Universidad de Oviedo

Programa de Doctorado en Materiales

Amplitudes de Dispersión en Teorías Confinantes y Holografía

Scattering Amplitudes in Confining Theories and Holography

TESIS DOCTORAL

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Scattering Amplitudes in Confining Theories and Holography

TESIS DOCTORAL

Director de tesis

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Universidad de Oviedo

RESUMEN DEL CONTENIDO DE TESIS DOCTORAL

1.- Título de la Tesis	
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RESUMEN (en español)

La correspondencia AdS/CFT o dualidad holográfica nos permite describir sistemas físicos en regímenes de acoplo fuerte donde las herramientas tradicionales a nuestro alcance como el análisis perturbativo no pueden ser aplicadas. Esta tesis está enfocada en el desarrollo de un método para calcular amplitudes de dispersión en teorías confinantes con duales holográficos mediante la correspondencia AdS/CFT. Estas teorías confinantes presentan un ''mas gap'' y un espectro de masas discreto.

Se ha desarrollado un método para calcular amplitudes de dispersión y longitudes de dispersión en procesos de dispersión elásticos utilizando la dualidad holográfica. Estas cantidades capturan las características básicas de la interacción en el proceso de dispersión, es decir si la interacción es atractiva o repulsiva. Este método consiste en la identificación los residuos de las funciones de Green de la teoría gravitatoria dual con las amplitudes en la teoría de campos. Dicho procedimiento fue aplicado a sistemas ya conocidos.

Estos modelos holográficos, obtenidos a partir de teoría de cuerdas, son duales a teorías de campos confinantes y presentan diferentes realizaciones geométricas. Aplicando nuestro método hemos obtenido en primer caso la relación entre la longitud de dispersión y la masa de las partículas que participan en la interacción, así como la relación entre la longitud de dispersión y la dimensión conforme de los operadores que crean dichas partículas y su espectro de masas. Finalmente hemos obtenido el valor de los coeficientes del lagrangiano quiral, así como el valor de la constante de desintegración del pion aplicando nuestro método a un modelo que incluye dichas partículas y presenta las simetrías de la teoría quiral.

La tesis contiene siete capítulos. El primer capítulo consiste en una introducción donde se expone el contexto de este trabajo. En el Capítulo 2 se hace un repaso de varios temas relacionados con la teoría cuántica de campos necesarios para la tesis. El en el Capítulo 3 se presenta la dualidad holográfica y sus principales características. En el Capítulo 4 se presenta y se desarrolla un método para calcular la amplitud y la longitud de dispersión dentro del marco de trabajo de la correspondencia AdS/CFT. En el Capítulo 5 se aplica este método a tres modelos diferentes con el objetivo de obtener la longitud de dispersión y el espectro de masas. En el Capítulo 6 se aplica dicho método al proceso de dispersión de piones para obtener su amplitud de dispersión. Finalmente, las conclusiones están dada en el Capítulo 7.

Universidad de Oviedo

RESUMEN (en Inglés)

The AdS/CFT correspondence or holographic duality allows us to describe physical systems in strongly coupled regimes where the traditional tools at our disposal such as perturbative analysis cannot be applied. This thesis is focused on the development of a method to compute scattering amplitudes in theories confining with holographic duals using the AdS/CFT correspondence. These confining theories present a "mass gap" and a discrete mass spectrum.

A method has been developed to calculate scattering amplitudes and scattering lengths in elastic scattering processes using holographic duality. These quantities capture the basic characteristics of the interaction in the dispersion process, i.e. whether the interaction is attractive or repulsive. This method consists of identifying the residues of Green's functions in dual gravitational theory with the amplitudes in field theory. This procedure was applied to already known systems.

These holographic models, obtained from string theory, are dual to confining field theories and present different geometric realizations. Applying our method, we have obtained in the first case the relationship between the scattering length and the mass of the particles participating in the interaction, as well as the relationship between the scattering length and the conformal dimension of the operators that create these particles and their mass spectrum. Finally, we have obtained the value of the coefficients of the chiral lagrangian as well as the value of the pion decay constant by applying our method to a model that includes these particles and presents the symmetries of the chiral theory.

The thesis contains seven chapters. The first chapter consists of an introduction where the context of this work is exposed. Chapter 2 reviews several topics related to quantum field theory necessary for the thesis. Chapter 3 presents the holographic duality and its main characteristics. Chapter 4 presents and develops a method for calculating the scattering amplitude and the scattering length within the framework of AdS/CFT correspondence. In Chapter 5 this method is applied to three different models to obtain the scattering length and mass spectrum. In Chapter 6 this method is applied to the process of dispersion of pions to obtain its scattering amplitude. Finally, the conclusions are given in Chapter 7.

SR. PRESIDENTE DE LA COMISIÓN ACADÉMICA DEL PROGRAMA DE DOCTORADO EN MATERIALES

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Contents

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Chapter 1

Introduction

The context of this thesis is that of the gauge/gravity duality, which made its appearance in String Theory, and gave a completely new way of understanding theories of gravity, quantum field theories and the relations between them.

The development of Quantum Field Theory (QFT) has been one of the biggest achievements in physics during the past century like that of Genereal Relativity (GR). Both theories have predicted with accuracy an astonishing amount of natural phenomena. The predictions provided by QFT and the Standard Model have met the experimental results with a precision unprecedented in science. However the Standard Model is just an effective theory that need a UV completion an a deep understanding of the full theory is still needed to give answer to some of the most fundamental questions of nature.

As we said the Standard Model is not the end of the story and despite it success the predictive power of QFT is limited to a set of regimes and phenomena. The usual approach consists of a pertubative analysis of the theory to obtain scattering amplitudes. This requires in most of the cases that the value of the coupling constant is small and that's why for strongly coupled theories the perturbative approach fails and alternative techniques are need. That's the reason the nature of the strong interaction was the hardest and latest to be understood, and in the seventies, Quantum Chromodynamics (QCD) appeared as the suitable QFT to account for the strong nuclear forces. And it was in the context of trying to understand the strong interactions where Sting Theory first appeared.

Strong nuclear forces are well described at high energies by the perturbative expansion of QCD, the microscopic theory that has as dynamical degrees of freedom quarks and gluons. At low energies QCD becomes strongly coupled and an effective description becomes a necessity. Mesons and baryons (hadrons) are the observed physical bound states at low energies and a standard approach is to construct an effective field theory with hadrons as dynamical degrees of freedom.

Bound with the effective theory does not mean that all the microscopic details are coarse-grained beyond reach. One can gain access to the microscopic physics as they are encoded in short-range interactions among the low energy degrees of freedom in the effective theory. It is even possible to do a systematic expansion of these interactions in terms of the microscopic length scale over a wavelength , the lowest order contribution being determined by the scattering length. This important text-book quantity can, in principle, be extracted from the zero momentum limit of the scattering amplitude involving the interacting degrees of freedom; for reviews on this approach, see [1–3]. Generically, a negative scattering length $a_s < 0$ implies that the interaction is attractive, while a positive value $a_s > 0$ may signal either a repulsive interaction (for small a_s) or the existence of bound states in an attractive potential (for large a_s).

In many cases, the details of the interactions for a given channel are not very important, and for an effective description at low energies it is enough to know the scattering length a_s and the range of interaction. However, a direct computation of the scattering amplitudes in the microscopic theory is typically not conceivable, especially at strong coupling. This also has some inherent limitations for heavier states in particular.

Due to confinement, strong interactions are short-ranged and in many cases can be approximated by contact interactions, which are determined by the scattering length measured in the experiment, or alternatively extracted from finite size corrections in lattice QCD [4–6]. In general, there are also interactions within the hadronic theory produced by hadron exchange. Hadronic matter is not the only example of a physical system with short-range interactions. Those abound in condensed matter systems, for instance in the descriptions of the effective potential for cold atoms at unitarity [7–9]. So there are many important physical systems whose complete description is out of reach of traditional approaches, typically because they are strongly coupled.

A promising alternative is to use the the AdS/CFT correspondence or holographic duality, which is well-suited for this task, as the gravity side becomes weakly coupled when the dual theory is strongly coupled. This duality establishes an equivalence between string theory in an AdS geometry and a QFT living on its boundary, with the peculiarity of being a strong/weak coupling duality. So this feature provides us with a powerful tool to understand strongly coupled field theories by means of calculations in the string theory side or its classical low energy description, supergravity.

The duality appeared in 1997 when Maldacena proposed the AdS/CFT correspondence [10]. The key objects in thee discovery of this duality are the D-branes, non-perturbative extended objects of different dimensionalities. These objects can be studied from the closed string point of view as emitters of closed strings or from the open string point of view as endpoints to which the strings can be attached. It is that dual picture that allowed Maldacena to formulate the AdS/CFT correspondence. The conjecture goes as follows. Taking D3-branes in flat space time and after applying some particular limits the two dual pictures of the branes emerge. One is the closed string picture described by type IIB closed String Theory in $AdS_5 \times S^5$ spacetime and the other is the open string picture described by the gauge theory on the worldvolume of the D3-brane, which is the (3+1)-dimensional $\mathcal{N} = 4$ supersymmetric Yang-Mills gauge theory, which is a conformal field theory. Both descriptions are thought to be equivalent and this was the first known example which realized the idea of the existence of a relationship between String Theory and gauge theories.

Also, since the inception of the gauge/gravity, or holographic, duality there has been a conscious effort to understand Quantum Chromodynamics (QCD) in the non-perturbative regime [11]. We do not know if the holographic dual of QCD exists and, even if it is the case, there is likely no weakly coupled gravity description. To date, quantitative predictions have usually been based on a phenomenological approach where weakly coupled holographic models are fitted to known QCD data obtained through experiments or other non-perturbative approaches. The most developed model in this respect is presumably V-QCD [12].

The main goal of this thesis is the development of a method to compute scattering amplitudes using the holographic duality and then apply it to some models with different holographic duals and finally obtain the value of the Low Energy Coefficients (LEC's) for pions that appear in the chiral lagrangian. We compute two-to-two scattering amplitudes and scattering lengths by identifying the residues of the Green's functions in the gravity theory with the amplitudes in the field theory using the relation between the one-point function and the regularized canonical momentum given by the holographic dictionary. With this method in hand, first we apply it to three different confining models with a discrete spectrum and a mass gap (the hard wall model [13], the GPPZ flow [14] and the AdS_6 soliton [11]), each one of them with different geometry in the gravity side, in order to obtain the spectrum of masses, the scattering length and it's functional dependence on the masses of the particles and on the conformal dimension of the operators that create them and to find out if the differences in the gravity description affect the final result of the scattering length. Lastly we apply our method to the Sakai-Sugimoto model [15, 16] to obtain the pion decay constant and the LEC's from the two-to-two pion low energy scattering amplitude which we compute in the holographic dual through tree-level Witten diagrams and compare those values with the ones existing in the literature.

About this thesis

This thesis is based on the papers [17], [18] and [19] it is divided on two different parts, the first one is an introduction on theoretical aspects and the second one is focused on the computation of scattering processes using holography, and has the following structure:

- The second chapter is a review on some relevant field theory topics for our work, the computation of scattering amplitudes in Quantum Field Theory, the large N limit in Yang-Mills gauge theories and its relation with String Theory and the Chiral Effective Theory, what happens to it in the large N limit and how are the dynamics of the pion field.

- The third chapter is an introduction of the AdS/CFT correspondence, first we see how it arises in the framework of String Theory, how the dictionary between both sides works in terms of fields/operators and also in terms of symmetries and how to compute correlation functions and then we review some string theory models with a confining geometry dual, the GPPZ flow, the AdS_6 soliton, the Witten-Yang-Mills model and the Witten-Sakai-Sugimoto model on which, later on, we will study the scattering process.

- In the fourth chapter we present a method to compute the two-to-two amplitudes and the scattering length using the holographic duality. Our method is based on the identification of the residues of Green's functions in the gravity dual with the amplitudes in the field theory. We will start reviewing the derivation of the scattering length from the four-point function in four spacetime dimensions. Then we will present the method to compute the scattering amplitude and the scattering length between four scalar particles using holography. We will allow a general value of the spacetime dimension and show that it is enough to extract the residues of poles in Green's functions of classical fields in the gravitational theory.

- In the fifth chapter we apply our method to three different models with confining duals computing the main contribution to the scattering length between two spin-zero particles in strongly coupled theories using the gauge/gravity duality. First for illustration purposes we will apply the method to the hard wall model, computing a contribution to the scattering length. Then we study two different theories with a mass gap: a massive deformation of $\mathcal{N} = 4$ super Yang-Mills theory ($\mathcal{N} = 1$) and a non-supersymmetric five-dimensional theory compactified on a circle. These cases have a different realization of the mass gap in the dual gravity description: the former is the well-known GPPZ singular solution and the latter a smooth AdS_6 soliton geometry.

- In the sixth chapter we apply our method to compute the pion scattering amplitude in the Witten-Sakai-Sugimoto model and obtain the pion decay constant and coefficients of fourth derivative terms in the chiral Lagrangian for massless quarks. We extract these quantities from the two- pion scattering amplitude, which we compute directly in the holographic dual through tree-level Witten diagrams.

Part I

Theoretical Foundations

Chapter 2

Review of Field Theory Topics

In this chapter we will present some key topics of Quantum Field Theory relevant for this thesis like the computation of scattering amplitudes, the large N limit and the Chiral Perturbation Theory (ChPT).

2.1 Scattering Amplitudes

Here we will review the computation of scattering amplitudes within the framework of quatum field theory using correlation fuctions and the LSZ reduction formula (named after the the three German physicists Harry Lehmann. Kurt Symanzik and Wolfhart Zimmerman). The main reference for this chapter is [20].

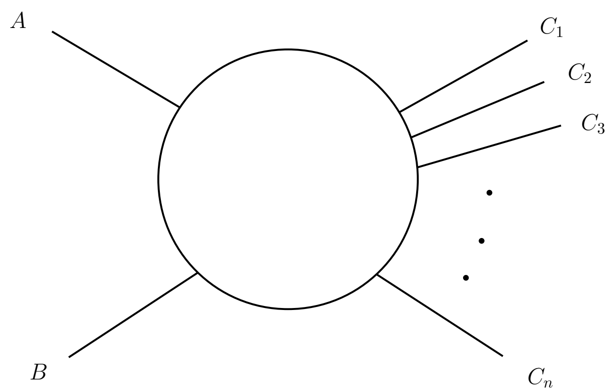


Figure 2.1.1: General scattering process

The most general scattering of two particles is of the form shown in Fig2.1.1 where two ingoing particles A and B collide and produce an arbitrary number of outgoing particles labeled as $C_1, C_2, C_3, \dots C_n$

$$A + B \to C_1 + C_2 + C_3 + \dots + C_n. \tag{2.1.1}$$

In a low energy scattering process which takes part in the elastic region, where the number and type of the particles doesn't change, the previous reaction reads

$$A + B \to A' + B', \tag{2.1.2}$$

where the particles satisfy the on-shell conditions

$$p_A^2 = m_A^2,\ p_B^2 = m_B^2,\ p_{A'}^2 = m_A^2,\ p_{B'}^2 = m_B^2. \tag{2.1.3}$$

Being p_A and p_B the 4-momenta of the particles before the collision and $p_{A'}$ and $p_{B'}$ the 4-momenta after it. The total momentum in such collision is conserved so

$$p_A + p_B - p_{A'} - p_{B'} = 0. \tag{2.1.4}$$

In a scattering process like the one described before we are interested in how the incoming particles given by $|i\rangle$ evolve into the outgoing particles $|f\rangle$. So the operator of interest is the scattering operator S. This operator maps the incoming states into the outgoing states

$$|f\rangle = S\,|i\rangle\,. \tag{2.1.5}$$

It encodes the information about the scattering process and it is defined as the limit of the time evolution operator for infinite times

$$S = \lim_{t_i, t_f \to \mp\infty} U\left(t_f, t_i\right). \tag{2.1.6}$$

The time evolution operator $U\left(t_f, t_i\right)$ expressed in the interaction picture, assuming the Hamiltonian is time independent, takes the system from an initial time t_i to a final time t_f and it is defined as

$$U\left(t_f, t_i\right) = e^{iH_0 t_f} e^{-i(H_0+V)(t_f - t_i)} e^{-iH_0 t_i}, \tag{2.1.7}$$

where H_0 is the free particle hamiltonian.

The S is matrix defined as

$$S_{fi} = \langle f|\,S\,|i\rangle\,, \tag{2.1.8}$$

where usually S is decomposed in two parts

$$S = \mathbf{1} + i\mathcal{T}, \tag{2.1.9}$$

the first one is the non-interacting part given by the identity matrix and the second one is the interacting part that encodes the dynamics of the process and is called the $\mathcal{T}$-matrix.

The matrix elements of S should reflect 4-momentum conservation so they always have to contain a factor $\delta^4\left(p_f - p_i\right)$. Extracting this factor the S-matrix element can be written as

$$\langle f|\,S\,|i\rangle = \langle f|\,i\rangle + (2\pi)^4\,\delta^4\left(p_f + p_i\right)\,i\mathcal{M}_{fi}, \tag{2.1.10}$$

with

$$i\mathcal{T}_{fi} = (2\pi)^4\,\delta^4\left(p_f + p_i\right)\,i\mathcal{M}_{fi}. \tag{2.1.11}$$

where $\mathcal{M}_{fi}$ is the scattering amplitude and from which one can obtain measurable quantities like the cross section, decay rates and the scattering length.

In order to obtain the S-matrix elements we use the LSZ reduction formula which reads

$$\langle p_1 \ldots p_k|\,S\,|p_{k+1} \ldots p_n\rangle = i^n \int \left(\prod_{i=1}^{n} d^4 x_n e^{ip_n \cdot x_n} \frac{p_i^2 + m^2}{\sqrt{Z_n}}\right)_{p_i\ \text{on-shell}} \langle \phi\left(x_1\right), \ldots, \phi\left(x_n\right)\rangle\,, \tag{2.1.12}$$

where Z_n factors correspond with the residues of the poles of the two-point correlation function. It relates correlation functions to the S-matrix elements and assumes that all the particles have the same mass. The factors $\left(p_i^2 + m^2\right)$ vanish once the momenta are placed on-shell, $p_i^2 = -m^2$. This means we only get a non-zero answer for diagrams contributing to the n-point function which have propagators for each external leg.

The last step in this process is to know how to compute correlation functions. Correlation functions are vacuum expectation values of time-ordered products of field operators and can be interpreted physically as the amplitude for propagation of particles, in the case of the two point function it represents the amplitude for propagation of a particle between points x and y. They are key objects in quantum field theories and via the LSZ reduction formula they can be used to compute observables such as S-matrix elements.

For a scalar field theory with a single field $\phi\left(x\right)$ and a vacuum state $|\Omega\rangle$ the n-point correlation function is the vacuum expectation value of the time-ordered products of n field operators in the Heisenberg picture

$$\langle \phi\left(x_1\right), \ldots, \phi\left(x_n\right)\rangle = \langle \Omega|\,T\left\{\phi\left(x_1\right), \ldots, \phi\left(x_n\right)\right\}|\Omega\rangle\,, \tag{2.1.13}$$

where $T\left\{\ldots\right\}$ is the time-ordering operator which orders the field operators so that earlier time field operators appear to the right of later time field operators. Transforming the fields and states into the interaction picture, this is rewritten as

$$\langle \phi\left(x_1\right), \ldots, \phi\left(x_n\right)\rangle = \frac{\langle 0|\,T\left\{\phi\left(x_1\right), \ldots, \phi\left(x_n\right) e^{iS_I[\phi]}\right\}|0\rangle}{\langle 0|\,e^{iS_I[\phi]}\,|0\rangle}, \tag{2.1.14}$$

where $|0\rangle$ is the ground state of the free theory and S_I is the part of the action that encodes the interaction

$$S_I = \int dt\,\left(-V_I\right), \quad V_I = e^{iH_0 t} V e^{-iH_0 t}. \tag{2.1.15}$$

Expanding $e^{iS_I[\phi]}$ the n-point correlation function becomes a sum of correlation functions which can be evaluated using Wick's theorem. A diagrammatic way to represent the resulting sum is via Feynman diagrams where each term can be evaluated using the Feynman rules.

In terms of diagrams the previous equations can be written as

$$\langle\phi(x_1),\ldots,\phi(x_n)\rangle=\frac{(\text{sum of all vacuum bubble diagrams})\times(\text{sum of all diagrams with no bubbles})}{(\text{sum of all vacuum bubble diagrams})}. \tag{2.1.16}$$

The first terms cancels with the normalization factor in the denominator so the n-point correlation function is the sum of all Feynman diagrams excluding vacuum bubbles

$$\langle\phi(x_1),\ldots,\phi(x_n)\rangle=\langle 0|\,T\left\{\phi(x_1),\ldots,\phi(x_n)\,e^{iS[\phi]}\right\}|0\rangle_{\text{no bubbles}}\,. \tag{2.1.17}$$

This sum also include disconnected diagrams, which are diagrams where at least one external leg is not connected to all other external legs through some connected path. Excluding these disconnected diagrams instead defines connected n-point correlation functions

$$\langle\phi(x_1),\ldots,\phi(x_n)\rangle_c=\langle 0|\,T\left\{\phi(x_1),\ldots,\phi(x_n)\,e^{iS[\phi]}\right\}|0\rangle_{\text{connected}}\,. \tag{2.1.18}$$

These contain all the information that full correlation functions contain since any disconnected diagram is a product of connected diagrams.

In the path integral formulation, n-point correlation functions are written as

$$\langle\phi(x_1),\ldots,\phi(x_n)\rangle=\frac{\int\mathcal{D}\phi\phi(x_1),\ldots,\phi(x_n)\,e^{iS[\phi]}}{\int\mathcal{D}\phi e^{iS[\phi]}}. \tag{2.1.19}$$

They can be evaluated using the partition function $Z[J]$

$$Z[J]=\int\mathcal{D}\phi e^{iS[\phi]+i\int d^dxJ(x)\phi(x)}, \tag{2.1.20}$$

where $J(x)$ is the scalar field source current so the n-point correlation function can be computed as the functional derivative of the partition function with respect to the sources

$$\langle\phi(x_1),\ldots,\phi(x_n)\rangle=(-i)^n\,\frac{1}{Z[J=0]}\left.\frac{\delta^nZ[J]}{\delta J(x_1)\ldots\delta J(x_n)}\right|_{J=0}. \tag{2.1.21}$$

2.2 Large N Limit

Gauge theories like Yang-Mills theory with gauge group $SU(N)$ allow a perturbative expansion in the computation of observables in powers of $1/N$ and simplify considerably in the limit when $N\to\infty$. This was first pointed out by 't Hooft in 1974 in [21] and it is called the large N limit. It provides us with a perturbative framework to study for example QCD where one extrapolates from the physical value for the number of colors, $N=3$, to the limit $N\to\infty$.

The action for Yang-Mills theory with gauge group $SU(N)$ is

$$S=-\frac{1}{2g_{YM}^2}\int d^4x\mathrm{Tr}\left(F^{\mu\nu}F_{\mu\nu}\right). \tag{2.2.1}$$

We define the 't Hooft coupling as

$$\lambda=g_{YM}^2N, \tag{2.2.2}$$

which will be kept fixed when taking the limit $N\to\infty$. So in terms of this new coupling the action (2.2.1) reads

$$S=-\frac{N}{2\lambda}\int d^4x\mathrm{Tr}\left(F^{\mu\nu}F_{\mu\nu}\right). \tag{2.2.3}$$

In order to analyze Feynman diagrams at large N the double line notation will be used. As usual quarks will carry a color index i with $i=1,\ldots,N$. On the other hand, gluons that usually are described by a single color index which takes the values $a=1,\ldots,N^2-1$ will now be described by two indices j and k

$$A_\mu^a\to\left(A_\mu\right)^j_k\quad\left(A_\mu\right)^j_j=0, \tag{2.2.4}$$

where $j, k = 1, \ldots, N$ so each gluon field is an $N \times N$ matrix.

The gluon propagator then reads

$$\left\langle A^i_{\mu j}(x) A^k_{\nu l}(y) \right\rangle = \Delta_{\mu\nu}(x-y) \left(\delta^i_l \delta^k_j - \frac{1}{N} \delta^i_j \delta^k_l \right), \tag{2.2.5}$$

where $\Delta_{\mu\nu}(x)$ is the usual Abelian gauge field propagator.

As we mentioned with this new notation the gauge fields carry two color indices so we can represent each gluon as if it was a quark-antiquark pair. A solid line is drawn for each color index. The Feynman diagram for the gluon propagator is shown in 2.2.1.

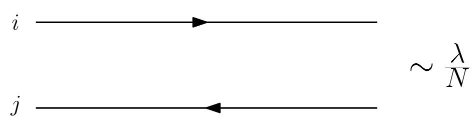
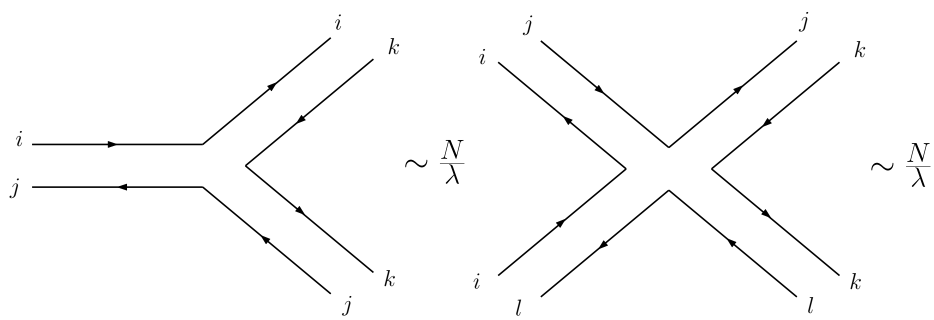


Figure 2.2.1: Gluon propagator in the double line notation.

The propagator scales as λ/N as can be read off from the action (2.2.3).

Expanding the action we can obtain the cubix and quartic vertex, depicted in 2.2.2.



(a) Cubic vertex. (b) Quartic vertex.

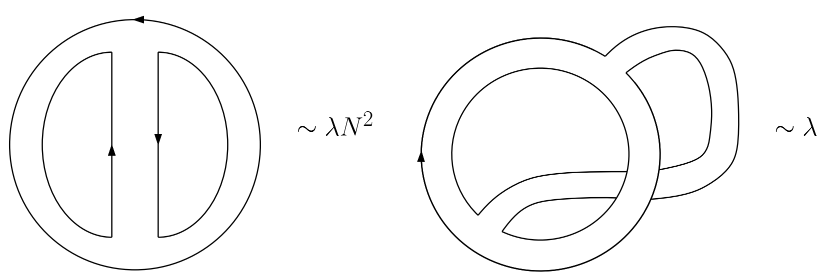
Figure 2.2.2: Some vertices in the double line notation.

