Obstructions to the existence of trapped submanifolds in relativistic spacetimes

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Abstract

Obstructions to the existence of trapped submanifolds in spacetimes of arbitrary dimension are given. These obstructions are obtained under natural geometric assumptions, which can be applied to initial data set for Einstein equations, assuring the absence of trapped submanifolds in its development. We highlight that for several of our results the existence of symmetries in the spacetime is not necessary.

Keywords: Compact spacelike submanifold; causal mean curvature; Cauchy problem in General Relativity.

1 Introduction

Physical and geometric interest come together in the study of trapped spacelike submanifold embedded in a relativistic spacetime. The notion of trapped surfaces was introduced by Penrose for the case of compact (without boundary) spacelike surfaces embedded in four-dimensional spacetimes, being the original concept given in terms of the signs or the vanishing of the so-called null expansions (see [15], [13]). More recently, a characterization about the causal orientation of the mean curvature vector field of the spacelike embedded surface were widely adopted by most authors. This one allows the generalization to arbitrary dimensional spacelike submanifolds embedded in spacetimes with dimension $n \geq 4$ (see [17] for more details).

We now recall the most frequently used nomenclature. An embedded 2-codimensional spacelike submanifold S is called a *future* (resp. *past*) *trapped surface* if its mean curvature vector field $\vec{H}$ is timelike and future pointing (resp. past pointing). When $\vec{H}$ is causal and future pointing over all the spacelike submanifold and timelike at least at a point of S, the submanifold is called *nearly future trapped*, and correspondingly for *nearly past trapped*. When $\vec{H}$ is causal and future-pointing everywhere, and non-zero at least at a point of S, the submanifold is said to be *weakly future trapped*, similarly for *weakly past trapped*. Finally, the submanifold S is called *marginally future trapped* if $\vec{H}$ is lightlike and future pointing all over S and non-zero at least at a point of S, and analogously for the past case. The case $\vec{H} \equiv \vec{0}$ corresponds to extremal or symmetric submanifolds. As usual in this context, the vector $\vec{0}$ is consider causal, future or past.

From the classical notion of trapped surface, Hawking and Penrose showed the existence of singularities in the evolution of a Cauchy initial data set containing a trapped surface (see [12]). However, the most recent notions mentioned above have been gaining relevance significantly. So, for example, marginally trapped surfaces play an important role in the study of the weak cosmic censorship conjecture (see [1]). Also, cases with arbitrary dimension and codimension have been recently considered obtaining interesting results (see for example [10]).

In this work we obtain several obstructions for the existence of trapped (in a broad sense) compact spacelike submanifolds of arbitrary dimension and codimension, embedded in a relevant class of spacetimes such as the stationary and globally hyperbolic ones. Alternatively, we can get different classes of initial data set for Einstein's equation such that a development of that data no admits a trapped compact spacelike submanifolds.

2 Preliminaries

We denote by $(M, \overline{g})$ an arbitrary m-dimensional *spacetime*, namely, a connected m-dimensional oriented and time-oriented Lorentzian manifold provided with the Lorentzian metric tensor $\overline{g}$ (see [16]).

Recall that an isometric immersion $x : S^n \to M^m$, $2 \leq n < m$, in a spacetime $(M, \overline{g})$ is *spacelike* if the induced metric via x is Riemannian. In this situation, S is said to be an (immersed) spacelike submanifold of M (see, [18, Def. 1.27]). However, from a physical point of view the concept of submanifold in a spacetime usually corresponds to the case of embedding submanifold, i.e., S^n is a topological subspace of M and x is the inclusion map. Although some of the geometric results that appear in this work can be formulated for the more general framework of immersed submanifolds, our general interest is centered in the case of embedded submanifolds. All the submanifolds considered in this work are supposed connected and compactness is understood without boundary.

