\mathcal{X}}(u, \alpha), M_{\mathcal{X}}(u, \alpha)]$, with $u = -1$ in the left column and $u = 1$ in the right column.

Analogously, the thick green line in the left plot is the part of set G for which $s_G(-1, \alpha) \in [-4, -1]$. This corresponds to $\alpha \in [0, .25]$ which results in a Lebesgue measure of .25. In the right plot, the thick green line is the part of G such that $s_G(1, \alpha) \in [2, 5]$. It corresponds to $\alpha \in [0, 1]$, whose Lebesgue measure is 1. These measures add again up to 5/4 but now their minimum is 1/4. Thus, making use of (4.9) and (4.10),

$$D_{mS}(R; \mathcal{X}) = D_{mS}(G; \mathcal{X}) = 5/8 \text{ and } D_{FS}(R; \mathcal{X}) = 1/2 \neq 1/4 = D_{FS}(G; \mathcal{X}).$$

(ii) Let $R, G \in \mathcal{F}_c(\mathbb{R}^p)$ be defined, for any $t \in \mathbb{R}$, by

$$R(t) := (-t/2 + 5/4)I_{[1/2,5/2]}(t),$$
$$G(t) := (t/4 - 1/2)I_{[2,6]}(t).$$

The corresponding α-levels are $R_\alpha = [1/2, 5/2 - 2 \cdot \alpha]$ and $G_\alpha = [4 \cdot \alpha + 2, 6]$. Thus, for each $\alpha \in [0, 1]$,

$$s_R(-1, \alpha) = -1/2,\ s_R(1, \alpha) = 5/2 - 2 \cdot \alpha,\ s_G(-1, \alpha) = -4 \cdot \alpha - 2 \text{ and } s_G(1, \alpha) = 6.$$

As in the previous case, we compute the Lebesgue measures of the α's for which these support functions are in the intervals established in (4.9) and (4.10). We illustrate it making use of the bottom row of Fig. 1. In the left plot, $s_R(-1, \alpha) \notin [-4, -1]$ for any $\alpha \in [0, 1]$; consequently, the Lebesgue measure is 0. There is, however, a thick red line in the right plot, representing the part of R such that $s_R(1, \alpha) \in [2, 5]$. This corresponds to $\alpha \in [0, .25]$, with a Lebesgue measure of .25. For G, one has $s_G(1, \alpha) \notin [2, 5]$, which results in zero Lebesgue measure, and no thick green line in the bottom right plot of Fig. 1. This time, for each $\alpha \in [0, .5]$, it is satisfied that $s_G(-1, \alpha) \in [-4, -1]$, resulting in a Lebesgue measure of .5. Thus, for both R and G the minimum Lebesgue measure is 0. Taking into account (4.5) and (4.6),

$$D_{mS}(R; \mathcal{X}) = 1/8 \neq 1/4 = D_{mS}(G; \mathcal{X}) \text{ and } D_{FS}(R; \mathcal{X}) = 0 = D_{FS}(G; \mathcal{X}).$$

This example shows the relevant differences and similarities between D_{mS} and D_{FS}. Let us comment them further, making use of the plots in Fig. 1. Focusing on case (i), top row plots, we have that R and G take the same D_{mS} depth value because the average of the amount of α's corresponding to the thick red lines between the two plots is the same as the average corresponding to the thick green lines. However, none of those amounts is the same, which is depicted by D_{FS}, providing different depth values. It gives smaller depth value to G because the amount of α's corresponding to one of the thick green lines is the smallest among the four. In case (ii), bottom row plots, we have that R and G take the same D_{FS} value because R and G result both in only one thick line each. D_{mS} depicts a difference between R and G. This difference is appreciated in the bottom row plots: the thick green line, associated to G, is larger than the thick red line, associated to R. Thus, D_{mS} gives a higher depth value to G than to R. As commented before, the difference is due to the distinct way in which they summarize the information. One can argue that D_{mS} is potentially better because it uses more information by computing the average. On the other hand, it can also be argued that D_{FS} will extract the relevant information in certain problems.

5. Properties of D_{mS}, D_{FS}, and D_{nS}

In this section, we will study whether the adaptations of simplicial depth to the fuzzy setting are semilinear and geometric depth functions [8].

Theorem 5.2 collects properties of the simplicial depth functions D_{mS} and D_{FS}. Its proof is based on Proposition 5.1 and proofs for the simplicial band depth [18, Theorems 1 and 2]. The result is valid for $\mathcal{H} \subset C^0[\mathcal{F}_c(\mathbb{R}^p)]$, namely fuzzy random variables all whose support functionals are continuous random variables. Note that, in order to define directly a notion of continuous fuzzy random variables, one would need first a reference measure with respect to which those variables would have a density function. In absence of such a measure (which would play the role of the Lebesgue measure in $\mathbb{R}^p$), the reduction to real random variables via the support function is more operative.

Proposition 5.1. *Let $\mathcal{X} \in L^0[\mathcal{F}_c(\mathbb{R}^p)]$, $U \in \mathcal{F}_c(\mathbb{R}^p)$ and $F_{u,\alpha}$ be the cumulative distribution function of the real random variable $s_{\mathcal{X}}(u,\alpha)$ for any $(u,\alpha) \in \mathbb{S}^{p-1} \times [0,1]$. Then,*

$$\mathbb{P}\Big(s_U(u,\alpha) \in [m_{\mathcal{X}}(u,\alpha), M_{\mathcal{X}}(u,\alpha)]\Big) = 1 - \Big[1 - F_{u,\alpha}(s_U(u,\alpha))\Big]^{p+1} - \Big[F_{u,\alpha}(s_U(u,\alpha)) - \mathbb{P}(s_{\mathcal{X}}(u,\alpha) = s_U(u,\alpha))\Big]^{p+1}$$

for any $(u,\alpha) \in \mathbb{S}^{p-1} \times [0,1]$. If $\mathcal{X} \in C^0[\mathcal{F}_c(\mathbb{R}^p)]$, the above expression reduces to

$$\mathbb{P}\Big(s_U(u,\alpha) \in [m_{\mathcal{X}}(u,\alpha), M_{\mathcal{X}}(u,\alpha)]\Big) = 1 - \Big[1 - F_{u,\alpha}(s_U(u,\alpha))\Big]^{p+1} - \Big[F_{u,\alpha}(s_U(u,\alpha))\Big]^{p+1}$$

for any $(u,\alpha) \in \mathbb{S}^{p-1} \times [0,1]$.

Theorem 5.2. *When computed with respect to a F-symmetric random variable $\mathcal{X} \in C^0[\mathcal{F}_c(\mathbb{R}^p)]$, the functions $D_{mS}(\cdot;\mathcal{X})$ and $D_{FS}(\cdot;\mathcal{X})$ satisfy P1, P2, P3a and P3b for the ρ_r metric for any $r \in (1,\infty)$.*

In general, D_{mS} and D_{FS} violate P4a, as shown by the following example. They also violate P4b, since P4b implies P4a [8, Proposition 5.8].

Example 5.3. Let us consider the fuzzy random variable $\mathcal{X} : \Omega \to \mathcal{F}_c(\mathbb{R})$ such that

$$\mathbb{P}(\omega \in \Omega : \mathcal{X}(\omega) = \mathrm{I}_{\{1\}}) = \mathbb{P}(\omega \in \Omega : \mathcal{X}(\omega) = \mathrm{I}_{\{-1\}}) = 1/2.$$

Let $\mathcal{X}_1 = \mathrm{I}_{\{1\}}$ and $\mathcal{X}_2 = \mathrm{I}_{\{-1\}}$ be two independent observations of $\mathcal{X}$.