Where each vertex contributes with a factor of N/λ.

A general diagram consists of an arbitrary number of propagators, interactions vertices and color index loops so it will give a contribution

$$\text{diagram} \sim \left(\frac{\lambda}{N}\right)^{\#\text{propagators}} \left(\frac{N}{\lambda}\right)^{\#\text{vertices}} (N)^{\#\text{color index loops}}. \tag{2.2.6}$$

To ilustrate this we will present some examples of diagrams in 2.2.3 and 2.2.4.



(a) (b)

Figure 2.2.3: Vacuum bubbles.

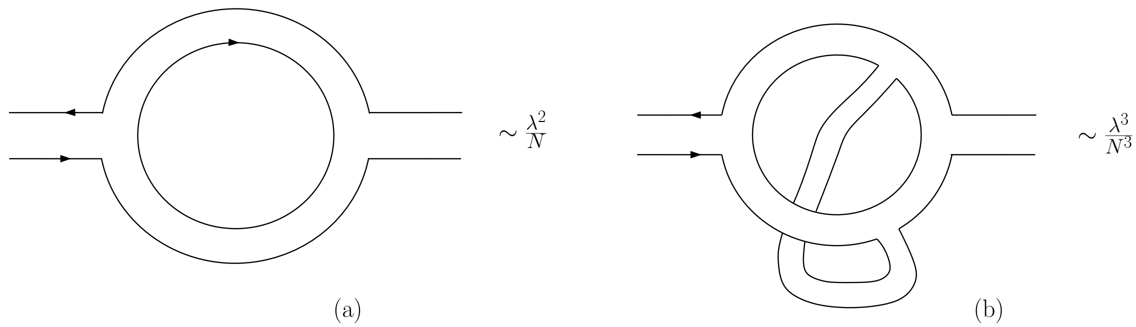


Figure 2.2.4: Gluon propagator.

In the first case, Figure 2.2.3, we have two vacuum bubbles both with one gluon exchange but with different color index loop structure. In the second case, Figure 2.2.4, we have the gluon propagator (a) and the gluon propagator with one loop correction (b). We see in both cases that the second diagram is supressed by a factor $1/N^2$ relative to the first so in the limit $N \to \infty$, while keeping the 't Hooft couplig λ fixed, the second ones will be sub-dominant. The dominant diagrams like Figure 2.2.3(a) and Figure 2.2.4(a) are called planar diagrams and the sub-dominant ones like Figure 2.2.3(b) and Figure 2.2.4(b) are called non-planar diagrams. So in the large N limit one only needs to sum over the planar diagrams.

Now we are going to explain why they are called planar and non-planar diagrams. The main idea is that the planar diagrams, like the one in 2.2.3(a), can be inscribed on the surface of a sphere as we see in Figure 2.2.5.

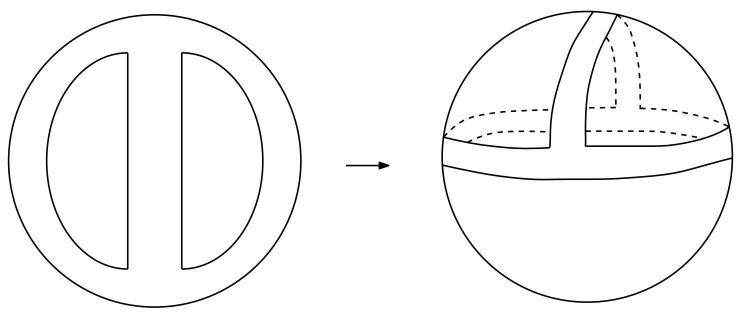


Figure 2.2.5: Planar diagram.

To recover the original diagram from the one depicted on the sphere one just needs to remove the bottom face of the sphere and proyect the remaining half on the plane.

On the other hand, non planar diagrams must be drawn on other kind of surfaces. As we see in Figure 2.2.6 the diagram depicted in 2.2.3(b) must be inscribed on a torus instead on a sphere.

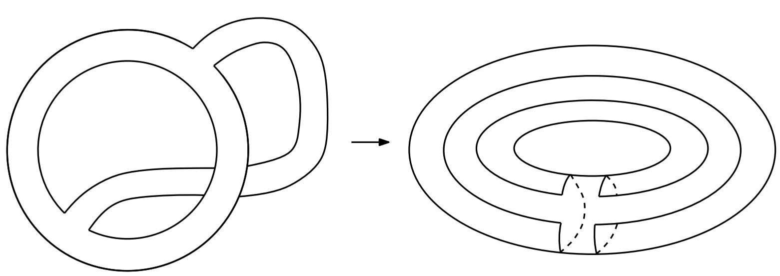


Figure 2.2.6: Non-planar diagram

Now we can introduce the Euler characteristic χ defined by

$$\chi = 2 - 2g - h, \tag{2.2.7}$$

being g the genus of the surface which corresponds to the number of handles and h the number of holes. In this case we are interested in closed surfaces where the diagrams can be inscribed so the relation (2.2.7) for the Euler characteristic reduces to

$$\chi = 2 - 2g. \tag{2.2.8}$$

In Figure 2.2.7 we show some surfaces and the corresponding values of g.

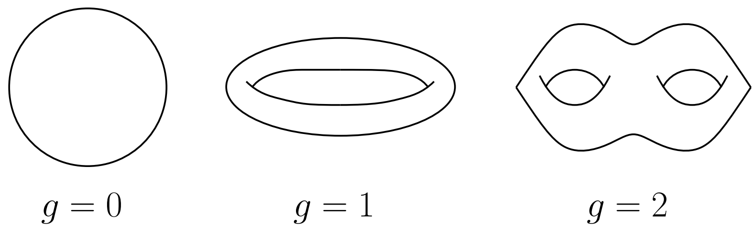


Figure 2.2.7: Examples of surfaces with different genus.

Then, given the genus of the surface where the diagram can be inscribed, its contribution which is weighted by its topology, can be computed in terms of the Euler characteristic as

$$\text{diagram} \sim N^{\chi}. \tag{2.2.9}$$

Thus the large N expansion of any process may be written as

$$\sum_{g=0}^{\infty} N^{2-2g}, \tag{2.2.10}$$

so the computation will be dominated by the surfaces of maximal χ or minimal genus.

We see that the large N limit of Yang-Mills consists of a sum over surfaces of different topologies, a feature also present in string pertubation theory where the sum is over the different topologies of the worldsheets. The scattering of two strings receives contributions from worldsheets of the form

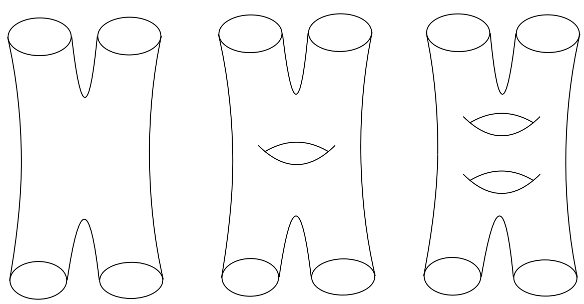


Figure 2.2.8: Worldsheets corresponding to the contributions to the scattering of two strings.

where the Euler characteristic (or the genus) also plays an important role because the pertubative expansion of a process is given by the sum

$$\sum_{\text{topologies}} g_s^{-(2-2g)}, \tag{2.2.11}$$

where g_s is the string coupling. Doing the comparison between (2.2.10) and (2.2.11) one can make the identification

$$g_s \sim \frac{1}{N}, \tag{2.2.12}$$

so this suggests a relation between weakly coupled string theory and the large N limit of the Yang-Mills theory.

2.3 Chiral Perturbation Theory

QCD, the theory of strong interactions, consists of Yang-Mills gauge theory coupled to fermions. These fermions are known as quarks and they have six different flavors, each flavor having different mass. If we are interested in studying hadron physics at energies below ~ 1 GeV we can ignore the charm, bottom and top quarks. Also the masses of the up and down quarks are of few MeV which is much less than $\Lambda_{QCD} \sim 200$ MeV so we can make the approximation that those quarks are massless. So the remaining theory with only two flavors of quarks will exhibit a global $U(2) \times U(2)$ global symmetry in addition to the $SU(3)$ color gauge symmetry. This symmetry is often denoted as $U(2)_L \times U(2)_R$ and treats the left- and right-handed fields differently and is known as chiral symmetry. Most of this content is based on [22] and for reviews on this topic see [2, 23, 24].

Gauge theories with massless fermions exhibit chiral symmetry as we mentioned, under this symmetry one can rotate the left-handed and the right-handed components of the fields independently and the theory remains the same, this is described by the transformation

$$q_L \to e^{i\theta_L} q_L, \quad q_R \to q_R, \tag{2.3.1}$$

or

$$q_L \to q_L, \quad q_R \to e^{i\theta_R} q_R. \tag{2.3.2}$$

The fermionic part of the Lagrangian of that theory reads in the general case when the theory has N_f flavors of fermions

$$\mathcal{L} = \sum_{i=1}^{N_f} i\bar{q}_i \not{D} q_i, \tag{2.3.3}$$

where $\not{D} q = \not{\partial} q - i\gamma^\mu A_\mu q$. Here $i = 1, \ldots, N_f$ labels the species of quark and it is referred as a flavor index. The theory symmetry group is manifest when we decompose the fermionic kinetic terms into left-handed and right-handed parts

$$\sum_{i=1}^{N_f} i\bar{q}_i \not{D} q_i = \sum_{i=1}^{N_f} i\bar{q}_{i_L} \not{D} q_{i_L} + i\bar{q}_{i_R} \not{D} q_{i_R}. \tag{2.3.4}$$

The symmetry of the Lagrangian is given by the $U(N_f)_L \times U(N_f)_R$ symmetry group. It can be divided into the diagonal and off-diagonal components, the first one treats the left-handed and the right-handed parts equally and its known as vector symmetry $U(N_f)_V$ and the second one treats them differently and is know as axial symmetry $U(N_f)_A$. The flavor can be decomposed into[1]

$$SU(N_f)_L \times SU(N_f)_R \times U(1)_V \times U(1)_A, \tag{2.3.5}$$

where the vector symmetry $U(1)_V$ acts as

$$q_L \to e^{i\theta_L} q_L, \quad q_R \to e^{i\theta_R} q_R, \quad \theta_L = \theta_R = \theta_V \tag{2.3.6}$$

and the axial group $U(1)_A$ transforms the fermions as

$$q_L \to e^{i\theta_L} q_L, \quad q_R \to e^{i\theta_R} q_R, \quad \theta_L = -\theta_R = \theta_A. \tag{2.3.7}$$

Where the current associated to the axial symmetry is not conserved by quantum effects so the symmetry is said to be anomalous.

The non-Abelian symmetry, $SU(N_f)_L \times SU(N_f)_R$, turns out to be spontaneously broken by the formation of the quark condensate giving the following breaking pattern

$$SU(N_f)_L \times SU(N_f)_R \to SU(N_f)_V. \tag{2.3.8}$$

This means that, although the Lagrangian is invariant, the vacuum is not. There is a quark condensate

$$\langle 0| \bar{q}_R q_L |0\rangle \neq 0. \tag{2.3.9}$$

The existence of quark condensate is telling us that the vacuum of space is populated by quark-anti-quark pairs.

[1] Actually it can be decomposed as $U(N_f) = \big(SU(N_f) \times U(1)\big)/\mathbb{Z}_{N_f}$.

The Nambu-Goldstone bosons associated to the broken $SU(N_f)$ axial symmetry are the pions. Massless particles formed by bound states of the original quarks. There are $N_f^2 - 1$ Goldstone bosons, one for each broken generator of the symmetry.

We would like to understand the dynamics of the Goldstone modes. In the case of a generic number of flavors N_f the pions can be parametrized by a $SU(N_f)$ matrix Σ that reads

$$\Sigma = \exp\left(\frac{i}{f_\pi}\pi \cdot T\right) = \exp\left(\frac{i}{f_\pi}\pi^a T^a\right), \tag{2.3.10}$$

where T^a are the generators of the $SU(N_f)$ and the component fields π^a, labeled $a = 1, \ldots, N_f^2 - 1$, are the pions.

At low energies the form of the action is entirely determined by the symmetries of the theory which is invariant under

$$SU(N_f)_L \times SU(N_f)_R, \tag{2.3.11}$$

and the $\Sigma(x)$ field transforms under this symmetry as

$$\Sigma \longrightarrow U_L \Sigma U_R^\dagger. \tag{2.3.12}$$

At leading order in a derivative expansion the most general Lagrangian is given by a single term that describes the dynamics of pions

$$\mathcal{L} = -\frac{f_\pi^2}{4}\mathrm{Tr}\left(\partial_\mu \Sigma^\dagger \partial^\mu \Sigma\right), \tag{2.3.13}$$

which is the chiral Lagrangian. This Lagrangian contains an infinite number of interaction terms packaged in a simple form by the demands of symmetry. These interactions can be seen more explicitly rewriting it in terms of the pion fields, at quartic order reads

$$\mathcal{L} = -\mathrm{Tr}\left(\partial_\mu \boldsymbol{\pi} \cdot \partial^\mu \boldsymbol{\pi}\right) - \frac{2}{3f_\pi^2}\mathrm{Tr}\left(\left(\boldsymbol{\pi} \cdot \partial_\mu \boldsymbol{\pi}\right)^2 - \boldsymbol{\pi} \cdot \boldsymbol{\pi}\left(\partial_\mu \boldsymbol{\pi} \cdot \partial^\mu \boldsymbol{\pi}\right)\right) + \ldots, \tag{2.3.14}$$

where $\boldsymbol{\pi} = \pi^a T^a$.

To lowest order it suffices to use the Lagrangian (2.3.13) but we can include higher derivative terms in it. At next order the Lagrangian introduces new parameters L_i which must be determined from experiment and the standard name for them is low energy constants or LECs. The new Lagrangian can involve four derivative interactions, there are four possible independent chiral-invariant terms with four derivatives

$$\begin{gathered} \mathrm{Tr}\left(\partial_\mu \Sigma^\dagger \partial^\mu \Sigma \partial_\nu \Sigma^\dagger \partial^\nu \Sigma\right), \quad \mathrm{Tr}\left(\left(\partial_\mu \Sigma^\dagger \partial_\nu \Sigma\right)^2\right), \\ \left(\mathrm{Tr}\left(\partial_\mu \Sigma^\dagger \partial_\nu \Sigma\right)\right)^2, \quad \left(\mathrm{Tr}\left(\partial_\mu \Sigma^\dagger \partial^\mu \Sigma\right)\right)^2. \end{gathered} \tag{2.3.15}$$

So the chiral lagrangian up to fourth order is

$$\begin{aligned} \mathcal{L} = -\tfrac{f_\pi^2}{4}\mathrm{Tr}\left(\partial_\mu \Sigma^\dagger \partial^\mu \Sigma\right) + L_1\left(\mathrm{Tr}\left(\partial_\mu \Sigma^\dagger \partial^\mu \Sigma\right)\right)^2 + L_2\left(\mathrm{Tr}\left(\partial_\mu \Sigma^\dagger \partial_\nu \Sigma\right)\right)^2 \\ + L_3 \mathrm{Tr}\left(\partial_\mu \Sigma^\dagger \partial^\mu \Sigma \partial_\nu \Sigma^\dagger \partial^\nu \Sigma\right) + \tilde{L}_3 \mathrm{Tr}\left(\left(\partial_\mu \Sigma^\dagger \partial_\nu \Sigma\right)^2\right). \end{aligned} \tag{2.3.16}$$

In the case of $SU(3)$ these expressions are not linearly independent, it can be shown that

$$\mathrm{Tr}\left(\left(\partial_\mu \Sigma^\dagger \partial_\nu \Sigma\right)^2\right) = \tfrac{1}{2}\left(\mathrm{Tr}\left(\partial_\mu \Sigma^\dagger \partial^\mu \Sigma\right)\right)^2 + \left(\mathrm{Tr}\left(\partial_\mu \Sigma^\dagger \partial_\nu \Sigma\right)\right)^2 - 2\mathrm{Tr}\left(\partial_\mu \Sigma^\dagger \partial^\mu \Sigma \partial_\nu \Sigma^\dagger \partial^\nu \Sigma\right), \tag{2.3.17}$$

leaving only three independent terms. Then the $SU(3)$ chiral Lagrangian is

$$\mathcal{L} = -\tfrac{f_\pi}{4}^2 \mathrm{Tr}\left(\partial_\mu \Sigma^\dagger \partial^\mu \Sigma\right) + L_1\left(\mathrm{Tr}\left(\partial_\mu \Sigma^\dagger \partial^\mu \Sigma\right)\right)^2 L_2\left(\mathrm{Tr}\left(\partial_\mu \Sigma^\dagger \partial_\nu \Sigma\right)\right)^2 + \tilde{L}_3 \mathrm{Tr}\left(\left(\partial_\mu \Sigma^\dagger \partial_\nu \Sigma\right)^2\right). \tag{2.3.18}$$

Finally, considering two massless quarks, $N_f = 2$, the symmetry of the Lagrangian will be given by a $U(2)_L \times U(2)_R$ symmetry group which can be decomposed into

$$SU(2)_L \times SU(2)_R \times U(1)_V \times U(1)_A, \tag{2.3.19}$$

and the breaking pattern of the chiral part will be

$$SU(2)_L \times SU(2)_R \to SU(2)_V. \tag{2.3.20}$$

Then the Goldstone bosons corresponding to the three broken generators of the $SU(2)_A$ symmetry are the three pions. In the real world, because of the non-vanishing and differing masses of the quarks, $SU(2)_L \times SU(2)_R$, is only an approximate symmetry and therefore the pions are not massless and they are pseudo-Goldstone bosons.

Being $T^a = \frac{1}{2}\sigma^a$ the $SU(2)$ generators the Lagrangian 2.3.14 reads

$$\mathcal{L}_{int} = -\frac{1}{2}\partial_\mu \pi \cdot \partial^\mu \pi - \frac{1}{6f_\pi^2}\left((\partial_\mu \pi \cdot \pi)^2 - \pi \cdot \pi\,(\partial_\mu \pi \cdot \partial^\mu \pi)\right). \tag{2.3.21}$$

In the $SU(2)$ case , we have another identity

$$\mathrm{Tr}\left(\left(\partial_\mu \Sigma^\dagger \partial_\nu \Sigma\right)^2\right) = \frac{1}{2}\left(\mathrm{Tr}\left(\partial_\mu \Sigma^\dagger \partial^\mu \Sigma\right)\right)^2, \tag{2.3.22}$$

so the Lagrangian with quartic derivative interactions reads

$$\mathcal{L} = -\frac{f_\pi}{4}^2 \mathrm{Tr}\left(\partial_\mu \Sigma^\dagger \partial^\mu \Sigma\right) + L_1\left(\mathrm{Tr}\left(\partial_\mu \Sigma^\dagger \partial^\mu \Sigma\right)\right)^2 + L_2\left(\mathrm{Tr}\left(\partial_\mu \Sigma^\dagger \partial_\nu \Sigma\right)\right)^2. \tag{2.3.23}$$

Then the lagrangian expanded to fourth order in the pion field and derivatives is

$$\begin{aligned}\mathcal{L} = &-\frac{1}{2}\partial_\mu \pi \cdot \partial^\mu \pi - \frac{1}{6f_\pi^2}\left((\partial_\mu \pi \cdot \pi)^2 - \pi \cdot \pi\,(\partial_\mu \pi \cdot \partial^\mu \pi)\right) \\ &+ \frac{4L_1}{f_\pi^4}(\partial_\mu \boldsymbol{\pi} \cdot \partial^\mu \boldsymbol{\pi})^2 + \frac{4L_2}{f_\pi^4}(\partial_\mu \boldsymbol{\pi} \cdot \partial_\nu \boldsymbol{\pi})^2 + O(\partial^2 \pi^6)\ .\end{aligned} \tag{2.3.24}$$

2.3.1 Axial current

The axial current can be obtained by first adding gauge fields for the right-handed and left-handed symmetries by promoting the derivative to a covariant derivative with the left-handed and right-handed gauge fields L_μ and R_μ

$$\partial_\mu \Sigma \longrightarrow D_\mu \Sigma = \partial_\mu \Sigma + iL_\mu \Sigma - i\Sigma R_\mu\ . \tag{2.3.25}$$

Considering the $SU(2)$ generators $\tau^a = \frac{1}{2}\sigma^a$, $a = 1, 2, 3$, with σ^a the Pauli matrices, the components of the left-handed and right-handed currents are

$$\begin{aligned}J_L^{a\mu} &= \frac{\delta \mathcal{L}}{\delta L_\mu^a} = -\frac{if_\pi^2}{4}\mathrm{Tr}\left(\left(\Sigma \partial^\mu \Sigma^\dagger - \partial^\mu \Sigma \Sigma^\dagger\right)\tau^a\right) + O(\partial^3) \\ J_R^{a\mu} &= \frac{\delta \mathcal{L}}{\delta R_\mu^a} = \frac{if_\pi^2}{4}\mathrm{Tr}\left(\left(\partial^\mu \Sigma^\dagger \Sigma - \Sigma^\dagger \partial^\mu \Sigma\right)\tau^a\right) + O(\partial^3)\ .\end{aligned} \tag{2.3.26}$$

Using the pion field exponential parametrization (2.3.10) the axial current in this case is

$$J_5^{a\mu} = J_L^{a\mu} - J_R^{a\mu} = -f_\pi \partial^\mu \pi^a + \frac{2}{3f_\pi}\left((\boldsymbol{\pi} \cdot \boldsymbol{\pi})\partial^\mu \pi^a - \frac{1}{2}\partial^\mu(\boldsymbol{\pi} \cdot \boldsymbol{\pi})\pi^a\right) + \ldots\ . \tag{2.3.27}$$

2.3.2 Chiral lagrangian in the large N limit

After presenting the basic features of the large N limit in Yang-Mills theories with gauge group $SU(N)$ it is relevant to mention what happens when we apply this limit to the chiral theory.

We saw that in the case of arbitrary number of flavors the chiral lagrangian is

$$\begin{aligned}\mathcal{L} = -\frac{f_\pi^2}{4}\mathrm{Tr}\left(\partial_\mu \Sigma^\dagger \partial^\mu \Sigma\right) &+ L_1\left(\mathrm{Tr}\left(\partial_\mu \Sigma^\dagger \partial^\mu \Sigma\right)\right)^2 + L_2\left(\mathrm{Tr}\left(\partial_\mu \Sigma^\dagger \partial_\nu \Sigma\right)\right)^2 \\ &+ L_3 \mathrm{Tr}\left(\partial_\mu \Sigma^\dagger \partial^\mu \Sigma \partial_\nu \Sigma^\dagger \partial^\nu \Sigma\right) + \tilde{L}_3 \mathrm{Tr}\left(\left(\partial_\mu \Sigma^\dagger \partial_\nu \Sigma\right)^2\right).\end{aligned} \tag{2.3.28}$$

This lagrangian presents terms with different numbers of traces taken over flavor indices. Each trace corresponds to a quark loop which contributes with a factor $1/N$ to the given diagram. In the large N limit, terms with two traces will be supressed relative to those with only one trace. Then the dominant terms in (2.3.28) are the single trace terms so in the large N limit, for arbitrary number of flavors, the lagrangian can be reduced to

$$\mathcal{L} = \mathcal{L}_0 + L_3^{SU(N_f)}\mathrm{Tr}\left(\partial_\mu \Sigma^\dagger \partial^\mu \Sigma \partial_\nu \Sigma^\dagger \partial^\nu \Sigma\right) + \tilde{L}_3^{SU(N_f)}\mathrm{Tr}\left(\left(\partial_\mu \Sigma^\dagger \partial_\nu \Sigma\right)^2\right). \tag{2.3.29}$$

Where the large-N_c scaling of the coefficients is $f_\pi^2 \sim L_i \sim N_c$ [25]. The relation between the coefficients in (2.3.29) and the ones in lagrangians (2.3.18) and (2.3.23), for $N_f = 3$ and $N_f = 2$ respectively, can be obtained using (2.3.17) and (2.3.22) giving

$$L_1^{SU(3)} = \frac{1}{2}\tilde{L}_3^{SU(N_f)},\ L_2^{SU(3)} = \tilde{L}_3^{SU(N_f)}\ L_3^{SU(3)} = L_3^{SU(N_f)} - 2\tilde{L}_3^{SU(N_f)}, \tag{2.3.30}$$

and

$$L_1^{SU(2)} = \frac{L_3^{SU(N_f)} - \tilde{L}_3^{SU(N_f)}}{2},\ L_2^{SU(2)} = \tilde{L}_3^{SU(N_f)}. \tag{2.3.31}$$

2.3.3 Pion scattering amplitude

Let us now discuss how the scattering amplitude can be extracted from the pion lagrangian (2.3.24) when $N_f = 2$ and $N_c \to \infty$. The elastic scattering amplitude for two pions

$$\pi^a(p_a) + \pi^b(p_b) \longrightarrow \pi^c(p_c) + \pi^d(p_d) \tag{2.3.32}$$

is given by the T-matrix element

$$\mathcal{T}_{ab,cd} = (2\pi)^4\delta^{(4)}(p_a + p_b - p_c - p_d)\mathcal{M}_{ab,cd}\ . \tag{2.3.33}$$

The function $\mathcal{M}$ is determined by a single scalar function $A(s,t,u) = A(s,u,t)$ defined by the isospin decomposition (see, *e.g.*, [23,26,27])

$$\mathcal{M}_{ab,cd} = \delta_{ab}\delta_{cd}A(s,t,u) + \delta_{ac}\delta_{bd}A(t,s,u) + \delta_{ad}\delta_{bc}A(u,t,s)\ , \tag{2.3.34}$$

where s,t,u are the Mandelstam variables

$$s = -(p_a + p_b)^2\ ,\ t = -(p_a - p_c)^2,\quad u = -(p_a - p_d)^2\ . \tag{2.3.35}$$

These variables encode the different scattering processes and correspond to the three channels as depicted in Fig. 2.3.1. At $O(p^4)$, the original derivation by Weinberg produces at tree level [28,29]

$$A(s,t,u) = \frac{s}{f_\pi^2} + \frac{8L_1}{f_\pi^4}s^2 + \frac{4L_2}{f_\pi^4}(t^2 + u^2)\ . \tag{2.3.36}$$

For massless pions there are additional logarithmic contributions to the amplitude that are introduced by pion loop corrections. The relevant pion diagram has two quartic vertices with two derivatives, that from the pion Lagrangian (2.3.24) have a large-N_c scaling $\sim 1/f_\pi^4 \sim 1/N_c^2$. The tree-level contributions on the other hand have a scaling $\sim L_i/f_\pi^4 \sim 1/N_c$. Therefore, in the large-N_c limit the pion loop contributions are relatively suppressed. The same statement applies to other meson loop corrections, they are suppressed in the large-N_c limit.