The extrinsic geometry of a submanifold S in a spacetime M is encoded by its second fundamental form $\mathrm{II} : \mathfrak{X}(S) \times \mathfrak{X}(S) \to \mathfrak{X}^{\perp}(S)$, given by

$$\mathrm{II}(V, W) := \left(\overline{\nabla}_V W\right)^{\perp} ,$$

where $\overline{\nabla}$ denotes the Levi-Civita connection of the metric $\overline{g}$. The *mean curvature vector field* can be defined according to several conventions. As it is usual in General Relativity, we define it as minus the metric contraction (without dividing by the dimension of the submanifold) of the second fundamental form, i.e.,

$$\vec{H} = -\sum_{i=1}^{n} \mathrm{II}(E_i, E_i),$$

where $\{E_1, ..., E_n\}$ denotes a local orthonormal frame on S. Note that this choice of the negative sign is opposite to the one usually taken in Differential Geometry.

When the immersed submanifold $x : S \to M$ is a spacelike hypersurface (i.e. $m = n+1$) and the induced metric is Riemannian, the time-orientation of M allows to take a global unitary timelike vector field N on S pointing to the future.

Let us represent by g the induced metric on the spacelike hypersurface S and, by ∇ its induced Levi-Civita connection. The Gauss and Weingarten formulae of S are respectively

$$\overline{\nabla}_V W = \nabla_V W - g(AV, W)N \,,$$

$$AV = -\overline{\nabla}_V N \,,$$

for all $V, W \in \mathfrak{X}(S)$, where A is the shape operator associated with N. Then, the *mean curvature function* associated with N is given, according to our convention, by $H := \mathrm{trace}(A)$.

The mean curvature function is identically zero if and only if the spacelike hypersurface is, locally, a critical point of the n-dimensional area functional for compactly supported normal variations. A spacelike hypersurface with $H = 0$ is called a *maximal* hypersurface.

As is well-known, spacelike hypersurfaces in general, and of constant mean curvature in particular, are interesting initial data sets to study the Einstein equation (see, for instance [7, Chap. 8]). In fact, let $(M, \overline{g})$ be an $(n+1)$-dimensional spacetime and denote by $\overline{\mathrm{Ric}}$ and $R(\overline{g})$ its Ricci tensor and its scalar curvature, respectively. Consider a *stress-energy tensor field T* on $\overline{M}$, namely a 2-covariant symmetric tensor which satisfies some reasonable conditions from a physical viewpoint (say $T(v, v) \geq 0$ for any timelike vector v, see for example [16, Section 3.3]). It is said that the spacetime $(M, \overline{g})$ is *an exact solution to the Einstein equation with zero cosmological constant and source T*, if the spacetime satisfies

$$\overline{\mathrm{Ric}} - \frac{1}{2}\mathrm{R}(\overline{g})\overline{g} = T. \tag{1}$$

If (1) holds, then the following constraint equations are satisfied on each spacelike hypersurface S in M

$$R(g) - \mathrm{trace}(A^2) + \mathrm{trace}(A)^2 = \varphi, \tag{2}$$

$$\mathrm{div}(A) - \nabla \mathrm{trace}(A) = X, \tag{3}$$

where g is the Riemannian metric on S induced by $\overline{g}$, $R(g)$ its scalar curvature, and $\varphi \in C^\infty(S)$ and X a 1-form depend on the stress energy tensor T. We remark that equations (2) and (3) are respectively obtained from the classical Gauss and Codazzi equations for the spacelike hypersurface $x : S \to M$.

Conversely, given $\varphi \in C^\infty(S)$ and $X \in \mathfrak{X}(S)$, they can be seen as differential equations with unknown g and A. Thus, an initial data set for the Cauchy problem in General Relativity is given by a triple (S, g, A), where (S, g) is an n-dimensional Riemannian manifold and $A : \mathfrak{X}(S) \to \mathfrak{X}(S)$ is a $(1,1)$-tensor field, self-adjoint with respect to g, which satisfies the constraint equations (2) and (3).

