Spacelike surfaces which admit a nondegenerate null normal section in a Lorentzian space form

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Abstract

In this paper, we develop a formula for spacelike surfaces in a 4-dimensional Lorentzian space form which involves its mean curvature vector field and the Gauss curvature of the induced metric and the Gauss curvature of the second fundamental form associated to a non-degenerate null normal section. By means of this formula, we stablish several sufficient conditions for compact spacelike surfaces with constant Gauss curvature to have a null umbilical direction. As another application, we give a new proof of the Liebmann rigidity theorem in Euclidean, hemispherical and hyperbolic spaces, and in the De Sitter spacetime.

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1 Introduction

Consider a spacelike surface S in a 4-dimensional spacetime. At least locally, there are two future-directed null vector fields orthogonal to it, one of them ξ, pointing out to a direction will be called inwards, and the other one, η, outwards. At each event $p \in S$, the shape operator associated to ξ, A_ξ (see definition in (1) for details), acting on a tangent vector $v \in T_pS$, measures how does the inwards directed family of light rays (orthogonal to S) converge or diverge in the direction of v. Hence, if the mean curvature function associated to A_ξ is positive (resp. negative), then the inwards light rays tends to converge (resp. diverge) *in average*.

Other physically relevant function is the *mean null curvature function* of a spacelike surface S, defined as $\phi_S = (\mathrm{tr}A_\xi)(\mathrm{tr}A_\eta)$, where ξ, η are future-directed null vectors orthogonal to S with $\langle \xi, \eta \rangle = -1$. The surface S is called *future converging* when $\phi_S > 0$ and $\mathrm{tr}A_\xi > 0$, i.e., when both families of light rays ortogonal to it converge. If, in addition, S is compact, then S is a *trapped surface*. These surfaces are the precursors of the singularities in gravitational collapse (see, for instance, [6], [7], [14]).

In this work, we are interested in spacelike surfaces such that at least one of its null-second fundamental forms II_ξ (or II_η), defined in (Eq. 5), is non degenerate. It physically means that, in each event of S, there does not exist any direction $v \in T_pM$ such that the inwards directed light rays neither approach, nor separate, nor rotate. As a consequence, II_ξ defines a new metric on the surface, which may be definite or not. If II_ξ is positive definite (resp. negative definite), then inwards directed light rays orthogonal to S converge (resp. diverge) along any direction at each event of S. In the indefinite case, the new metric is Lorentzian and the inwards directed light rays converge in some directions and diverge in others. An analogous interpretation may be done for the null direction η, but in this case the light rays are now directed outwards. Our first aim precisely consists in to find some sufficient conditions to assure that both null-second fundamental forms are non degenerate (Prop. 2.1).

On the other hand, it is natural to ask ourselves about what relation is there between both metrics on such surfaces. With this aim, we find a formula which relate the Gauss curvatures of S when is endowed with the first and the second fundamental forms (Eq. 17). This formula widely generalizes the given in [2] and [3] for surfaces in the 3-dimensional De Sitter spacetime, and in [12] for surfaces in 4-dimensional Lorentz-Minkowski spacetime through a light cone. By means of this new formula, we stablish some integral conditions to characterize the null umbilical directions for compact spacelike surfaces a null second fundamental form positive definite (Th. 4.1) and marginally trapped surfaces (Th. 4.4) in a Lorentzian space form.

Finally, we study spacelike surfaces immersed in a totally umbilical hypersurface of a Lorentzian space form and, by using the previous characterization of umbilical directions of compact spacelike inmmersions, we give new proofs of the rigidity Liebmann theorem for surfaces on the Euclidean, hemispherical and hyperbolic 3-spaces (Th. 5.3, Th. 5.4 and Th. 5.6) and in the 3-dimensional De Sitter spacetime (Th. 5.7).

2 Preliminaries

Let $x : M^2 \longrightarrow \overline{M}_1^4(c)$ be a spacelike immersion of a 2-dimensional (connected) manifold M^2 into a 4-dimensional Lorentzian space form $\overline{M}_1^4(c)$ of constant sectional curvature c. Denote by $\langle\, ,\, \rangle$ for the Lorentzian metric of $\overline{M}_1^4(c)$ as well as the induced on M^2 via x. We write ∇ and $\overline{\nabla}$ the Levi-Civita connections of M^2 and $\overline{M}_1^4(c)$, respectively, and let $\nabla^\perp$ be the connection on the normal bundle of the submanifold. The Gauss and Weingarten formulas of x are

$$\overline{\nabla}_X Y = \nabla_X Y + \mathrm{II}(X,Y) \qquad \text{and} \qquad \overline{\nabla}_X \xi = -A_\xi X + \nabla^\perp_X \xi, \tag{1}$$

for any tangent vector fields X, Y on M^2 and a normal vector field ξ. The shape (or Weingarten) operator A_ξ is related to the second fundamental form II by

$$\langle A_\xi X, Y\rangle = \langle \mathrm{II}(X,Y), \xi\rangle.$$

The mean curvature vector field is given by $\mathbf{H} = \frac{1}{2}\,\mathrm{tr}_{\langle\, ,\, \rangle}\mathrm{II}$, and the Gauss and Codazzi equations of x are respectively,

$$R(X,Y)Z = c\,\{\langle Y,Z\rangle X - \langle X,Z\rangle Y\} + A_{\mathrm{II}(Y,Z)}X - A_{\mathrm{II}(X,Z)}Y \tag{2}$$

$$(\nabla_X \mathrm{II})(Y,Z) = (\nabla_Y \mathrm{II})(X,Z), \tag{3}$$

where R stands for the curvature tensor of the induced metric and

$$(\nabla_X \mathrm{II})(Y,Z) = \nabla^\perp_X \mathrm{II}(Y,Z) - \mathrm{II}(\nabla_X Y, Z) - \mathrm{II}(Y, \nabla_X Z),$$

for any tangent vector fields X, Y, Z on M^2. For each normal vector field ξ, the Codazzi equation provides us that,

$$(\nabla_X A_\xi)Y - (\nabla_Y A_\xi)X = A_{\nabla^\perp_X \xi}Y - A_{\nabla^\perp_Y \xi}X. \tag{4}$$

Assume ξ is globally defined and denote by II_ξ the symmetric tensor field on M^2,

$$\mathrm{II}_\xi(X,Y) = -\langle A_\xi X, Y\rangle = -\langle \mathrm{II}(X,Y), \xi\rangle. \tag{5}$$

When II_ξ is nondegenerate everywhere on M^2, we will say that ξ is a nondegenerate normal section [4, p. 59]. At any $p \in M^2$, A_ξ is self-adjoint, hence, there exists an orthonormal basis of tangent vectors e_1, e_2 to M^2 consisting of eigenvectors of A_ξ, that is $A_\xi(e_i) = \lambda_i e_i$. The eigenvalues λ_1, λ_2 are the principal curvatures and the eigenvectors e_1, e_2 the principal directions of the normal direction ξ. The Casorati curvature of the normal direction ξ is defined by,

$$C_\xi = \frac{\mathrm{tr}(A_\xi^2)}{2} = \frac{1}{2}(\lambda_1^2 + \lambda_2^2). \tag{6}$$

Clearly $C_\xi = 0$ if and only if the normal direction ξ is geodesic.