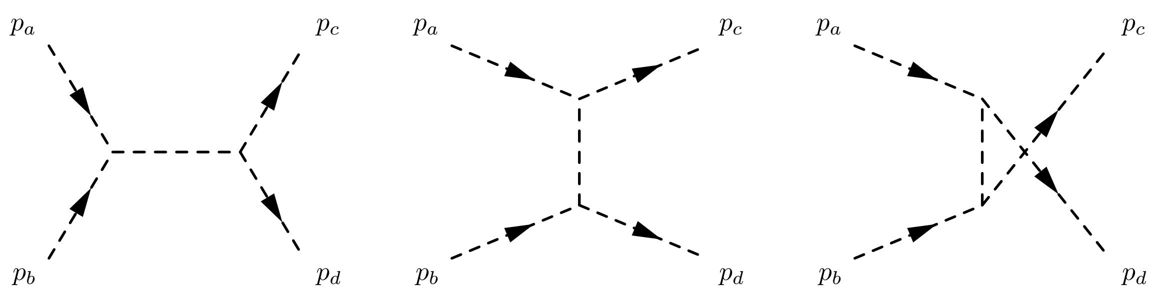


Figure 2.3.1: The three kinematically non-identical 2-to-2 elastic scattering processes. All propagating particles are pions, hence all legs are represented by dashed lines. Left: s-channel, Middle: t-channel, and Right: u-channel.

In order to compare to gauge/gravity models we will consider the axial current correlators for values of the momenta where the pions are on-shell. We start with the two-point function of the axial current, which will be proportional to the pion propagator[2]

$$\left\langle J_5^{a\mu}(-p)J_5^{b\nu}(p)\right\rangle \underset{p^2\to 0}{\approx} f_\pi^2 p^\mu p^\nu \left\langle \pi^a(-p)\pi^b(p)\right\rangle = -if_\pi^2\delta^{ab}\frac{p^\mu p^\nu}{p^2}\ . \tag{2.3.37}$$

[2]In principle there are additional terms depending on the pion field in (2.3.27) that introduce pion loop corrections to the axial current correlator, however, as for the scattering amplitude, these are suppressed in the large-N_c limit.

This leads to the Ward identity for current conservation

$$-ip_\mu \left\langle J_5^{a\mu}(-p) J_5^{b\nu}(p) \right\rangle \underset{p^2\to 0}{\approx} -f_\pi^2 p^\nu \delta^{ab} \ . \tag{2.3.38}$$

However, in the absence of an anomaly and quark masses, the axial current must be conserved and one would have expected this Ward identity to vanish. As we will see, this issue is solved in the holographic model when the two-point function correlator is computed and we find a vanishing value (see 6.5.8). The reason is that there is missing contact term in landing on (2.3.38) whose origin could be understood from contributions to the axial current other than the gradient of the pion field.

Similarly, the four-point function of the axial current will be proportional to the one of the pions

$$\left\langle J_5^{a\mu_a}(p_a) J_5^{b\mu_b}(p_b) J_5^{c\mu_c}(-p_c) J_5^{d\mu_d}(-p_d) \right\rangle \underset{p_i^2\to 0}{\approx} f_\pi^4 p_a^{\mu_a} p_b^{\mu_b} p_c^{\mu_c} p_d^{\mu_d} \left\langle \pi^a(p_a)\pi^b(p_b)\pi^c(-p_c)\pi^d(-p_d) \right\rangle \ . \tag{2.3.39}$$

From the pion correlator we are interested just in the leading pole contribution, which gives us the pion scattering amplitude through the LSZ reduction formula

$$\left\langle \pi^a(p_a)\pi^b(p_b)\pi^c(-p_c)\pi^d(-p_d) \right\rangle \underset{p_i^2\to 0}{\approx} (2\pi)^4 \delta^{(4)}(p_a+p_b-p_c-p_d) \frac{i\mathcal{M}_{ab,cd}}{p_a^2 p_b^2 p_c^2 p_d^2} \ , \tag{2.3.40}$$

where $\mathcal{M}_{ab,cd}$ in this expression is the amplitude (2.3.34).

2.3.4 Vector boson contributions

Our discussion thus far has been focusing on the chiral effective action at low energies, where heavier mesons have been integrated out.[3] Whether the vector bosons have been integrated out or not affects the value of the LECs in the pion action. Namely, starting from an action

$$\mathcal{L}_{UV} = \mathcal{L}_\pi(f_\pi, L_1, L_2) + \mathcal{L}_\mathcal{V} \ , \tag{2.3.41}$$

where $\mathcal{L}_\pi(f_\pi, L_1, L_2)$ is (2.3.23) with the coefficients f_π, L_1, and L_2 and the last term $\mathcal{L}_\mathcal{V}$ is the action for the vector bosons. At sufficiently low energies, the vector bosons can be integrated out, resulting in an action that only contains the pions but with modified coefficients

$$\mathcal{L}_{IR} = \mathcal{L}_\pi(\hat{f}_\pi, \hat{L}_1, \hat{L}_2) \ . \tag{2.3.42}$$

Let us show this explicitly for the case when the structure of the effective action follows the Hidden Local Symmetry (HLS) approach [30,31], which is commonly discussed in the context of holographic models, where it arises from gauge symmetries in the gravity dual.

In the HLS approach the pion matrix is factorized

$$\Sigma(x) = \xi_L^\dagger(x)\xi_R(x) \ . \tag{2.3.43}$$

In addition to the left- and right-moving flavor symmetries there is an emergent gauge symmetry, so that the fields $\xi_{L,R}$ transform as

$$\xi_L(x) \to h(x)\xi_L(x)U_L^\dagger \ , \ \xi_R(x) \to h(x)\xi_R(x)U_R^\dagger \ . \tag{2.3.44}$$

The massive vector boson is identified with the gauge field for the hidden symmetry and the form of the action is constrained by demanding local gauge invariance. The HLS action is directly related to the non-linear realization of local chiral transformations introduced by Weinberg [32] by going to the unitary gauge (see, *e.g.*, [31])

$$\xi_L^\dagger(x) = \xi_R(x) = e^{\frac{i\pi^a(x)}{f_\pi}\frac{\sigma^a}{2}} \ . \tag{2.3.45}$$

Let us denote $\boldsymbol{V}_\mu = V_\mu^a \sigma^a$ as the gauge field associated to the vector boson and $\boldsymbol{V}_{\mu\nu} = \partial_\mu \boldsymbol{V}_\nu - \partial_\nu \boldsymbol{V}_\mu + g\boldsymbol{V}_\mu \times \boldsymbol{V}_\nu$ as its field strength with g an effective gauge coupling. In addition to the terms shown in (2.3.24), the effective action has the vector boson contributions

$$\mathcal{L}_V = -\frac{1}{4}\boldsymbol{V}_{\mu\nu}\cdot\boldsymbol{V}^{\mu\nu} - \frac{1}{2}m_V^2\left(\boldsymbol{V}_\mu - \frac{g_{V\pi\pi}}{m_V^2}\boldsymbol{\pi}\times\partial_\mu\boldsymbol{\pi}\right)^2 \ , \tag{2.3.46}$$

[3] However, in most holographic examples the effective action includes massive vector bosons coupled to the pions.

where m_V is the mass of the vector boson and $g_{V\pi\pi}$ is the coupling to the pion field. The expansion of the invariant mass term for the vector boson includes a cubic coupling with the pions and a $O(\partial^2\pi^4)$ coupling of the same form as the terms shown in the first line of (2.3.24).[4], see, *e.g.*, [31,33]. However, in the pion scattering amplitude this additional vertex contribution is canceled out by the leading contribution from vector boson exchange, so the $O(p^2)$ pion scattering amplitude does not change. In general, there can be vector bosons transforming as adjoint fields of the chiral symmetry with different couplings where this cancelation does not happen, and additional couplings to the vector bosons that appear at $O(p^4)$ and so can also affect the amplitude.

There are, however, $O(p^4)$ contributions that add up to the contributions shown in (2.3.36). The tree-level contribution to the pion scattering amplitude from the vector meson exchange has the form [34,35]

$$A_V(s,t,u) = \frac{g_{V\pi\pi}^2}{m_V^2}\left[\frac{t(s-u)}{m_V^2-t}+\frac{u(s-t)}{m_V^2-u}\right] \ . \tag{2.3.47}$$

Expanding for small momenta $m_V^4 \gg s^2, t^2, u^2$, and using the fact that here the pions are massless particles yields

$$s(t+u) = -s^2, \quad 2ut = s^2 - t^2 - u^2 \ . \tag{2.3.48}$$

Therefore, the vector boson contribution to the low energy scattering amplitude reduces to

$$A_V(s,t,u) = \frac{g_{V\pi\pi}^2}{m_V^4}\left[t^2+u^2-2s^2\right] \ . \tag{2.3.49}$$

We can compare this expression with Weinberg's amplitude (2.3.36). We see that these contributions can be added to terms with the same dependence in the momentum, in such a way that one obtains effective values of the LECs that are shifted $L_i \to \hat{L}_i = L_i + \Delta L_i^V$ at energies below the vector meson mass. Therefore, integrating out the massive vector boson will shift the LECs of the chiral Lagrangian (2.3.24) to

$$\Delta L_2^V = -\Delta L_1^V = \frac{f_\pi^4}{4}\frac{g_{V\pi\pi}^2}{m_V^4} \ . \tag{2.3.50}$$

Furthermore, one could have several massive vector bosons $\boldsymbol{V}_{i\,\mu}$, $i = 1, 2, \ldots$ with similar couplings to the pions, in this case the shift in the LECs will have contributions from all of them

$$\Delta L_2^V = -\Delta L_1^V = \frac{f_\pi^4}{4}\sum_i \frac{g_{V_i\pi\pi}^2}{m_{V_i}^4} \ . \tag{2.3.51}$$

We could be in a situation in which the LECs in the pion action vanish or are much smaller than the vector boson contribution. In this case, after integrating out the vector bosons, the LECs would automatically satisfy the relation $\hat{L}_2 = -\hat{L}_1$, which corresponds to the Skyrme model [36–38]. An $O(p^4)$ coupling between the vector boson and the pions

$$\sim z_4\, \boldsymbol{V}^{\mu\nu}\cdot\partial_\mu\boldsymbol{\pi}\times\partial_\nu\boldsymbol{\pi} \ , \tag{2.3.52}$$

does not modify this relation, see (4.38) of [33], where z_4 is introduced in (4.27).

[4]Any additional free parameters in the HLS model are absorbed in the coefficients of the gauge-fixed effective action in such a way that the HLS model is equivalent to the non-linear sigma model for tree-level on-shell amplitudes

Chapter 3

AdS/CFT Correspondence

3.1 The Correspondence

The AdS/CFT correspondence is one of the best understood examples of the gauge/gravity duality. Originally proposed by Maldacena in 1997 in [10] and also in [39,40], it brings together aspects of string theory, quantum field theory and general relativity. Also, it is a very useful tool to understand strongly coupled quantum field theories. This correspondence establishes a duality between string theories, which are gravitational theories and quantum field theories without gravity, it relates strongly coupled quantum field theories to classical gravity theories, which are way more tractable than the former ones. The quantum field theory can be interpreted as living in the boundary of spacetime, for this reason the correspondence is also referred to as holography. A remarkable scientific implication of this is that we can study and understand the dynamics of strongly coupled theories through gravity, something that is not accesible by the usual perturbative methods in quantum field theory. So it can be applied to strongly coupled sistems such as low energy quantum chromodynamics, the theory of strong interactions in the Standard Model. The content of this chapter is mainly based on [41]. Here we will present the correspondence itselft and how it is derived from string theory. We will also explore the relation between gravity fields and operators, the matching symmetries of both theories and finally we will ilustrate how is the holographic computation of correlations functions in the case of scalar fields.

We will see how the correspondence arises in the context of string theory, more precisely the connection between type IIB string theory compactified on $AdS_5 \times S^5$ and $\mathcal{N} = 4$ super-Yang-Mills theory.

Starting with the type IIB string theory setup, we consider a stack of N parallel D3 branes in ten dimensional Minkowski space which are sitting very closed to each other. String theory on this background contains two kinds of perturbative excitations , closed and open strings. The closed strings are the excitations moving through all space and the open strings are excitations of the D-branes on which the string ends. This perturbative scenario is valid when

$$g_{YM}^2 N \sim g_s N \ll 1. \tag{3.1.1}$$

At low energies, $E < 1/l_s$, only massless string states can be excited. The massless closed strings are described by the low-energy effective Lagrangian of type IIB supergravity and the massless open strings are described by the low-energy effective Lagrangian of $\mathcal{N} = 4$ $U(N)$ super-Yang-Mills theory. The complete effective action of the massless modes will have the form

$$S = S_{bulk} + S_{brane} + S_{int}. \tag{3.1.2}$$

S_{bulk} is the action of ten dimensional supergravity, the brane action S_{brane} is defined on the (3+1) dimensional brane worldvolume and contains the $\mathcal{N} = 4$ super-Yang-Mills Lagrangian and S_{int} describes the interactions between the brane modes and the bulk modes.

The action of the closed strings is the action of ten dimensional supergravity and schematically reads

$$S_{bulk} = \frac{1}{2\kappa^2} \int d^{10}x \sqrt{-g} e^{-2\phi} \left(R + 4\partial_M \phi \partial^M \phi \right) + \ldots, \tag{3.1.3}$$

with $M = 0, 1, \ldots, 9$. Where g_{AB} is the metric, ϕ the dilaton, R the curvature scalar and $\kappa \propto g_s \alpha'^2$. Expanding the metric as $g_{AB} = \eta_{AB} + \kappa h_{AB}$, being h_{AB} the metric fluctuations, we get at lowest order in h_{AB}

$$S_{bulk} \sim -\frac{1}{2} \int d^{10}x \partial_M h_{AB} \partial^M h^{AB} + \mathcal{O}(\kappa). \tag{3.1.4}$$

The actions S_{brane} and S_{int} are derived from the DBI action. Performing an expansion of $e^{-\phi}$ and $g_{AB} = \eta_{AB} + \kappa h_{AB}$, they read at leading order in α'

$$S_{brane} = -\frac{1}{2\pi g_s}\int d^4x \mathrm{Tr}\left(\frac{1}{2}F_{\mu\nu}F^{\mu\nu} - \sum_i D_\mu\phi^i D^\mu\phi^i + \frac{1}{2\pi g_s}\sum_{ij}\left[\phi^i,\phi^j\right]^2 + \mathcal{O}\left(\alpha'\right)\right), \tag{3.1.5}$$

$$S_{int} = -\frac{\kappa}{8\pi g_s}\int d^4x\phi F_{\mu\nu}F^{\mu\nu} + \dots \tag{3.1.6}$$

Where the scalars and gauge fields are $\phi^i = \phi^{ia}T^a$ and $A_\mu = A^a_\mu T^a$. Then we take the limit $\alpha' \to 0$. The $\mathcal{O}\left(\alpha'\right)$ terms in S_{brane} vanish so it becomes the bosonic part of the $\mathcal{N} = 4$ super-Yang-Mills theory action. Making the identification

$$2\pi g_s = g^2_{YM}. \tag{3.1.7}$$

Also since $\kappa \propto \alpha'^2$,this limit also implies $\kappa \to 0$ so the $\mathcal{O}\left(\kappa\right)$ terms in S_{bulk} vanish so the remaining part is just the action of free supergravity in ten dimensional Minkowski space. Finally S_{int} which is of order κ vanishes for $\alpha' \to 0$ yielding to the decoupling of open and closed strings.

Summarising, in the low energy limit the D3-brane setup consists of closed strings propagating in the bulk and open strings which are excitations of the brane, being the open and closed modes both massless. Then by taking the limit $\alpha' \to 0$ both kinds of strings decouple from each other and we end with $\mathcal{N} = 4$ super-Yang-Mills theory describing the dynamics of the open strings and the closed strings are effectively described by supergravity in ten dimensional Minkowski space as shown in 3.1.1.

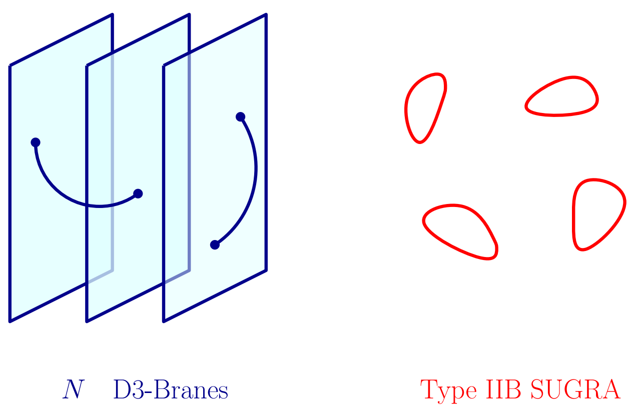


Figure 3.1.1: String setup in the perturbative limit.

In order to get to the other side of the correspondence one needs to change the perspective from which the setup is considered. Now we consider the D-branes as massive charged objects. The D3-branes are a solution of the supergravity equations of motion that reads

$$ds^2 = f^{-1/2}\left(-dt^2 + dx_1^2 + dx_2^2 + dx_3^2\right) + f^{1/2}\left(dr^2 + r^2 d\Omega_5^2\right), \tag{3.1.8}$$

$$f = 1 + \frac{L^4}{r^4}, \quad L^4 \equiv 4\pi g_s \alpha'^2 N. \tag{3.1.9}$$

The classical supergravity description is only valid when the radius of curvature L becomes large compared to the string length

$$\frac{L^4}{l_s^4} \sim g_s N \sim g^2_{YM} N \gg 1. \tag{3.1.10}$$

This background consists of two different regions depending on the value of r. If $r \gg L$ the function f can be approximated by $f \sim 1$ so the metric (3.1.8) reduces to ten dimensional flat spacetime. On the other hand if $r \ll L$ the function f is given by $f \sim {R^4}/{r^4}$ so the metric becomes

$$ds^2 = \frac{r^2}{L^2}\eta_{\mu\nu}dx^\mu dx^\nu + \frac{L^2}{r^2}dr^2 + L^2 d\Omega_5^2, \tag{3.1.11}$$

which is the geometry of $AdS_5 \times S^5$. Introducing a new coordinate $z = L^2/r$ the metric reads

$$ds^2 = \frac{L^2}{z^2}\left(\eta_{\mu\nu}dx^\mu dx^\nu + dz^2\right) + L^2 d\Omega_5^2. \tag{3.1.12}$$

So now the conformal boundary is located at $z = 0$.

We will have closed strings propagating in two regions of spacetime, some propagating in ten dimensional Minkowski space and others propagating in the near horizon region of the branes whose geometry corresponds to $AdS_5 \times S^5$. Taking the limit $\alpha' \to 0$ the closed strings living in the two different regions decouple from each other. This is because the excitations that live very close to $r = 0$ can't escape the gravitational potential and propagate in the asymptotic region. So in the low energy limit we can have closed strings propagating in flat spacetime described by type IIB supergravity and closed strings propagating in the near horizon region described by the $AdS_5 \times S^5$solution of IIB supergravity being both types of closed strings decoupled from each other as depicted in 3.1.2.

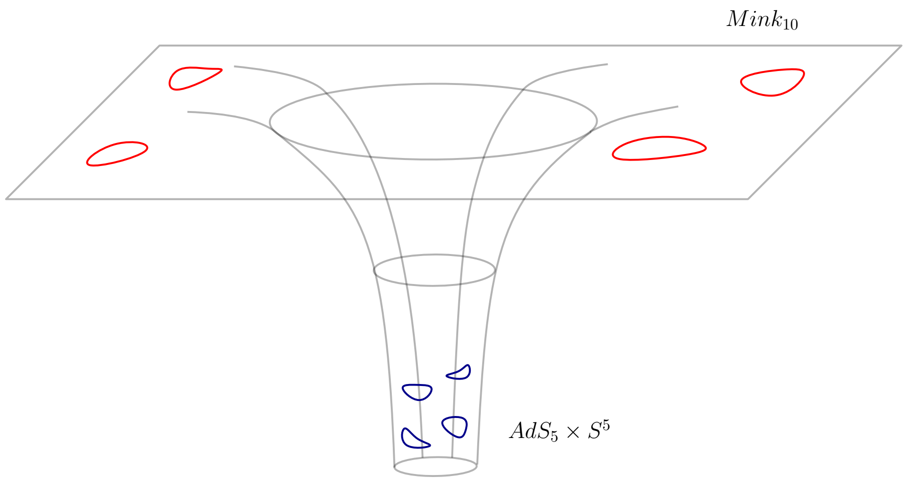


Figure 3.1.2: String setup in the supergravity limit.

From both perspectives we end with two decoupled theories in the low energy limit . In both cases one of the decoupled systems is supergravity in flat space so the correspondence identifies the second system that appears in both descriptions leading to the conjecture that $\mathcal{N} = 4$ $U(N)$ super-Yang-Mills theory in 3+1 dimensions is dual to type IIB superstring theory on $AdS_5 \times S^5$.

This is summarised in the table (3.1.1)

• $g_s N \ll 1$	$\mathcal{N} = 4$ Super Yang-Mills on $\mathbb{R}^{3,1}$ Type IIB Supergravity on $\mathbb{R}^{9,1}$
• $g_s N \gg 1$	Type IIB Supergravity on $AdS_5 \times S^5$ Type IIB Supergravity on $\mathbb{R}^{9,1}$

Table 3.1.1: Resulting theories after applying both limits to the stack of N D3-branes.

3.1.1 Field/Operator Dictionary

The AdS/CFT correspondence establishes a one-to-one relation between operators in $\mathcal{N} = 4$ $U(N)$ super-Yang-Mills theory and the fields of type IIB string theory on $AdS_5 \times S^5$. This field-operator map allows the AdS/CFT correspondence to be formulated as relation between generating functionals in field theory and supergravity.

The dictionary is formulated in the following way, the generating functional $W\left[\phi_{(0)}\right]$ for the Green's functions of the conformal field theory living on the d-dimensional boundary of the $(d+1)$-dimensional AdS space is given in terms of the fields $\phi_{(0)}$ which play the role of the sources of the operators $\mathcal{O}$ and the field theory partition function $Z\left[\phi_{(0)}\right]$ reads

$$Z_{gauge}\left[\phi_{(0)}\right] = e^{-W\left[\phi_{(0)}\right]} = \left\langle \exp\left(\int d^d x \phi_{(0)}(x)\,\mathcal{O}(x)\right)\right\rangle_{CFT}. \tag{3.1.13}$$

On the other hand, the AdS side of the duality is governed by the supergravity action $S_{sugra}[\phi]$, where the ϕ fields live in the five-dimensional Anti-de Sitter spacetime and the gravity partiton function reads

$$Z_{sugra}[\phi] = e^{-S_{sugra}[\phi]}. \tag{3.1.14}$$

The AdS/CFT conjecture states that the classical supergravity action is the generating functional for the field theory Green's function of operators $\mathcal{O}$. The relation is

$$W\left[\phi_{(0)}\right] = S_{sugra}[\phi]\big|_{\lim_{z\to 0}\phi(x,z)\sim\phi_{(0)}(x)}. \tag{3.1.15}$$

So the field theory generating functional is identified with the supergravity classical action where the ϕ fields take the boundary values $\phi_{(0)}$.

In the strongest form of the correspondence, *i.e.* for any value of N and g_s, the partition function of the gravity side will be the one of type IIB string theory. Then, the correspondence states that

$$Z_{string}\left[\phi_{(0)}\right] = \left\langle \exp\left(\int d^d x \phi_{(0)}(x)\,\mathcal{O}(x)\right)\right\rangle_{CFT}. \tag{3.1.16}$$

With Z_{string} not known explicitly.

3.1.2 Dictionary of Symmetries

We saw that the AdS/CFT correspondence is a strong/weak coupling duality. In the large N limit it relates the region of weak field theory coupling $\lambda = g_{YM}^2 N$ in the SYM theory to the region of high curvature in string theory and vice versa. One of the properties of these theories that do not depend on the coupling are the symmetries so they can be compared to test the duality.

Let us examine more closely the matching of symmetries on both sides of the correspondence. The $\mathcal{N}=4$ Super Yang-Mills theory is invariant under $SO(4,2)\times SO(6)$, understood as the four-dimensional conformal group times the R-symmetry of the theory. It preserves $\mathcal{N}=4$ supersymmetry so there are sixteen supercharges and is also conformal so in addition there are sixteen superconformal supercharges. All of these symmetries form the supergroup $PSU(2,2|4)$ under which $\mathcal{N}=4$ Super Yang-Mills theory is invariant. The bosonic subgroup of $PSU(2,2|4)$ is given by $SU(2,2)\sim SO(4,2)$ and $SU(4)\sim SO(6)$ and the fermionic part is generated by the supercharges and the superconformal supercharges.

In the string theory side on $AdS_5\times S^5$we have the isometry groups of AdS_5 and S^5 which are given by $SO(4,2)$ and $SO(6)$ respectively. These coincide with the bosonic subgroup of $PSU(2,2|4)$. $AdS_5\times S^5$ is a maximally supersymmetric solution of type IIB string theory so it has 32 Killing spinors which generate the fermionic isometries. Also these coincide with the fermionic component of $PSU(2,2|4)$. So string theory on $AdS_5\times S^5$ preserves $PSU(2,2|4)$ symmetry.

We see then that the global symmetry groups on both sides of the correspondence agree. In the field theory sixteen ordinary supersymmetries as well as sixteen special conformal supersymmetries match with the isometries generated by the 32 Killing spinors of $AdS_5\times S^5$. So we conclude that the symmetries of $\mathcal{N}=4$ Super Yang-Mills theory in flat spacetime and of type IIB string theory on $AdS_5\times S^5$ match.

3.1.3 Correlation Functions

We saw in 3.1.1 that there is a map between the generating functional of the field theory and the supergravity action. This relation is the starting point for the holographic calculation of correlation functions of gauge invariant operators. As we explained for each operator $\mathcal{O}_i$ on the field theory side exists a corresponding source $\phi^i_{(0)}$. In order to obtain correlation functions from the generating functional $W\left[\phi^i_{(0)}\right]$ one has to take derivatives with respect to the sources $\phi^i_{(0)}$ as follows

$$\langle \mathcal{O}_1(x_1)\,\mathcal{O}_2(x_2)\ldots\mathcal{O}_n(x_n)\rangle_{CFT} = -\left.\frac{\delta^n W}{\delta\phi^1_{(0)}(x_1)\,\delta\phi^2_{(0)}(x_2)\ldots\delta\phi^n_{(0)}(x_n)}\right|_{\phi^i_{(0)}=0}. \tag{3.1.17}$$

The main technical idea behind the AdS/CFT correspondence is that the boundary values of supergravity fields act as sources for gauge invariant operators in the field theory so each operator $\mathcal{O}_i$ corresponds to a supergravity field $\phi^i(x,z)$. We will write the bulk fields generically as $\phi^i(x,z)$ with value proportional to $\phi^i_{(0)}(x)$ for $z\to 0$. The goal is to extremize the action $S_{sugra}[\phi]$ subject to these boundary conditions. So one has to solve the equations of motion in the bulk subject to Dirichlet boundary conditions on the boundary and evaluate the action on the solution. Having the relation (3.1.15) one can write

$$W[\phi_0] \simeq \text{extremum}_{\phi|_{z=0} \sim \phi_{(0)}} S_{sugra}[\phi], \tag{3.1.18}$$

so the generator of Green's functions in the gauge theory is the on-shell supergravity action. Then, taking variational derivatives with respect to the source $\phi^i_{(0)}$, one obtains the correlation functions.