Global hyperbolicity is an assumption for physically reasonable spacetimes. Using a well-known characterization, we can recall that a spacetime is globally hyperbolic if it admits a Cauchy hypersurface, i.e., a $\mathcal{C}^0$ spacelike hypersurface which is crossed exactly once by any inextensible timelike curve. Indeed, if the spacetime is globally hyperbolic, an embedded smooth Cauchy hypersurface may be found (see, [11], [3], [4], [5]).

A solution to the Cauchy problem of the Einstein equation (1) corresponding to the initial data (S, g, A) is a spacetime $(\overline{M}, \overline{g})$, such that (S, g) is an (embedded) Cauchy spacelike hypersurface and the tensor field A coincides with the shape operator of the embedding. In this setting, the spacetime $(\overline{M}, \overline{g})$ is called a development of the given initial data set. In [6], Choquet and Geroch shown that given an initial data set for the Einstein's equation satisfying the constrain conditions, there exists a *development of that data set*, which is maximal in the sense that it is an extension of every other development.

3 Set up

For the sake of making this work self-contained, we will reformulate in an intrinsic language and for arbitrary spacelike submanifolds, an integral formula previously obtained in [2, For. 4] for the case of surfaces in 4-dimensional spacetimes.

Let $x : S^n \to M^m$, $n < m$, be an immersed spacelike submanifold in a spacetime $(M, \overline{g})$. Let $\{E_i\}_{i=1}^{n}$ and $\{N_j\}_{j=1}^{m-n}$ be local orthonormal frames in the tangent vector bundle TS and in the normal vector bundle $T^{\perp}S$, respectively. Given a smooth vector field X on the immersion, we define the operator $\mathrm{div}_S : \mathfrak{X}(M) \longrightarrow C^{\infty}(S)$, given by

$$\mathrm{div}_S X := \sum_i \overline{g}\left(\overline{\nabla}_{E_i} X, E_i\right).$$

At any point $p \in S$, we can split $X(p) = X(p)^{\top} + X(p)^{\perp}$, being $X^{\top}$ and $X^{\perp}$ the tangent and normal projections relative a S. So,

$$\begin{aligned} \mathrm{div}_S X^{\perp} &= \sum_i \overline{g}\left(\overline{\nabla}_{E_i} X^{\perp}, E_i\right) = \sum_{i,j} \epsilon_j\, \overline{g}\left(X^{\perp}, N_j\right) \overline{g}\left(\overline{\nabla}_{E_i} N_j, E_i\right) \\ &= \overline{g}\left(X, \vec{H}\right), \end{aligned}$$

where $\epsilon_j = \overline{g}(N_j, N_j)$. Hence,

$$\mathrm{div}_S\,(X) = \mathrm{div}_S\left(X^{\perp}\right) + \mathrm{div}_S\left(X^{\top}\right) = \overline{g}\left(X, \vec{H}\right) + \mathrm{div}\left(X^{\top}\right), \tag{4}$$

where div denotes the divergence operator on (S, g), and $\vec{H}$ is the mean curvature vector field of x. We should point out that the operator div_S defined as (4) is well-defined, i.e., it is independent of the chosen local orthonormal frame.

In particular, if the submanifold S is compact (without boundary), then making use of the Gauss theorem, we can obtain the following integral formula,

$$\int_S \left\{\mathrm{div}_S\,(X) - \overline{g}\left(X, \vec{H}\right)\right\} dV_g = 0, \tag{5}$$

where dV_g is the Riemannian volume element of (S, g).

4 Obstructions to the existence of trapped submanifolds for stationary spacetimes

The notion of symmetry is essential in Physics. In General Relativity, an infinitesimal symmetry is usually based on the assumption of the existence of an one-parameter group of transformations generated by a Killing vector field or, more generally, a conformal Killing vector field. In fact, an usual simplification for the search of exact solutions to the Einstein equation is to assume the existence, a priori, of such an infinitesimal symmetry ([8], [9]). Although different causal characters for the infinitesimal symmetry may be assumed, the timelike choice is natural, since the integral curves of a timelike infinitesimal symmetry provide a privileged class of observers or test particles in the spacetime.