Clearly, $\mathcal{X}$ is F-symmetric with respect to $A = \mathrm{I}_{\{0\}}$. Let $B \in \mathcal{F}_c(\mathbb{R})$ be defined by

$$B(t) := (-t/2 + 1/2) I_{(0,1]}(t) + I_{\{0\}}(t), \quad t \in \mathbb{R}.$$

Thus

$$B_\alpha = [0, 1-2\alpha] \text{ for } \alpha \in [0,1/2] \text{ and } B_\alpha = \{0\} \text{ for } \alpha \in [1/2,1].$$

Additionally,

$$s_B(-1,\alpha) = 0 \text{ for all } \alpha \in [0,1] \text{ and } s_B(1,\alpha) = 0 \text{ for all } \alpha \in [1/2,1].$$

Taking into account the definition of D_{FS}, for all $n \in \mathbb{N}$

$$D_{FS}(A + n \cdot B; \mathcal{X}) \geq 1/2; \text{ consequently, } \lim_{n\to\infty} D_{FS}(A + n \cdot B; \mathcal{X}) > 0.$$

Analogously, $D_{mS}(A + n \cdot B; \mathcal{X}) \geq 1/2$ for all $n \in \mathbb{N}$, whence

$$\lim_{n\to\infty} D_{mS}(A + n \cdot B; \mathcal{X}) > 0.$$

Property P4a considers sequences of fuzzy sets of the form $\{A + n \cdot B\}_n$. By restricting the selection of the fuzzy set B to the family

$$\mathfrak{B} := \Big\{ B \in \mathcal{F}_c(\mathbb{R}^p) : \forall u \in \mathbb{S}^{p-1}, \exists C_u \subseteq [0,1] \text{ with } \nu(C_u) = 1 \\ \text{such that } s_B(u,\alpha) \neq 0 \; \forall \alpha \in C_u \Big\},$$

the following result holds for D_{FS} and D_{mS}, which is in line with property P4a. Property P4b, however, considers a general sequence of fuzzy sets $\{A_n\}_n$, not allowing for this adaptation.

Proposition 5.4. *For any $\mathcal{X} \in L^0[\mathcal{F}_c(\mathbb{R}^p)]$ and $B \in \mathfrak{B}$, we have that*

- $\lim_n D_{FS}(A + n \cdot B; \mathcal{X}) = 0$, *for any $A \in \mathcal{F}_c(\mathbb{R}^p)$ maximizing $D_{FS}(\cdot; \mathcal{X})$.*
- $\lim_n D_{mS}(A + n \cdot B; \mathcal{X}) = 0$, *for any $A \in \mathcal{F}_c(\mathbb{R}^p)$ maximizing $D_{mS}(\cdot; \mathcal{X})$.*

The following result is for D_{nS}.

Theorem 5.5. *For any $\mathcal{X} \in L^0[\mathcal{F}_c(\mathbb{R}^p)]$, $D_{nS}(\cdot; \mathcal{X})$ satisfies P1, P4a and P4b for the d_r distances for any $r \in [1, \infty]$ and for the ρ_r distances for any $r \in [1, \infty)$.*

For property P2, intuitively, the notion of symmetry to be considered would make use of the central symmetry of the support function of a fuzzy set in every $u \in \mathbb{S}^{p-1}$ and $\alpha \in [0, 1]$. We do not consider properties P3a and P3b because the multivariate simplicial depth does not generally satisfy the analog property, (M3). Because of these reasons and that naive simplicial fuzzy depth is not one of our recommended fuzzy depth, we do not pursue these properties further.

6. Empirical simplicial depths

Given $\mathcal{H} \subseteq L^0[\mathcal{F}_c(\mathbb{R}^p)]$, let $\mathcal{X} \in \mathcal{H}$ be a fuzzy random variable and $\mathcal{X}_1, \ldots, \mathcal{X}_n$ be independent and identically distributed random variables with distribution $\mathbb{P}_{\mathcal{X}}$. Let $\mathfrak{X}$ be a fuzzy random variable corresponding to the empirical distribution associated to $\mathcal{X}_1, \ldots, \mathcal{X}_n$. That is, $\mathfrak{X}$ takes on as values the observed values $\mathcal{X}_1(\omega), \ldots, \mathcal{X}_n(\omega)$ (possibly repeated) with probability n^{-1}. The simplicial depths associated with this empirical distribution are the empirical or sample simplicial depths.

In Section 6.1, we provide the explicit definitions in $\mathcal{F}_c(\mathbb{R})$, in order to illustrate the behavior of our three proposals. For ease of comparison with Tukey depth, we use in Section 6.3 the same dataset as in [8]. It is interesting to point out that the behavior is similar, in spite of the distribution not being from $C^0[\mathcal{F}_c(\mathbb{R}^p)]$, as assumed in some of our theoretical results (Theorem 5.2). In order to illustrate the case of fuzzy random variables with continuously distributed support functionals, we use in Section 6.2 a synthetic sample from a fuzzy random variable in $C^0[\mathcal{F}_c(\mathbb{R}^p)]$.

6.1. Empirical definitions for $\mathcal{F}_c(\mathbb{R})$

From (4.5) and (4.6), D_{mS} and D_{FS} have in common that their computation involves the function

$$F_A(u) := \int_{[0,1]} \mathbb{P}\Big(s_A(u,\alpha) \in [m_{\mathcal{X}}(u,\alpha), M_{\mathcal{X}}(u,\alpha)] \Big) \, \mathrm{d}\nu(\alpha),$$

with $u \in \mathbb{S}^0 = \{-1, 1\}$. To establish the empirical simplicial and modified simplicial fuzzy depth, making use of $\mathcal{X}_1, \ldots, \mathcal{X}_n$, we calculate $F_A(u)$ for the fuzzy random variable $\mathfrak{X}$ as $\binom{n}{2}^{-1} L_n^A(u)$ with

$$L_n^A(u) := \sum_{i=1}^{n} \sum_{j \geq i}^{n} L_{i,j,u}^A, \tag{6.1}$$

$$L_{i,j,u}^A = \nu(\{\alpha \in [0,1] : s_A(u,\alpha) \in S_{i,j}(u,\alpha)\}), \tag{6.2}$$

and $S_{i,j}(u,\alpha) := [\min\{s_{\mathcal{X}_i}(u,\alpha), s_{\mathcal{X}_j}(u,\alpha)\}, \max\{s_{\mathcal{X}_i}(u,\alpha), s_{\mathcal{X}_j}(u,\alpha)\}]$.

The modified simplicial fuzzy depth based on $\mathcal{J} \subseteq \mathcal{F}_c(\mathbb{R})$ of a fuzzy set $A \in \mathcal{J}$ with respect to $\mathfrak{X}$ is then

$$D_{mS}(A; \mathfrak{X}) = \int_{\mathbb{S}^0} \binom{n}{2}^{-1} L_n^A(u)\, \mathrm{d}\mathcal{V}_1(u) = 2^{-1} \binom{n}{2}^{-1} \left[L_n^A(1) + L_n^A(-1) \right] \tag{6.3}$$

and the simplicial fuzzy depth based on $\mathcal{J} \subseteq \mathcal{F}_c(\mathbb{R})$ of a fuzzy set $A \in \mathcal{J}$ with respect to $\mathfrak{X}$ is

$$D_{FS}(A; \mathfrak{X}) = \inf_{u \in \mathbb{S}^0} \binom{n}{2}^{-1} L_n^A(u) = \binom{n}{2}^{-1} \min\left\{ L_n^A(1), L_n^A(-1) \right\}. \tag{6.4}$$

Similarly, the naive simplicial fuzzy depth based on $\mathcal{J} \subseteq \mathcal{F}_c(\mathbb{R})$ of a fuzzy set $A \in \mathcal{J}$ with respect to $\mathfrak{X}$ is

$$D_{nS}(A; \mathfrak{X}) = \binom{n}{2}^{-1} \sum_{i=1}^{n} \sum_{j \geq i}^{n} I_{i,j}^A, \tag{6.5}$$

where $I_{i,j}^A$ equals 1 if $s_A(u,\alpha) \in \left[\min\{s_{\mathcal{X}_i}(u,\alpha), s_{\mathcal{X}_j}(u,\alpha)\}, \max\{s_{\mathcal{X}_i}(u,\alpha), s_{\mathcal{X}_j}(u,\alpha\} \right]$ for every $(u,\alpha) \in \mathbb{S}^0 \times [0,1]$, and 0 otherwise.

6.2. Simulated data

We draw a sample ($n = 100$) from a trapezoidal fuzzy random variable in $C^0[\mathcal{F}_c(\mathbb{R}^p)]$. To construct it, we follow the procedure in [29]. Let X_1, X_2, X_3, X_4 be independent and continuous real-valued random variables. In particular, let X_1 be normally distributed with zero mean and standard deviation 10 and X_2, X_3 and X_4 be each chi-squared distributed with 1 degree of freedom. Set

$$\mathcal{X} = \mathrm{Tra}(X_1 - X_2 - X_3, X_1 - X_2, X_1 + X_2, X_1 + X_2 + X_4) \tag{6.6}$$

which is well-defined since the random variables X_2, X_3 and X_4 take non-negative values. By construction,

$$s_{\mathcal{X}}(-1,\alpha) = -(X_1 - X_2 - (1-\alpha)X_3)$$

and

$$s_{\mathcal{X}}(1,\alpha) = X_1 + X_2 + (1-\alpha)X_4,$$

which are continuous variables for each $\alpha \in [0,1]$. Accordingly, $\mathcal{X} \in C^0[\mathcal{F}_c(\mathbb{R})]$ as required by Theorem 5.2.