3.1.3.1 Two-point Function

To illustrate how the previous procedure works we are going to compute the two-point function $\langle \mathcal{O}(x)\mathcal{O}(y)\rangle$ of gauge invariant operators $\mathcal{O}$ of the d-dimensional field theory. For simplicity we are going to consider a scalar operator $\mathcal{O}$ with conformal dimension Δ which is dual to a scalar field ϕ in the $(d+1)$-dimensional gravity theory. The action reads in Euclidean signature

$$S[\phi] = \frac{C}{2}\int dz d^d x \sqrt{g}\left(g^{MN}\partial_M\phi\partial_N\phi + m^2\phi^2\right), \tag{3.1.19}$$

where the mass of the scalar field satisfies $m^2L^2 = \Delta(\Delta - d)$ with R the AdS radius of curvature.

The equation of motion of the action (3.1.19) reads

$$\left(\Box_g - m^2\right)\phi = 0, \quad \Box_g\phi = \frac{1}{\sqrt{\phi}}\partial_M\left(\sqrt{g}g^{MN}\partial_N\phi\right), \tag{3.1.20}$$

being the metric

$$ds^2 = \frac{L^2}{z^2}\left(dz^2 + \delta_{\mu\nu}dx^\mu dx^\nu\right). \tag{3.1.21}$$

Performing a Fourier decomposition in the field theory directions x^μ using a plane wave ansatz of the form

$$\phi(x,z) = \int \frac{d^d p}{(2\pi)^d} e^{ip^\mu x_\mu}\phi(p,z), \tag{3.1.22}$$

the equation of motion for the modes $\phi(p,z)$ becomes

$$z^2\partial_z^2\phi(p,z) - (d-1)z\partial_z\phi(p,z) - \left(m^2L^2 + p^2z^2\right)\phi(p,z) = 0, \tag{3.1.23}$$

where $p^2 = \delta_{\mu\nu}p^\mu p^\nu$.

The equation of motion (3.1.23) is a Bessel equation which has two independent solutions in terms of modified Bessel functions

$$\phi_p(z) = A_p z^{d/2}K_\nu(pz) + B_p z^{d/2}I_\nu(pz), \tag{3.1.24}$$

where $\nu = \Delta - d/2 = \sqrt{d^2/4 + m^2L^2}$.

We have to impose regularity in the interior and since $I_\nu(pz)$ diverges for $z\to\infty$ we have to omit this solution by setting $B_p = 0$. In contrast $K_\nu(pz)$ decays exponentially for $z\to\infty$ thus it is regular at the interior and for $z\to 0$ the solution behaves in the boundary as

$$\phi(p,z) \sim A_p z^{d/2-\nu} = A_p z^{d-\Delta}. \tag{3.1.25}$$

Having the solution in hand the next step is to insert it in the action, which can be simplified by integrating by parts

$$S[\phi] = -\frac{C}{2}\int d^d x\sqrt{g}g^{zz}\phi\partial_z\phi\Big|_{z=\epsilon}. \tag{3.1.26}$$

The integrand of (3.1.26) has to be evaluated at both limits of integration, ie $z\to\infty$ and $z=0$. Using the solution (3.1.24), having imposed regularity, the integrand of (3.1.26) vanishes for $z\to\infty$. At the lower limit $z=0$ the action diverges so we have to regularise it by omitting the region $0<z<\epsilon$ and imposing the boundary conditions at $z=\epsilon$ where ϵ is small. So we have to match $\phi(p,z)$ to $\phi_{(0)}(p)$ at $z=\epsilon$, then the correct normalised solution for $\phi(p,z)$ reads

$$\phi(p,z) = \frac{z^{d/2}K_\nu(pz)}{\epsilon^{d/2}K_\nu(\epsilon z)}\phi_{(0)}(p)\,\epsilon^{d-\Delta}. \tag{3.1.27}$$

Writing the action (3.1.26) in momentum space

$$S\left[\phi\right]=-\frac{CL^{d-1}}{2\epsilon^{d-1}}\int\frac{d^{d}p}{\left(2\pi\right)^{d}}\left(2\pi\right)^{d}\delta^{d}\left(p+q\right)\phi\left(p,z\right)\partial_{z}\phi\left(q,z\right)\Big|_{z=\epsilon}, \tag{3.1.28}$$

and using the solution (3.1.27)we can expres the action just in terms of the sources $\phi_{(0)}$. Then using (3.1.17) and (3.1.15) we get for the two-point function

$$\begin{aligned}\left\langle \mathcal{O}\left(p\right)\mathcal{O}\left(q\right)\right\rangle =-\left(2\pi\right)^{2d}\frac{\delta^{2}S\left[\phi_{(0)}\right]}{\delta\phi_{(0)}(-p)\delta_{(0)}(-q)} &= -\frac{\left(2\pi\right)^{d}\delta^{d}(p+q)CL^{d-1}}{\epsilon^{2\Delta-d-1}}\frac{d}{dz}\ln\left(z^{d/2}K_{\nu}\left(pz\right)\right)\Big|_{z=\epsilon}\\ &= -\frac{\left(2\pi\right)^{d}\delta^{d}(p+q)CL^{d-1}}{\epsilon^{2\Delta-d}}\left(\frac{d}{2}+\frac{\epsilon pK_{\nu}'(p\epsilon)}{K_{\nu}(p\epsilon)}\right).\end{aligned} \tag{3.1.29}$$

The next step is to take the limit $\epsilon\to0$ so we have to expand the Bessel function for small arguments. The expansion depends on whether ν is a positive integer or not, in this case ν will be a positive integer. After performing the expansion we get

$$\begin{aligned}\left\langle \mathcal{O}\left(p\right)\mathcal{O}\left(q\right)\right\rangle =\left(2\pi\right)^{d}\delta^{d}\left(p+q\right)CL^{d-1}&\left(\frac{\beta_{0}+\beta_{1}\epsilon^{2}p^{2}+\ldots\beta_{\nu}(\epsilon p)^{2(\nu-1)}}{\epsilon^{2\Delta-d}}\right.\\ &\left.-\frac{(-1)^{\nu+1}}{2^{2(\nu-1)}\Gamma(\nu)^{2}}p^{2\nu}\ln\left(\epsilon p\right)\left(1+\mathcal{O}\left(\epsilon^{2}\right)\right)\right),\end{aligned} \tag{3.1.30}$$

where the coefficients β_i are functions of ν.

The terms on the first line of (3.1.30) correspond to scheme dependent contact terms and in the second line in the limit $\epsilon\to0$ only the first term involving the logarithm of the momentum contributes then the correlator is

$$\left\langle \mathcal{O}\left(p\right)\mathcal{O}\left(q\right)\right\rangle =-\left(2\pi\right)^{d}\delta^{d}\left(p+q\right)CL^{d-1}\frac{\left(-1\right)^{\nu+1}}{2^{2(\nu-1)}\Gamma\left(\nu\right)^{2}}p^{2\nu}\ln\left(\epsilon p\right). \tag{3.1.31}$$

And coming back to the position space the correlator becomes

$$\left\langle \mathcal{O}\left(x\right)\mathcal{O}\left(y\right)\right\rangle =CL^{d-1}\frac{\Gamma\left(\Delta\right)}{\Gamma\left(\Delta-d/2\right)}\frac{2\Delta-d}{\pi^{d/2}\left|x-y\right|^{2\Delta}}, \tag{3.1.32}$$

where $\Delta=\nu-d/2$.

3.2 Confining geometries

Holography is a great tool to study conformal gauge theories and certainly has to play a role in the understanding of non conformal, realistic theories. Those theories exhibit qualitative properties such as confinement and have a mass gap and a discrete spectrum of massive particles. So gravity models that aim to be dual to theories like QCD should present those features. We will now describe how these features can be realized through a gravitational dual. A criterion for determining whether a gauge theory shows confinement is obtained from the Wilson loop, when the loop presents an area law behaviour the theory confines. Even though we will not discuss the behaviour of Wilson loops here we will consider that holographic models are confining when there is a discrete spectrum of normalizable modes that map to particles, glueballs, or mesons in the field theory dual[1], and that spectrum has been computed in [17,18]. In the models we are interested in, the mass gap, which is also a generic feature of any confining gauge theory, arises because the geometry ends smoothly at a finite value of the holographic coordinate. In these type of theories, the gravity dual has a geometry that either ends in a singular manner or smoothly if a cycle in the compact space collapses to zero size. Also in a theory with mass gap and discrete spectrum, we expect poles in the two point functions corresponding to the physical states. The holographic models that we will review here are the Witten Yang-Mills model [11], the Sakai-Sugimoto model [15,16], the $\mathcal{N}=1^{*}$ super-Yang-Mills dual to the GPPZ geometry [14] and the AdS_6 soliton [43]. There are more examples in the bibliography like the Klebanov-Strassler model [44], the Maldacena-Nuñez model [45] and the 2+1 models [42,46].

[1]There are examples of models with this kind of spectrum that are not confining, see [42].

3.2.1 GPPZ Flow

The GPPZ solution found by Girardello, Petrini, Porrati, and Zaffaroni [14] is an exact, $\mathcal{N}=1$ supersymmetric kink solution of 5-dimensional gauged supergravity. The solution was proposed as the gravity dual of $\mathcal{N}=4$ super-Yang-Mills with a massive deformation which corresponds to $\mathcal{N}=1^*$ super-Yang-Mills. There are also more proposals of gravity duals of $\mathcal{N}=1^*$ super-Yang-Mills [47, 48] and uplifts to ten dimensions of the GPPZ solution [49, 50]. Although it does not capture all of the expected features of the field theory, the kink solution is a true deformation of $\mathcal{N}=4$, that exhibits confinement (the Wilson loop exhibits an area law behaviour), magnetic screening and there is a gaugino condensate.

In the field theory side, the matter content of $\mathcal{N}=4$ super-Yang-Mills is split in a $\mathcal{N}=1$ vector multiplet and three chiral multiplets in the adjoint representation. An equal mass is given for the three chiral superfields X_i in such a way that the global symmetry group is broken to $SU(3)$ and supersymmetry is broken to $\mathcal{N}=1$.

$$\int d^2\theta m_{ij} \mathrm{Tr} X_i X_j + \mathrm{c.c}, \; m_{ij} = m\,\delta_{ij} \tag{3.2.1}$$

where, in general, m_{ij} is a complex, symmetric matrix, in such a way that the global symmetry group is broken to $SU(3)$ and supersymmetry is broken to $\mathcal{N}=1$. At weak coupling the theory flows to pure $\mathcal{N}=1$ super-Yang-Mills at energy scales much below the mass of the chiral multiplets. The weakly coupled theory is confining and this property holds in the strongly coupled theory, as two-point functions of gauge-invariant operators show poles for a discrete spectrum of massive states [51, 52].

In this case the background geometry is a a solution of five-dimensional supergravity truncated to a single scalar coupled to a metric of the form

$$ds^2 = dr^2 + e^{2A(r)}\eta_{\mu\nu}dx^\mu dx^\nu. \tag{3.2.2}$$

Being the action for the scalar field

$$S = \int d^5x\sqrt{-g}\left(\frac{R}{4} - \frac{1}{2}g^{MN}\partial_M\phi\partial_N\phi - V\left(\phi\right)\right). \tag{3.2.3}$$

The solution is given by

$$\varphi_B = \frac{\sqrt{3}}{2}\log\frac{1+\sqrt{1-u}}{1-\sqrt{1-u}}, \quad e^{2A(r)} = \frac{u}{1-u} \quad 1-u = e^{-2r}, \tag{3.2.4}$$

Then metric can be written as

$$ds^2 = \frac{du^2}{4\left(1-u\right)^2}du^2 + \frac{u}{1-u}\eta_{\mu\nu}dx^\mu dx^\nu. \tag{3.2.5}$$

Therefore, $u \to 1$ corresponds to the boundary of the bulk, which is asymptotically AdS_5 and $u \to 0$ is the origin of the bulk spacetime.

3.2.2 AdS_6 Soliton

The AdS_6 background geometry is dual to a 5-dimensional CFT. After compactifying one direction with supersymmetry-breaking conditions, it should describe a 4-dimensional confining theory at low energies. At strong coupling there is no separation between the confinement and the compactification scales, so glueballs and Kaluza-Klein modes have similar masses. Nevertheless, the theory is effectively four-dimensional with a mass gap.

The geometry of the AdS_6 soliton can be obtained by a double Wick rotation of a black brane solution [43]

$$ds^2 = \frac{L^2}{z^2}\left(\frac{dz^2}{f\left(z\right)} + f\left(z\right)d\tau^2 + \eta_{\mu\nu}dx^\mu dx^\nu\right), \quad f\left(z\right) = 1 - \frac{z^5}{z_\Lambda^5}, \tag{3.2.6}$$

where τ is the compact coordinate $\tau \sim \tau + 2\pi/M_{KK}$, with $M_{KK} = 5/(2z_\Lambda)$ the compactification scale. The space ends at a finite value of the radial coordinate $z = z_\Lambda$ and the scale of confinement is given by $\Lambda = 1/z_\Lambda$.

While we will not discuss an explicit holographic dual field theory, this geometry can in principle be embedded in string theory, as deformations of AdS_6 solutions of Type II supergravity [53, 54].

The AdS_6 soliton geometry is a close cousin of the Witten-YM model [11], extensively used to mimic QCD in applications of holography. The model consists of a non-supersymmetric compactification of D4-branes along a circle, and becomes pure Yang-Mills at low energies and weak coupling. The holographic dual geometry is also the AdS_6 soliton in an appropriately chosen frame, but in addition there is background dilaton, to which other fields may be coupled [55]. Alternatively, the Witten-YM model can be obtained from the compactification of $M5$-branes along a two-torus. In the holographic dual description the geometry is AdS_7 compactified along two spatial directions.

3.2.3 Witten-Yang-Mills Model

The Witten-Yang-Mills model is a proposal for studying ordinary large N gauge theories in four dimensions without supersymmetry via string theory. This model exhibit properties such as confinement and a mass gap and also the $1/N$ scaling in the string coupling constant.

The $\mathcal{N}=4$ super Yang-Mills theory is very different from real QCD so in order to study the latter one Witten proposed in [11] a gravity dual for a four dimensional Yang-Mills theory without supersymmetry.

The way he proposed to study a four-dimensional gauge theory is via compactification of a certain six-dimensional theory with (0,2) supersymmetry. This proposal arises from the so-called M2 and M5 branes of M-theory, whose low-energy limit are solutions of 11-dimensional supergravity. The near horizon geometries of these branes are $AdS_4 \times S^7$ and $AdS_7 \times S^4$, respectively. The dual of M-theory on $AdS_7 \times S^4$ is the 6-dimensional superconformal low-energy theory of M5 branes.

So the starting point for obtaining a four dimensional Yang-Mills theory without supersymmetry is the $(2,0)$ superconformal theory in six dimensions realized on N parallel coinciding M5-branes. The compactification of this theory on a circle of radius R_1 gives a five-dimensional theory whose low-energy effective theory is the maximally supersymmetric $SU(N)$ gauge theory, with gauge coupling $g_5^2 = 2\pi R_1$. M-theory compactified on a supersymmetry preserving circle leads to 10-dimensional superstring theory of type IIA with the M5 branes turned into D4 branes.

To obtain the four dimensional Yang-Mills theory without supersymmetry, one compactifies the 5-dimensional theory further on another S^1 of radius R_0. The dimensionless gauge coupling constant g_4 in four dimensions is given by $g_4^2 = g_5/(2\pi R_0) = R_1/R_0$. To break supersymmetry one imposes anti-periodic boundary conditions on the fermions around the second S^1. All modes not protected by gauge symmetry become massive in this case leaving pure Yang-Mills theory as low-energy effective theory. The 5-dimensional supersymmetric field theory thus turns into ordinary 4-dimensional Yang-Mills theory. On the other hand, the background geometry of string theory becomes

$$ds^2 = \left(\frac{U}{R}\right)^{3/2}\left[\eta_{\mu\nu}dx^\mu dx^\nu + f(U)\,d\tau^2\right] + \left(\frac{R}{U}\right)^{3/2}\left[\frac{dU^2}{f(U)} + U^2 d\Omega_4^2\right], \tag{3.2.7}$$

with

$$f(U) = 1 - \frac{U_{KK}^3}{U^3}, \quad M_{KK} = \frac{3}{2}\frac{U_{KK}^{1/2}}{R^{3/2}}, \quad e^\phi = g_s\left(\frac{U}{R}\right)^{3/4}, \quad F_4 = dC_3 = \frac{2\pi N_c}{V_4}\epsilon_4. \tag{3.2.8}$$

Which is the type IIA supergravity background sourced by the D4-branes at zero temperature when $N_c \to \infty$. Where $R^3 = \pi g_s N_c l_s^3$ and the Kaluza-Klein mass is $M_{KK} = (3/2)\,U_{KK}^{1/2} R^{3/2}$. The τ direction has periodicity $2\pi/M_{KK}$ and ϵ_4 is the volume form of a unit S^4, of volume $V_4 = 8\pi^2/3$. The map relating the field theory quantities and the parameters in the D4 geometry is

$$R^3 = \frac{1}{2}\frac{\lambda_{YM} l_s^2}{M_{KK}}, \quad U_{KK} = \frac{2}{9}\lambda_{YM} M_{KK} l_s^2, \quad g_s N_c = \frac{1}{2\pi}\frac{\lambda_{YM}}{M_{KK} l_s}, \tag{3.2.9}$$

where l_s is the string length and $\lambda_{YM} = g_{YM} N_c$ is the 't Hooft coupling of the (3+1)-dimensional dual Yang-Mills theory.

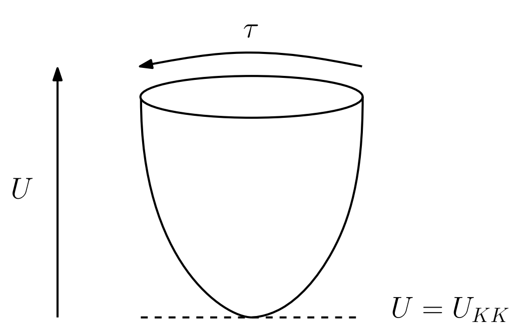


Figure 3.2.1: The cigar topology of the Witten model

The factor $f(U)$ in the metric implies that the circle collapses to zero size at $U = U_{KK}$, where the geometry ends. U is the holographic direction and corresponds to an energy scale of the field theory with $U = \infty$ corresponding to the conformal boundary. The lower bound $U \geq U_{KK}$ sets a minimal energy scale for states in the dual field theory encoding the mass gap of Yang-Mills in the confining phase.

The topology of the subspace formed by U and the reduced direction τ is that of a cigar with the circle shrinking to zero size at $U = U_{KK}$ as depicted in Figure 3.2.1.

3.2.4 Witten-Sakai-Sugimoto Model

The Witten-Sakai-Sugimoto model [15,16] is a generalization of the Witten model mentioned before and it is the holographic dual of four dimensional, large N QCD with massless flavors which are included by introducing N_f probe D8-branes in the geometry induced by D4-branes [15,56].

The model is built by placing N_f probe D8-branes into a D4 background in type IIA string theory giving a geometry which is the near horizon limit of a nonsupersymmetric (3+1)-dimensional D-brane intersection of N_c D4-branes and N_f D8-$\overline{\text{D8}}$ D-brane pairs. The brane configuration is given in the following table

	0	1	2	3	τ	5	6	7	8	9
D4	$\times$	$\times$	$\times$	$\times$	$\times$	$\cdot$	$\cdot$	$\cdot$	$\cdot$	$\cdot$
D8	$\times$	$\times$	$\times$	$\times$	$\cdot$	$\times$	$\times$	$\times$	$\times$	$\times$
$\overline{\text{D8}}$	$\times$	$\times$	$\times$	$\times$	$\cdot$	$\times$	$\times$	$\times$	$\times$	$\times$

Table 3.2.1: D-brane intersection in the WSS model. Branes are extended along the directions marked with $\times$.

In this configuration the N_c D4-branes are compactified in the τ direction along a circle with supersymmetry-breaking boundary conditions, the D8- and $\overline{\text{D8}}$-branes are transverse to the D4-branes and sit at separate points.[2] The radial coordinate U transverse to the D4-branes is bounded from below $U \geq U_{KK}$.

In the near-horizon limit where the D4-branes are replaced by the geometry displayed above, each D8 and $\overline{\text{D8}}$, that sit at separated points in the τ direction as $U \to \infty$, join at a finite value of the radial coordinate and form a single object. So as $U \to U_{KK}$ the radius of the circle shrinks to zero and at a finite point $U = U_0$ the D8/$\overline{\text{D8}}$ branes merge forming a single object as depicted in Fig 3.2.2.

The chiral symmetry is realized as the gauge symmetry of the N_f D8 $-$ $\overline{\text{D8}}$ pairs. When they merge resulting on a single D8-brane only one factor of $U(N_f)$ survives as the gauge symmetry, so this mechanism is interpreted as the holographic realization of the spontaneous breaking of the $U(N_f)_L \times U(N_f)_R$ chiral symmetry.

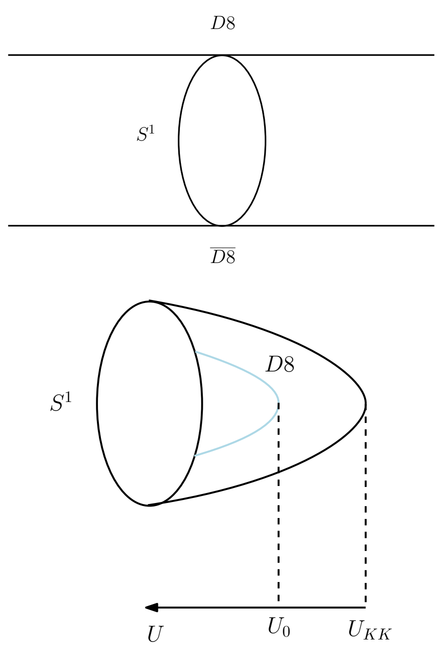


Figure 3.2.2: Sketch of D8- and$\overline{\text{D8}}$-branes.

[2]That asymptotic separation between the D8 and $\overline{\text{D8}}$ can be changed, producing different values of the quark condensate.

From the 4-8 strings and the 4-$\bar{8}$strings (the open strings with one end attached to the D4-brane and the other end to the D8-brane and $\overline{\text{D8}}$-brane respectively) N_f flavors of massless fermions are obtained which are interpreted as quarks in QCD. The chirality of the fermions created by 4-8 strings is opposite to that created by the 4-$\bar{8}$ strings. Therefore the $U\left(N_f\right)_{D8} \times U(N_f)_{\bar{D}8}$ gauge symmetry of the N_f D8-$\bar{\text{D}}$8 pairs is interpreted as the $U\left(N_f\right)_L \times U(N_f)_R$ chiral symmetry of QCD.

The action for the D8-branes in the string frame

$$S_{\text{D8}} = S_{\text{DBI}} + S_{\text{WZ}} \tag{3.2.10}$$

consists of Dirac-Born-Infeld (DBI) and Wess-Zumino (WZ) actions, schematically

$$\begin{aligned} S_{\text{DBI}} &= -T_8 \int_{D8} d^9x\, e^{-\phi}\, \text{STr}\sqrt{-\det\left(G_{MN} + 2\pi\alpha'\, F_{MN}\right)} & (3.2.11) \\ S_{\text{WZ}} &= \frac{1}{3!(2\pi)^3}\int_{D8} C_3 \wedge \text{STr}\, F^3, & (3.2.12) \end{aligned}$$

where $T_8 = 1/((2\pi)^8 l_s^9)$ is the tension of the D8-brane. F_{MN} is the field strength of the non-Abelian $U(N_f)$ gauge field living on the brane, while G_{MN} is the induced metric $G_{MN} = g_{\mu\nu}(X)\partial_M X^\mu \partial_N X^\nu$, with $g_{\mu\nu}$ the 10d background metric (3.2.7). The embedding functions X^μ are also $N_f \times N_f$ Hermitian matrices. STr stands for symmetrized trace, this prescription for the trace is unambiguous up to $O(F^4)$ in the gauge fields [57–60]. At higher orders there can be additional corrections in $\alpha' = l_s^2$, but fortunately we will not need them. The WZ term captures the axial anomaly of the dual field theory [15].

Part II

Holographic Computation of Scattering

Chapter 4

Scattering in Holography

In this chapter we present the features of the scattering amplitudes computation in d-dimensional, large-N, strongly coupled theories with a $(d+1)$-dimensional holographic dual. In holographic models that are dual to a confining theory there is a gapped, discrete spectrum of normalizable modes that map to particles, glueballs, or mesons in the field theory dual. In particular we will focus on a scalar gauge-invariant single-trace operator $\mathcal{O}$. Examples could be a glueball operator such as $\mathcal{O} = \mathrm{tr}F^2$, with F the field strength of the gauge fields, or $\mathcal{O} = \overline{q}q$, where q corresponds to a quark operator. Scattering between these particles can be obtained by applying LSZ reduction formulas to correlators of gauge-invariant operators with the quantum numbers of the particles involved. In the holographic calculation this implies that it is enough to identify the leading pole contributions in bulk correlators when the momenta are taken on-shell. The corresponding residues then determine the scattering amplitudes.

A (real-valued) gauge-invariant single-trace local operator will map to a scalar field Φ in the holographic gravity dual. When the operator acts on the vacuum, it creates scalar excitations, belonging to a tower of states all with the same quantum numbers and different masses m_a, $a = 1, 2, 3, \ldots$. In the large-N limit these states behave approximately as almost free particles, with interactions that can be trated perturbatively in a $1/N$ expansion. Here we will restrict to cases where the scalar field is treated as a probe, so backreaction on the metric and on other fields will be neglected.