Recall that a spacetime admitting a globally defined timelike Killing vector field is called stationary. In this family of spacetimes is well-known the absence of marginally trapped, nearly trapped or trapped compact imbedded submanifolds (see [14, Th. 1]).

In the following result we describe a class of initial data set for the Cauchy problem of the Einstein's equations, whose development can not be a stationary spacetime.

Theorem 1 *Let (S, g, A) be an initial data set for the Cauchy problem of the Einstein's equation. Assume any of the hypothesis,*

(i) the Riemannian manifold S admits a compact submanifold P, whose mean curvature vector field $\vec{h}$ in S satisfies

$$\left\|\vec{h}\right\| < \left|\mathrm{trace}_{|_P} A\right| , \tag{6}$$

where $\mathrm{trace}_{|_P} A$ *is the trace of the restriction of A to TP.*

(ii) the Riemannian manifold S admits a compact submanifold P, whose mean curvature vector field $\vec{h}$ in S has no zeros and satisfies

$$\left\|\vec{h}\right\| \leq \left|\mathrm{trace}_{|_P} A\right| . \tag{7}$$

Then, a development spacetime $(M, \tilde{g})$ of this data set cannot be a stationary spacetime.

Proof.

We reason by reductio ad absurdum. So, suppose that a such compact submanifold exists. Consider $\{E_1, ..., E_k\}$ a local orthonormal frame on P, and let us extend it to a local orthonormal tangent frame $\{E_1, ...E_k, U_{k+1}, ..., U_n\}$ on S. The mean curvature vector field $\vec{H}$ of P in the spacetime M is given by

$$\begin{aligned} -\vec{H} &= \sum_{i=1}^{k} II_P(E_i, E_i) = \sum_{i=1}^{k} (\overline{\nabla}_{E_i} E_i)^{\perp} \\ &= \sum_{j=k+1}^{n} \sum_{i=1}^{k} g(\overline{\nabla}_{E_i} E_i, U_j) U_j - \sum_{i=1}^{k} g(\overline{\nabla}_{E_i} E_i, N) N, \end{aligned}$$

where N denotes the future unitary normal vector field on S in the development M. Taking in account this equality it is clear the relation

$$\vec{H} = \vec{h} + \left(\mathrm{trace}_{|_P} A\right) N . \tag{8}$$

Using (8), we know in the first case that $\vec{H}$ is a timelike vector field. Now, making use of integral formula (5) for a timelike Killing vector field, the contradiction appears. Taking into account that the vector field $\vec{H}$ must be causal and time-oriented, an analogous reasoning holds for (ii).

□

Remark 2 *Note that Theorem 1 presents a clear obstruction on an initial data set to belong to the class of initial data set, whose Cauchy development spacetime is stationary.*

An alternative and interesting formulation of the previous result is given as follows,

Corollary 3 *A spacetime admitting a spacelike hypersurface S which satisfies the hypotheses (i) or (ii), being A its shape operator, cannot admit a global timelike Killing vector field.*

Remark 4 On the optimality hypotheses of the Theorem 1. Consider the Lorentzian product manifold $M = \mathbb{R} \times \mathbb{S}^3$, where $\mathbb{S}^3$ denote the unitary Riemannian 3-dimensional sphere. Let $t_0 \in \mathbb{R}$ and take us $S = \{t_0\} \times \mathbb{S}^3$. If we take a copy of the sphere $\mathbb{S}^2$ embedded in $\mathbb{S}^3$, then we have a clear counterexample for the Theorem 1 when the assumption (i) does not hold. To deny the assumption (ii), consider the upper hyperboloid leaf $\mathbb{H}^3$ in $\mathbb{L}^4$ and take us a suitable totally umbilical round sphere in the hyperbolic space $\mathbb{H}^3$.