From now on we assume $\overline{M}_1^4(c)$ is time orientable, i.e., there exists a globally defined timelike vector field $\mathbf{Z}$ on $\overline{M}_1^4(c)$ (see [15, pp.345–346] for instance). Now, we take the normal component of $\mathbf{Z}$, $\mathbf{Z}^\perp$, and we define the following future (with the same orientation that $\mathbf{Z}$) unit timelike vector field orthogonal to M^2,

$$\mathbf{N} = \frac{1}{\sqrt{-\langle \mathbf{Z}^\perp, \mathbf{Z}^\perp\rangle}}\,\mathbf{Z}^\perp \in \mathfrak{X}^\perp(M^2).$$

From now on we suppose $\overline{M}_1^4(c)$ and M^2 to be orientable. Hence, we may construct a new unit future timelike vector field $\mathbf{E} \in \mathfrak{X}^\perp(M^2)$, such that, at each point $p \in M^2$, (N_p, E_p) be a orthonormal basis and for all positive oriented basis of T_pM^2, (e_1, e_2), then (N_p, E_p, e_1, e_2) be a positive oriented orthonormal basis of $T_{x(p)}M$. Thus, there exist two independent future null vector fields $\xi, \eta \in \mathfrak{X}(M^2)$ defined as follows,

$$\xi = \frac{1}{\sqrt{2}}(\mathbf{N} + \mathbf{E}), \qquad \eta = \frac{1}{\sqrt{2}}(\mathbf{N} - \mathbf{E}),$$

which trivialize the normal bundle of M^2 and satisfy $\langle \xi,\, \eta\rangle = -1$. Moreover, such a pair of null sections is essentially unique, i.e., for any $(\xi',\, \eta')$ satisfying the previous conditions it is holds that $\xi' = f\,\xi$ and $\eta' = \frac{1}{f}\,\eta$ for some differentiable function f on M^2.

The following formula holds for II,

$$\mathrm{II}(X,Y) = \mathrm{II}_\eta(X,Y)\xi + \mathrm{II}_\xi(X,Y)\eta, \tag{7}$$

for every $X, Y \in \mathfrak{X}(M^2)$. In particular,

$$\mathbf{H} = -\langle \mathbf{H}, \eta \rangle \xi - \langle \mathbf{H}, \xi \rangle \eta. \tag{8}$$

Contracting in (2) we obtain,

$$\operatorname{Ric}(Y, Z) = c\langle Y, Z\rangle + 2\langle A_{\mathbf{H}} Y, Z\rangle + \langle (A_\xi A_\eta + A_\eta A_\xi) Y, Z\rangle, \tag{9}$$

and

$$(K - c)Id = \Big(- 2\langle \mathbf{H}, \eta\rangle Id + A_\eta \Big) A_\xi + \Big(- 2\langle \mathbf{H}, \xi\rangle Id + A_\xi \Big) A_\eta,$$

where K is the Gauss curvature of M^2. Taking into account that $\mathrm{tr} A_\xi = 2\langle \mathbf{H}, \xi\rangle$ and analogously for η, we obtain that,

$$(K - c)Id = A_\xi A_\eta + A_\eta A_\xi + 2A_{\mathbf{H}}, \tag{10}$$

Therefore,

$$2(K - c) = 4\langle \mathbf{H}, \mathbf{H}\rangle + 2\mathrm{tr}(A_\xi A_\eta) = 4\langle \mathbf{H}, \mathbf{H}\rangle - \langle \mathrm{II}, \mathrm{II}\rangle, \tag{11}$$

where, as usual, $\langle \mathrm{II}, \mathrm{II}\rangle_p = \Sigma_{i,j=1}^2 \langle \mathrm{II}(e_i, e_j), \mathrm{II}(e_i, e_j)\rangle$ for $\{e_1, e_2\}$ an orthonormal basis of $T_p M^2$.

In order to obtain a sufficient condition which asserts that the null normal sections ξ and η are nondegenerate we give the following result.

Proposition 2.1. *Let $x : M^2 \longrightarrow \overline{M}_1^4(c)$ be a spacelike immersion. If the following inequality is satisfied,*

$$\varphi := (K - c)^2 - 4(K - c)\langle \mathbf{H}, \mathbf{H}\rangle + 4\det(A_{\mathbf{H}}) > 0,$$

then the null normal sections ξ and η are nondegenerate.

Proof. Assume $\det(A_\xi)$ vanishes at $p \in M^2$. Let e_1, e_2 be the principal directions of the null normal direction ξ with $A_\xi(e_1) = 0$ and $A_\xi(e_2) = \lambda e_2$. A direct computation shows that $\det(A_\xi A_\eta + A_\eta A_\xi) = -\lambda^2 \langle A_\eta(e_1), e_2\rangle^2$. But taking account (10), we get,

$$\det\Big((K - c)Id - 2A_{\mathbf{H}}\Big) = (K - c)^2 - 4(K - c)\langle \mathbf{H}, \mathbf{H}\rangle + 4\det(A_{\mathbf{H}}) \leq 0,$$

which contradicts our assumption. $\square$

Remark 2.2. Note that $\varphi(p) = 0$ if and only if $K(p) - c$ is an eigenvalue of $2A_{\mathbf{H}}$. On the other hand, assume that μ is a log-harmonic function defined on a simply-connected domain $U \subset \mathbb{R}^2$. In [6], it is shown that $\Big(U, \frac{1}{\mu}(dx^2+dy^2)\Big)$ is a flat surface which can be isometrically immersed in the 4-dimensional Lorentz-Minkowski space $\mathbb{L}^4$. Moreover, its second fundamental form satisfies,

$$\mathrm{II}(\partial_x, \partial_x) = \Big(\frac{1}{\mu} - 1\Big)\mathbf{H}, \quad \mathrm{II}(\partial_x, \partial_y) = 0, \quad \mathrm{II}(\partial_y, \partial_y) = \Big(\frac{1}{\mu} + 1\Big)\mathbf{H},$$

where $\mathbf{H} = (1, 1, 0, 0)$. Taking $\eta = \mathbf{H}$ it is clear that the null normal section η is degenerate and $\det(A_\xi) = 1/\mu^2 - 1$. Therefore, for suitable choices of $\mu > 0$, we obtain that ξ is nondegenerate. In this case, a direct computation shows that $\varphi = 0$ and thus Proposition 2.1 can not be weakened to $\varphi \geq 0$.