The tower of massive states is captured by the two-point function of the scalar operator, in momentum space $p = (\omega, \mathbf{k})$

$$\langle \mathcal{O}(p)\mathcal{O}(p')\rangle = (2\pi)^4\delta^{(4)}(p+p')G(p). \tag{4.0.1}$$

When the momentum is put on-shell at one of the masses $\omega \to \varepsilon_a(\mathbf{k}) = \sqrt{\mathbf{k}^2 + m_a^2}$, there is a pole singularity in the two-point function

$$G(p) \sim \frac{-iZ_a}{p^2 + m_a^2} + \cdots \tag{4.0.2}$$

We can extract the residues of these poles from a LSZ-like reduction formula

$$Z_a = i \lim_{\omega\to\varepsilon_a} (p^2 + m_a^2)G(p). \tag{4.0.3}$$

Since the width of these states is suppressed in the large-N limit, it is possible to define asymptotic non-interacting states and compute the scattering matrix. The S-matrix can be split in the non-interacting part and the interacting part, or T-matrix,

$$S = \mathbb{I} + i\mathcal{T}. \tag{4.0.4}$$

In a two-to-two scattering of the spin-zero particles with masses m_n and spatial momenta $\mathbf{k}_n$, $n = 1, 2, 3, 4$, the T-matrix element

$$i\mathcal{T}_{12\to34} = \langle m_3, \mathbf{k}_3;\, m_4, \mathbf{k}_4|\; i\mathcal{T}\; |m_1, \mathbf{k}_1;\, m_2, \mathbf{k}_2\rangle\,, \tag{4.0.5}$$

can be obtained from a connected four-point function using the LSZ reduction formula ($\varepsilon_n \equiv \sqrt{\mathbf{k}_n^2 + m_n^2}$)

$$i\mathcal{T}_{12\to34} \equiv \lim_{\omega_n\to\varepsilon_n} \left(\prod_{n=1}^{4} \frac{p_n^2 + m_n^2}{Z_n^{1/2}}\right) \langle \mathcal{O}(p_1)\mathcal{O}(p_2)\mathcal{O}(p_3)\mathcal{O}(p_4)\rangle_c\,. \tag{4.0.6}$$

To define the scattering length one should first recall that momentum states are defined with different normalization in the relativistic and non-relativistic approaches. Taking this into account, we extract the following factor from the matrix elements

$$S_{12\to 34} = 4(\varepsilon_1\varepsilon_2\varepsilon_3\varepsilon_4)^{1/2}\widetilde{S}_{12\to 34}. \tag{4.0.7}$$

Here $\widetilde{S}_{12\to 34}$ will become the non-relativistic scattering matrix. Let us now focus on the case of elastic scattering between two particles of possibly different masses $m_1 = m_3$ and $m_2 = m_4$. Momentum conservation implies that the T-matrix contribution should take the form

$$i\widetilde{\mathcal{T}}_{12\to 34} = (2\pi)^4\delta(\varepsilon_1+\varepsilon_2-\varepsilon_3-\varepsilon_4)\delta^{(3)}(\mathbf{k}_1+\mathbf{k}_2-\mathbf{k}_3-\mathbf{k}_4)i\widetilde{\mathcal{M}}. \tag{4.0.8}$$

In the non-relativistic limit $\varepsilon_n \approx m_n + \frac{\mathbf{k}_n^2}{2m_n}$. Going to the center of mass frame $\mathbf{k}_1 = -\mathbf{k}_2 = \mathbf{k}$ and integrating over the final momenta of the particle of mass m_1 or m_2

$$i\widetilde{\mathcal{T}}_{\mathbf{k}\to\mathbf{k}'} = i(2\pi)\delta\left(\frac{\mathbf{k}'^2}{2m_{12}} - \frac{\mathbf{k}^2}{2m_{12}}\right)\widetilde{\mathcal{M}}(\mathbf{k}'-\mathbf{k}), \tag{4.0.9}$$

where $\mathbf{k}' = \mathbf{k}_3 = -\mathbf{k}_4$, $m_{12} = m_1 m_2/(m_1+m_2)$ is the reduced mass and we have assumed that the scattering amplitude depends only on the exchanged momentum between the particles. The formula above can be understood as the Born approximation to the scattering of a particle of mass m_{12} in a potential, the Fourier transform of the potential being $\widetilde{V}(\mathbf{q}) = -\widetilde{\mathcal{M}}(\mathbf{q})$. It follows that the scattering amplitude of the outgoing wave is

$$f(\mathbf{q}) = \frac{m_{12}}{2\pi}\widetilde{\mathcal{M}}(\mathbf{q}). \tag{4.0.10}$$

Therefore, the scattering length is

$$a_s = -\lim_{\mathbf{q}\to\mathbf{0}} \frac{m_{12}}{2\pi}\widetilde{\mathcal{M}}(\mathbf{q}). \tag{4.0.11}$$

4.1 Scattering amplitude in holography

We will assume that a large-N strongly coupled gauge theory in d spacetime dimensions has a holographic dual description. The dual will in general consist of Einstein gravity in $d+1$ dimensions coupled to a scalar field Φ dual to $\mathcal{O}$

$$S = \frac{1}{16\pi G_N}\int d^{d+1}x\sqrt{-g}\left(R - (\partial\Phi)^2 - 2V(\Phi)\right). \tag{4.1.1}$$

In order for the theory to be well defined in the UV, we will assume that the potential has a critical point $V'(0) = 0$. In this case there is a pure AdS_{d+1} solution, dual to a UV fixed point, where the scalar is constant $\Phi = 0$. The metric is

$$ds^2_{d+1} = g_{MN}dx^M dx^M \underset{z\to 0}{\longrightarrow} \frac{L^2}{z^2}\left(dz^2 + \eta_{\mu\nu}dx^\mu dx^\nu\right). \tag{4.1.2}$$

Here g_{MN}, $M, N = 0, 1, \ldots, d$, is the $d+1$ dimensional metric, that we will assume is asymptotically AdS_{d+1} with radius L. We will use capital latin letters for spacetime indices in the bulk of the gravity dual and greek letters for the field theory directions so in these coordinates x^μ, $\mu = 0, 1, \ldots, d-1$ span the directions of the field theory dual.

We will consider scalar potentials admitting an analytic expansion for small values of the field Φ then the potential expanded around the UV critical point will have the form

$$V(\Phi) \simeq -\frac{d(d-1)}{2L^2} + \frac{1}{2}m^2\Phi^2 + \frac{v_4}{4L^2}\Phi^4 + \cdots \tag{4.1.3}$$

The mass of the scalar m^2 determines the dimension Δ of the dual operator $m^2L^2 = \Delta(\Delta - d)$. Let us introduce the following parametrization of the dimension of the dual operator

$$\Delta = \frac{d}{2} + \nu. \tag{4.1.4}$$

Where $-1 < \nu < d/2$ is the range of allowed values for relevant operators satisfying the unitarity bound and the mass is $m^2L^2 = \nu^2 - \frac{d^2}{4}$. The quartic term introduces a contact interaction for the scalar in the bulk, proportional to v_4.

With this action, the asymptotic expansion of the scalar close to the AdS_{d+1} boundary is in general of the form

$$\Phi(x,z) = \left(\frac{z}{L}\right)^{\frac{d}{2}-\nu}\left[A_0(x) + \frac{z}{L}A_1(x) + \cdots + \left(\frac{z}{L}\right)^{2\nu}\left(C_\nu(x) + \frac{z}{L}C_{\nu,1}(x) + \cdots\right)\right], \tag{4.1.5}$$

where the first part corresponds to the non-normalizable solution and the second part to the normalizable solution. If $\frac{d}{2} + \nu$ is an integer, then the expansion of the non-normalizable solution may include also logarithmic terms $\log(z/L)$.

The expansion of the metric is

$$g_{MN}(x,z) = \frac{L^2}{z^2}\left(\eta_{MN} + \frac{z^2}{L^2}H_{2MN}(x) + \cdots + \frac{z^d}{L^d}H_{dMN}(x) + \cdots\right). \tag{4.1.6}$$

The equations of motion fix the coefficients in the non-normalizable solution of the scalar A_1, etc and the coefficients of the metric $H_{n\,MN}$ for $n < d$ (as well as the coefficients of logarithmic terms). All these coefficients are local functionals of A_0 and its derivatives along the field theory directions.

4.1.1 Solutions and propagators of the scalar field

We will restrict to cases where the scalar field is treated as a probe, so backreaction on the metric and on other fields will be neglected. The bulk action will have the form (4.1.1).

The linearized equation of motion for the scalar field is

$$\frac{1}{\sqrt{-g}}\partial_M\left(\sqrt{-g}g^{MN}\partial_N\Phi\right) - V'(\phi = 0) = 0. \tag{4.1.7}$$

Using the metric (4.1.2) and the potential (4.1.3) the equation of motion becomes

$$(\Box - m^2)\Phi = \frac{1}{\sqrt{-g}}\partial_M\left(\sqrt{-g}g^{MN}\partial_N\Phi\right) - m^2\Phi = 0. \tag{4.1.8}$$

Solutions to the equation of motion stemming from (**??**) can be constructed perturbatively using the bulk-to-boundary $K(x,x';z)$ and and bulk-to-bulk $G(x,x';z,z')$ propagators by including higher order terms of the potential (4.1.3). The bulk-to-boundary propagator determines the linearized solution for any boundary condition

$$(\Box_{z,x} - m^2)K = 0, \quad K(x,x';z) \underset{z\to 0}{\longrightarrow} z^{d-\Delta}\delta^{(d)}(x-x'). \tag{4.1.9}$$

The bulk-to-bulk propagator on the other hand is the Green's function defined by the differential equation

$$(\Box_{z,x} - m^2)G = \frac{1}{\sqrt{-g}}\delta^{(d)}(x-x')\delta(z-z'), \quad G(x,x';z,z') \underset{z\to 0}{\longrightarrow} z^{\Delta}. \tag{4.1.10}$$

In order to accommodate the AdS soliton model that we study later, we will assume Poincaré invariance of the full geometry along a subset of the field theory directions $d_{eff} \le d$ and remove all explicit dependence from the remaining field theory directions. We will thus split the metric as follows

$$ds^2_{d+1} = g_{zz}dz^2 + g_{xx}\eta_{\mu\nu}dx^\mu dx^\nu + g_{ij}dy^i dy^j, \tag{4.1.11}$$

where now x^μ, $\mu = 0, 1, \ldots, d_{eff} - 1$ and y^i, $i = d_{eff}, \ldots, d-1$. Written in this way, the plane spanned by the vectors x^μ retain the Poincaré symmetry, while directions y^i are orthogonal to this plane. We proceed by expanding the scalar field and the propagators in plane waves

$$\Phi(x,z) = \int \frac{d^{d_{eff}}p}{(2\pi)^{d_{eff}}}\phi(p,z)e^{ip\cdot x}. \tag{4.1.12}$$

For a confining theory, there will be a discrete set of normal modes satisfying conditions of regularity at the origin and being normalizable at the boundary

$$p^2 = -M_n^2, \phi = \varphi_n(z), n = 0, 1, 2, 3, \ldots. \tag{4.1.13}$$

The spectrum of masses M_n^2 corresponds to the spectrum of massive states in the dual field theory associated to the dual scalar operator. The equation of motion for the scalar can be put in Sturm-Liouville form

$$\partial_z\left(\sqrt{-g}g^{zz}\partial_z\phi\right) - m^2\sqrt{-g}\phi + \sqrt{-g}g^{xx}M^2\phi = 0. \tag{4.1.14}$$

We can identify $\rho(z) = \sqrt{-g}g^{xx}$ with the weight, so that normal modes can be chosen to form an orthonormal basis

$$\int dz\, \rho(z)\varphi_n(z)\varphi_m(z) = \delta_{nm}. \tag{4.1.15}$$

The integration is over the whole allowed range of the radial coordinate. The basis of normal modes can be used to write an expression for the bulk-to-bulk propagator

$$G(p; z, z') = -\sum_n \frac{\varphi_n(z)\varphi_n(z')}{p^2 + M_n^2}. \tag{4.1.16}$$

4.1.2 n-point functions

In order to compute the T-matrix element (4.0.6) we need both the residues of the poles in the two-point and the four-point functions, in the limit where the external momenta are on-shell. The correlators can be computed from the one-point function in the presence of a external source j for the scalar operator

$$\langle\mathcal{O}(x)\rangle_j = \frac{\int D\mathcal{A}\,\mathcal{O}(x)\, e^{iS_{YM}+i\int d^dx j(x)\mathcal{O}(x)}}{\int D\mathcal{A}\, e^{iS_{YM}+i\int d^dx j(x)\mathcal{O}(x)}}. \tag{4.1.17}$$

Here S_{YM} is the action of the dual field theory, depending on the fields $\mathcal{A}$ and $D\mathcal{A}$ is the field theory path integral measure. Note that the operator $\mathcal{O}(x)$ is a functional of the fields $\mathcal{A}$. Higher order correlation functions are obtained from the one-point function as

$$\langle\mathcal{O}(x_1)\cdots\mathcal{O}(x_n)\rangle = \frac{1}{n}\sum_{k=1}^{n}\prod_{\substack{a=1\\a\neq k}}^{n}\left(\frac{1}{i}\frac{\delta}{\delta j(x_a)}\right)\langle\mathcal{O}(x_k)\rangle_j\Big|_{j=0}. \tag{4.1.18}$$

An external source is realized in the gravity dual as a boundary condition for the scalar field. This fixes $A_0(x) = L^{\frac{d}{2}-\nu}j(x)$:

$$\Phi(x, z\to 0) = z^{\frac{d}{2}-\nu}j(x), \tag{4.1.19}$$

The one-point function can be computed using the canonical momentum of the dual scalar Φ [52, 61]

$$\pi_\Phi = \sqrt{-g}g^{zz}\partial_z\Phi. \tag{4.1.20}$$

The regularized momentum $\pi_\Phi^{reg} = \pi_\Phi + \pi_\Phi^{c.t.}$ can be obtained from a variation of the action including counterterms. The one-point function is obtained as the finite result from [52, 62]

$$\langle\mathcal{O}(x)\rangle_j = \mathcal{N}\frac{1}{L^{\frac{d}{2}+\nu-1}}\lim_{z\to 0}\left(\frac{z}{L}\right)^{\frac{d}{2}-\nu}\pi_\Phi^{reg} = \mathcal{N}\frac{2\nu C_\nu(x)}{L^{\frac{d}{2}+\nu}} + \text{local terms}. \tag{4.1.21}$$

In the special case $\nu = 0$, the overall coefficient is 2 instead of 2ν. The dimensionless factor $\mathcal{N} = L^{d-1}/(16\pi G_N)$ is proportional to the number of degrees of freedom of the field theory, $\mathcal{N}\sim N^2$ for a large-N gauge theory in $d = 4$. The local terms are proportional to powers of j or its derivatives and are in general scheme dependent (terms depending on $\log j$ might be present as well). They will drop from the LSZ reduction formula, so we can neglect them.

In order to compute the correlators we expand around a background solution and solve the equations of motion derived from the action (4.1.1) with the boundary condition (4.1.19). We then evaluate the solution on the regularized momentum and find the expectation value from (4.1.21). We construct the classical solution by introducing a small parameter ϵ and rescaling the source $j\to\epsilon j$. Then we can do an expansion in powers of ϵ

$$\Phi = \phi + \epsilon\phi^{(1)} + \epsilon^2\phi^{(2)} + \cdots, g_{MN} = h_{MN} + \epsilon h_{MN}^{(1)} + \epsilon^2 h_{MN}^{(2)} + \cdots. \tag{4.1.22}$$

In this expansion ϕ, h_{MN} are the background values, $\phi^{(1)}$ should satisfy the boundary condition (4.1.19) and the remaining terms should not change the boundary conditions for the scalar field of the metric. This implies that

$$\lim_{z\to 0}\frac{1}{z^{\frac{d}{2}-\nu}}\phi^{(n)}=0, n>1, \lim_{z\to 0} z^2 h_{MN}^{(m)}=0, m>0. \tag{4.1.23}$$

Each term in the expansion is a homogeneous functional of the source j, and we will assume that the background only depends on the radial coordinate $\partial_\mu\phi=\partial_\mu h_{MN}=0$. For the computation of the two- and four- point functions we need just to expand (4.1.21) up to $O(\epsilon^3)$.

At each order, the solution will be obtained as the convolution of a Green's function with the sources

$$\phi^{(n)}(z,x)=\int d^dx_1\cdots d^dx_n\; G^{(n)}(z,x;x_1,\cdots,x_n)j(x_1)\cdots j(x_n), \tag{4.1.24}$$

$$h_{MN}^{(n)}(z,x)=\frac{L^2}{z^2}\int d^dx_1\cdots d^dx_n G_{MN}^{(n)}(z,x;x_1,\cdots,x_n)j(x_1)\cdots j(x_n). \tag{4.1.25}$$

The boundary conditions demand that

$$\lim_{z\to 0} G^{(1)}(z,x;x_1)\sim z^{\frac{d}{2}-\nu}\delta^{(d)}(x-x_1), \tag{4.1.26}$$

$$\lim_{z\to 0}\frac{1}{z^{\frac{d}{2}-\nu}}G^{(n)}(z,x;x_1,\cdots,x_n)=0, \tag{4.1.27}$$

$$\lim_{z\to 0} G_{MN}^{(n)}(z,x;x_1,\cdots,x_n)=0. \tag{4.1.28}$$

It will be convenient to take Fourier transforms and work in momentum space,

$$\widetilde{\phi}^{(n)}(z,p)=\int\frac{d^dp_1}{(2\pi)^d}\cdots\frac{d^dp_n}{(2\pi)^d}\;\widetilde{G}^{(n)}(z,p;-p_1,\cdots,-p_n)\widetilde{j}(p_1)\cdots\widetilde{j}(p_n), \tag{4.1.29}$$

$$\widetilde{h}_{MN}^{(n)}(z,p)=\frac{L^2}{z^2}\int\frac{d^dp_1}{(2\pi)^d}\cdots\frac{d^dp_n}{(2\pi)^d}\widetilde{G}_{MN}^{(n)}(z,p;-p_1,\cdots,-p_n)\widetilde{j}(p_1)\cdots\widetilde{j}(p_n). \tag{4.1.30}$$

Expanding for $z\to 0$, we should find

$$\widetilde{G}^{(1)}(z,p;-p1)\simeq z^{\frac{d}{2}-\nu}(2\pi)^d\delta^{(d)}(p-p_1)+\cdots+\left(\frac{z}{L}\right)^{\frac{d}{2}+\nu}\widetilde{G}_\nu^{(1)}(p;-p_1)+\cdots, \tag{4.1.31}$$

$$\widetilde{G}^{(n)}(z,p;-p_1,\cdots,-p_n)\simeq\left(\frac{z}{L}\right)^{\frac{d}{2}+\nu}\widetilde{G}_\nu^{(n)}(p;-p_1,\cdots,-p_n)+\cdots. \tag{4.1.32}$$

Therefore, the contribution to the coefficient that determines the one-point function in (4.1.21) is

$$\widetilde{C}_\nu^{(n)}(p)=\int\frac{d^dp_1}{(2\pi)^d}\cdots\frac{d^dp_n}{(2\pi)^d}\;\widetilde{G}_\nu^{(n)}(p;-p_1,\cdots,-p_n)\widetilde{j}(p_1)\cdots\widetilde{j}(p_n). \tag{4.1.33}$$

4.1.3 Residues and scattering amplitude

The quartic term in the scalar potential gives a contribution to the scattering amplitude that can be interpreted as being produced by a Witten diagram with four scalar legs joining at a single point in the bulk. The relevant quantity is

$$\widetilde{G}^{(3)}(z,p_1;-p_2,-p_3,-p_4)\propto(2\pi)^d\delta^{(d)}(p_1-p_2-p_3\cdot-p_4)\,v_4\int dz'\,\sqrt{-g}G(p_1;z,z')K(p_2;z')K(p_3;z')K(p_4;z'). \tag{4.1.34}$$

From (4.1.18) and (4.1.33) we can deduce the following expressions for the two and four point functions

$$\langle\mathcal{O}(p_1)\mathcal{O}(p_2)\rangle=-\frac{i\nu\mathcal{N}}{L\Delta}\left(\widetilde{G}_\nu^{(1)}(p_1;-p_2)+\widetilde{G}_\nu^{(1)}(p_2;-p_1)\right), \tag{4.1.35}$$

$$\langle \mathcal{O}(p_1)\mathcal{O}(p_2)\mathcal{O}(p_3)\mathcal{O}(p_4)\rangle = \frac{i\nu\mathcal{N}}{2L\Delta}\sum_{k=1}^{4}\widetilde{G}^{(3)}_{\nu}(p_k;-p_{n_1},-p_{n_2},-p_{n_3})\Big|_{n_i=1,2,3,4;ni\neq k} \tag{4.1.36}$$

When the momenta are taken on-shell, both bulk-to-boundary and bulk-to-bulk propagators have poles

$$K(p;z)\xrightarrow[p^2\to -M_n^2]{}\frac{c_n\varphi_n(z)}{p^2+M_n^2},\quad G(p;z,z')\xrightarrow[p^2\to -M_n^2]{}-\frac{\varphi_n(z)\varphi_n(z')}{p^2+M_n^2}. \tag{4.1.37}$$

And we find that the kernels are of the form

$$\widetilde{G}^{(1)}(z;p_1;-p_2)\xrightarrow[p_i^2\to -M_{n_i}^2]{}(2\pi)^d\delta^{(d)}(p_1-p_2)\frac{\Gamma^{(1)}_{n_1}(z,p_1)}{p_1^2+M_{n_1}^2}, \tag{4.1.38}$$

$$\widetilde{G}^{(3)}(z,p_1;-p_2,-p_3,-p_4)\xrightarrow[p_i^2\to -M_{n_i}^2]{}(2\pi)^d\delta^{(d)}(p_1-p_2-p_3\cdot-p_4)\frac{\Gamma^{(3)}_{n_1;n_2,n_3,n_4}(z)}{\prod_{i=1}^4(p_i^2+M_{n_i}^2)}. \tag{4.1.39}$$

Where the last expression is the leading pole contribution to the amplitude. Close to the boundary, the residue of the bulk-to-boundary propagator has the form

$$c_n\varphi_n(z)\xrightarrow[z\to 0]{}\frac{Z_n}{2\nu\mathcal{N}}\,z^{\Delta}, \tag{4.1.40}$$

where $\mathcal{N}=L^{d-1}/(16\pi G_N)$ is a dimensionless normalization factor proportional to the number of degrees of freedom in the dual field theory. The residues can be directly read from the boundary expansion $z\to 0$ by isolating the poles

$$\Gamma^{(1)}_{n_1}\simeq z^{\Delta}\frac{Z_{n_1}}{2\nu\mathcal{N}}+\cdots,\Gamma^{(3)}_{n_1;n_2,n_3,n_4}\simeq z^{\Delta}\frac{\mathcal{Z}_{n_1;n_2,n_3,n_4}}{2\nu\mathcal{N}}+\cdots. \tag{4.1.41}$$

The factor $Z_{n_!}$ can be identified with the residue of the massive pole in the two-point function of the dual scalar operator, explicitly

$$Z_{n_1}=2\nu\mathcal{N}\,c_n\times\lim_{z\to 0}\frac{\varphi_{n_1}(z)}{z^{\Delta}}. \tag{4.1.42}$$

The factor $\mathcal{Z}_{n_1;n_2,n_3,n_4}$ is the leading pole contribution to the four-point function of the dual scalar operator. Through the LSZ reduction formula it will determine the scattering amplitude. Its explicit form is

$$\mathcal{Z}_{n_1;n_2,n_3,n_4}=-2\nu v_4\mathcal{N}\,c_{n_2}c_{n_3}c_{n_4}\kappa_{n_1,n_2,n_3,n_4}\times\lim_{z\to 0}\frac{\varphi_{n_1}(z)}{z^{\Delta}}, \tag{4.1.43}$$

where the overlap κ is defined as

$$\kappa_{n_1,n_2,n_3,n_4}=\int dz'\sqrt{-g}\prod_{i=1}^{4}\varphi_{n_i}(z'). \tag{4.1.44}$$

With the residues (4.1.42) and (4.1.43), using the LSZ reduction formula (4.0.6), the contribution of the quartic term to the scattering amplitude is

$$\mathcal{M}_{n_1,n_2,n_3,n_4}=\frac{1}{4}\sum_{i=1}^{4}\frac{\mathcal{Z}_{n_i;\{n_k\neq n_i\}}}{(Z_{n_1}Z_{n_2}Z_{n_3}Z_{n_4})^{1/2}}. \tag{4.1.45}$$

If $d_{eff}=4$, the scattering length for two-to-two elastic scattering can be determined directly from the previous amplitude using the formula derived in [17]

$$a_s=-\frac{\mathcal{M}_{n_1=n_3,n_2=n_4}}{8\pi(M_{n_1}+M_{n_2})}. \tag{4.1.46}$$

Chapter 5

Scattering Lenght In Confining Theories

In the following sections, we will restrict to states created by operators that have a dual description as classical fields in gravity, typically glueballs and mesons. Some states, baryons for instance, map to heavy objects such as wrapped branes on the gravity side [63]. The effective potential for these types of objects has been studied for instance in the Sakai-Sugimoto model using solitonic configuration on flavor branes [64,65]. A similar analysis is the calculation of glueball decay rates using three-particle scattering amplitudes in the GPPZ model [66], while for very high energy scattering a description in terms of strings may be more appropriate [67].

5.1 Scattering in a hard wall model

We will illustrate our approach with a simple example. We will treat the scalar field as a probe, neglecting the backreaction on the metric, *i.e.*, it will remain fixed in this case. For simplicity, we consider AdS_{d+1} (4.1.2) metric in the whole bulk, but in order to mimic confinement we will introduce a hard wall [13] at a finite value of the radial coordinate z. This means that the space abruptly ends, with the radial coordinate taking values on the interval $0 < z < 1/\Lambda$. A Dirichlet condition for the fields at the hard wall makes the spectrum of fluctuations discrete and gapped, with the gap set by the IR scale Λ.