5 Obstructions to the existence of trapped submanifolds in absence of symmetries

In the previous Theorem 1, the presence of a infinitesimal symmetry given by a timelike vector field is essential. Nevertheless, in what follows we obtain several obstructive results, being noticeable that no special symmetry on the spacetime is assumed. So, we will show that the existence of a certain spacelike hypersurface S in the spacetime assures the non-existence of compact spacelike submanifolds with strictly causal mean curvature around S. We can transfer this fact imposing suitable conditions on a set of initial data for the Cauchy problem in General Relativity, which obstruct the existence of compact spacelike submanifolds with strictly causal mean curvature in a development of that data.

Let S be a spacelike hypersurface embedded in the spacetime M and take us N an unitary timelike future normal vector field on S. We can define a natural extension $\overline{N}$ of N on an open tubular neighbourhood of S in M. Indeed, given an arbitrary point $p \in S$ we consider the unique geodesic $\phi_p(t) = \exp_p(tN_p)$ with velocity N_p at p. Therefore, the map $\phi_p(t)$ define the flow of the extension $\overline{N}$. Due to the construction itself, it is clear that the 1-form metrically equivalent to the vector field $\overline{N}$ is closed. As a direct consequence we have that

$$\overline{g}\left(\overline{\nabla}_X \overline{N}, Y\right) = \frac{1}{2}\left(\mathcal{L}_{\overline{N}}\overline{g}\right)(X, Y)$$

for $X, Y \in \mathfrak{X}(M)$. Hence, the $(1,1)$-tensor field $\overline{A} : \mathfrak{X}(M) \to \mathfrak{X}(M)$ defined by $\overline{g}\left(\overline{A}X, Y\right) = -\frac{1}{2}\left(\mathcal{L}_{\overline{N}}\overline{g}\right)(X, Y)$ extends in a canonical way the shape operator of S.

We denote by $\phi_{(-\epsilon,\epsilon)}S$ the open subset in the spacetime M given by the points q of M such that $q = \phi_p(r)$ for $p \in S$ and $r \in (-\epsilon, \epsilon)$. Physically, it is the portion of the spacetime which is Cauchy-development forwards and backwards from S up to a quantity of ϵ. Analogously we can define $\phi_{(-\epsilon,\epsilon)}\Omega$, being $\Omega \subset S$ a domain in S.

Theorem 5 *Let (S, g, A) be an initial data set for the Einstein's Equation such that the tensor field A is negative (resp. positive) definite. Then, for any compact domain $\Omega \subset S$, there exists $\epsilon > 0$ such that there is no compact spacelike submanifold with future (resp. past) causal mean curvature in $\phi_{(-\epsilon,\epsilon)}\Omega$, included the extremal case with $\vec{H} = \vec{0}$.*

Proof. Consider a geodesic $\gamma_p : (-\delta, \delta) \to M$, with $\gamma_p(0) = p \in \Omega \subset S$ and $\gamma_p'(0) = N_p$. On γ_p we consider the function $\eta := \det(\overline{A} \circ \gamma_p)$. Then, there exists a positive number $\delta^+ < \delta$, such that the sign of η is constant on $(-\delta^+, \delta^+)$ and this implies that $\overline{A}$ is negative definite on that interval. Otherwise, it would have a zero eigenvalue and so its determinant would be null, which is a contradiction.

Now, since Ω is compact, we can find another positive constant ϵ such that for any geodesic $\gamma_q : (-\epsilon, \epsilon) \to M$, $q \in \Omega$, the same holds. Thus, $\overline{A}$ is negative definite on $\phi_{(-\epsilon,\epsilon)}\Omega$.

The proof finishes using formula (5) for the timelike vector field $\overline{N}$, whose divergence is positive. The proof is similar when $\overline{A}$ is positive defined.

□

Previous results has an interesting consequence if we attend to closed models.