On the other hand, the polynomial $P(t) = t^2 - 4\langle H, H\rangle t + 4\det(A_\xi)$ has, at any point, non-negative discriminant as a consequence of the Schwarz inequality. Thus, the assumption $P(K - c) > 0$ does not follows from a condition on $P(t)$ independent of $K - c$.

Corollary 2.3. *Let $x : M^2 \longrightarrow \overline{M}_1^4(c)$ be a spacelike immersion. Assume that M^2 is extremal ($\mathbf{H} = 0$) and not totally geodesic, then the null normal sections ξ and η are nondegenerate, with II_η and II_ξ Lorentzian metrics on M^2.*

Proof. It is a direct consequence of (11). The Lorentzian signature is deduced from $\mathrm{tr}A_\xi = 2\langle \mathbf{H}, \xi\rangle = 0$.

□

Recall that a spacelike immersion $x : M^2 \longrightarrow \overline{M}_1^4(c)$ is called pseudo-umbilical when $A_{\mathbf{H}} = \rho Id$ for $\rho \in C^\infty(M)$.

Corollary 2.4. *Let $x : M^2 \longrightarrow \overline{M}_1^4(c)$ be a pseudo-umbilical spacelike immersion. If $K(p) \neq c + 2\rho(p)$ at every point $p \in M^2$, then ξ and η are nondegenerate null normal sections*

Proof. Taking into account that $\det(A_{\mathbf{H}}) = \rho^2$ and $\langle \mathbf{H}, \mathbf{H}\rangle = \rho$ the result is a direct consequence of Proposition 2.1.

□

Proposition 2.5. *Let $x : M^2 \longrightarrow \overline{M}_1^4(c)$ be a spacelike immersion with null normal vector field ξ and $\boldsymbol{H}$ the mean curvature vector field of M^2. Then,*

$$\det(A_\xi) = 2\langle \mathbf{H}, \xi\rangle^2 - C_\xi. \tag{12}$$

Proof. The characteristic equation for the shape operator A_ξ,

$$A_\xi^2 - (\mathrm{tr}A_\xi)A_\xi + (\det A_\xi)Id = 0,$$

implies that,

$$2(\det A_\xi) = (\mathrm{tr}A_\xi)^2 - (\mathrm{tr}A_\xi^2).$$

Taking into account that $\mathrm{tr}A_\xi = 2\langle \mathbf{H}, \xi\rangle$ the result easily follows. $\square$

3 Gauss curvature of II_ξ

Assume ξ is a nondegenerate null normal section on M^2 into $\overline{M}_1^4$. In this case, II_ξ given by (5) provides with a new metric on M^2. This section is devoted to obtain an explicit formula for the Gauss curvature of the metric II_ξ.

Let D denote the Levi-Civita connection of the metric tensor II_ξ. The difference tensor L between the Levi-Civita connections D and ∇ is given by,

$$L(X, Y) = D_X Y - \nabla_X Y, \tag{13}$$

for all $X, Y \in \mathfrak{X}(M^2)$. From the Koszul formula for II_ξ and (4) we have,

$$L(X, Y) = \frac{1}{2} A_\xi^{-1}\Big[(\nabla_X A_\xi)Y + A_{\nabla^\perp_Y \xi}X + B(X, Y)\Big],$$

where $\langle B(X, Y), Z\rangle = -\langle A_{\nabla^\perp_Z \xi}X, Y\rangle$ for all $X, Y, Z \in \mathfrak{X}(M^2)$.

Since $\langle \xi, \xi\rangle = 0$ we obtain that $\langle \nabla^\perp_X \xi, \xi\rangle = 0$ for every $X \in \mathfrak{X}(M^2)$. Therefore there exists a 1-form ω such that,

$$\nabla^\perp_X \xi = \omega(X)\xi. \tag{14}$$

We define $\Theta \in \mathfrak{X}(M^2)$ by $\langle \Theta, X\rangle = \omega(X)$ for every $X \in \mathfrak{X}(M^2)$. That is, Θ is the vector field $\langle , \rangle$-metrically equivalent to ω. A direct computation shows that,

$$B(X, Y) = \mathrm{II}_\xi(X, Y)\Theta,$$

for every $X, Y \in \mathfrak{X}(M^2)$. Therefore the symmetric difference tensor L can be written as follows,

$$L(X, Y) = \frac{1}{2} A_\xi^{-1}\Big[(\nabla_X A_\xi)Y\Big] + \frac{1}{2}\omega(Y)X + \frac{1}{2}\mathrm{II}_\xi(X, Y)A_\xi^{-1}\Theta. \tag{15}$$

Remark 3.1. The curvature of the normal connection satisfies,

$$R^{\perp}(X,Y)\xi = d\omega(X,Y)\xi, \quad R^{\perp}(X,Y)\eta = -d\omega(X,Y)\eta.$$

Therefore, for every normal vector field $\alpha \in \mathfrak{X}^{\perp}(M^2)$ we obtain that,

$$R^{\perp}(X,Y)\alpha = d\omega(X,Y)\bar{\alpha},$$

where $\bar{\alpha} = \langle \alpha, \xi \rangle \eta - \langle \alpha, \eta \rangle \xi$. It should be pointed out that if we put $J\alpha = \bar{\alpha}$ then $\langle J\alpha, J\alpha \rangle = -\langle \alpha, \alpha \rangle$ and $J^2 = Id$.

Now consider the Riemannian curvature tensor R^{ξ} of II_{ξ} which can be decomposed as follows

$$R^{\xi} = R + Q_1 + Q_2$$

where,

$$Q_1(X,Y)Z = (D_X L)(Y,Z) - (D_Y L)(X,Z),$$
$$Q_2(X,Y)Z = L(Y, L(X,Z)) - L(X, L(Y,Z)),$$

and $X, Y, Z \in \mathfrak{X}(M^2)$. Therefore we get the following formula for the Gauss curvature K^{ξ} of II_{ξ},

$$2K^{\xi} = \mathrm{tr}_{\mathrm{II}_{\xi}}(\mathrm{Ric}) + \mathrm{tr}_{\mathrm{II}_{\xi}}(\widehat{Q_1}) + \mathrm{tr}_{\mathrm{II}_{\xi}}(\widehat{Q_2}) \tag{16}$$

where $\widehat{Q_i}(X,Y) = \mathrm{tr}\{Z \mapsto Q_i(Z,X)Y\}$ $i = 1, 2$, and for a symmetric $(0,2)$ tensor T, $\mathrm{tr}_{\mathrm{II}_{\xi}} T$ is the ordinary trace of the $(1,1)$-tensor $\overline{T}$ defined by $\mathrm{II}_{\xi}(\overline{T}(X), Y) = T(X,Y)$.