The equations of motion for the scalar field are ($\Box \equiv \eta^{\mu\nu}\partial_\mu\partial_\nu$)

$$z^2\partial_z^2\Phi - (d-1)z\partial_z\Phi + z^2\Box\Phi - L^2V'(\Phi) = 0. \tag{5.1.1}$$

We will take the scalar potential to be of the form (4.1.3), and do the expansion (4.1.22) for the scalar field with a vanishing background field $\phi = 0$. The equation of motion at $O(\epsilon^n)$ is

$$z^2\partial_z^2\phi^{(n)} - (d-1)z\partial_z\phi^{(n)} + z^2\Box\phi^{(n)} - m^2L^2\phi^{(n)} = W^{(n)}, \tag{5.1.2}$$

where the inhomogeneous terms read, up to $O(\epsilon^3)$,

$$W^{(1)} = W^{(2)} = 0, \quad W^{(3)} = v_4(\phi^{(1)})^3. \tag{5.1.3}$$

Fixing the mass of the scalar to $m^2L^2 = \nu^2 - \frac{d^2}{4}$, the boundary conditions at the hard wall and the asymptotic AdS boundary are

$$\phi^{(n)}\Big|_{z=1/\Lambda} = 0, \quad \phi^{(1)} \underset{z\to 0}{\sim} z^{\frac{d}{2}-\nu}j(x), \quad \phi^{(n)} \underset{z\to 0}{\sim} O(z^{\frac{d}{2}+\nu}), n>1. \tag{5.1.4}$$

With these conditions $\phi^{(2)} = 0$. We will now go to momentum space via Fourier transformation

$$\widetilde{\phi}^{(1)}(p,z) = \int d^dx e^{-ip\cdot x}\phi^{(1)}(z,x). \tag{5.1.5}$$

The equations of motion are

$$\left(z^2\partial_z^2 - (d-1)z\partial_z + z^2M^2 + \frac{d^2}{4} - \nu^2\right)\widetilde{\phi}^{(n)}(p,z) = \widetilde{W}^{(n)}(p,z), \tag{5.1.6}$$

where we have defined $M^2 = -p^2$ and

$$\widetilde{W}^{(3)}(p,z)=(2\pi)^d\delta^{(d)}(p-p_1-p_2-p_3)v_4\int\frac{d^dp_1}{(2\pi)^d}\frac{d^dp_2}{(2\pi)^d}\frac{d^dp_3}{(2\pi)^d}\widetilde{\phi}^{(1)}(p_1,z)\widetilde{\phi}^{(1)}(p_2,z)\widetilde{\phi}^{(1)}(p_3,z). \tag{5.1.7}$$

The solutions for $\widetilde{\phi}^{(1)}$ and $\widetilde{\phi}^{(3)}$ can be written as

$$\widetilde{\phi}^{(1)}(p,z)=K(p,z)\widetilde{j}(p),\quad \widetilde{\phi}^{(3)}(p,z)=\int_0^{1/\Lambda}dz_1\frac{L^{d+1}}{z_1^{d+1}}\Sigma(p,z,z_1)\widetilde{W}^{(3)}(p,z_1). \tag{5.1.8}$$

Where we have introduced the bulk-to-boundary (K) and the bulk-to-bulk (Σ) propagators. The equations of motion for each of the propagators is

$$\left(z^2\partial_z^2-(d-1)z\partial_z+z^2M^2+\frac{d^2}{4}-\nu^2\right)K(p,z)=0 \tag{5.1.9}$$

$$\left(z^2\partial_z^2-(d-1)z\partial_z+z^2M^2+\frac{d^2}{4}-\nu^2\right)G(p,z,z_1)=\frac{z^{d+1}}{L^{d+1}}\delta(z-z_1). \tag{5.1.10}$$

They must satisfy the boundary conditions

$$K(p,1/\Lambda)=0,\quad K(p,z)\underset{z\to 0}{\sim}z^{\frac{d}{2}-\nu},\quad G(p,1/\Lambda,z_1)=0,\quad G(p,z,z_1)\underset{z\to 0}{\sim}O(z^{\frac{d}{2}+\nu}). \tag{5.1.11}$$

Then, the expansions for $z\to 0$ should be of the form

$$K(p,z)\simeq z^{\frac{d}{2}-\nu}+\dots+\left(\frac{z}{L}\right)^{\frac{d}{2}+\nu}K_\nu(p)+\dots \tag{5.1.12}$$

$$G(p,z,z_1)\simeq\left(\frac{z}{L}\right)^{\frac{d}{2}+\nu}G_\nu(p,z_1)+\dots. \tag{5.1.13}$$

Direct comparison with (4.1.33) shows that

$$\tilde{G}_\nu^{(1)}(p;-p_1)=(2\pi)^d\delta^{(d)}(p-p_1)K_\nu(p_1)$$

$$\tilde{G}_\nu^{(3)}(p;-p_1,-p_2,-p_3)=(2\pi)^d\delta^{(d)}(p-p_1-p_2-p_3)\,v_4\int_0^{1/\Lambda}dz_1\tfrac{L^{d+1}}{z_1^{d+1}}G_\nu(p,z_1)K(p_1,z_1)K(p_2,z_1)K(p_3,z_1). \tag{5.1.14}$$

5.1.1 Bulk-to-boundary and bulk-to-bulk propagators

The equation of motion that the propagators must satisfy can be put in Sturm-Liouville form. Multiplying the equation by $1/(\Lambda z)^{d+1}$, using as variable $u=\Lambda z$ and defining $\mu=M/\Lambda$,

$$\partial_u\left(\frac{1}{u^{d-1}}\partial_uK(p,u)\right)+\left(\frac{\frac{d^2}{4}-\nu^2}{u^{d+1}}+\frac{\mu^2}{u^{d-1}}\right)K(p,u)=0 \tag{5.1.15}$$

$$\partial_u\left(\frac{1}{u^{d-1}}\partial_u\Sigma(p,u,u1)\right)+\left(\frac{\frac{d^2}{4}-\nu^2}{u^{d+1}}+\frac{\mu^2}{u^{d-1}}\right)G(p,u,u_1)=\frac{\Lambda}{(L\Lambda)^{d+1}}\delta(u-u_1). \tag{5.1.16}$$

The constant factor multiplying the delta function can be absorbed in the normalization of Σ. The eigenvalues of the associated Sturm-Liouville problem are μ^2, and the weight function is $w(u)=1/u^{d-1}$.

The bulk-to-boundary propagator is easy to find as a solution to the homogeneous equation satisfying the Dirichlet boundary condition and $K(p,u)\sim(u/\Lambda)^{\frac{d}{2}-\nu}$ when $u\to 0$:

$$K(p,u)=-\frac{\pi\mu^\nu}{2^\nu\Gamma(\nu)\Lambda^{\frac{d}{2}-\nu}}u^{d/2}\left(Y_\nu(\mu u)-c_\mu J_\nu(\mu u)\right),\quad c_\mu=\frac{Y_\nu(\mu)}{J_\nu(\mu)}. \tag{5.1.17}$$

Let us consider solutions to the homogeneous equation satisfying a Dirichlet condition at the hard wall and a normalizable condition at the AdS boundary. These solutions only exist for a discrete set of values of M, corresponding to the zeroes of the Bessel function J_ν:

$$\varphi_a(u) = c_a u^{d/2} J_\nu(\mu_a u), \quad c_a = \frac{\sqrt{2}}{J_{\nu+1}(\mu_a)}, \quad J_\nu(\mu_a) = 0, \quad a = 1, 2, 3, \dots. \tag{5.1.18}$$

The values of $M_a = \mu_a \Lambda$ are the masses of glueball or meson states created by the operator dual to the scalar field. When we compute the residues of the two-point function and scattering amplitudes we will take the momenta to be on-shell on one of these masses. The solutions are normalizable as long as $\nu > -1$, which corresponds to the unitarity bound.

The functions φ_a form an orthonormal set

$$\int_0^1 \frac{du}{u^{d-1}} \varphi_a(u) \varphi_b(u) = \delta_{ab}. \tag{5.1.19}$$

Then, the bulk-to-bulk propagator can be written as the usual expansion in the basis of solutions to the boundary problem

$$G(p, u, u_1) = \frac{\Lambda}{(L\Lambda)^{d+1}} \sum_{a=1}^{\infty} \frac{1}{\mu^2 - \mu_a^2} \varphi_a(u) \varphi_a(u_1). \tag{5.1.20}$$

5.1.2 Residues and scattering amplitude

When the momenta are taken on-shell to one of the masses of the scalar glueballs/mesons $\mu \to \mu_a$, the poles in the bulk-to-bulk propagator are manifest. In the bulk-to-boundary propagator they are implicit in the coefficient c_μ,

$$c_\mu \simeq \frac{Y_\nu(\mu_a)}{J_{\nu+1}(\mu_a)(\mu_a - \mu)} = \frac{Y_\nu(\mu_a)}{\sqrt{2}(\mu_a - \mu)} c_a \simeq \frac{\sqrt{2}\mu_a Y_\nu(\mu_a)}{\mu_a^2 - \mu^2} c_a. \tag{5.1.21}$$

Then, we can approximate the bulk-to-boundary propagator by the single pole contribution

$$K(p, u) \underset{p^2 \to -M_a^2}{\simeq} \frac{\pi \mu_a^{\nu+1} Y_\nu(\mu_a)}{2^{\nu - \frac{1}{2}} \Gamma(\nu) \Lambda^{\frac{d}{2} - \nu}} \frac{1}{\mu_a^2 - \mu^2} \varphi_a(u), \tag{5.1.22}$$

and similarly for the bulk-to-bulk propagator

$$G(p, u, u_1) \underset{p^2 \to -M_a^2}{\simeq} \frac{\Lambda}{(L\Lambda)^{d+1}} \frac{1}{\mu^2 - \mu_a^2} \varphi_a(u) \varphi_a(u_1). \tag{5.1.23}$$

Expanding now the normalizable solutions for $u \to 0$,

$$\varphi_a(u) \simeq c_a \frac{\mu_a^\nu}{2^\nu \Gamma(\nu+1)} u^{\frac{d}{2}+\nu} + \dots, \tag{5.1.24}$$

and taking into account that $M^2 = -p^2$, we obtain

$$K_\nu(p) \underset{p^2 \to -M_a^2}{\simeq} \Lambda^{2(\nu+1)} \frac{\pi \mu_a^{2\nu+1} Y_\nu(\mu_a)}{2^{2\nu - \frac{1}{2}} \nu \Gamma(\nu) 2} \frac{c_a}{p^2 + M_a^2} \tag{5.1.25}$$

$$G_\nu(p, u_1) \underset{p^2 \to -M_a^2}{\simeq} -\Lambda^{2+\nu-\frac{d}{2}} \frac{\mu_a \nu}{2^\nu \nu \Gamma(\nu) L^{d+1}} \frac{c_a}{p^2 + M_a^2} \varphi_a(u_1). \tag{5.1.26}$$

Comparing with the formula (4.1.41) we can extract the value of residues in the two-point function of the dual operator, for a glueball/meson of mass M_a

$$Z_a = \mathcal{N} \Lambda^{2(\nu+1)} z_a, \quad z_a = \frac{\pi \mu_a^{2\nu+1} Y_\nu(\mu_a)}{2^{2\nu - \frac{3}{2}} \Gamma(\nu)^2} c_a. \tag{5.1.27}$$

It is convenient to use this expression for the residue to simplify the formula of the bulk-to-boundary propagator:

$$K(p, u) \underset{p2 \to -Ma2}{\simeq} \Lambda^{\nu+2-\frac{d}{2}} \frac{2^{\nu-1} \Gamma(\nu) z_a}{c_a \mu_a^\nu} \frac{1}{p^2 + M_a^2} \varphi_a(u), \tag{5.1.28}$$

Combining (4.1.41) and (5.1.14) we get for the residue of the four-point function

$$\mathcal{Z}_{a;bcd} = -\mathcal{N}\Lambda^{8-d+4\nu}4^{\nu-1}\Gamma(\nu)^2 v_4 z_b z_c z_d \frac{c_a\mu_a^\nu}{c_b\mu_b^\nu c_c\mu_c^\nu c_d\mu_d^\nu}\int_0^1 \frac{du_1}{u_1^{d+1}}\varphi_a(u_1)\varphi_b(u_1)\varphi_c(u_1)\varphi_d(u_1). \tag{5.1.29}$$

Then, the contribution of the quartic term in the scalar potential to the scattering amplitude (4.1.45) is

$$\mathcal{M}_{v_4\,abcd} = -\frac{v_4}{\mathcal{N}}4^{\nu-2}\Gamma(\nu)^2\Lambda^{4-d}\kappa_{abcd}\sum_{k=a,b,c,d}\left(\frac{z_{n_1}z_{n_2}z_{n_3}}{z_k}\right)^{1/2}\frac{c_k\mu_k^\nu}{c_{n_1}\mu_{n_1}^\nu c_{n_2}\mu_{n_2}^\nu c_{n_3}\mu_{n_3}^\nu}\Big|_{n_i=a,b,c,d;n_i\neq k}, \tag{5.1.30}$$

where we have defined the overlap between the solutions of the scalar field as

$$\kappa_{abcd} = \int_0^1 \frac{dv}{v^{d+1}}\varphi_a(v)\varphi_b(v)\varphi_c(v)\varphi_d(v). \tag{5.1.31}$$

When $v \to 0$ the integrand has the following behavior

$$\frac{1}{v^{d+1}}\varphi_a(v)\varphi_b(v)\varphi_c(v)\varphi_d(v) \sim v^{d+4\nu-1}. \tag{5.1.32}$$

The integral is convergent as long as $d + 4\nu > 0$. Unitarity restricts $\nu > -1$, but the condition that the amplitude is finite imposes a stronger restriction for $d < 4$,

$$\nu > -\frac{d}{4} \Rightarrow \Delta > \frac{d}{4}. \tag{5.1.33}$$

If instead of a quartic term in the potential of the scalar field we had a term $\sim v_K\Phi^K$, $K > 4$, then there would be a contribution to the scattering amplitude similar to (5.1.31), but involving a convolution of K of the modes

$$\mathcal{M}_{v_K} \sim \int_0^1 \frac{dv}{v^{d+1}}\prod_{i=1}^K \varphi_{a_i}(v) \equiv \kappa_K. \tag{5.1.34}$$

This can also be interpreted as the (holographic) wavefunction overlap of all the modes involved in the scattering. This integral is convergent for $(d/2+\nu)K > d$, which gives the condition for a finite K-particle scattering amplitude

$$\nu > \frac{d}{K} - \frac{d}{2} \Rightarrow \Delta > \frac{d}{K}. \tag{5.1.35}$$

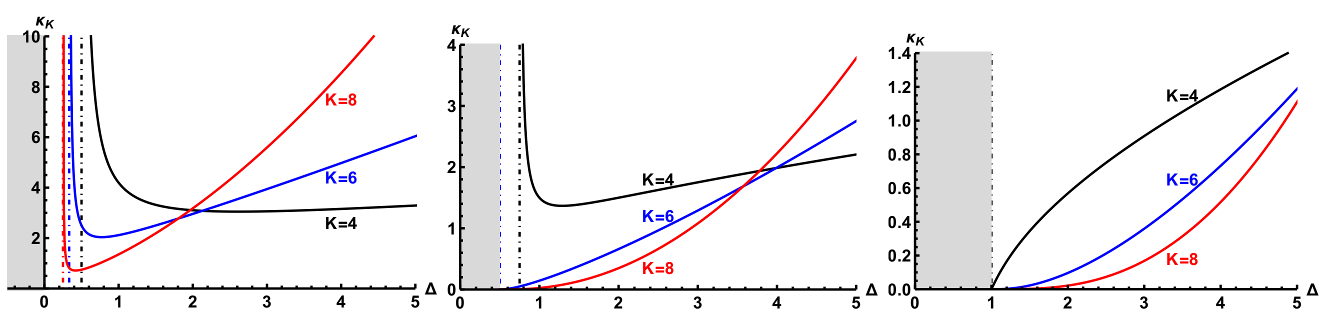


Figure 5.1.1: We depict the overlap κ_K as defined in (5.1.34) for various dimensions: $d = 2$ (Left), $d = 3$ (Middle), and $d = 4$ (Right). The three curves correspond to the overlap of $K = 4, 6, 8$ modes. The shaded gray is the region excluded by unitarity (5.1.33) and the vertical dot-dashed lines are the more stringent constraints (5.1.35) where applicable.

We see that larger values of K impose a looser constraint, they become more restrictive than the unitarity bound for $d < 2K/(K-2)$. In Fig. 5.1.1 we have plotted the overlap for various dimensions and for varying number of modes, showing that the corresponding amplitude diverges as the bound (5.1.35) is approached, if the bound is above the unitarity bound.

5.1.3 Scattering length

We have derived a formula for the scattering length in (4.1.46) when $d = 4$. Since the scattering amplitude (5.1.30) does not depend on the spatial momentum, we can read directly the result for the contribution of the quartic term in the scalar potential in units of the IR scale

$$\Lambda_{a_s}(v_4) = \frac{v_4}{16\pi\mathcal{N}} 4^{\nu-1}\Gamma(\nu)^2 \frac{\kappa_{aabb}}{\mu_a + \mu_b} \left(\frac{|z_a|}{(c_a\mu_a^\nu)^2} + \frac{|z_b|}{(c_b\mu_b^\nu)^2} \right), \tag{5.1.36}$$

where

$$\kappa_{aabb} = \int_0^1 \frac{du}{u^5} \varphi_a(u)^2 \varphi_b(u)^2. \tag{5.1.37}$$

This can be further simplified to

$$\Lambda_{a_s}(v_4) = \frac{v_4}{32\mathcal{N}} \kappa_{aabb} \frac{\mu_a |Y_\nu(\mu_a) J_{\nu+1}(\mu_a)| + \mu_b |Y_\nu(\mu_b) J_{\nu+1}(\mu_b)|}{\mu_a + \mu_b}. \tag{5.1.38}$$

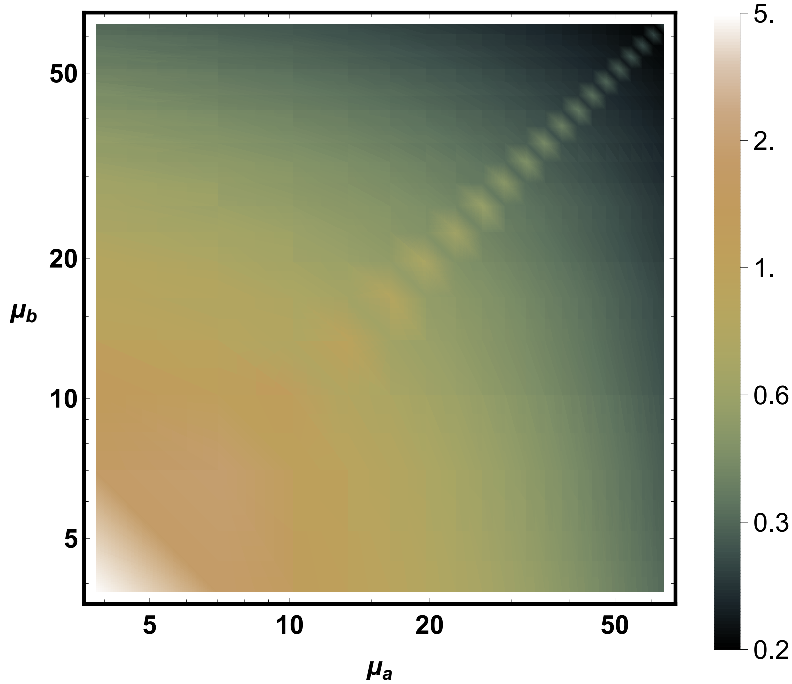


Figure 5.1.2: Contribution of the quartic term in the scalar potential to the scattering length $10^3 \times \mathcal{N}\Lambda a_s(v_4)/v_4$ as a function of the masses $\mu_a = M_a/\Lambda$, $\mu_b = M_b/\Lambda$ of the particles involved in the scattering for $\nu = 1$. Notice that the axes scale logarithmically.

The overlap (5.1.37) is clearly positive $\kappa_{aabb} > 0$, so the sign of the contribution to the scattering length is determined by the sign of the coefficient v_4 in the quartic term of the potential. For a large enough value of $|v_4|$ this contribution would dominate and determine the sign of the full scattering length. In this case the interaction between the scalar glueballs or hadrons associated to the operator $\mathcal{O}$ would be repulsive for $v_4 > 0$ and attractive for $v_4 < 0$. In general we expect the overlap to decrease when the mass difference is larger, as the solutions for the scalar field associated to each mass will be peaked at different positions in the radial direction. A smaller scattering length implies weaker interactions between states of different mass. We have confirmed the expected trend of $a_s(v_4)$ by computing numerically the scattering length for a large set of masses; see Fig. 5.1.2. In Fig. 5.1.3 we, on the other hand, show how the scattering length a_s depends on the scaling dimension of the scalars involved in the process. For small scaling dimension the scattering length increases, reaching a maximum and then slowly decreasing indefinitely.

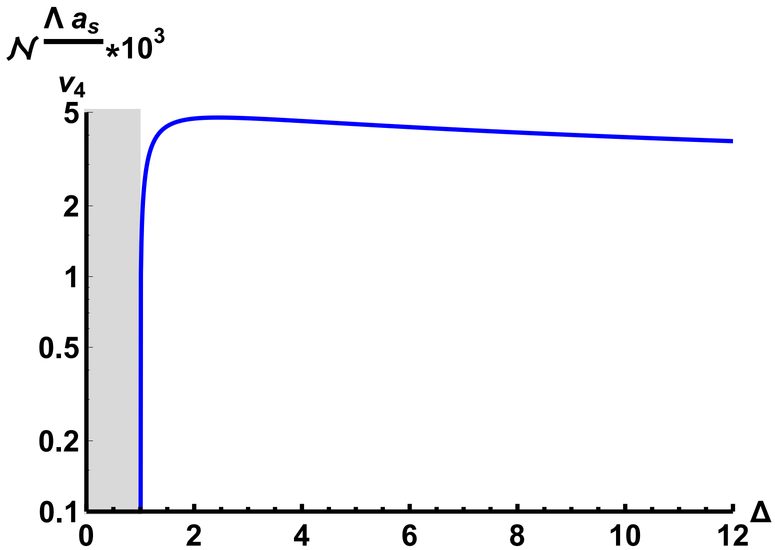


Figure 5.1.3: We depict the contribution of the quartic term in the scalar potential to the scattering length $10^3 \times \mathcal{N}\Lambda a_s(v_4)/v_4$ for the lowest mass states as a function of the scaling dimension Δ. The shaded gray region is excluded by the unitarity constraint (5.1.33), which in this case is no weaker than (5.1.35).

5.2 Scattering in the dual to $\mathcal{N} = 1^*$ super Yang-Mills

Here we will not study the most general case of the model presented in (3.2.1), but restrict to the simpler subset of vanishing gaugino condensate. In this case the background geometry is a a solution of five-dimensional supergravity truncated to a single scalar coupled to the metric. In a convenient set of coordinates the metric is $(0 < u < 1)$

$$ds^2_{4+1} = L^2 \left(\frac{du^2}{4(1-u)^2} + \frac{u}{1-u}\eta_{\mu\nu}dx^\mu dx^\nu \right). \tag{5.2.1}$$

The metric can be put in the form given in (4.1.2) by a change of coordinates

$$u = 1 - \frac{z^2}{z_\Lambda^2}, \quad x_\mu \to \frac{1}{z_\Lambda}x_\mu. \tag{5.2.2}$$

Therefore, $u \to 1$ corresponds to the boundary of the bulk, which is asymptotically AdS_5. The other limit $u \to 0$ is the origin of bulk spacetime $z \to z_\Lambda$. The scale of confinement is $\Lambda = 1/z_\Lambda$; in the following we will set $z_\Lambda = 1$, so all dimensionful quantities are given in units of Λ.

5.2.1 Scalar solutions

Although there are several scalar fields in the supergravity action, their action involves coupling the background metric to the scalar field that will subsequently make the analysis more involved. This will affect the spectrum of normal modes and moreover introduce cubic couplings [68, 69]. These technical complications may obscure the physics we are primarily interested in, which is how the geometric realization of confinement affects to the scattering amplitudes. Thus, in order to facilitate the analysis and the comparison with other models, we will study a family of probe scalar fields, decoupled from the background scalar and with a quartic potential as presented in the general analysis.

A scalar operator of dimension $\Delta = 2 + \nu$ is dual to a scalar field of mass

$$m^2L^2 = \Delta(\Delta - 4) = \nu^2 - 4. \tag{5.2.3}$$

The linearized equation of motion for the scalar field is

$$\phi'' + \frac{2-u}{1-u}\phi' + \left(\frac{M^2}{4(1-u)} - (\nu^2 - 4)\frac{u}{4(1-u)^2} \right)\phi = 0. \tag{5.2.4}$$

Regular solutions are given in terms of hypergeometric functions

$$\phi_M(u) = (1-u)^{\frac{2+\nu}{2}} {}_2F_1\left(1+\frac{\nu}{2}-\frac{1}{2}\sqrt{\nu^2+M^2-4}, 1+\frac{\nu}{2}+\frac{1}{2}\sqrt{\nu^2+M^2-4}; 2; u\right). \tag{5.2.5}$$

The bulk-to-boundary propagator $K_\nu(M;u)$ is proportional to (5.2.5), normalized to have the right asymptotic behavior

$$K_M(u) \underset{u\to 1}{\longrightarrow} 1\cdot(1-u)^{\frac{2-\nu}{2}} = z^{2-\nu}. \tag{5.2.6}$$

Then,

$$K_M(u) = \frac{\Gamma(1-\nu)}{\pi\csc(\pi\nu)}\Gamma\left(1+\frac{\nu}{2}-\frac{1}{2}\sqrt{\nu^2+M^2-4}\right)\Gamma\left(1+\frac{\nu}{2}+\frac{1}{2}\sqrt{\nu^2+M^2-4}\right)\phi_M(u). \tag{5.2.7}$$

Poles in the bulk-to-boundary propagator correspond to the spectrum of normalizable modes. In this case poles appear when the argument of the middle Γ function in (5.2.7) is a non-positive integer

$$1+\frac{\nu}{2}-\frac{1}{2}\sqrt{\nu^2+M^2-4} = -n, \quad n = 0,1,2,\ldots. \tag{5.2.8}$$

This gives the following mass spectrum

$$M_n^2 = 4(n^2+(2+\nu)n+2+\nu). \tag{5.2.9}$$

At these values the scalar solutions (5.2.5) become

$$\phi_{M_n}(u) = (1-u)^{\frac{2+\nu}{2}} {}_2F_1(-n, n+\nu+2; 2; u). \tag{5.2.10}$$

The normal modes are $\varphi_n = \alpha_n\phi_{M_n}(u)$, where

$$\alpha_n = \sqrt{2(n+1)(\nu+n+1)(\nu+2n+2)}. \tag{5.2.11}$$

They form an orthonormal basis respect to the weight $\rho(u) = u/(2(1-u)^2)$

$$\int_0^1 du\,\rho(u)\varphi_n(u)\varphi_m(u) = \delta_{nm}. \tag{5.2.12}$$

The computation is straightforward and we have relegated the details in Appendix A. The bulk-to-bulk propagator is determined via (4.1.16).