The choice of the χ_1^2 distribution for X_3, X_4 is because it is very skewed (Pearson coefficient: $2\sqrt{2}$). That allows us to realize how the depth is affected not just by the location of the core of the trapezoidal fuzzy set but also by the slopes of its sides.

To illustrate the performance of the different depth functions, let $\mathcal{X}_1, \ldots, \mathcal{X}_{100}$ be independent copies distributed as $\mathcal{X}$. With some abuse of notation, each $\mathcal{X}_i$ will also denote the observed trapezoidal fuzzy set, represented in each of the plots of Fig. 2. Thus, we illustrate the performance of each of our three proposals by computing the depths of each $\mathcal{X}_i$ with respect to the corresponding empirical fuzzy random variable $\mathfrak{X}$. Naive simplicial depth, D_{nS}, is illustrated in the top row of Fig. 2, modified simplicial depth, D_{mS}, in the middle row and simplicial fuzzy depth, D_{FS}, in the bottom row. The plots in the first column represent the five trapezoidal fuzzy values having the largest depth values. These are colored from red (highest depth) to yellow (high depth). The middle column of Fig. 2 presents a zoom to

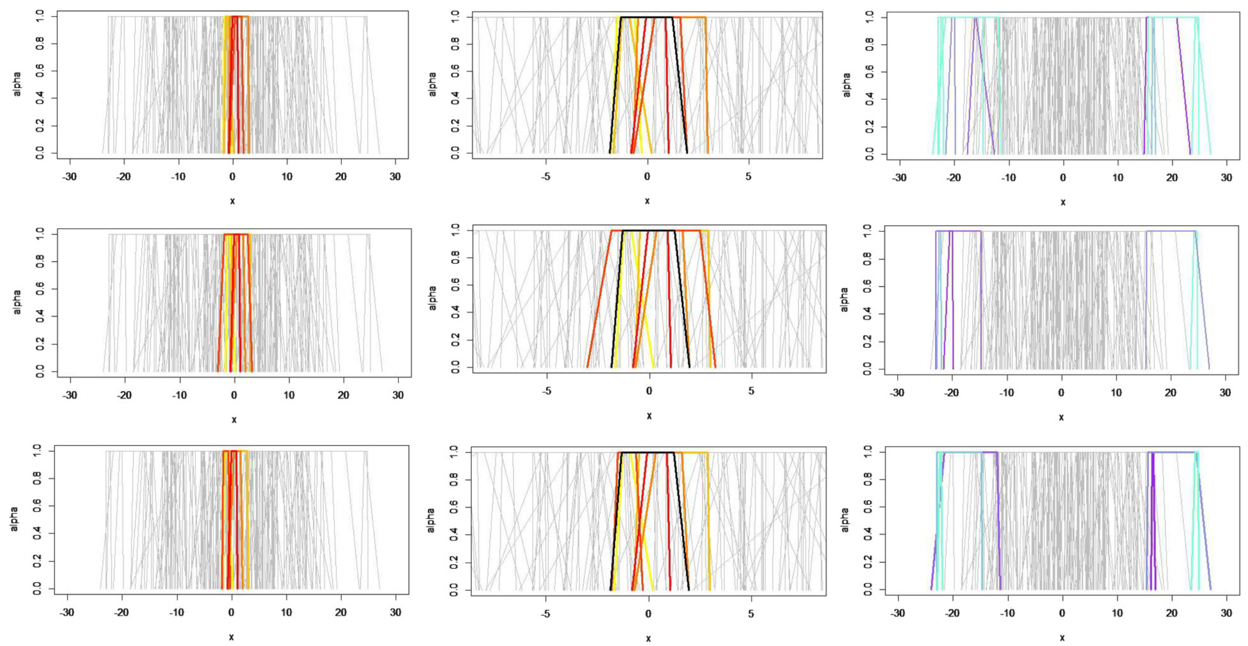

Fig. 2. Illustration of the empirical naive simplicial fuzzy depth, D_{nS}, (top row), the empirical modified simplicial fuzzy depth, D_{mS}, (middle row) and the empirical simplicial fuzzy depth, $D_{FS,}$ (bottom row) over a sample of trapezoidal fuzzy sets of size 100 drawn from $\mathcal{X}$ in (6.6). The sample is plotted in gray. The color in the first and second column plots represents the trapezoidal fuzzy sets in the sample corresponding to the 5 larger depth values, with the second column being a zoom of the first in the interval $[-8, 8]$; in order to better observe the different depth values. Colors range from red (highest depth) to yellow (high depth) in the first column. In addition, in the second column the median fuzzy set [29] is highlighted in black. The third column represents the trapezoidal fuzzy sets with the 5 minimal depth values for the same depth functions. Depth values are shown through the colors, which range from aqua marine blue (lowest depth) to violet (low depth).

the central part of each plot. A zoom of each of these plots highlighting the deepest sets is in the central column of the figure.

We also represent, plotted in black in the middle column of Fig. 2, the median fuzzy set estimator, M, with respect to the sample $\mathcal{X}_1, \ldots, \mathcal{X}_{100}$, as defined in [29]. The median M is not necessarily one of the sample fuzzy sets; and in the particular case of Fig. 2, it is not. The maximizers of the depth functions D_{nS}, D_{mS} and D_{FS} provide alternative definitions of a median fuzzy set. They are in the vicinity of M.

The right column of Fig. 2 shows the trapezoidal fuzzy sets with the minimal 5 depth values for the three different proposals of simplicial depth. The trapezoidal fuzzy sets with minimal depth are the ones furthest to the left and right, as expected. It is observable from the plots that the three definitions order the sets with minimal depth in a similar way. The main difference lies in that D_{nS} gives a high number of ties (observe the many sets in aquamarine blue in the last column of the first row). The reason for this is that D_{nS} is a sum of indicator functions (6.5) while the other two proposals make use the Lebesgue measure [(6.2), (6.3) and (6.4)]. Thus, it is generally more convenient to use the proposals D_{mS} and D_{FS} instead of D_{nS}, the latter being inappropriate for some applications like classification. The use of a sum of indicator functions versus the Lebesgue measure also explains that D_{nS} results in smaller depth values than D_{mS} or D_{FS}.

The main difference between the D_{mS} and D_{FS} depths of a fuzzy set A is that the former takes the average of $L_n^A(u)$ in (6.1) and the latter its minimum. Thus, a fuzzy number A with, for instance,

$$L_n^A(1) \text{ close to } L_n^M(1) \text{ and } L_n^A(-1) \text{ far from } L_n^M(-1)$$

does not take a maximal depth value with D_{FS} but can take it with D_{mS}. This is observed in the central column of Fig. 2.

A similar phenomenon is observed with the fuzzy numbers taking minimal depth values. The bottom row right column plot in Fig. 2 shows that there exist fuzzy numbers in the sample with minimal depth for D_{FS}, some are on

Table 1
Absolute frequency in the sample of each trapezoidal fuzzy set T_i, $i = 1, \dots, 9$, represented in Fig. 3.

T_1	T_2	T_3	T_4	T_5	T_6	T_7	T_8	T_9
22	16	39	36	85	22	35	12	12

the left side of the plot and the others on the right side. Among the ones on the left there are those that have, for instance,

$$L_n^A(-1) \text{ far from } L_n^M(-1) \text{ while } L_n^A(1) \text{ is not as far from } L_n^M(1).$$

Analogously, among the ones on the right there those that have, for instance,

$$L_n^A(1) \text{ far from } L_n^M(1) \text{ while } L_n^A(-1) \text{ is not as far from } L_n^M(-1).$$

As observable in the central row right column plot in Fig. 2, these fuzzy numbers does not necessarily take minimal depth value with D_{mS}, as this depth function takes the average between $L_n^A(1)$ and $L_n^A(-1)$.

6.3. Real data

We use the *Trees* dataset (from the `SAFD` R package for Statistical Analysis of Fuzzy Data), which was first used in [4]. It is from a reforestation project in the region of Asturias (Northern Spain) by the INDUROT forest institute at the University of Oviedo. The project takes into account three species of trees: birch (*Betula celtiberica*), sessile oak (*Quercus petraea*) and rowan (*Sorbus aucuparia*).

The most relevant variable is *quality*, whose observations are trapezoidal fuzzy sets coming from an expert subjective assessment of height, diameter, leaf structure and other features. In Fig. 3, quality is in the x-axis in the range 1–5, from low to perfect quality. Membership is represented in the y-axis.