Lemma 3.2. *The trace with respect to* II_{ξ} *of the Ricci tensor* Ric *is given by,*

$$\mathrm{tr}_{\mathrm{II}_{\xi}}(\mathrm{Ric}) = \mathrm{tr}_{\mathrm{II}_{\xi}}(K \cdot \langle \ , \ \rangle) = \frac{-2K\langle \mathbf{H}, \xi \rangle}{\det(A_{\xi})}.$$

Proof. Let $\{E_1, E_2\}$ be a $\langle \ , \ \rangle$-orthonormal basis of T_pM^2, such that $A_{\xi}(E_i) = \lambda_i E_i$ at a point $p \in M^2$. Consider now $F_i = |\lambda_i|^{-1/2} E_i$, $i = 1, 2$, then $\{F_1, F_2\}$ is a II_{ξ}-orthonormal basis of T_pM^2 with $\varepsilon_i = \mathrm{II}_{\xi}(F_i, F_i) = -\lambda_i/|\lambda_i|$. A direct computation gives now the result. $\square$

Now observe that the vector field $-A_{\xi}^{-1}\Theta$ is metrically equivalent to ω with respect to II_{ξ}. The following result relates this vector field with the second right term of (16).

Lemma 3.3. *The trace with respect to* II_{ξ} *of the tensor* $\widehat{Q_1}$ *is given by,*

$$\mathrm{tr}_{\mathrm{II}_{\xi}}(\widehat{Q_1}) = \mathrm{div}_{\mathrm{II}_{\xi}}(A_{\xi}^{-1}\Theta).$$

Proof. Consider $\{E_1, E_2\}$ and $\{F_1, F_2\}$ constructed as previously. Extend F_1 and F_2 near the point p as a local be a $\langle\ ,\ \rangle$-orthonormal II_ξ-orthonormal frame $\{F_1, F_2\}$ satisfying $(D_{F_i}F_j)_p = 0$.

A direct computation shows that,

$$\Big[\mathrm{tr}_{\mathrm{II}_\xi}(\widehat{Q_1})\Big]_p = \varepsilon_1\varepsilon_2\Big[\mathrm{II}_\xi\Big(F_2, Q_1(F_2, F_1)F_1)\Big)_p + \mathrm{II}_\xi\Big(F_1, Q_1(F_1, F_2)F_2)\Big)_p\Big].$$

On the other hand,

$$\mathrm{II}_\xi\Big(F_2, Q_1(F_2, F_1)F_1)\Big)_p = (F_2)_p\mathrm{II}_\xi\Big(F_2, L(F_1, F_1)\Big) - (F_1)_p\mathrm{II}_\xi\Big(F_2, L(F_2, F_1)\Big),$$

and taking into account (21) we obtain,

$$\Big[\mathrm{tr}_{\mathrm{II}_\xi}(\widehat{Q_1})\Big]_p = -\varepsilon_1(F_1)_p\omega(F_1) - \varepsilon_2(F_2)_p\omega(F_2) = \Big[\mathrm{div}_{\mathrm{II}_\xi}(A_\xi^{-1}\Theta)\Big]_p$$

□

Lemma 3.4. *The trace with respect to* II_ξ *of the tensor* $\widehat{Q_2}$ *is given by,*

$$\begin{aligned}\mathrm{tr}_{\mathrm{II}_\xi}(\widehat{Q_2}) = \ & \mathrm{II}_\xi(L, L) - \frac{1}{4\det(A_\xi^2)}\mathrm{II}_\xi\Big(\nabla^{\mathrm{II}_\xi}(\det(A_\xi)), \nabla^{\mathrm{II}_\xi}(\det(A_\xi))\Big) \\ & +\omega\Big(A_\xi^{-1}\Theta + \tfrac{\nabla^{\mathrm{II}_\xi}(\det(A_\xi))}{2\det(A_\xi)}\Big).\end{aligned}$$

Proof. Let $\{E_1, E_2\}$ be a local $\langle\ ,\ \rangle$-orthonormal frame at a point $p \in M^2$ such that $A_\xi(E_i) = \lambda_i E_i$ for $i = 1, 2$ at $p \in M^2$. Then, construct $\{F_1, F_2\}$ as in Lemma 3.3.

A direct computation shows,

$$\begin{aligned}\Big[\mathrm{tr}_{\mathrm{II}_\xi}(\widehat{Q_2})\Big]_p = \ & \varepsilon_1\varepsilon_2\Big[\mathrm{II}_\xi\Big(L(F_1, L(F_2, F_1)) - L(F_2, L(F_1, F_1)), F_2\Big) \\ & +\mathrm{II}_\xi\Big(L(F_2, L(F_2, F_1)) - L(F_1, L(F_2, F_2)), F_1\Big)\Big]_p.\end{aligned}$$

Now, from (21) we obtain,

$$\mathrm{II}_\xi(L(X, Y), Z) - \mathrm{II}_\xi(L(X, Z), Y) = \omega(Y)\mathrm{II}_\xi(X, Z) - \omega(Z)\mathrm{II}_\xi(X, Y),$$

and therefore,

$$\begin{aligned}\Big[\mathrm{tr}_{\mathrm{II}_\xi}(\widehat{Q_2})\Big]_p = \ & \varepsilon_1\varepsilon_2\Big[\mathrm{II}_\xi\Big(L(F_1, F_2), L(F_1, F_2)\Big) - \varepsilon_1(\omega(F_2))^2 \\ & -\mathrm{II}_\xi\Big(L(F_2, F_2), L(F_1, F_1)\Big) - \varepsilon_2\omega(L(F_1, F_1))\end{aligned}$$

$$+\mathrm{II}_\xi\Big(L(F_1,F_2),L(F_1,F_2)\Big)-\varepsilon_2(\omega(F_1))^2$$
$$-\mathrm{II}_\xi\Big(L(F_2,F_2),L(F_1,F_1)\Big)-\varepsilon_1\omega(L(F_2,F_2))\Big].$$