5.2.2 Scattering for low mass states

Using the values (5.2.9), the leading pole in the bulk-to-boundary propagator (5.2.7) is of the form (4.1.37), with

$$c_n = \frac{4(2+\nu+2n)(-1)n}{n!\alpha_n}\frac{\Gamma(1-\nu)}{\pi\csc(\pi\nu)}\Gamma(2+\nu+n). \tag{5.2.13}$$

Therefore, the residue of the two-point function (4.1.42) is

$$Z_n = 2\nu\mathcal{N}\,c_n\alpha_n\,{}_2F_1(-n, n+\nu+2; 2; 1). \tag{5.2.14}$$

For the lowest mass $n = 0$, we have that

$$\alpha_0 = \sqrt{2(\nu+1)(\nu+2)}, \quad c_0 = 4\nu(\nu+1)(\nu+2)/\alpha_0, \tag{5.2.15}$$

and the residue is

$$Z_0 = 8\mathcal{N}\nu^2(\nu+1)(\nu+2). \tag{5.2.16}$$

The residue of the leading pole in the four-point function is given by (4.1.43),

$$\mathcal{Z}_{n_1;n_2,n_3,n_4} = -2\nu v_4\mathcal{N}\,c_{n_2}c_{n_3}c_{n_4}\kappa_{n_1,n_2,n_3,n_4}\,\alpha_{n_1}\,{}_2F_1(-n_1, n_1+\nu+2; 2; 1), \tag{5.2.17}$$

where the overlap is

$$\kappa_{n_1,n_2,n_3,n_4} = \int_0^1 du\, \frac{u^2}{2(1-u)^3} \prod_{i=1}^{4} \varphi_{n_i}(u). \tag{5.2.18}$$

If all the scatterers have equal masses M_0, then this simplifies to

$$\kappa_{0,0,0,0} = \int_0^1 du\, \frac{u^2}{2(1-u)^3}\varphi_0^4(u) = \frac{\alpha_0^4}{2}\int_0^1 du\, u^2(1-u)^{1+2\nu} = \frac{\alpha_0^4}{4(\nu+1)(\nu+2)(2\nu+3)}. \tag{5.2.19}$$

Then,

$$\mathcal{Z}_{0;0,0,0} = -64\mathcal{N}\, v_4\, \frac{\nu^4(\nu+1)^3(\nu+2)^3}{2\nu+3}. \tag{5.2.20}$$

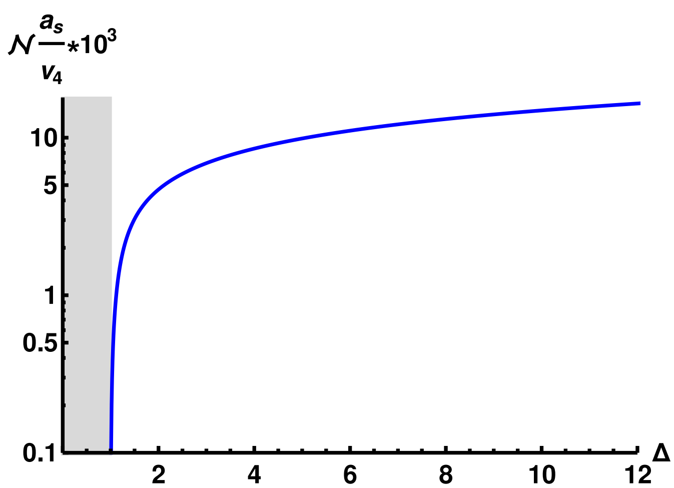
Let us proceed with our calculation to extract the scattering amplitudes for equal mass lowest modes. For this it is enough to use the formulas (4.1.46) and (4.1.45) by plugging in the values for the residues (5.2.16) and (5.2.20). We therefore find the amplitude

$$\mathcal{M}_{0;0,0,0} = -\frac{v_4}{\mathcal{N}}\frac{(\nu+1)(\nu+2)}{2\nu+3}, \tag{5.2.21}$$

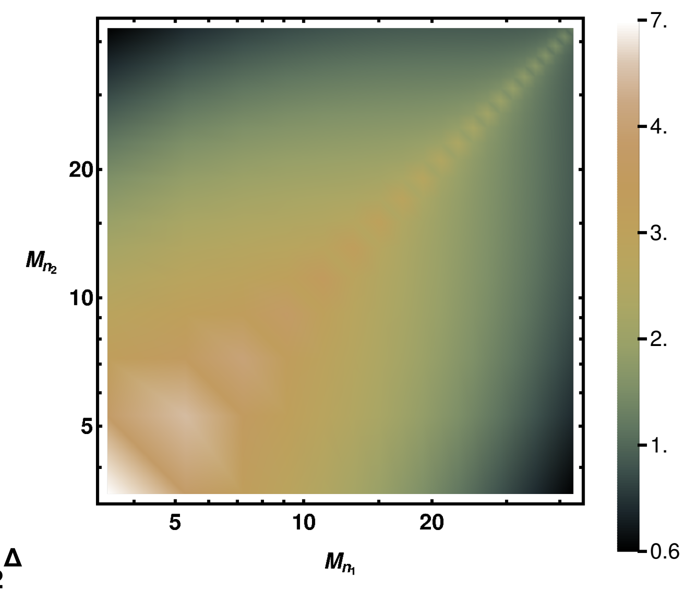
and the scattering length

$$a_s = \frac{v_4}{\mathcal{N}}\frac{(\nu+1)(\nu+2)^{1/2}}{32\pi(2\nu+3)}. \tag{5.2.22}$$

As shown in Appendix A, this expression is also valid for $\Delta < \frac{d}{2}$, extrapolating it to values $-1 < \nu < 0$. We have represented the result for the scattering length in Fig. 5.2.1a. We have also computed the scattering length for $\nu = 1$ and scatterers of different mass, the results are represented in Fig. 5.2.1b.



(a) Rescaled scattering length in units of the confinement scale. The horizontal axis is the conformal dimension of the scalar operator that creates the particles involved in the scattering, in this case the scattering is for the particles of lowest mass at each value of the conformal dimension. Notice the log-linear scale.



(b) Density plot of the scattering length $10^3 \times \mathcal{N}a_s/v_4$ in units of the confinement scale. The vertical and horizontal axes scale logarithmically and indicate the masses of the particles involved in the scattering, in units of the confinement scale, for states created by a scalar operator of fixed conformal dimension $\Delta = 3$ ($\nu = 1$).

Figure 5.2.1: Scattering length in $\mathcal{N} = 1^*$ SYM

5.3 Scattering in non-supersymmetric theories

Having discussed scattering in a concrete confining background geometry, it is interesting to raise the question of how universal the results are and, in particular, how sensitive the scattering length is quantitatively to different conformal symmetry breaking mechanisms so here we will compute again the scattering lenght but in the model presented in (3.2.2)

5.3.1 Normalizable solutions for probe scalar fields

A scalar operator of dimension $\Delta = \frac{5}{2} + \nu$ is dual to a scalar field of mass

$$m^2L^2 = \Delta(\Delta - 5) = \nu^2 - \frac{25}{4}. \tag{5.3.1}$$

Note that Δ is the conformal dimension in the five-dimensional CFT. In the effective four-dimensional field theory the effective dimension of the dual operator will in general be different. Let us proceed with computing the spectrum in the following. The spectra of fluctuations in AdS_6 soliton geometries has been worked out previously in [70–72] as well as in [73, 74], where interesting extensions are investigated. However, as in the previous section, we will focus on probe scalars and their spectra for arbitrary conformal dimension as represented in (5.3.1).

The linearized equation of motion for the scalar field is

$$\phi''(z) + \left(\frac{f'}{f} - \frac{4}{z}\right)\phi'(z) + \left(\frac{M^2}{f} - \frac{\nu^2 - \frac{25}{4}}{z^2 f}\right)\phi(z) = 0. \tag{5.3.2}$$

The Sturm-Liouville form of this equation gives a weight $\rho(z) = 1/z^4$. We do not know an analytic solution to this equation, so we will resort to a numerical calculation to compute the scattering amplitude and the scattering length. Close to the AdS_6 boundary, the scalar field has a leading order behavior $\phi \sim z^{\frac{5}{2}-\nu}$. We will factor out this dependence so that the scalar solution goes to a constant at the boundary. Defining $\phi = z^{\frac{5}{2}-\nu}\chi$ the equation of motion becomes

$$\chi'' + \left(\frac{f'}{f} + \frac{1-2\nu}{z}\right)\chi'(z) + \left(\frac{M^2}{f} + \left(\frac{5}{2} - \nu\right)\frac{f'}{zf} + \frac{\nu^2 - \frac{25}{4}}{z^2}\left(1 - \frac{1}{f}\right)\right)\chi(z) = 0. \tag{5.3.3}$$

As before, we will fix $z_\Lambda = 1$ and express all quantities in units of Λ. We then impose regularity at the origin $z = 1$ and do numerical shooting to the boundary. The details about the numerical calculation can be found in Appendix B.

For a given mass M, if the value of the solution χ_M at the boundary is non-zero $\chi_M(0) \neq 0$, the bulk-to-boundary propagator can be defined as

$$K_M(z) = z^{\frac{5}{2}-\nu}\frac{\chi_M(z)}{\chi_M(0)}. \tag{5.3.4}$$

The spectrum of normal modes is determined by the values M_n for which $\chi_{M_n}(0) = 0$, which can be found numerically; see Appendix B. We have also derived an analytic estimate for the masses using a WKB approximation; see Appendix C

$$M_n^2 = \frac{\pi^2}{\xi^2}(n+1)(n+\nu) + O(n0), \quad n \geq 0, \tag{5.3.5}$$

where $\xi \approx 1.25$. For GPPZ we obtain that the difference between the exact and WKB result is ν- and n- independent $M^2_{\text{exact}} - M^2_{WKB} = 4$. Comparison between the WKB values and numerical values in the AdS_6 soliton show also very good agreement, especially as the value of ν is increased. The spectrum for the first few modes and different values of ν is represented in Fig. 5.3.1.

The fully normalizable solutions are given by

$$\varphi_n(z) \equiv \alpha_n\phi_n(z) = \alpha_n\, z^{\frac{5}{2}-\nu}\chi_{M_n}(z), \tag{5.3.6}$$

where the coefficients α_n are chosen in such a way that the normal modes have unit norm, forming a complete orthonormal set

$$\int_0^1 \frac{dz}{z^4}\varphi_n(z)\varphi_m(z) = \delta_{nm}, \quad \alpha_n = \left(\int_0^1 \frac{dz}{z^4}\varphi_n(z)^2\right)^{-1/2}. \tag{5.3.7}$$

Some numerical values of the normalization coefficients are given in Appendix B. The bulk-to-bulk propagator is determined by this basis as in (4.1.16).

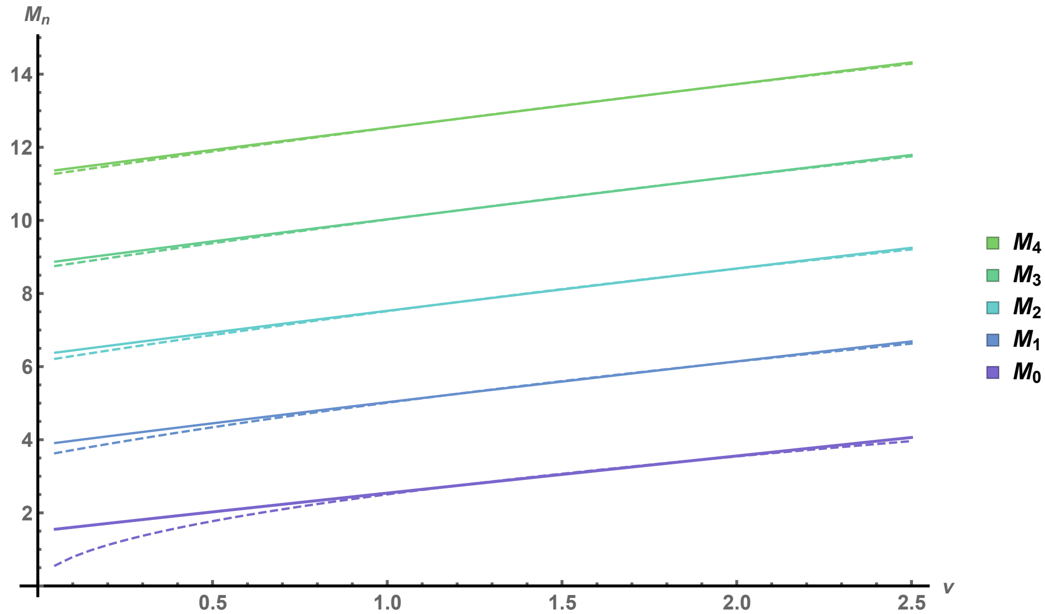


Figure 5.3.1: Masses of the lower modes in units of the confinement scale. The solid curves correspond to the numerical results while the dashed curve stem from the WKB approximation obtained in Appendix C.

5.3.2 Scattering for low mass states

Using the values given in Table B.0.1, the leading pole in the bulk-to-boundary propagator (5.3.4) is of the form (4.1.37). In order to compute the residue numerically it is convenient to first evaluate numerically the limit[1]

$$k_n = \frac{1}{(2\nu)!} \lim_{M \to M_n} (M_n^2 - M_2) \times \frac{\partial_z^{2\nu} \chi_M(z)}{\chi_M(0)}\Big|_{z=0}. \tag{5.3.8}$$

Then, the coefficient relating the residue to the normal mode is $c_n = k_n/\alpha_n$, and the residue of the two-point function of the dual operator is determined by

$$Z_n = 2\nu \mathcal{N} k_n. \tag{5.3.9}$$

Some numerical results for k_n are presented in Tables B.0.1 and B.0.2 in Appendix B.

The residue of the leading pole in the four-point function is given by (4.1.43), evaluating the limit it becomes

$$\mathcal{Z}_{n_1;n_2,n_3,n_4} = -2\nu v_4 \mathcal{N} c_{n_2} c_{n_3} c_{n_4} \alpha_{n1} k_{n1} \kappa_{n_1,n_2,n_3,n_4}, \tag{5.3.10}$$

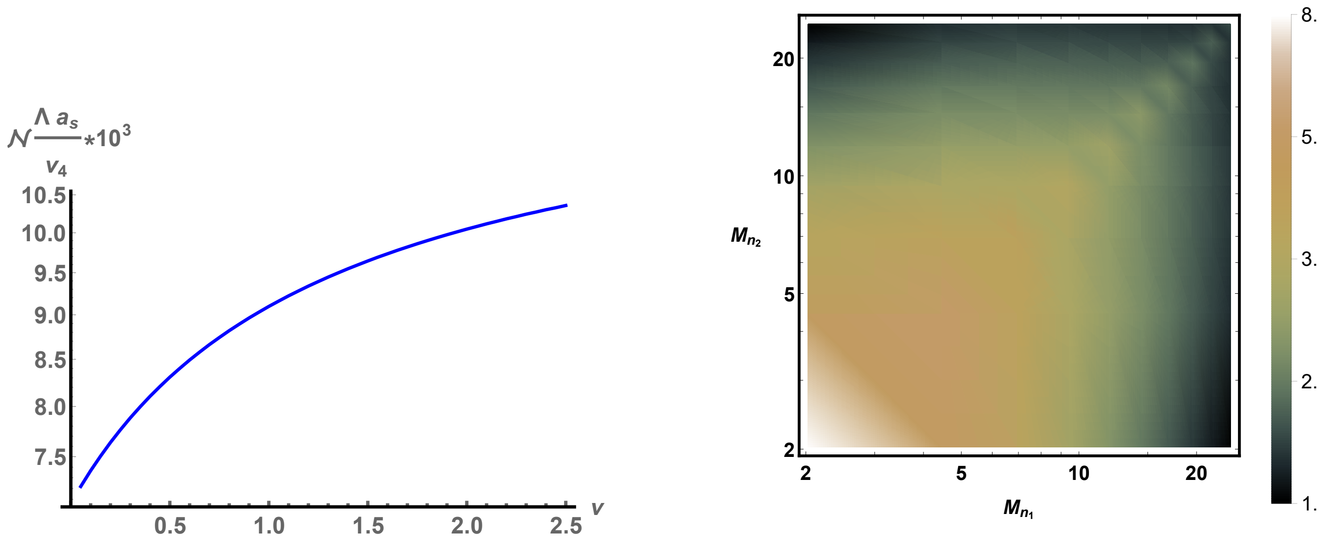
where the overlap κ is defined as

$$\kappa_{n_1,n_2,n_3,n_4} = \int_0^1 \frac{dz}{z^6} \prod_{i=1}^{4} \varphi_{n_i}(z). \tag{5.3.11}$$

We list some numerical values of the overlap in Appendix B.

We proceed to compute the scattering length and the scattering amplitude for scatterers of equal and lowest mass. For this it is enough to use the formulas (4.1.46) and (4.1.45) plugging the values we have found for the residues (5.3.9) and (5.3.10). The results for the lowest mass and different values of the conformal dimensions are represented in Figure 5.3.2a, while the results for scattering with different masses and fixed conformal dimensions can be found in Fig. 5.3.2b.

[1] We explain how to generalize this calculation when ν is not a half-integer in Appendix B.



(a) Rescaled scattering length in units of the confinement scale. The horizontal axis is ν, that up to a constant shift is the conformal dimension of the scalar operator that creates the particles involved in the scattering. In this case the scattering is for the particles of lowest mass at each value of the conformal dimension.

(b) Density plot of the scattering length $10^3 \times \mathcal{N} a_s/v_4$ in units of the confinement scale. The vertical and horizontal axes scale logarithmically and indicate the masses of the particles involved in the scattering, in units of the confinement scale, for states created by a scalar operator of fixed conformal dimension (corresponding to $\nu = 1/2$).

Figure 5.3.2: Scattering length in the non-supersymmetric holographic mode

5.4 Discussion

We have presented a method to compute the scattering length of the effective interaction between color singlet states (glueballs or hadrons) in a large-N strongly coupled gauge theory with a holographic dual. We have focused on the states created by a scalar operator of fixed dimension Δ and performed explicitly the calculation in various models. It is, in principle, straightforward to extend our analysis to operators of different spins or to consider the scattering process between particles created by operators of different dimensions.

First we illustrated the power of our method by finding the main contribution to the scattering length of a contact interaction in the two-to-two elastic scattering of spin-0 particles. We considered the hard wall model [13], consisting of a AdS_5 geometry with a sharp cutoff by imposing Dirichlet boundary conditions on the bulk fields. The position of the cutoff was identified as the scale of confinement, hence determining the masses of the particles in the field theory. As a model it is quite crude, and one might wonder whether the scattering amplitudes are very sensitive to the way the confinement scale is introduced.

We have found that the condition of having a finite amplitude puts a lower bound on the dimensions of the scalar operator. The constraint stems from the properties of the solution in the asymptotic region, so we expect it to be universal for any model that is asymptotically AdS. We have argued that the bound is above the unitarity bound for theories in $d < 2K/(K-2)$ spacetime dimensions if there is a polynomial term of order K in the scalar potential of the dual theory. Note that for the case we have worked out explicitly, $K = 4$, the bound is above the value of scalar operators in some CFTs. For instance, in the $d = 3$ Ising model $\Delta_\sigma \simeq 0.518$ (see, *e.g.*, [75]), which is below 3/4 bound that we found in the holographic model. The same holds for the $d = 2$ Ising model, where $\Delta_\sigma = 1/8$, while the holographic bound is 1/4.

It is interesting to ask why the bound we are proposing is stronger than expected? It might be that the bound only applies to a restricted set of theories with holographic duals, in the large-N limit, or with a discrete spectrum of massive states. It would be especially interesting if the last option was true, as it would imply that a theory with an operator of low enough dimension cannot have a discrete gapped spectrum, going beyond conformal bootstrap bounds (see, *e.g.*, [76] for a review). Note that theories with a holographic dual that do not have a fixed point in the UV, *i.e.* with an asymptotic geometry different from AdS, might also avoid the bound. However, this will signal UV physics different from that of an ordinary field theory in d dimensions.

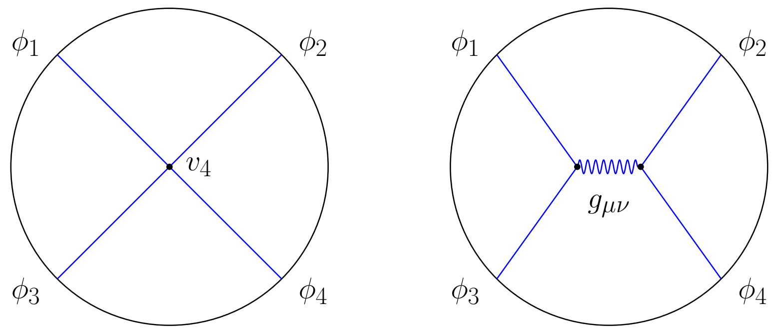


Figure 5.4.1: Witten diagrams corresponding to the contribution by the quartic interaction that we have computed (Left) and the contribution due to one graviton exchange (Right).

As our purpose was to point out the possibility of computing this observable in holography and to expose the method, we have restricted for simplicity to interactions that only involve the dual scalar field. Our calculation can be understood pictorially as a Witten diagram with four scalar legs joining at a single point in the bulk. We should mention that the calculation of four-point functions of scalar operators through Witten diagrams and other techniques in AdS/CFT has a long history, starting with [77, 78]. However, previous works have largely focused on conformal theories and using a representation in position space, or Mellin transforms that take advantage of conformal symmetry [79–82]. In some cases, but still restricted to conformal theories, correlators in momentum space have also been studied; for recent works, see [83–85].

It should be noted that the scalar field is coupled to the metric and maybe other fields as well, so there are more contributions to the effective interaction of scalar states and therefore to the scattering length. In particular, there should be a universal contribution corresponding to the exchange of one graviton between the two scalars, as in Fig. 5.4.1. It seems quite likely that the graviton exchange will produce a tensor contribution to the effective potential, analogous to the one observed in the nucleon-nucleon potential. It would be quite interesting to derive the effective potential for fermion fields and compare with fits of the nucleon-nucleon potential to experiments. Exchange diagrams would also introduce additional momentum dependence in the scattering amplitude, whereas the contact interaction in the effective potential we have studied reduces to a delta function in space.

Following this last line of thought, a direct comparison with QCD would be more consequential for holographic models that are designed to approximate it as closely as possible, such as the IHQCD or V-QCD models [?, 12]. It should be noted that baryons usually do not have a simple description in the holographic model as fields in the gravity action, although it is possible to avoid this issue by considering different types of large-N limits [?, ?, ?, ?].

Our results for the $\mathcal{N} = 1^*$ SYM [14] are represented in Fig. 5.2.1a, corresponding to the scattering length for particles of the lowest masses as a function of the conformal dimension of the dual operator, and in Fig. 5.2.1b, corresponding to the scattering length for fixed conformal dimension and different masses of the particles involved in the scattering process. The same quantities for the non-supersymmetric model [?] are represented in Fig. 5.3.2a and in Fig. 5.3.2b. They are qualitatively quite similar. There is a smooth increase of the scattering length with the conformal dimension along similar-looking curves. The scattering length is larger for scatterers of equal masses and decreases when any of the masses is increased. The conclusion is that the scattering length for this type of contribution is largely insensitive to the physics that produces the mass gap.[2] For the hard wall model one can observe a similar behavior for the scattering length as a function of the masses in Fig 5.1.2. The dependence with the conformal dimension, see Fig 5.1.3, is also quite close, although in the hard wall model it is not monotonically increasing, suggesting that the hard wall model may miss some of the relevant physics for large-dimension operators. A direct comparison of the scattering lengths in each model can be seen in Fig. 5.4.2. It is, however, important to notice that while the functional dependences are very similar between the models of the present work, the AdS_6 soliton geometry always results in larger scattering lengths than in GPPZ and the quantitative match between the models is only within $\mathcal{O}(1)$.

[2] From a technical point of view the similarity probably originates in the solutions to the Sturm-Liouville problem in each geometry being similar.

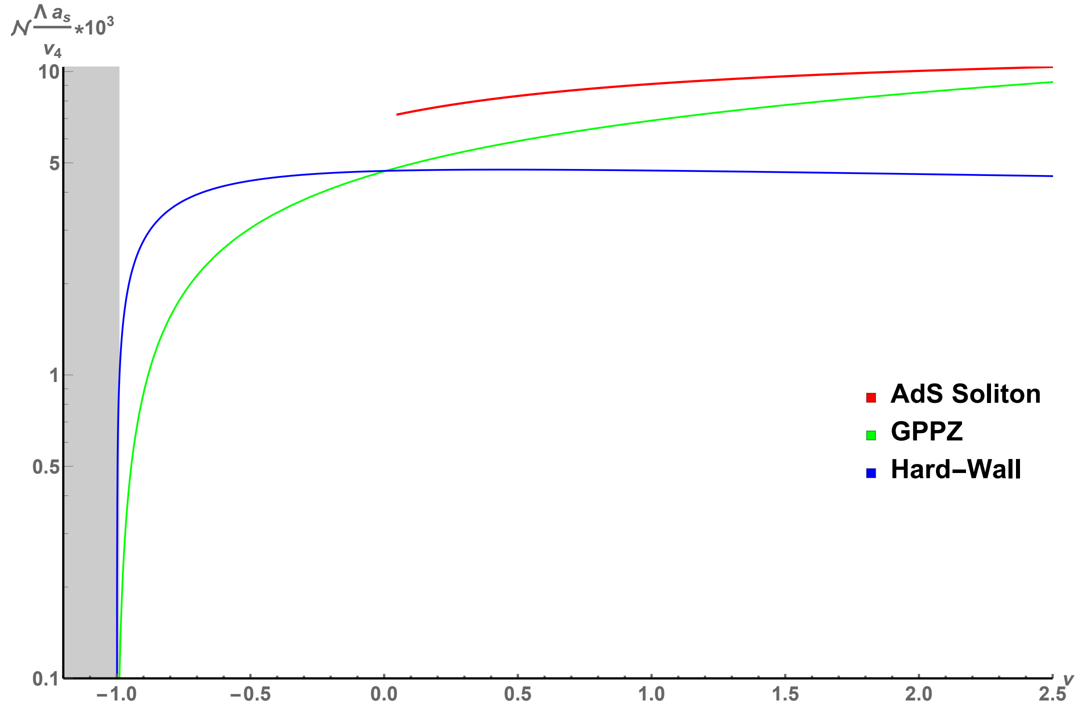


Figure 5.4.2: Scattering length for the duals to the AdS_6 soliton, GPPZ geometry, and hard wall, in units of the confinement scale as a function the dimension of the dual operator $\nu = \Delta - \frac{d}{2}$. $\nu = -1$ corresponds to the unitarity bound, where the scattering length vanishes in all cases. The scattering length is roughly of the same order of magnitude and shows the same increasing trend with ν for the AdS_6 soliton and GPPZ model, while the hard wall model starts decreasing at larger values of ν.

The dependence on the mass can be partially understood. This is because the scattering amplitude is proportional to the overlap between the modes in the gravity dual, which is bound to decrease as the masses become more apart, and the modes will have support on different regions. In field theory this would imply that particles of very different masses have weaker contact interactions. Although we do not have a clear cut explanation, a possible interpretation is that the particles with very different Compton wavelengths are less likely to scatter: imagine the one with smaller wavelength as a particle-like object of the size of its Compton wavelength and the other as a wave.