The dataset contains 9 different trapezoidal fuzzy values. Therefore the assumption in Theorem 5.2 that each support function has a continuous distribution is violated. From left to right, we denote them by $T_1, \dots, T_9$. These sets appear in the sample with a certain multiplicity, resulting in a sample $\mathcal{X}_1, \dots, \mathcal{X}_n$ of size $n = 279$. Table 1 shows the absolute frequency of the fuzzy sets in the sample. We denote by $\mathfrak{X}$ the fuzzy random variable corresponding to the empirical distribution associated to $\mathcal{X}_1, \dots, \mathcal{X}_n$.

One can observe from Fig. 3 that

$$s_{T_i}(1,\alpha) \geq s_{T_j}(1,\alpha) \text{ and } s_{T_i}(-1,\alpha) \leq s_{T_j}(-1,\alpha) \tag{6.7}$$

for each $\alpha \in [0,1]$ and $i, j \in \{1, \dots 9\}$ with $i \leq j$. In fact, the inequalities are strict except for the cases of T_4, T_5 and T_6, where

$$s_{T_4}(-1,0) = s_{T_5}(-1,0) \text{ and } s_{T_5}(1,0) = s_{T_6}(1,0). \tag{6.8}$$

Taking into account the sample version of D_{nS} in (6.5) and the fact that $I_{i,j}^A$ takes value 1 if

$$s_A(u,\alpha) \in [\min\{s_{\mathcal{X}_i}(u,\alpha), s_{\mathcal{X}_j}(u,\alpha)\}, \max\{s_{\mathcal{X}_i}(u,\alpha), s_{\mathcal{X}_j}(u,\alpha)\}]$$

for every $(u,\alpha) \in \mathbb{S}^0 \times [0,1]$ and 0 otherwise, the computation of $D_{nS}(T_i; \mathfrak{X})$ reduces to computing the simplicial depth in $\mathbb{R}$ of $s_{T_i}(u,\alpha)$ with respect to $s_{\mathfrak{X}}(u,\alpha)$ for some (u,α) where the inequalities in (6.7) are strict. By (6.8), this is the case of $(u,\alpha) = (1,1)$, for instance. Thus

$$D_{nS}(T_i; \mathfrak{X}) = SD(s_{T_i}(1,1); s_{\mathfrak{X}}(1,1))$$

for each $i \in \{1, \dots, 9\}$.

Making use of the ordering in (6.7), the identity $L_{i,j,u}^{T_k} = 1$ holds for each $k \in [i,j]$ and $L_{i,j,u}^{T_k} = 0$ otherwise. Considering the sample versions of D_{mS} and D_{FS} in (6.3) and (6.4), in this case

$$D_{nS}(T_i; \mathfrak{X}) = D_{mS}(T_i; \mathfrak{X}) = D_{FS}(T_i; \mathfrak{X})$$

for each $i \in \{1, \dots, 9\}$. Thus, in computing the depth of an element in the dataset with respect to the empirical fuzzy random variable, we obtain the same depth value independently of which of the three simplicial based fuzzy depths

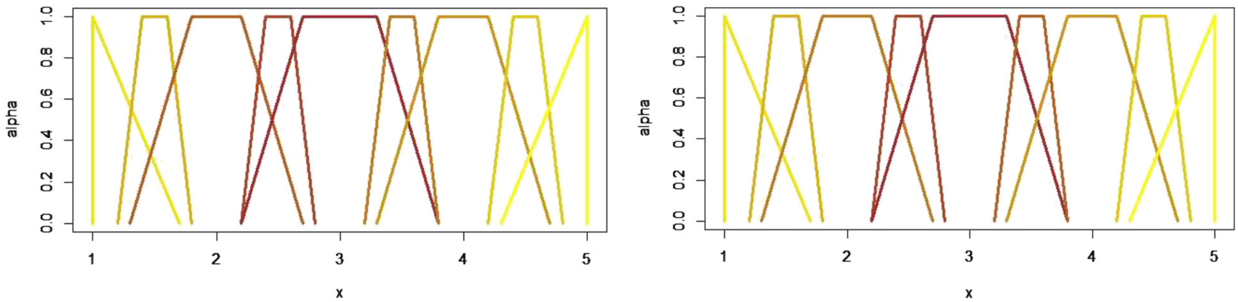

Fig. 3. Display of the *Trees* dataset. In the first column, colors are assigned based on the simplicial depth, ranging from brown (high depth) to yellow (low depth). The second column applies the same procedure to Tukey depth.

is used. The left plot of Fig. 3 represents in the color the depth values of each of the 9 distinct trapezoidal elements in the dataset.

From Fig. 3, the order induced in the dataset by the simplicial and Tukey fuzzy depth functions is similar. In fact, the only difference is T_3 and T_6. With simplicial fuzzy depths, T_3 is the third deepest set and T_6 is the fourth, with the opposite ranking for Tukey depth. Value T_3 has 39 repetitions in the sample while T_6 has only 22. Also, the 0-level and 1-level diameters for T_3 are greater than those of T_6. As a minimum is involved in the Tukey depth definition, an explanation for $D_{FT}(T_6; \mathcal{X}) > D_{FT}(T_3; \mathcal{X})$ is that the weight of T_3 in the sample is greater than the weight of T_6.

7. Discussion

Simplicial depth is one of the most widely used depth functions in multivariate statistics. It is built over the notion of simplex in $\mathbb{R}^p$. In the space of fuzzy sets, the notion of simplex is not an obvious one. With the characterization introduced in Proposition 3.1 of simplices in the multivariate space, we justify the notion of simplex in $\mathcal{K}_c(\mathbb{R}^p)$ and extend it to the fuzzy setting, working α-level by α-level (Definition 3.5). Making use of this notion, we propose a straightforward adaptation of simplicial depth to the fuzzy setting and two more elaborate definitions:

- Naive simplicial depth, D_{nS} in (4.1), generalizes multivariate simplicial depth. We proved some properties in Theorem 5.5 and showed that it may result in a high number of ties at zero, which is not desirable in some applications.
- Modified simplicial fuzzy depth, D_{mS} (Definition 4.3), improves the naive simplicial fuzzy depth analogously to the way modified band depth improves the band depth. Thus, the amount of elements in a space that take zero depth value is less with the modified simplicial fuzzy depth than with the naive simplicial depth.
- Simplicial depth, D_{FS} (Definition 4.5), transforms the modified simplicial fuzzy depth in the direction of the Tukey depth. This is done by applying the infimum over $\mathbb{S}^{p-1}$ instead of the expected value in the depth function formulation.

Although it is clear throughout the paper the authoritativeness of D_{mS} and D_{FS} over D_{nS}, there is not a clear winner between D_{mS} and D_{FS}. The practical similarities and differences between them are discussed in Example 4.7 and Subsection 6.2. Their properties are collected in Theorem 5.2 and Proposition 5.4. For some of these properties it is required of the fuzzy random variables to satisfy certain type of continuity. This is inherited from the fact that the multivariate simplicial depth requires of continuous distributions to satisfy the notion of multivariate depth. Thus, our three proposals do not satisfy the entirety of the desirable properties of semilinear and geometric depth function in [8] (Section 5). However, as we can see in the illustrations in Section 6, the behavior of the three proposals is similar in practice. As shown there, it is also similar to that of the Tukey fuzzy depth, despite Tukey fulfills both notions and the comparison is done with respect to a fuzzy random variable that does not satisfy the continuity required in Theorem 5.2.

For future work, it is interesting to study more instances in order to create a library of statistical depth functions for the fuzzy setting. Also, it would be interesting to study further properties of Tukey and simplicial depths, such as convergence of the sample depth to the population depth (consistency) and their continuity or semicontinuity properties.

Declaration of competing interest

The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

Data availability

The dataset is publicly available already

Acknowledgements

This research was supported by grant PID2022-139237NB-I00 funded by MCIN/AEI/10.13039/501100011033 and "ERDF A way of making Europe". L. González was also supported by grant MTM2017-86061-C2-2-P funded by MCIN/AEI/10.13039/501100011033 and "ERDF A way of making Europe". For Alicia Nieto-Reyes this research was also funded by grant number 21.VP67.64662 of the "Proyectos Puente 2022" from the Spanish "Consejería de Universidades, Igualdad, Cultura y Deporte del Gobierno de Cantabria". P. Terán is also supported by the Ministerio de Ciencia, Innovación y Universidades grant PID2019-104486GB-I00 and the Consejería de Empleo, Industria y Turismo del Principado de Asturias grant GRUPIN-IDI2018-000132.