Corollary 6 *Let (S, g, A) be an initial data set for the Einstein's Equation, such that the Riemannian manifold S is compact, and the tensor field A is negative (resp. positive) definite. Then, there exist two positive constants σ_i, $i = 1, 2$, such that there is no compact spacelike submanifold with future (resp. past) causal mean curvature, in $\phi_{(-\sigma_1,\sigma_2)}S$, included the extremal case with $\vec{H} = \vec{0}$. In particular, there is no compact marginally, weakly, nearly future (resp. past) trapped, or future (resp. past) trapped submanifold in $\phi_{(-\sigma_1,\sigma_2)}S$.*

Remark 7 Corollary 6 admits the following nice topological interpretation: any compact spacelike submanifold with future (resp. past) causal mean curvature is far from any simply-connected compact Cauchy hypersurface with negative (resp. positive) definite shape operator.

We may weak the assumption on the operator A, assuming an initial data set (S, g, A) with A negative semi-definite. Nevertheless, we can guarantee the (global) character of the operator $\overline{A}$ via a curvature assumption. In fact, let γ be an arbitrary integral curve of the geodesic timelike vector field $\overline{N}$ and consider a spacelike vector field X on γ, such that X commute with $\overline{N}$. Then

$$\begin{aligned} -\overline{g}\left(\overline{R}(\overline{N}, X)\overline{N}, X\right) &= \overline{g}\left(\overline{\nabla}_{\overline{N}}\overline{\nabla}_X\overline{N}, X\right) \\ &= \frac{1}{2}\gamma'\left(\left(\mathcal{L}_{\overline{N}}\overline{g}\right)(X, X)\right) - \overline{g}\left(\overline{\nabla}_X\overline{N}, \overline{\nabla}_X\overline{N}\right) . \end{aligned}$$

Since the last addend is always non-positive, the previous equality means that $\gamma'\left(\left(\mathcal{L}_{\overline{N}}\overline{g}\right)(X, X)\right)$ is non-negative if so is the sectional curvature of timelike planes in the spacetime. In particular, since $\left(\mathcal{L}_{\overline{N}}\overline{g}\right)(X, X)$ is non-negative at S (for any tangent vector X), then the same holds in the future of S. Thus, if the sectional curvature of timelike planes is non-negative, then $\left(\mathcal{L}_{\overline{N}}\overline{g}\right)$ is positive or semi-definite in the future of S. Therefore, we can state

Theorem 8 *A spacetime whose timelike sectional curvatures are non-negative and which admits an (intrinsic) initial data set with negative (resp. positive) semi-definite shape operator A, does not admit a compact spacelike submanifold in the future (resp. in the past) of the initial data set, with future (resp. past) causal mean curvature vector field, unless $\vec{H} = \vec{0}$. In particular, there is no compact marginally, weakly, nearly future (resp. past) trapped, or future (resp. past) trapped submanifold in the future $I^+(S)$ (resp. in the past $I^-(S)$) of the initial data set.*

Remark 9 Note that no special symmetry on (S, g) is assumed in previous results in this section, thus, the results can be applied to the case when S is any Cauchy hypersurface in a spacetime (which, a posteriori, can be regarded as a slice in a global orthogonal splitting (see [5]).

Taking into account this last remark, we can give, for example, a reformulation of the Theorem 8, as follows.

Corollary 10 *A globally hyperbolic spacetime whose timelike sectional curvatures are non-negative and which has a Cauchy hypersurface S with negative (resp. positive) semi-definite shape operator A, does not admit a compact spacelike submanifold in the future (resp. in the past) of S with future (resp. past) causal mean curvature vector field, unless $\vec{H} = \vec{0}$. In particular, there is no compact marginally, weakly, nearly future (resp. past) trapped, or future (resp. past) trapped submanifold in the future $I^+(S)$ (resp. in the past $I^-(S)$) of the Cauchy hypersurface S.*

Remark 11 Note that in Theorems 8 and 10 no topological conditions are assumed on the spacelike hypersurface.

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6 Data availability

Not applicacable.

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