Observe that,

$$\mathrm{II}_\xi(L,L)=\sum_{i,j}\varepsilon_i\varepsilon_j\mathrm{II}_\xi(L(F_i,F_j),L(F_i,F_j))$$

whereas,

$$\mathrm{II}_\xi(\mathrm{tr}_{\mathrm{II}_\xi}(L),\mathrm{tr}_{\mathrm{II}_\xi}(L))=\sum_{i,j}\varepsilon_i\varepsilon_j\mathrm{II}_\xi(L(F_i,F_i),L(F_j,F_j)),$$

where $\mathrm{tr}_{\mathrm{II}_\xi}(L)=\varepsilon_1L(F_1,F_1)+\varepsilon_2L(F_2,F_2)$ is the vector field obtained from the $\mathrm{II}_\xi-$contraction of L. Hence, we obtain the following formula,

$$\mathrm{tr}_{\mathrm{II}_\xi}(\widehat{Q_2})=\mathrm{II}_\xi(L,L)-\mathrm{II}_\xi(\mathrm{tr}_{\mathrm{II}_\xi}(L),\mathrm{tr}_{\mathrm{II}_\xi}(L))-\omega(\mathrm{tr}_{\mathrm{II}_\xi}(L))-\mathrm{II}_\xi(A_\xi^{-1}\Theta,A_\xi^{-1}\Theta).$$

We end the proof obtaining an explicit expression of the vector field $\mathrm{tr}_{\mathrm{II}_\xi}(L)$. A straightforward computation shows,

$$\mathrm{tr}_{\mathrm{II}_\xi}(L)=\frac{1}{2}A_\xi^{-1}\Theta+\frac{1}{2}\Big\{\varepsilon_1A_\xi^{-1}\Big[(\nabla_{F_1}A_\xi)F_1\Big]+\varepsilon_2A_\xi^{-1}\Big[(\nabla_{F_2}A_\xi)F_2\Big]\Big\}.$$

Until now, we have not used the concrete expressions for F_1 and F_2, which are needed to obtain the vector field $\mathrm{tr}_{\mathrm{II}_\xi}(L)$. In fact, for every $X\in\mathfrak{X}(M^2)$ we get,

$$X\det(A_\xi)=X(\langle A_\xi E_1,E_1\rangle\langle A_\xi E_2,E_2\rangle)$$
$$=\lambda_2\langle(\nabla_XA_\xi)E_1,E_1\rangle+\lambda_1\langle(\nabla_XA_\xi)E_2,E_2\rangle.$$

Using the Codazzi equation (4) we can rewrite the previous formula as follows,

$$X\det(A_\xi)=-\det(A_\xi)\Big[\varepsilon_1\langle(\nabla_{F_1}A_\xi)F_1,X\rangle+\varepsilon_2\langle(\nabla_{F_2}A_\xi)F_2,X\rangle-\omega(X)\Big],$$

and therefore,

$$X\det(A_\xi)=\det(A_\xi)\mathrm{II}_\xi\Big(A_\xi^{-1}\Big[\varepsilon_1(\nabla_{F_1}A_\xi)F_1+\varepsilon_2(\nabla_{F_2}A_\xi)F_2-\Theta\Big],X\Big).$$

Moreover,

$$\nabla^{\mathrm{II}_\xi}(\det(A_\xi))=\ -\det(A_\xi)A_\xi^{-1}\Theta+\det(A_\xi)\Big\{\varepsilon_1A_\xi^{-1}\Big[(\nabla_{F_1}A_\xi)F_1\Big]$$
$$+\varepsilon_2A_\xi^{-1}\Big[(\nabla_{F_2}A_\xi)F_2\Big]\Big\},$$

and thus,

$$\varepsilon_1 A_\xi^{-1}\Big[(\nabla_{F_1}A_\xi)F_1\Big] + \varepsilon_2 A_\xi^{-1}\Big[(\nabla_{F_2}A_\xi)F_2\Big] = \frac{\nabla^{\mathrm{II}_\xi}(\det(A_\xi))}{\det(A_\xi)} + A_\xi^{-1}\Theta.$$

Finally, for the vector field $\mathrm{tr}_{\mathrm{II}_\xi}(L)$ we have,

$$\mathrm{tr}_{\mathrm{II}_\xi}(L) = A_\xi^{-1}\Theta + \frac{\nabla^{\mathrm{II}_\xi}(\det(A_\xi))}{2\det(A_\xi)},$$

concluding the proof.

□

The three previous lemmas show our key result.

Theorem 3.5. *Let $x : M^2 \longrightarrow \overline{M}_1^4(c)$ be a spacelike immersion of a 2-dimensional orientable manifold M^2 into an 4-dimensional Lorentzian space form $\overline{M}_1^4(c)$ of constant sectional curvature c. Assume that ξ is a nondegenerate null normal section. Then the Gauss curvature K^ξ of the metric II_ξ satisfies,*

$$\begin{aligned} 2K^\xi = {} & \tfrac{-2K\langle \mathbf{H},\xi\rangle}{\det(A_\xi)} + \mathrm{div}_{\mathrm{II}_\xi}(A_\xi^{-1}\Theta) + \mathrm{II}_\xi(L,L) \\ & -\tfrac{1}{4\det(A_\xi^2)}\mathrm{II}_\xi\Big(\nabla^{\mathrm{II}_\xi}(\det(A_\xi)), \nabla^{\mathrm{II}_\xi}(\det(A_\xi))\Big) + \omega\Big(A_\xi^{-1}\Theta + \tfrac{\nabla^{\mathrm{II}_\xi}(\det(A_\xi))}{2\det(A_\xi)}\Big). \end{aligned} \tag{17}$$

Remark 3.6. *Note that* (17) *widely extends formula* (10) *in* [13].

Now, assume the nondegenerate normal null direction ξ is umbilical. Then, there exists a smooth function f on M such that $A_\xi = -e^{2f}Id$. Next, we will analyse each of the terms in (17).

Since $\mathrm{II}_\xi = e^{2f}\langle\ ,\ \rangle$, that is, II_ξ and $\langle\ ,\ \rangle$ are conformally related, we can deduce the following formulas.

$$d_\xi = e^{4f}, \qquad \nabla^{\mathrm{II}_\xi} d_\xi = e^{-2f}\nabla d_\xi = 4e^{2f}\nabla f,$$

where we have denoted by $d_\xi = \det(A_\xi)$. Thus, for the fourth term of the second member of (17), we have,

$$\frac{1}{4d_\xi^2}\mathrm{II}_\xi\Big(\nabla^{\mathrm{II}_\xi}d_\xi, \nabla^{\mathrm{II}_\xi}d_\xi\Big) = \frac{4}{e^{2f}}\|\nabla f\|^2. \tag{18}$$

Now from the Codazzi equation (4), taking into account that $A_\xi = -e^{2f}Id$, we have

$$-X(e^{2f})Y + Y(e^{2f})X = -e^{2f}\omega(X)Y + e^{2f}\omega(Y)X, \tag{19}$$

for all $X, Y \in \mathfrak{X}(M)$. Therefore, from (19), we arrive to

$$\omega(X) = 2X(f) \quad \text{for all} \quad X \in \mathfrak{X}(M). \tag{20}$$

Observe that $\Theta = 2\nabla f$. Therefore, formula (20) gives,

$$\omega\Big(A_\xi^{-1}\Theta + \frac{\nabla^{\mathrm{II}_\xi} d_\xi}{2d}\Big) = \omega(0) = 0,$$

and so the last term in (17) identically vanishes.