Appendix A. Proofs

Proof of Proposition 3.1. Define

$$\mathcal{C} := \left\{x \in \mathbb{R}^p : \langle u, x\rangle \in [m(u), M(u)] \text{ for all } u \in \mathbb{S}^{p-1}\right\}.$$

We prove first $S[x_1, \dots, x_{p+1}] \subseteq \mathcal{C}$. Let $x \in S[x_1, \dots, x_{p+1}]$. By (3.2), there exist $\lambda_1, \dots, \lambda_{p+1} \geq 0$ with $\sum_{i=1}^{p+1} \lambda_i = 1$ such that $x = \sum_{i=1}^{p+1} \lambda_i x_i$. For any fixed direction $u \in \mathbb{S}^{p-1}$, we have $\langle u, x\rangle = \sum_{i=1}^{p+1} \lambda_i \langle u, x_i\rangle$. As $\lambda_i \in [0, 1]$ for all $i = 1, \dots, p+1$, we have that $\langle u, x\rangle \in [m(u), M(u)]$; consequently, $x \in \mathcal{C}$.

Now let $x \in \mathcal{C}$ and assume by contradiction that $x \notin S[x_1, \dots, x_{p+1}]$. The simplex $S[x_1, \dots, x_{p+1}]$ and the set $\{x\}$ are closed, convex and bounded subsets of $\mathbb{R}^p$. By the Hyperplane Separation Theorem (see, e.g., [26]), there exist $u \in \mathbb{R}^p\backslash\{0\}$ and $b \in \mathbb{R}$ such that $\langle u, x\rangle > b$ and $\langle u, s\rangle < b$ for all $s \in S[x_1, \dots, x_{p+1}]$. This implies that $\langle u, x\rangle > \langle u, s\rangle$ for all $s \in S[x_1, \dots, x_{p+1}]$. Therefore $\langle \overline{u}, x\rangle > M(\overline{u})$ for the vector $\overline{u} := \|u\|^{-1}u \in \mathbb{S}^{p-1}$, then $x \in \mathcal{C}$, which leads to a contradiction. Thus $x \in S[x_1, \dots, x_{p+1}]$. □

Proof of Proposition 3.3. Let $A_1, \dots, A_{p+1} \in \mathcal{K}_c(\mathbb{R}^p)$ and $A \in \mathcal{K}_c(\mathbb{R}^p)$ be such that there exist $\lambda_1, \dots, \lambda_{p+1} \geq 0$ with $\sum_{i=1}^{p+1} \lambda_i = 1$ and $A = \sum_{i=1}^{p+1} \lambda_i \cdot A_i$. By (2.2), $s_A(u) = \sum_{i=1}^{p+1} \lambda_i \cdot s_{A_i}(u)$ for every $u \in \mathbb{S}^{p-1}$. Thus

$$m(u) = \left(\sum_{i=1}^{p+1} \lambda_i\right) m(u) \leq s_A(u) \leq \left(\sum_{i=1}^{p+1} \lambda_i\right) M(u) = M(u).$$

Then $A \in S_c[A_1, \dots, A_{p+1}]$. □

Proof of Proposition 3.8. For any $A \in \mathcal{F}_c(\mathbb{R})$, since $\mathbb{S}^0 = \{-1, 1\}$ we have

$$s_A(1, \alpha) = \sup A_\alpha, \quad s_A(-1, \alpha) = \sup\{-x \mid x \in A_\alpha\} = -\inf A_\alpha.$$

For any fixed α, inequality $m(u\alpha) \leq s_A(u, \alpha) \leq M(u\alpha)$ will hold for $u = 1$ if and only if

$$\min\{\sup(A_1)_\alpha, \sup(A_2)_\alpha\} \leq \sup A_\alpha \leq \max\{\sup(A_1)_\alpha, \sup(A_2)_\alpha\}$$

which, taking into account the assumption $A_1 \preceq A_2$, is equivalent to

$$\sup(A_1)_\alpha \le \sup A_\alpha \le \sup(A_2)_\alpha.$$

In its turn, the inequality will hold for $u = -1$ if and only if

$$\min\{-\inf(A_1)_\alpha, -\inf(A_2)_\alpha\} \le -\inf A_\alpha \le \max\{-\inf(A_1)_\alpha, -\inf(A_2)_\alpha\}$$

or, multiplying all terms by -1,

$$\max\{\inf(A_1)_\alpha, \inf(A_2)_\alpha\} \ge \inf A_\alpha \ge \min\{\inf(A_1)_\alpha, \inf(A_2)_\alpha\}.$$

This, again by the assumption $A_1 \preceq A_2$, is the same thing as

$$\inf(A_2)_\alpha \ge \inf A_\alpha \ge \inf(A_1)_\alpha.$$

The conjunction of those two conditions is just $A_1 \preceq A \preceq A_2$. Hence

$$S_F[A_1, A_2] = \{A \in \mathcal{F}_c(\mathbb{R}) : A_1 \preceq A \preceq A_2\}. \quad \square$$

Proof of Proposition 4.1. We need to show that the event

$$\Big\{s_A(u,\alpha) \in [m_{\mathcal{X}}(u,\alpha), M_{\mathcal{X}}(u,\alpha)] \text{ for all } (u,\alpha) \in \mathbb{S}^{p-1} \times [0,1]\Big\}$$
$$= \bigcap_{u \in \mathbb{S}^{p-1}} \bigcap_{\alpha \in [0,1]} \big\{s_A(u,\alpha) \in [m_{\mathcal{X}}(u,\alpha), M_{\mathcal{X}}(u,\alpha)]\big\}$$

is measurable.

First, for each fixed u, α and $i = 1, \ldots, p+1$, the mapping $s_{\mathcal{X}_i}(u,\alpha)$ is a random variable [12, Lemma 4]. Subsequently,

$$\Omega_{u,\alpha} := \big\{s_A(u,\alpha) \in [m_{\mathcal{X}}(u,\alpha), M_{\mathcal{X}}(u,\alpha)]\big\}$$
$$= \left(\bigcup_{i=1}^{p+1} \{s_{\mathcal{X}_i}(u,\alpha) \le s_A(u,\alpha)\}\right) \cap \left(\bigcup_{i=1}^{p+1} \{s_{\mathcal{X}_i}(u,\alpha) \ge s_A(u,\alpha)\}\right)$$

is measurable.

Taking a countable dense subset $D \subset [0,1]$ such that $0 \in D$, let us prove

$$\bigcap_{\alpha \in [0,1]} \Omega_{u,\alpha} = \bigcap_{\alpha \in D} \Omega_{u,\alpha} \quad \text{for each fixed } u \in \mathbb{S}^{p-1}. \tag{A.1}$$

Inclusion '$\supset$' is trivial. For the converse inclusion, assume for now $\alpha \in (0,1]$. We construct a sequence of elements of D converging to α from the left (which is why $\alpha > 0$ is needed). Indeed, for each $n \in \mathbb{N}$ with $n > \alpha^{-1}$ the open interval $\left(\alpha - n^{-1}, \alpha\right)$ contains some $\alpha_n \in D$ because D is dense. Since $\alpha - n^{-1} < \alpha_n < \alpha$, we have $\alpha_n \to \alpha^-$.

Notice the mapping $s_A(u,\cdot)$ is left continuous [19]. Similarly, for any arbitrary $\omega \in \Omega$, the $s_{\mathcal{X}_i(\omega)}(u,\cdot)$ are left continuous, whence $m_{\mathcal{X}}(u,\cdot)$ and $M_{\mathcal{X}}(u,\cdot)$ are too. For any $\omega \in \bigcap_{\alpha \in D} \Omega_{u,\alpha}$ we have

$$m_{\mathcal{X}}(u,\alpha_n) \le s_A(u,\alpha_n) \le M_{\mathcal{X}}(u,\alpha_n)$$

(please note the unspecified dependence of m and M on ω via the $s_{\mathcal{X}_i}$). By the left continuity, also

$$m_{\mathcal{X}}(u,\alpha) \le s_A(u,\alpha) \le M_{\mathcal{X}}(u,\alpha).$$

That means ω is in $\Omega_{u,\alpha}$ for each $\alpha \in (0,1]$. The case $\alpha = 0$ holds as well since we chose D with $0 \in D$. Accordingly, (A.1) holds. That proves that each $\bigcap_{\alpha \in [0,1]} \Omega_{u,\alpha}$, being a countable intersection of measurable events, is measurable.

The space $\mathbb{S}^{p-1} \subset \mathbb{R}^p$ is separable. Let us take a countable dense subset $D' \subseteq \mathbb{S}^{p-1}$. The proof will be complete if we show

$$\bigcap_{u \in \mathbb{S}^{p-1}} \bigcap_{\alpha \in [0,1]} \Omega_{u,\alpha} = \bigcap_{u \in D'} \bigcap_{\alpha \in [0,1]} \Omega_{u,\alpha},$$

since the left-hand side is the event we wish to prove measurable and the right-hand side is a countable intersection of measurable events. As before, only inclusion '$\subset$' needs to be proved. Let us fix an arbitrary $u^* \in \mathbb{S}^{p-1}$. Due to the density of D', there exists a sequence $\{u_n\}_n \subset D'$ such that $u_n \to u^*$ with $u_n \in D'$. Whenever $\omega \in \bigcap_{u\in D'} \bigcap_{\alpha\in[0,1]} \Omega_{u,\alpha}$,

$$m_{\mathcal{X}}(u_n, \alpha) \le s_A(u_n, \alpha) \le M_{\mathcal{X}}(u_n, \alpha) \text{ for all } \alpha \in [0, 1].$$

By the continuity of the support functions for fixed α [19], $u_n \to u^*$ implies

$$m_{\mathcal{X}}(u^*, \alpha) \le s_A(u^*, \alpha) \le M_{\mathcal{X}}(u^*, \alpha) \text{ for all } \alpha \in [0, 1].$$

That establishes

$$\bigcap_{u\in D'} \bigcap_{\alpha\in[0,1]} \Omega_{u,\alpha} \subseteq \bigcap_{\alpha\in[0,1]} \Omega_{u^*,\alpha}.$$

By the arbitrariness of u^*,

$$\bigcap_{u\in D'} \bigcap_{\alpha\in[0,1]} \Omega_{u,\alpha} \subseteq \bigcap_{u\in\mathbb{S}^{p-1}} \bigcap_{\alpha\in[0,1]} \Omega_{u,\alpha},$$

as wished. The proof is complete. □

Proof of Proposition 4.4. In order to show that both expressions defining D_{mS} make sense and are equal, and justify the claim that Fubini's theorem applies, we start by considering the following subset of the product measurable space $\Omega \times \mathbb{S}^{p-1} \times [0, 1]$:

$$Z := \left\{ (\omega, u, \alpha) \in \Omega \times \mathbb{S}^{p-1} \times [0, 1] : \min_{1\le i\le p+1} s_{\mathcal{X}_i(\omega)}(u, \alpha) \le s_A(u, \alpha) \le \max_{1\le i\le p+1} s_{\mathcal{X}_i(\omega)}(u, \alpha) \right\}.$$

Let us prove that Z is measurable, i.e., it is in the product σ-algebra of $\Omega \times \mathbb{S}^{p-1} \times [0, 1]$. Bear in mind that Z is *not* the event $\bigcap_u \bigcap_\alpha \Omega_{u,\alpha} \subseteq \Omega$ from the previous proof.

Given any fuzzy random variable $\mathcal{X}$, the support mapping

$$\tilde{s} : (\omega, u, \alpha) \in \Omega \times \mathbb{S}^{p-1} \times [0, 1] \mapsto s_{\mathcal{X}(\omega)}(u, \alpha) \in \mathbb{R}$$

is a random variable, by [12, Lemma 4] or [1, Proposition 4.6]. Denote by $\tilde{s}_{\mathcal{X}_i}$ the support mapping of each $\mathcal{X}_i$. Also consider the support mapping $\tilde{s}_A$ of A seen as a degenerate fuzzy random variable, namely $\tilde{s}_A(\omega, u, \alpha) = s_A(u, \alpha)$. Then

$$Z = \left(\bigcup_{i=1}^{p+1} \{\tilde{s}_{\mathcal{X}_i} \le \tilde{s}_A\} \right) \cap \left(\bigcup_{i=1}^{p+1} \{\tilde{s}_{\mathcal{X}_i} \ge \tilde{s}_A\} \right),$$

which is a measurable event since the $\tilde{s}_{\mathcal{X}_i}$ and $\tilde{s}_A$ are all random variables. And, accordingly, its indicator function $I_Z : \Omega \times \mathbb{S}^{p-1} \times [0, 1] \to \{0, 1\}$ is measurable (and integrable against probability measures, since it is bounded).

By Fubini's theorem,

$$\int_{\Omega\times\mathbb{S}^{p-1}\times[0,1]} I_Z \, \mathrm{d}(\mathbb{P} \otimes \mathcal{V}_p \otimes \nu) = \int_\Omega \int_{\mathbb{S}^{p-1}\times[0,1]} I_Z(\omega, u, \alpha) \, \mathrm{d}(\mathcal{V}_p \otimes \nu)(u, \alpha) \, \mathrm{d}\mathbb{P}(\omega)$$
$$= \int_{\mathbb{S}^{p-1}\times[0,1]} \int_\Omega I_Z(\omega, u, \alpha) \, \mathrm{d}\mathbb{P}(\omega) \, \mathrm{d}(\mathcal{V}_p \otimes \nu)(u, \alpha).$$

Now, for each $\omega \in \Omega$,

$$\begin{aligned} \textstyle\int_{\mathbb{S}^{p-1}\times[0,1]} I_Z(\omega, u, \alpha) \, \mathrm{d}(\mathcal{V}_p \otimes \nu) \quad (u, \alpha) &= (\mathcal{V}_p \otimes \nu) \big(\{(u, \alpha) \mid I_Z(\omega, u, \alpha) = 1\} \big) \\ &= (\mathcal{V}_p \otimes \nu) \big(\{(u, \alpha) \mid (\omega, u, \alpha) \in Z\} \big) \\ &= (\mathcal{V}_p \otimes \nu) \big(\{(u, \alpha) \mid m_{\mathcal{X}}(u, \alpha) \le s_A(u, \alpha) \le M_{\mathcal{X}}(u, \alpha)\} \big) \end{aligned}$$

whence the second term in the chain of identities is

$$\int_{\Omega} \int_{\mathbb{S}^{p-1}\times[0,1]} I_Z(\omega,u,\alpha)\,\mathrm{d}(\mathcal{V}_p\otimes\nu)(u,\alpha)\,\mathrm{d}\mathbb{P}(\omega)$$
$$= E\left[(\mathcal{V}_p\otimes\nu)\left(\{(u,\alpha)\mid m_{\mathcal{X}}(u,\alpha)\le s_A(u,\alpha)\le M_{\mathcal{X}}(u,\alpha)\}\right)\right].$$

Moreover, for each (u,α),

$$\int_{\Omega} I_Z(\omega,u,\alpha)\,\mathrm{d}\mathbb{P}(\omega)=\mathbb{P}\left(\{\omega\in\Omega\mid m_{\mathcal{X}}(u,\alpha)\le s_A(u,\alpha)\le M_{\mathcal{X}}(u,\alpha)\}\right)$$

whence the third term in the chain of identities is, applying again Fubini's theorem,

$$\begin{aligned}&\int_{\mathbb{S}^{p-1}\times[0,1]}\int_{\Omega} I_Z(\omega,u,\alpha)\,\mathrm{d}\mathbb{P}(\omega)\,\mathrm{d}(\mathcal{V}_p\otimes\nu)(u,\alpha)\\&=\int_{\mathbb{S}^{p-1}}\int_{[0,1]}\mathbb{P}(\{\omega\in\Omega\mid m_{\mathcal{X}}(u,\alpha)\le s_A(u,\alpha)\le M_{\mathcal{X}}(u,\alpha)\})\,\mathrm{d}\nu(\alpha)\,\mathrm{d}\mathcal{V}_p(u).\end{aligned}\tag{A.2}$$

Those are the expressions for $D_{mS}(A;\mathcal{X})$ in Definition 4.3 and (4.5), which are therefore well defined and equivalent since both equal $\int_{\Omega\times\mathbb{S}^{p-1}\times[0,1]} I_Z\,\mathrm{d}(\mathbb{P}\otimes\mathcal{V}_p\otimes\nu)$. □

Proof of Proposition 4.6. It is similar to the proof for the modified simplicial fuzzy depth, but fixing each individual $u\in\mathbb{S}^{p-1}$ and considering the measurable mapping $I_Z(\cdot,u,\cdot)$. □

Proof of Proposition 5.1. Define the events $Q:=\{m_{\mathcal{X}}(u,\alpha)\le s_U(u,\alpha)\}$ and $R:=\{M_{\mathcal{X}}(u,\alpha)\ge s_U(u,\alpha)\}$. Taking into account

$$\mathbb{P}(Q^c\cap R^c)\le\mathbb{P}(m_{\mathcal{X}}(u,\alpha)>M_{\mathcal{X}}(u,\alpha))=0,$$

we obtain

$$\mathbb{P}(Q\cap R)=1-\mathbb{P}(Q^c\cup R^c)=1-\mathbb{P}(Q^c)-\mathbb{P}(R^c).$$

Besides, since $\mathcal{X}_1,\dots,\mathcal{X}_{p+1}$ are independent and identically distributed random variables with distribution $\mathbb{P}_{\mathcal{X}}$,

$$s_{\mathcal{X}_1}(u,\alpha),\dots,s_{\mathcal{X}_{p+1}}(u,\alpha)$$

are independent random variables. Then

$$\mathbb{P}(Q^c)=\mathbb{P}(s_{\mathcal{X}_1}(u,\alpha)>s_U(u,\alpha))^{p+1}\text{ and }\mathbb{P}(R^c)=\mathbb{P}(s_{\mathcal{X}_1}(u,\alpha)<s_U(u,\alpha))^{p+1}.$$

The result follows. In the particular case that $\mathcal{X}\in C^0[\mathcal{F}_c(\mathbb{R}^p)]$, the random variable $s_{\mathcal{X}}(u,\alpha)$ is continuous, whence $\mathbb{P}(s_{\mathcal{X}_1}(u,\alpha)=s_U(u,\alpha))=0$. □

Proof of Theorem 5.2.