We analyse now the third term in (17). The metrics II_ξ and $\langle\ ,\ \rangle$ are conformally related and therefore the difference tensor (13) satisfies

$$L(X,Y) = X(f)Y + Y(f)X - \langle X, Y\rangle \nabla f. \tag{21}$$

Let $\{F_1, F_2\}$ be a (local) II_ξ-orthonormal frame and denote $L(F_i, F_j)$ by L_{ij}. A direct computation from (21) shows

$$L_{11} = 2F_1(f)F_1 - e^{-2f}\nabla f, \quad L_{22} = 2F_2(f)F_2 - e^{-2f}\nabla f,$$
$$L_{12} = L_{21} = F_1(f)F_1 + F_2(f)F_2.$$

Therefore,

$$\begin{aligned}\mathrm{II}_\xi(L,L) &= \mathrm{II}_\xi(L_{11},L_{11}) + 2\mathrm{II}_\xi(L_{12},L_{12}) + \mathrm{II}_\xi(L_{22},L_{22}) = \\ &= 4e^{-2f}\|\nabla f\|^2. \end{aligned}\tag{22}$$

Finally, we will compute the divergence term of (17). Taking into account that $\Theta = 2\,\nabla f$ and using the relation between two conformal metrics

$$\mathrm{div}_{\mathrm{II}_\xi}(X) = \mathrm{div}(X) + 2\,X(f),$$

we have that

$$\mathrm{div}_{\mathrm{II}_\xi}(A^{-1}\Theta) = -2\,e^{-2f}\Delta f. \tag{23}$$

Consequently, from (18), (22) and (23), and taking into account that $\mathrm{Tr}(A_\xi) = 2\langle \mathbf{H}, \xi\rangle$, equation (17) is reduced to the well-known formula that relates the Gauss curvature of two conformal metrics $\langle\cdot\,,\cdot\rangle$ and II_ξ,

$$K - K_\xi\, e^{2f} = \Delta f.$$

Moreover, since ξ is umbilical and by using of (14), we conclude that f is constant if and only if $\nabla^\perp_X \xi = 0$ (and, as consequence, $\nabla^\perp_X \eta = 0$). In other words, $\langle\cdot\,,\cdot\rangle$ and II_ξ are homothetic if and only if there is a parallel umbilical (nondegenerate) normal null section. This motives the study of when a normal null section is umbilical, which we present in the next section.

4 Applications

Let us denote by $d\mu$ for the canonical measure associated with the induced metric $\langle\ ,\ \rangle$, and by $d_\xi = \det(A_\xi)$.

Theorem 4.1. *Let $x : M^2 \longrightarrow \overline{M}^4(c)$ be a compact spacelike immersion. Assume ξ is a normal null section with II_ξ positive definite and K signed (i.e., $K > 0$ or $K < 0$). Then,*

$$\int_{M^2} \frac{K\langle \mathbf{H}, \xi\rangle}{d_\xi} d\mu_{\mathrm{II}_\xi} \leq -2\pi\chi(M) \qquad (K > 0), \quad \textit{or}$$

$$\int_{M^2} \frac{K\langle \mathbf{H}, \xi\rangle}{d_\xi} d\mu_{\mathrm{II}_\xi} \geq -2\pi\chi(M) \qquad (K < 0).$$

Moreover, equality holds if and only if ξ is an umbilical direction.

Proof. Assume $K > 0$. Since II_ξ is supposed to be definite positive, we have that at every point $p \in M^2$ the eigenvalues of A_ξ satisfy $\lambda_i < 0$ for $i = 1, 2$. Hence, $-\mathrm{tr}(A_\xi) = -2\langle \mathbf{H}, \xi\rangle \geq 2\sqrt{d_\xi}$ with equality holding at every point if and only if ξ is an umbilical direction. Note that since $K > 0$, then $K\langle \mathbf{H}, \xi\rangle \leq -K\sqrt{d_\xi}$ with equality at every point if and only if ξ is umbilical. Therefore, by using the Gauss-Bonnet Theorem we get,

$$\int_{M^2} \frac{K\langle \mathbf{H}, \xi\rangle}{d_\xi} d\mu_{\mathrm{II}_\xi} \leq -\int_{M^2} \frac{K}{\sqrt{d_\xi}} d\mu_{\mathrm{II}_\xi} = -\int_{M^2} K d\mu = -2\pi\chi(M)$$

and the equality holds if and only if ξ is umbilical.

The case $K < 0$ follows analogously.

□

Corollary 4.2. *Let $x : M^2 \longrightarrow \overline{M}^4(c)$ be a compact spacelike immersion. Assume ξ is a normal null section with II_ξ positive definite and K signed (i.e., $K > 0$ or $K < 0$). Suppose that d_ξ is constant and $\nabla^\perp \xi = 0$. Then, ξ is an umbilical direction.*

Proof. Assume $K > 0$. Directly from the formula (17), we have

$$2\int_{M^2} \frac{K\langle \mathbf{H}, \xi\rangle}{d_\xi} d\mu_{\mathrm{II}_\xi} = \int_{M^2} \left[\mathrm{II}_\xi(L, L) - 2K^\xi\right] d\mu_{\mathrm{II}_\xi} \geq -4\pi\chi(M),$$

and therefore the equality holds in Theorem 4.1.

A similar argument works for $K < 0$.

□

Corollary 4.3. *Let $x : M^2 \longrightarrow \overline{M}^4(c)$ be a compact spacelike immersion. Assume ξ and η are normal null sections with II_ξ and II_η positive definite and K signed (i.e., $K > 0$ or $K < 0$). Then,*

$$\int_{M^2} \Big[\frac{K\langle \mathbf{H}, \xi\rangle}{d_\xi^{1/2}} + \frac{K\langle \mathbf{H}, \eta\rangle}{d_\eta^{1/2}}\Big] d\mu \leq -4\pi\chi(M) \qquad (K > 0), \quad \textit{or}$$

$$\int_{M^2} \Big[\frac{K\langle \mathbf{H}, \xi\rangle}{d_\xi^{1/2}} + \frac{K\langle \mathbf{H}, \eta\rangle}{d_\eta^{1/2}}\Big] d\mu \geq -4\pi\chi(M) \qquad (K < 0).$$

Moreover, equality holds if and only if x is totally umbilical.