Property P1 for D_{mS} and D_{FS}. Let $M\in\mathcal{M}_{p\times p}(\mathbb{R})$ be a non-singular matrix and $A,B\in\mathcal{F}_c(\mathbb{R}^p)$. Let us consider independent and identically distributed random variables $\mathcal{X}_1,\dots,\mathcal{X}_{p+1}$ with distribution $\mathbb{P}_{\mathcal{X}}$ and denote, for any $u\in\mathbb{S}^{p-1}$ and $\alpha\in[0,1]$,

$$\begin{aligned}\bar{m}_{\mathcal{X}}(u,\alpha)&:=\min\{s_{M\cdot\mathcal{X}_1+B}(u,\alpha),\dots,s_{M\cdot\mathcal{X}_{p+1}+B}(u,\alpha)\}\\\bar{M}_{\mathcal{X}}(u,\alpha)&:=\max\{s_{M\cdot\mathcal{X}_1+B}(u,\alpha),\dots,s_{M\cdot\mathcal{X}_{p+1}+B}(u,\alpha)\}.\end{aligned}$$

From the properties of the minimum and maximum, and (2.1),

$$\begin{aligned}\bar{m}_{\mathcal{X}}(u,\alpha)&=\min\{s_{M\cdot\mathcal{X}_1}(u,\alpha),\dots,s_{M\cdot\mathcal{X}_{p+1}}(u,\alpha)\}+s_B(u,\alpha),\\\bar{M}_{\mathcal{X}}(u,\alpha)&=\max\{s_{M\cdot\mathcal{X}_1}(u,\alpha),\dots,s_{M\cdot\mathcal{X}_{p+1}}(u,\alpha)\}+s_B(u,\alpha).\end{aligned}$$

Defining the function

$$g : \mathbb{S}^{p-1} \to \mathbb{S}^{p-1} \text{ with } g(u) = \left(1/\left\| M^T \cdot u \right\|\right) M^T \cdot u$$

and making use of (2.3), we obtain

$$\bar{m}_{\mathcal{X}}(u,\alpha) = \left\| M^T \cdot u \right\| \cdot \min\left\{ s_{\mathcal{X}_1}(g(u),\alpha), \ldots, s_{\mathcal{X}_{p+1}}(g(u),\alpha) \right\} + s_B(u,\alpha),$$
$$\bar{M}_{\mathcal{X}}(u,\alpha) = \left\| M^T \cdot u \right\| \cdot \max\left\{ s_{\mathcal{X}_1}(g(u),\alpha), \ldots, s_{\mathcal{X}_{p+1}}(g(u),\alpha) \right\} + s_B(u,\alpha).$$

Similarly, $s_{M\cdot A+B}(u,\alpha) = \left\| M^T \cdot u \right\| \cdot s_A(g(u),\alpha)$. Consequently, as g is a bijective map,

$$\left\{ (u,\alpha) \in \mathbb{S}^{p-1} \times [0,1] : s_A(u,\alpha) \in [m_{\mathcal{X}}(u,\alpha), M_{\mathcal{X}}(u,\alpha)] \right\} =$$
$$\left\{ (u,\alpha) \in \mathbb{S}^{p-1} \times [0,1] : s_{M\cdot A+B}(u,\alpha) \in [\bar{m}_{\mathcal{X}}(u,\alpha), \bar{M}_{\mathcal{X}}(u,\alpha)] \right\}.$$

Thus $D_{mS}(A;\mathcal{X}) = D_{mS}(M \cdot A + B; M \cdot \mathcal{X} + B)$.

The proof for D_{FS} is analogous.

Property P2 for D_{mS} and D_{FS}. Let $\mathcal{X} \in C^0[\mathcal{F}_c(\mathbb{R}^p)]$ be F-symmetric with respect to some fuzzy set $A \in \mathcal{F}_c(\mathbb{R}^p)$. We begin by maximizing the integrand in (4.5), which, by Proposition 5.1 is $1 - [1 - F_{u,\alpha}(s_U(u,\alpha))]^{p+1} - [F_{u,\alpha}(s_U(u,\alpha))]^{p+1}$. That is equivalent to minimizing

$$[1 - F_{u,\alpha}(s_U(u,\alpha))]^{p+1} + [F_{u,\alpha}(s_U(u,\alpha))]^{p+1}. \tag{A.3}$$

Considering the function

$$f : [0,1] \to \mathbb{R} \text{ with } f(x) = (1-x)^{p+1} + x^{p+1}, \tag{A.4}$$

the expression in (A.3) is the composition of $F_{u,\alpha}$ and f. The function $F_{u,\alpha}$ is non-decreasing and f is strictly decreasing in $[0, 1/2]$ and strictly increasing in $[1/2, 1]$, with a minimum at $1/2$. Thus (A.3) is minimized at any $t \in \mathbb{R}$ such that $F_{u,\alpha}(t) = 1/2$ for all $u \in \mathbb{S}^{p-1}$ and $\alpha \in [0,1]$. By (2.4) and the assumption that $\mathcal{X} \in C^0[\mathcal{F}_c(\mathbb{R}^p)]$, the point $s_A(u,\alpha)$ satisfies that condition for each $u \in \mathbb{S}^{p-1}$ and $\alpha \in [0,1]$.

Since A maximizes the integrand in (4.5) and (4.6) for each (u,α), clearly it maximizes both $D_{mS}(\cdot,\mathcal{X})$ and $D_{FS}(\cdot,\mathcal{X})$.

Property P3a for D_{mS}. It suffices to prove $D_{mS}\left((1-\lambda)A + \lambda B;\mathcal{X}\right) - D_{mS}(B;\mathcal{X}) \geq 0$ with $B \in \mathcal{F}_c(\mathbb{R}^p)$ and $\lambda \in [0,1]$. Recall that $\mathcal{X} \in C^0[\mathcal{F}_c(\mathbb{R}^p)]$ is F-symmetric with respect to A. Thus, each $s_{\mathcal{X}}(u,\alpha)$ is a continuous random variable which is centrally symmetric with respect to $s_A(u,\alpha)$ and $F_{u,\alpha}(s_A(u,\alpha)) = 1/2$.

Set

$$x^{\lambda}_{u,\alpha} := (1-\lambda)s_A(u,\alpha) + \lambda s_B(u,\alpha). \tag{A.5}$$

By (4.5), Proposition 5.1 and the linearity of the support function,

$$\begin{aligned} &D_{mS}\left((1-\lambda)\cdot A + \lambda \cdot B;\mathcal{X}\right) - D_{mS}(B;\mathcal{X}) = \\ &\quad \int_{\mathbb{S}^{p-1}} \int_{[0,1]} \left\{ \left[1 - F_{u,\alpha}(s_B(u,\alpha))\right]^{p+1} + \left[F_{u,\alpha}(s_B(u,\alpha))\right]^{p+1} \right. \\ &\quad \left. - \left[1 - F_{u,\alpha}\left(x^{\lambda}_{u,\alpha}\right)\right]^{p+1} - \left[F_{u,\alpha}\left(x^{\lambda}_{u,\alpha}\right)\right]^{p+1} \right\} \mathrm{d}\nu(\alpha)\, \mathrm{d}\mathcal{V}_p(u). \end{aligned} \tag{A.6}$$

Consider again the function $f : [0,1] \to \mathbb{R}$ with $f(x) = (1-x)^{p+1} + x^{p+1}$. Now if $s_B(u,\alpha) \leq s_A(u,\alpha)$, we have $s_B(u,\alpha) \leq x^{\lambda}_{u,\alpha}$ and

$$F_{u,\alpha}(s_B(u,\alpha)) \leq F_{u,\alpha}\left(x^{\lambda}_{u,\alpha}\right) \leq 1/2.$$

Since f is decreasing in $[0, 1/2]$,

$$f(F_{u,\alpha}(s_B(u,\alpha))) \geq f\left(F_{u,\alpha}\left(x^{\lambda}_{u,\alpha}\right)\right).$$

That implies that the integrand in (A.6) is non-negative. The same conclusion is reached in the case $s_B(u,\alpha) \geq s_A(u,\alpha)$, using the fact that f is increasing in $[1/2, 1]$. Thus

$$D_{mS}\big((1-\lambda)A + \lambda B; \mathcal{X}\big) - D_{mS}(B; \mathcal{X}) \geq 0.$$

Property P3a for D_{FS}. Let $B \in \mathcal{F}_c(\mathbb{R}^p)$ and $\lambda \in [0, 1]$. Recall $\mathcal{X} \in C^0[\mathcal{F}_c(\mathbb{R}^p)]$ is F-symmetric with respect to A. Using (4.6) and $x^{\lambda}_{u,\alpha}$ as in (A.5),

$$\begin{aligned}&D_{FS}\big((1-\lambda)\cdot A + \lambda \cdot B\big) - D_{FS}(B; \mathcal{X}) = \\ &\inf_{u\in\mathbb{S}^{p-1}} \int_{[0,1]} \left(1 - \left(1 - F_{u,\alpha}\left(x^{\lambda}_{u,\alpha}\right)\right)^{p+1} - F_{u,\alpha}\left(x^{\lambda}_{u,\alpha}\right)^{p+1}\right) \mathrm{d}\nu(\alpha) - \\ &\inf_{u\in\mathbb{S}^{p-1}} \int_{[0,1]} \Big(1 - (1 - F_{u,\alpha}(s_B(u,\alpha)))^{p+1} - F_{u,\alpha}(s_B(u,\alpha))^{p+1}\Big) \mathrm{d}\nu(\alpha).\end{aligned}$$

Following the arguments in the proof of Property P3a for D_{mS},

$$\begin{aligned}&\int_{[0,1]} \left(1 - \left(1 - F_{u,\alpha}\left(x^{\lambda}_{u,\alpha}\right)\right)^{p+1} - F_{u,\alpha}\left(x^{\lambda}_{u,\alpha}\right)^{p+1}\right) \mathrm{d}\nu(\alpha) \geq \\ &\int_{[0,1]} \Big(1 - (1 - F_{u,\alpha}(s_B(u,\alpha)))^{p+1} - F_{u,\alpha}(s_B(u,\alpha))^{p+1}\Big) \mathrm{d}\nu(\alpha)\end{aligned}$$

for each $u \in \mathbb{S}^{p-1}$. The inequality is preserved if we take the infimum on both sides. Thus $D_{FS}\big((1-\lambda)\cdot A + \lambda \cdot B; \mathcal{X}\big) \geq D_{FS}(B; \mathcal{X})$.

Property P3b for D_{mS} and D_{FS}. From [8, Theorem 5.4], P3b is equivalent to P3a for any ρ_r metric with $r \in (1, \infty)$. $\square$

Proof of Proposition 5.4. Let $A, B \in \mathcal{F}_c(\mathbb{R}^p)$ be fuzzy sets such that A maximizes $D_{FS}(\cdot; \mathcal{X})$. Any C_u defined as appears in the definition of $\mathfrak{B}$ satisfies $C_u \subseteq [0, 1]$ and $\nu(C_u) = 1$. Thus,

$$D_{FS}(A + n \cdot B; \mathcal{X}) = \inf_{u\in\mathbb{S}^{p-1}} \int_{C_u} \mathbb{P}\left(s_{A+n\cdot B}(u,\alpha) \in [m_{\mathcal{X}}(u,\alpha), M_{\mathcal{X}}(u,\alpha)]\right) \mathrm{d}\nu(\alpha)$$

and, fixing an arbitrary $u \in \mathbb{S}^{p-1}$,

$$D_{FS}(A + n \cdot B; \mathcal{X}) \leq \int_{C_u} \mathbb{P}\left(s_{A+n\cdot B}(u,\alpha) \in [m_{\mathcal{X}}(u,\alpha), M_{\mathcal{X}}(u,\alpha)]\right) \mathrm{d}\nu(\alpha).$$

Using the Fatou's Lemma,

$$\begin{aligned}&\limsup_{n\to\infty} D_{FS}(A + n \cdot B; \mathcal{X}) \leq \\ &\limsup_{n\to\infty} \int_{C_u} \mathbb{P}\left(s_{A+n\cdot B}(u,\alpha) \in [m_{\mathcal{X}}(u,\alpha), M_{\mathcal{X}}(u,\alpha)]\right) \mathrm{d}\nu(\alpha) \leq \\ &\int_{C_u} \limsup_{n\to\infty} \mathbb{P}\left(s_{A+n\cdot B}(u,\alpha) \in [m_{\mathcal{X}}(u,\alpha), M_{\mathcal{X}}(u,\alpha)]\right) \mathrm{d}\nu(\alpha).\end{aligned} \tag{A.7}$$

By Proposition 5.1 and (2.1),

$$\begin{aligned}&\mathbb{P}\left(s_{A+n\cdot B}(u,\alpha)\in[m_{\mathcal{X}}(u,\alpha),M_{\mathcal{X}}(u,\alpha)]\right)=\\&1-[1-F_{u,\alpha}(s_A(u,\alpha)+n\cdot s_B(u,\alpha))]^{p+1}-[F_{u,\alpha}(s_A(u,\alpha)+n\cdot s_B(u,\alpha))]^{p+1}.\end{aligned} \tag{A.8}$$

As $F_{u,\alpha}$ is the distribution function of the real random variable $s_{\mathcal{X}}(u,\alpha)$, for each $\alpha\in C_u$, the limit of $F_{u,\alpha}(s_A(u,\alpha)+n\cdot s_B(u,\alpha))$ is 1 if $s_B(u,\alpha)>0$ and 0 if $s_B(u,\alpha)<0$. Since $B\in\mathfrak{B}$, we have $s_B(u,\alpha)\neq 0$ for all $\alpha\in C_u$. Using (A.8), whether $s_B(u,\alpha)$ is larger or smaller than 0 one obtains

$$\lim_{n\to\infty}\mathbb{P}\left(s_{A+n\cdot B}(u,\alpha)\in[m_{\mathcal{X}}(u,\alpha),M_{\mathcal{X}}(u,\alpha)]\right)=0,$$

for every $\alpha\in C_u$, which implies, by (A.7), that $\lim_n D_{FS}(A+n\cdot B;\mathcal{X})=0$.

The proof for D_{mS} is analogous. □

Proof of Theorem 5.5.

Property P1. The proof is analogous to that of P1 in Theorem 5.2.

Property P4b. Let $\mathfrak{d}:=\{d_r:r\in[1,\infty]\}\cup\{\rho_r:r\in[1,\infty)\}$ be the set of metrics of type d_r and ρ_r. Let us fix $d\in\mathfrak{d}$. Denoting by A a fuzzy set that maximizes $D_{nS}(\cdot;\mathcal{X})$, let $\{A_n\}_n$ be a sequence of fuzzy sets such that $\lim_n d(A,A_n)=\infty$. That implies, see [8, Proposition 8.3], that there exists $u_0\in\mathbb{S}^{p-1}$ and $\alpha_0\in[0,1]$ such that

$$\lim_n |s_{A_n}(u_0,\alpha_0)|=\infty. \tag{A.9}$$

By (4.4),

$$D_{nS}(A_n;\mathcal{X})\le\mathbb{P}\left(s_{A_n}(u_0,\alpha_0)\in[m_{\mathcal{X}}(u_0,\alpha_0),M_{\mathcal{X}}(u_0,\alpha_0)]\right),$$

which, by Proposition 5.1, results in

$$\begin{aligned}D_{nS}(A_n;\mathcal{X})\le&1-[1-F_{u_0,\alpha_0}(s_{A_n}(u_0,\alpha_0))]^{p+1}\\&-[F_{u_0,\alpha_0}(s_{A_n}(u_0,\alpha_0))-\mathbb{P}(s_{\mathcal{X}_1}(u_0,\alpha_0)=s_{A_n}(u_0,\alpha_0))]^{p+1}.\end{aligned}$$

Taking limits and using (A.9) and the properties of the cumulative distribution function, we obtain $\lim_n D_{nS}(A_n;\mathcal{X})=0$.

Property P4a. According to [8, Proposition 5.8], P4b implies P4a for the metrics d_r and ρ_r for any $r\in[1,\infty)$. □

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