We end this section with an application to the so-called marginally trapped surfaces. Recall that a compact spacelike surface M^2 in a 4-dimensional Lorentzian manifold is called *marginally trapped* when its mean curvature vector field $\mathbf{H}$ is null everywhere, i.e., $\langle \mathbf{H}, \mathbf{H}\rangle = 0$ and $\mathbf{H}$ nowhere vanishes. From the formula (8) we have

$$\langle \mathbf{H}, \mathbf{H}\rangle = -2\,\langle \mathbf{H},\, \xi\rangle\langle \mathbf{H},\, \eta\rangle = -\frac{1}{2}\,\mathrm{tr}(A_\xi)\,\mathrm{tr}(A_\eta),$$

and, from the discussion of uniqueness (up to a positive function) of the null normal sections ξ, η which satisfy $\langle \xi,\, \eta\rangle = -1$, we may equivalent define that M^2 is marginally trapped if $\mathrm{tr}(A_\xi) = 0$ and $\mathrm{tr}(A_\eta) > 0$ [7]. In this case, the natural choice for the null normal section ξ in Theorem 4.1 would be $\xi = \mathbf{H}$. However, from Proposition 2.5 it follows that the metric $\mathrm{II}_{\mathbf{H}}$ is Lorentzian and, therefore, the assumptions of Theorem 4.1 are not satisfied.

We recall [10] that a marginally trapped surface cannot be contained in a region of the spacetime where there exists a timelike Killing vector field. Thus, we consider now the case $\overline{M}_1^4(c)$ where $c > 0$.

Theorem 4.4. *Let $x : M^2 \longrightarrow \overline{M}_1^4(c)$, $c > 0$, be a (compact) marginally trapped surface. Assume its mean curvature vector $\mathbf{H}$ is nondegenerate, null and satisfies that $\det(A_{\mathbf{H}})$ is a (necessarily nonzero) constant. Then,*

$$\int_{M^2} \Big[\mathrm{II}_{\mathbf{H}}(L, L) - \mathrm{II}_{\mathbf{H}}(A_{\mathbf{H}}^{-1}\Theta, A_{\mathbf{H}}^{-1}\Theta)\Big] d\mu_{\mathrm{II}_{\mathbf{H}}} = 0.$$

Proof. This equality directly follows from an integration of (17) over M^2 and taking into account that every compact Lorentzian surface is a topological torus.

□

5 Surfaces in totally umbilical hypersurfaces

In this section all the manifolds are assumed to be orientable.

Let $i : P^3_\varepsilon \to \overline{M}^4_1(c)$ be a semi-Riemannian hypersurface with sign ε, that is, $\varepsilon = +1$ when P^3_ε inherits a Lorentzian metric and $\varepsilon = -1$ when P^3_ε inherits a Riemannian metric. Assume $i : P^3_\varepsilon \to \overline{M}^4_1(c)$ is totally umbilical with second fundamental form

$$\mathrm{II}^i = \alpha\langle , \rangle \mathbf{T}, \tag{24}$$

where $\alpha \in \mathbb{R}$ and $\mathbf{T}$ is a unit normal vector field on P^3_ε which satisfies $\langle \mathbf{T}, \mathbf{T}\rangle = \varepsilon$. Therefore, the hypersurface P^3_ε inherits a Riemannian or Lorentzian metric of constant sectional curvature $c + \varepsilon\alpha^2$.

Assume now $\phi : M^2 \to P^3_\varepsilon$ is a spacelike immersion. The composition $x = i \circ \phi$ is a spacelike immersion in $\overline{M}^4_1(c)$ with null normal sections,

$$\xi = \frac{1}{\sqrt{2}}(\mathbf{N} + \mathbf{T}), \quad \eta = \frac{-\varepsilon}{\sqrt{2}}(-\mathbf{N} + \mathbf{T}), \tag{25}$$

where $\mathbf{N}$ is the unit normal vector field along the immersion ϕ which satisfies $\langle \mathbf{N}, \mathbf{N}\rangle = -\varepsilon$. In that follows we will write II^x for the second fundamental form of the immersion x and analogously for ϕ and i. Other geometric elements will be distinguished in a similar way. We have $A^i = \varepsilon\alpha \cdot Id$ and

$$\mathrm{II}^x = \mathrm{II}^\phi + \mathrm{II}^i,$$

[4, Chap. 3]. On the other hand, $K = c + \varepsilon\alpha^2 - \varepsilon \det(A^\phi_N)$ and $\mathrm{tr}(A^\phi_N) = 2\langle \mathbf{H}^\phi, \mathbf{N}\rangle = -2\varepsilon H^\phi$, where H^ϕ denotes the mean curvature function of the inmersion ϕ. Now a direct computation shows that,

$$A^x_\xi = \frac{1}{\sqrt{2}}\Big(A^\phi_{\mathbf{N}} + \varepsilon\alpha Id\Big), \quad A^x_\eta = \frac{-\varepsilon}{\sqrt{2}}\Big(- A^\phi_{\mathbf{N}} + \varepsilon\alpha Id\Big),$$

and therefore,

$$\det(A^x_\xi) = \frac{1}{2}\Big(\det(A^\phi_N) + \varepsilon\alpha\mathrm{tr}(A^\phi_N) + \alpha^2\Big) = \frac{1}{2}\Big(\varepsilon(c-K) - 2\alpha H^\phi + 2\alpha^2\Big), \tag{26}$$

$$\det(A^x_\eta) = \frac{1}{2}\Big(\det(A^\phi_N) - \varepsilon\alpha\mathrm{tr}(A^\phi_N) + \alpha^2\Big) = \frac{1}{2}\Big(\varepsilon(c-K) + 2\alpha H^\phi + 2\alpha^2\Big), \tag{27}$$

Remark 5.1. Since A^x_ξ and A^x_η commute, the normal curvature tensor of the immersion x vanishes identically. Therefore, there exists at any T_pM^2 an orthonormal basis $\{e_1, e_2\}$ which diagonalizes simultaneously every A^x_Z for $Z \in \mathfrak{X}^{\perp_x}(M^2)$ [4, Proposition 4.1.2].

Lemma 5.2. *Let $\phi : M^2 \to P^3_\varepsilon$ be an immersion of a surface (spacelike if $\varepsilon = 1$) in P^3_ε. If $i : P^3_\varepsilon \to \overline{M}^4_1(c)$ is a totally umbilical hypersurface, then $\nabla^{\perp_x}\xi = \nabla^{\perp_x}\eta = 0$.*

Proof. Since $\nabla^{\perp_x}_v \xi = \omega(v)\xi$ a straightforward computation from (25) shows,

$$\omega(v) = -\langle \nabla^{\perp_x}_v \xi, \eta \rangle = \langle \nabla^{\perp_x}_v \mathbf{T}, \mathbf{N} \rangle = 0.$$

□

Now we apply our technique to give new proofs of results in [2].

Theorem 5.3. (*Liebmann classical rigidity theorem* [9]) *Let $\phi : M^2 \longrightarrow \mathbb{E}^3$ be a compact connected surface in the 3-dimensional Euclidean space. If the Gauss curvature of M^2 is a positive constant K, then M^2 is a totally umbilical round sphere.*

Proof. From the Gauss-Bonnet Theorem, the surface M^2 is topologically an sphere. Let $i : \mathbb{E}^3 \hookrightarrow \mathbb{L}^4$ be the usual totally geodesic embedding at $t = 0$. Since the scalar α in (24) is zero, we deduce that

$$\det(A^x_\xi) = \det(A^x_\eta) = \frac{K}{2},$$

making use of (26). Therefore A^x_ξ and A^x_η are positive definite (otherwise, we make a change of ξ or η to $-\xi$ or $-\eta$). Now, Lemma 5.2 and Corollary 4.2 can be called to ensure that the null normal sections ξ and η are umbilical and so, M^2 is a totally umbilical round sphere in $\mathbb{L}^4$ and so also in $\mathbb{E}^3$.

□

Theorem 5.4. *Let $\phi : M^2 \longrightarrow \mathbb{S}^3_+$ be a compact connected surface in the 3-dimensional north hemisphere $\mathbb{S}^3_+$. If the Gauss curvature of M^2 is a constant ($K > 1$), then M^2 is a totally umbilical round sphere.*

Proof. Again here we have that M^2 is a topological sphere. Let $i : \mathbb{S}^3 \hookrightarrow \mathbb{S}^4_1$ be the totally geodesic embedding at $t = 0$, where we have considered $\mathbb{S}^4_1$ as the warped product $\mathbb{R} \times_{\cosh(t)} \mathbb{S}^3$. Since the scalar α in (24) is 0, making use of (26) we deduce that

$$\det(A^x_\xi) = \det(A^x_\eta) = \frac{1}{2}(K - 1) > 0.$$

The last inequality is due to on a surface in a hemisphere an elliptic point p_0 is always reached (then, $K(p_0) > 1$). Therefore A^x_ξ and A^x_η are positive definite (otherwise, we make a change of ξ or η to $-\xi$ or $-\eta$). Now, Lemma

5.2 and Corollary 4.2 can be called again to ensure that the null normal sections ξ and η are then umbilical and so, M^2 is a totally umbilical round sphere in $\mathbb{S}^3_+$.

$\square$

Remark 5.5. Let $\phi : M^2 \longrightarrow \mathbb{S}^3$ be a minimal compact connected surface of (arbitrary) Gauss curvature K and $i : \mathbb{S}^3 \to \mathbb{S}^4_1$ the above embedding. From Theorem 4.1 and (26), we derive that there is at least a point $p \in M^2$ with satisfies $K(p) = 0$ or at least a point with $K(p) = 1$. In particular, if K is constant, we can deduce the classical result $K = 0$ or $K = 1$.

Consider the 4-dimensional anti De sitter spacetime of sectional curvature -1,

$$\mathbb{H}^4_1 = \{v \in \mathbb{R}^5 \ : \ -v_0^2 - v_1^2 + v_2^2 + v_3^2 + v_4^2 = -1\}.$$

The 3-dimensional hyperbolic space $\mathbb{H}^3$, the complete simply connected Riemannian manifold with sectional curvature -1, may be realized as the following totally geodesic spacelike hypersurface of $\mathbb{H}^4_1$,

$$\mathbb{H}^3 = \{v \in \mathbb{H}^4_1 : \ v_0 = 0, \ v_1 > 0\}.$$

Theorem 5.6. *Let $\phi : M^2 \longrightarrow \mathbb{H}^3$ be a compact connected surface in the 3-dimensional hyperbolic space. If the Gauss curvature K is a positive constant, then M^2 is a totally umbilical round sphere.*

Proof. We have that, one more time, M^2 is a topological sphere. Let $i : \mathbb{H}^3 \hookrightarrow \mathbb{H}^4_1$ be the above embedding. Since the scalar α in (24) is 0, making use of (26) we deduce that

$$\det(A^x_\xi) = \det(A^x_\eta) = \frac{1}{2}(K+1).$$

Therefore A^x_ξ and A^x_η are positive definite (otherwise, we make a change of ξ or η to $-\xi$ or $-\eta$). Now, Lemma 5.2 and Corollary 4.2 can be called one more time to ensure that the null normal sections ξ and η are umbilical and so, M^2 is a totally umbilical round sphere in $\mathbb{H}^3$.

$\square$

We end this article obtaining a new proof of [3, Th. 12].

Theorem 5.7. *Let $\phi : M^2 \longrightarrow \mathbb{S}^3_1$ be a compact connected spacelike surface with constant positive Gauss curvature $K < 1$. Then M^2 is a totally umbilical round sphere.*

Proof. From the assumption on K, we have that M^2 is a topological sphere. Denote by $i : \mathbb{S}^3_1 \hookrightarrow \mathbb{S}^4_1$ the usual totally geodesic immersion, and consider as bellow $x = i \circ \phi$. Since $\alpha = 0$, we deduce that

$$\det(A^x_\xi) = \det(A^x_\eta) = \frac{1}{2}(1 - K).$$

Therefore A^x_ξ and A^x_η are positive definite. Lemma 5.2 and Corollary 4.2 imply that the null normal sections ξ and η are umbilical. Therefore, M^2 is a totally umbilical round sphere in $\mathbb{S}^3_1$.

□

Remark 5.8. Observe that the totally geodesic embeddings $\mathbb{L}^3 \hookrightarrow \mathbb{L}^4$ and $\mathbb{H}^3_1 \hookrightarrow \mathbb{H}^4_1$ have not been considered previously because there exists no compact spacelike surface in $\mathbb{L}^3$ (see for instance [8]) neither in the anti De Sitter spacetime $\mathbb{H}^3_1$ (see for instance [1, Cor. 3]).

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