As a function of the conformal dimension, we observe in the $\mathcal{N} = 1^*$ SYM and hard wall models that the scattering length vanishes when the unitarity bound is saturated, conforming to expectations, as this point corresponds to free particles. It is then natural that the scattering becomes stronger as the dimension increases above the unitarity bound. For the AdS_6 soliton we did not reach the unitarity bound but indeed we observe that the scattering length increases with the conformal dimension. The overall sign of the scattering length depends on whether the interaction is repulsive (positive) or attractive (negative). In the plots, this sign is determined by the factor v_4 that defines the contact interaction in the gravity dual.

Chapter 6

Holographic calculation of the pion scattering amplitude

After computing the scattering lenght and spectrum of masses of three different models with confining duals we compute the pion scattering amplitude in the Witten-Sakai-Sugimoto model and extract from two-pion scattering amplitude, which we compute directly in the holographic dual through tree-level Witten diagrams, the pion decay constant and coefficients of fourth derivative terms in the chiral Lagrangian for massless quarks.

The observables that are relevant in the construction of a holographic model are the LECs. These constants enter in the effective QCD action and determine the interactions among hadrons and, most importantly, their values can be inferred from experiments. ChPT provides a systematic approach to characterize the LECs based on the approximate flavor symmetry of the microscopic QCD Lagrangian and the spontaneous breaking of this symmetry by a chiral condensate in the QCD vacuum. A holographic model that aspires to quantitatively counterfeit QCD observables in the confined phase should therefore be able to reproduce the chiral effective action with values of the LECs that match the experimental observations.

Being able to extract the LECs from the holographic model is fundamental. An early proposal on how to construct the effective action was within the Witten-Sakai-Sugimoto (WSS) model [11, 15, 16] by unveiling the action in the gravity side for modes of the fields dual to mesons. Later works in other models followed a similar approach [13, 86–99]. We follow a different path, we extract the LECs from the low-energy scattering amplitude of pions, that we compute directly from Witten diagrams in the gravity dual following the method developed in [17, 18]. To be definite, we focus on the WSS model with two flavors of massless quarks and find that the coefficients originating from pion self-interactions in the gravity dual were misidentified.

The holographic dictionary instructs to map gauge-invariant operators to fields in the gravity dual. The pion should be understood as a mode produced by the axial current operator. We can thus obtain the pion scattering amplitudes from axial current correlators via an LSZ reduction formula where we have to identify the massless poles appearing in the correlators. The pion propagator will be determined by the two-point function of the axial current and the $2 \to 2$ scattering amplitude henceforth by the four-point function.

From (6.2.13), the expectation value (vev) of the axial current can be computed from the asymptotic values of the canonical momentum conjugate to the gauge fields

$$\langle J^{\mu}_{5\,a} \rangle = \left(\lim_{Z\to\infty} + \lim_{Z\to -\infty} \right) \pi^{\mu}_{a} \ . \tag{6.0.1}$$

We can extract higher order correlators from the vev by taking variations with respect to an external axial gauge field

$$\langle J^{\mu\,a}_{5}(x) J^{\mu_1\,a_1}_{5}(y_1) \cdots J^{\mu_n\,a_n}_{5}(y_n) \rangle = (-i)^n \prod_{i=1}^{n} \frac{\delta}{\delta A^{a_i}_{5\,\mu_i}(y_i)} \langle J^{\mu}_{5\,a}(x) \rangle \ , \tag{6.0.2}$$

where the external gauge fields are identified as the asymptotic values of the gauge fields (6.2.1). Taking into acount that

$$\int d^4x A^{a}_{5\,\mu}(x) J^{\mu\,a}_{5}(x) = \int \frac{d^4q}{(2\pi)^4} A^{a}_{5\,\mu}(-q) J^{\mu\,a}_{5}(q) \ , \tag{6.0.3}$$

the analogous formula in momentum space reads

$$\langle J_5^{\mu\, a}(p) J_5^{\mu_1\, a_1}(q_1) \cdots J_5^{\mu_n\, a_n}(q_n) \rangle = (-i)^n \prod_{i=1}^{n} \frac{\delta}{\delta A^{a_i}_{5\, \mu_i}(-q_i)} \langle J^{\mu}_{5\, a}(p) \rangle \ . \tag{6.0.4}$$

6.1 The holographic model

In this section we are going to present the explicit setup used within the WSS model to derive the pion scattering amplitude at strong coupling.

In the following we will take $N_f = 2$ so the Abelian and non-Abelian components of the gauge field on the D8-branes are split according to

$$A_M = a_M \frac{\mathbb{I}_2}{2} + A^a_M \frac{\sigma^a}{2} \ . \tag{6.1.1}$$

We will denote the Abelian field strength as $f_{\mu\nu} = \partial_\mu a_\nu - \partial_\nu a_\mu$. For the non-Abelian part, we will distinguish between the linear and non-linear part of the field strength as follows

$$F^a_{\mu\nu} = f^a_{\mu\nu} + \epsilon^{abc} A^b_\mu A^c_\nu \ , \quad f^a_{\mu\nu} = \partial_\mu A^a_\nu - \partial_\nu A^a_\mu \ . \tag{6.1.2}$$

For an antipodal D8-brane embedding it is convenient to introduce a change of coordinates $(\tau, U) \to (y, z)$ with

$$y = r\cos\theta \ , \ z = r\sin\theta \ , \ \theta = M_{\rm KK}\tau \ , \ U^3 = U^3_{\rm KK} + U_{\rm KK} r^2 \ . \tag{6.1.3}$$

The D8-brane will be localized at $y = 0$ and extended along the z direction. The induced metric on the D8-brane in this case reads

$$ds_8^2 = \frac{4}{9}\left(\frac{R}{U_z}\right)^{3/2} \frac{U_{\rm KK}}{U_z} dz^2 + \left(\frac{U_z}{R}\right)^{3/2} \eta_{\mu\nu} dx^\mu dx^\nu + R^{3/2} U_z^{1/2} d\Omega_4^2 \ , \tag{6.1.4}$$

where

$$U_z^3 = U^3_{\rm KK} + U_{\rm KK} z^2 \ . \tag{6.1.5}$$

It will be convenient to introduce dimensionless coordinates $Z = z/U_{\rm KK}$, $x^\mu = \hat{x}^\mu / M_{\rm KK}$ and define

$$u(Z) = (1 + Z^2)^{1/3} \ . \tag{6.1.6}$$

Then, the induced metric in the dimensionless quantities is $G_{MN} = L^2 \hat{G}_{MN}$, where

$$d\hat{s}_8^2 = u(Z)^{1/2} \left[\frac{dZ^2}{u(Z)^3} + u(Z)\eta_{\mu\nu} d\hat{x}^\mu d\hat{x}^\nu + \frac{9}{4} d\Omega_4^2\right] \ , \ L^2 = \frac{4}{9} R^{3/2} U^{1/2}_{\rm KK} = \frac{4}{27}\lambda_{\rm YM} l_s^2 \ . \tag{6.1.7}$$

The DBI action can be split according to the factors of the field strength, expanding up to $O(F^4)$,

$$S_{\rm DBI} = -\tilde{T}_8 \int d^4\hat{x} dZ\, u(Z)^2 \left(2 + \left(\frac{\pi\alpha'}{L^2}\right)^2 \mathcal{L}^{[2]}_{\rm DBI} + \left(\frac{\pi\alpha'}{L^2}\right)^4 \mathcal{L}^{[4]}_{\rm DBI} + \dots\right) \ , \tag{6.1.8}$$

where

$$\tilde{T}_8 = \frac{3}{2g_s} V_4 R^3 L^6 T_8 = \frac{N_c \lambda^3_{\rm YM}}{3^9 \pi^5} \ , \ \frac{\pi\alpha'}{L^2} = \frac{27\pi}{4\lambda_{\rm YM}} \ . \tag{6.1.9}$$

The Lagrangian densities, in terms of dimensionless gauge fields, coordinates, and the metric read

$$\begin{aligned} \mathcal{L}^{[2]}_{\rm DBI} &= \frac{1}{2} F^a_{MN} F^{a\, MN} + \frac{1}{2} f_{MN} f^{MN} \\ \mathcal{L}^{[4]}_{\rm DBI} &= -\frac{1}{6}\Big[F^{a\, MA} F^a_{NA} F^b_{MB} F^{b\, NB} + \frac{1}{2} F^{a\, MN} F^{a\, AB} F^b_{MB} F^b_{AN} \\ &\qquad -\frac{1}{8}\left(F^{a\, MN} F^a_{MN} F^{b\, AB} F^b_{AB} + 2F^{a\, MN} F^{a\, AB} F^b_{MN} F^b_{AB}\right)\Big] + O(f^2 F^2, f^4) \ . \end{aligned} \tag{6.1.10}$$

In the quartic action (6.1.10) we omitted writing the explicit form of the Abelian and mixed terms since at tree level they do not contribute to a quartic interaction with non-Abelian fields in the external legs.

The Wess-Zumino action is non-vanishing due to the background four-form

$$\frac{1}{2\pi}\int_{S^4} F_4 = N_c \tag{6.1.11}$$

in such a way that the Wess-Zumino action is proportional to a five-dimensional Chern-Simons term for the gauge fields on the brane

$$S_{\rm WZ} = \frac{N_c}{24\pi^2}\int_{M^4\times\mathbb{R}} \omega_5(A) = \int d^4\hat{x}dZ\,\mathcal{L}_{\rm WZ}\ . \tag{6.1.12}$$

For purely non-Abelian $SU(2)$ fields, the Chern-Simons term vanishes. However, there are mixed terms between Abelian and non-Abelian components

$$\mathcal{L}_{\rm WZ} = \frac{N_c}{32\pi^2}\epsilon^{MNLPQ} a_M \partial_N A_L^a \partial_P A_Q^a + O(A^4 a)\ . \tag{6.1.13}$$

6.2 Chiral symmetry and pion mode

Let us discuss the realization of chiral symmetry and the existence of a massless mode corresponding to the pion. Considering only fields that are constant on the S^4, there are two boundary values that have to be specified for the gauge field on the D8-branes, each of them mapping to background values for the $U(N_f)_L$ and $U(N_f)_R$ gauge fields

$$\lim_{Z\to+\infty} A_\mu(x,Z) = L_\mu(x) = V_\mu(x) + A_{5\,\mu}(x)\ ,\ \lim_{Z\to-\infty} A_\mu(x,Z) = R_\mu(x) = V_\mu(x) - A_{5\,\mu}(x)\ , \tag{6.2.1}$$

where V_μ are the $U(N_f)_V$ (vector) and $A_{5\,\mu}$ the $U(N_f)_A$ (axial) gauge fields. The set of gauge transformations $U(x,Z)$ of the field on the D8-branes

$$A_M = U^{-1}A_M U - iU^{-1}\partial_M U\ , \tag{6.2.2}$$

generate gauge transformations of the background left- and right-handed gauge fields in the dual field theory

$$\lim_{Z\to+\infty} U(x,Z) = U_L(x)\ ,\ \lim_{Z\to-\infty} U(x,Z) = U_R(x)\ . \tag{6.2.3}$$

In the $A_Z = 0$ gauge the set of allowed bulk gauge transformations is reduced to Z-independent transformations $U_{L,R}(x)$, and the global subgroup are constant transformations. These global transformations are identified with the $U(N_f)_L \times U(N_f)_R$ flavor group of the dual field theory. Note that, when the sources for the flavor currents are turned off $\lim_{Z\to\pm\infty} A_\mu = 0$, the global transformations do not change the boundary values of the gauge field as expected. As we mentioned, chiral symmetry is spontaneously broken $U(N_f)_L \times U(N_f)_R \to U(N_f)_V$ in the dual field theory, so there should be a mode on the D8-brane that corresponds to massless pions. We will elucidate this below.

From the quadratic action in (6.1.10) we obtain the following set of linearized equations,

$$\begin{aligned}&\partial_Z\left[u(Z)^3 f^a_{Z\mu}\right] + \frac{1}{u(Z)}\eta^{\alpha\beta}\partial_\alpha f^a_{\beta\mu} = 0\\ &\eta^{\alpha\beta}\partial_\alpha f^a_{\beta Z} = 0\ ,\end{aligned} \tag{6.2.4}$$

where $f^a_{MN} = \partial_M A^a_N - \partial_N A^a_M$. We will split the gauge potential in transverse, longitudinal, radial, and gauge parts:

$$A^a_\mu = A^{\perp\,a}_\mu + \partial_\mu A^{\parallel\,a} + \partial_\mu C^a\ ,\ A^a_Z = B^a_Z + \partial_Z C^a\ ,\ \eta^{\mu\nu}\partial_\mu A^{\perp\,a}_\nu = 0\ . \tag{6.2.5}$$

The second equation in (6.2.4) imposes the conditions ($\partial^2 = \eta^{\alpha\beta}\partial_\alpha\partial_\beta$)

$$\partial^2 B^a_Z = \partial^2\partial_Z A^{\parallel\,a}\ . \tag{6.2.6}$$

Then, either $\partial^2 A^a_M = 0$, or B_Z and $A^{\parallel\,a}$ are pure gauge and can be absorbed in C^a. In the case when the first condition is true, corresponding to a massless mode, the first equation in (6.2.4) becomes

$$\partial_Z\left[u(Z)^3(\partial_Z A^a_\mu - \partial_\mu A^a_Z)\right] = 0\ . \tag{6.2.7}$$

The solutions are, up to gauge transformations $C^a(x,Z)$,

$$A^a_\mu(x,Z) = \hat{V}^a_\mu(x) + \hat{A}^a_{5\,\mu}(x)\frac{2}{\pi}\arctan(Z)\ ,\ A^a_Z(x,Z) = 2\varphi^a(x)\phi_0(Z) = \frac{2}{\pi}\frac{\varphi^a(x)}{1+Z^2}\ . \tag{6.2.8}$$

If we set $\hat{V}^a_\mu = \hat{A}_{5\,\mu} = 0$ the mode is normalizable, so there is a massless particle in the dual field theory, which will be identified with the pion. This in fact is an exact solution of the $O(F^2)$ action. The normalizable solution has the field strength

$$F^a_{\mu Z} = 2\partial_\mu \varphi^a \phi_0(Z)\ ,\ F^a_{\mu\nu} = 0\ . \tag{6.2.9}$$

Let us show now that indeed φ^a is the pion field up to an overall normalization. Following the usual prescription of the holographic dictionary, we first compute the canonical momentum conjugate to the gauge field

$$\pi^\mu_a = \frac{\delta S_{D8}}{\delta\left(\partial_Z A^a_\mu\right)}\ . \tag{6.2.10}$$

From (6.1.8),

$$\pi^\mu_a = -\tilde{T}_8 \left(\frac{\pi\alpha'}{L^2}\right)^2 u(Z)^2 \left[\Pi^{[2]\,\mu}_a + \left(\frac{\pi\alpha'}{L^2}\right)^2 \Pi^{[4]\,\mu}_a + \dots\right]\ , \tag{6.2.11}$$

where, using (6.1.10),

$$\begin{aligned} \Pi^{[2]\,\mu}_a &= 2F^{a\,Z\mu} \\ \Pi^{[4]\,\mu}_a &= \frac{1}{6}\left[4F^b_{L\nu}\left(F^{a\,\mu L}F^{b\,Z\nu} + F^{b\,\mu L}F^{a\,Z\nu}\right) + 4F^a_{L\nu}F^{b\,\mu L}F^{b\,Z\nu}\right. \\ &\qquad \left. + \left(F^{a\,Z\mu}F^b_{NL}F^{b\,NL} + 2F^{b\,Z\mu}F^{b\,NL}F^a_{NL}\right)\right]\ . \end{aligned} \tag{6.2.12}$$

The expectation values of the flavor currents $J^{a\,\mu}_L$ and $J^{a\,\mu}_R$ are obtained from the boundary values of the canonical momentum[1]

$$\langle J^{\mu\,a}_L\rangle = \lim_{Z\to+\infty} \pi^\mu_a\ ,\ \langle J^{\mu\,a}_R\rangle = -\lim_{Z\to-\infty} \pi^\mu_a\ . \tag{6.2.13}$$

To leading order in the field, and restoring units in the x^μ directions,

$$\langle J^{\mu\,a}_5\rangle = \langle J^{\mu\,a}_L\rangle - \langle J^{\mu\,a}_R\rangle \simeq f_\pi^2 \eta^{\mu\nu}\partial_\nu\varphi^a\ ,\ \langle J^{\mu\,a}_V\rangle = \langle J^{\mu\,a}_L\rangle + \langle J^{\mu\,a}_R\rangle \simeq 0\ , \tag{6.2.14}$$

where f_π^2 is given in (6.3.12). This shows that φ^a is proportional to the pion field, since the axial current is proportional to its gradient. Our next goal is to find the pion effective action. Usually this has been done by identifying φ^a as the pion field and using the D8-brane action integrated over the holographic radial coordinate Z as the effective action for the field φ^a. Within this general idea there are two different approaches, an *off-shell* approach where φ^a is taken to be an arbitrary function and an *on-shell* approach where it is a massless field $\partial^2\varphi^a = 0$. The off-shell approach is the one used originally in the WSS model [15, 16], and has been also employed in other phenomenological models [13, 87, 88, 90–92, 94, 96, 97], while the on-shell approach was introduced in [95] for the AdS/QCD model of [87].

6.3 Effective action for the pion field and vector mesons

For N_f flavors, the pion field is a $N_f \times N_f$ unitary matrix $U = e^{2i\Pi/f_\pi}$, with Π a Hermitean matrix. In addition, there is a tower of massive vector meson fields v^n_μ, $n = 1, 2, 3, \dots$. Higher n corresponds to higher mass. From the dual gravity point of view, n labels Kaluza-Klein modes in the holographic radial direction. The effective action (in the absence of sources) was found to be of the form

$$\begin{aligned} \mathcal{L} = &-\operatorname{tr}\left(\partial_\mu\Pi\partial^\mu\Pi\right) - \frac{1}{3f_\pi^2}\operatorname{tr}\left[\Pi, \partial_\mu\Pi\right]^2 + \frac{1}{2e_S^2 f_\pi^4}\operatorname{tr}\left[\partial_\mu\Pi, \partial_\nu\Pi\right]^2 \\ &+ \sum_n 2\operatorname{tr}\left(\partial_{[\mu}v_{\nu]}\right)^2 + m^2_{v^n}\left(v^n_\mu\right)^2 + \frac{2b_{v^n\pi\pi}}{f_\pi^2}\operatorname{tr}\left(\partial_{[\mu}v_{\nu]}\left[\partial^\mu\Pi, \partial^\nu\Pi\right]\right)\ , \end{aligned} \tag{6.3.1}$$

where $e_S^2 f_\pi^2 \simeq 0.51$. Focusing on the quartic terms in the pion field $O(\partial^2\Pi^4)$, the naïve LECs in this action[2] would be the same as for the Skyrme model [36–38]. For $N_f = 3$ the LECs satisfy the relations

$$L_2^{SU(3)} = 2L_1^{SU(3)}\ ,\ L_3^{SU(3)} = -3L_2^{SU(3)},\quad L_1^{SU(3)} = \frac{1}{32e_S^2}\ . \tag{6.3.2}$$

[1] The relative sign stems from the variation of the D8-brane on-shell action $\delta S_{on-shell} = \int d^4x \left(\left\langle J^{\mu\,a}_L\right\rangle \delta L^a_\mu + \left\langle J^{\mu\,a}_R\right\rangle \delta R^a_\mu\right)$, with the left current at the upper limit of the radial integration and the right current at the lower limit.

[2] Their precise definition can be found in Eqs. (2.3.23) and (2.3.22).

The first relation actually always holds in the large-N_c limit, so there are only two independent coefficients (this is true for any value of N_f). For $SU(2)$ the term proportional to L_3 can be recast as the term proportional to L_1, so that for the WSS model

$$L_1^{SU(2)} = L_1^{SU(3)} + \frac{1}{2}L_3^{SU(3)} = -L_2^{SU(3)} = -L_2^{SU(2)} . \tag{6.3.3}$$

However, the action (6.3.1) is not in the standard form, in particular, the last term capturing the coupling between the vector mesons and the pions is not the expected one from hidden local symmetry (HLS) considerations. In order to correct for this, one redefines the vector meson fields as follows

$$\hat{v}_\mu^n = v_\mu + \frac{b_{v^n\pi\pi}}{2f_\pi^2}\left[\Pi, \partial_\mu \Pi\right] . \tag{6.3.4}$$

After this redefinition the action becomes

$$\mathcal{L} = -\mathrm{tr}\left(\partial_\mu \Pi \partial^\mu \Pi\right) + \sum_n 2\mathrm{tr}\left(\partial_{[\mu}\hat{v}_{\nu]}\right)^2 + m_{\hat{v}^n}^2 (v_\mu^n)^2 - 2g_{v^n\pi\pi}\mathrm{tr}\left(\hat{v}_\mu^n\left[\Pi, \partial^\mu \Pi\right]\right) . \tag{6.3.5}$$

The basic idea of the off-shell approach is to expand the field in "Kaluza-Klein" modes of the holographic radial direction, which, excluding the massless mode, are of the form

$$A_\mu(x, Z) = \sum_n A_\mu^{(n)}(x)\psi_n(Z) \ , \ A_Z^{(n)} = 0 . \tag{6.3.6}$$

Here the functions ψ_n are eigenfunctions of the radial derivative part of the equations

$$u(Z)\partial_Z\left[u(Z)^3\partial_Z\psi_n(Z)\right] = -m_n^2\psi_n(Z) \ , \ \lim_{Z\to\pm\infty}\psi_n(Z) = 0 , \tag{6.3.7}$$

where m_n^2 determine the masses of mesons in the dual field theory. Then, from (6.2.4)

$$\partial^2 A_\mu^{(n)} - m_n^2 A_\mu^{(n)} = 0 \ , \ \eta^{\alpha\beta}\partial_\alpha A_\beta^{(n)} = 0 . \tag{6.3.8}$$

Introducing the Kaluza-Klein expansion, together with the massless solution back in the action (6.1.8) and integrating over the radial coordinate results in an action for the 4D fields $A_\mu^{(n)}$ and φ. This is to be interpreted as the effective action for the meson fields in the dual field theory, with interactions determined by non-quadratic terms. In this derivation the 4D fields are off-shell, *i.e.*, the equations of motion (6.3.8) are not imposed. In previous derivations only the $O(F^2)$ terms were kept, while higher α' corrections to the DBI action were neglected. This restricts the terms in the action to be at most quartic in the fields (for pions and vector mesons) and to have at most four derivatives in the field theory directions. Terms $O(F^4)$ can contribute at the same order of derivatives and fields, so they must be included if one is interested in computing the value of the LECs at finite 't Hooft coupling.

The off-shell Kaluza-Klein expansion is essentially the approach applied to the WSS construction in [15, 16] to derive the meson effective action. The action (6.3.1) was actually computed in a slightly different way. Allowing the boundary conditions of the gauge field to be fixed only up to boundary gauge transformations, it is possible to apply a gauge transformation to the solution (6.2.8) (with $\hat{V}_\mu = \hat{A}_{5\,\mu} = 0$) such that the radial component vanishes and the pion field φ^a is moved to the components of the gauge potential along the field theory directions. Beyond the linear approximation, this amounts to fixing $A_Z = 0$ and introducing instead a pure gauge configuration for the boundary gauge fields, that is taken to be

$$\hat{V}_\mu = -\frac{i}{2}\Sigma^{-1}\partial_\mu\Sigma \ , \ \hat{A}_{5\,\mu} = -\frac{i}{2}\Sigma^{-1}\partial_\mu\Sigma . \tag{6.3.9}$$

We then identify $\Sigma(x)$ as the $SU(N_f)$ matrix of the pions. In this case the field strengths are

$$F_{Z\mu} = -i\Sigma^{-1}\partial_\mu\Sigma\,\phi_0(Z) \ , \ F_{\mu\nu} = -i\left[\Sigma^{-1}\partial_\mu\Sigma, \Sigma^{-1}\partial_\nu\Sigma\right](\psi_0(Z) - 1)\psi_0(Z) \tag{6.3.10}$$

with

$$\psi_0(Z) = \frac{1}{2} + \frac{1}{\pi}\arctan(Z) , \tag{6.3.11}$$

and we have used $\psi_0'(Z) = \phi_0(Z)$. Plugging (6.3.10) back in the action (6.1.8) and keeping only $O(F^2)$ terms one finds (6.3.1) up to quartic order in the fields. The terms that involve only the pion have the form (2.3.23) with

$$\begin{aligned} f_\pi^2 &= 8M_{\mathrm{KK}}^2\frac{N_c\lambda_{\mathrm{YM}}}{4^2 3^3\pi^3}\int_{-\infty}^{\infty} dZ u^3\phi_0^2 = \frac{N_c\lambda_{\mathrm{YM}}}{54\pi^4}M_{\mathrm{KK}}^2 \\ L_2^{SU(2)} &= -L_1^{SU(2)} = 2\frac{N_c\lambda_{\mathrm{YM}}}{4^2 3^3\pi^3}\int_{-\infty}^{\infty} dZ\frac{(\psi_0 - 1)^2\psi_0^2}{u} = \frac{N_c\lambda_{\mathrm{YM}}}{6^3\pi^7}b \ , \ b \approx 15.25 . \end{aligned} \tag{6.3.12}$$

One should note that since the boundary values of the gauge potentials do not vanish, this actually corresponds to having a non-zero source proportional to the derivatives of the pion field, in such a way that the pion field enters as a "spurion". In order to demonstrate this we will introduce external gauge fields $\hat{\mathcal{V}}_\mu$, $\hat{\mathcal{A}}_{5\,\mu}$ coupled to the axial and vector currents by modifying the boundary conditions in (6.3.9)

$$\hat{V}_\mu = \Sigma^{-1}\hat{\mathcal{V}}_\mu\Sigma - \frac{i}{2}\Sigma^{-1}\partial_\mu\Sigma \ , \ \hat{A}_{5\,\mu} = \Sigma^{-1}\hat{\mathcal{A}}_{5\,\mu}\Sigma - \frac{i}{2}\Sigma^{-1}\partial_\mu\Sigma \ . \tag{6.3.13}$$

Then, a simultaneous transformation of the pion field and the boundary gauge fields leaves the boundary values of the bulk gauge field invariant.

Expanding up to quartic order in the fields, the shift (6.3.4) removes the source terms depending on the pion field from the effective action and terms quartic in the pion field go away, so the only contributions depending on the pion field left are quadratic or interaction terms involving vector mesons. That the spurion action is able to capture the LECs can be understood from the fact that external gauge fields should be dressed by the physical pion field (once massive vector mesons have been integrated out) in the same way as they are for the spurion in (6.3.13). However, even when the massive vector mesons have not been integrated out, there are terms at $O(F^4)$ in the action (6.1.8) not included in the original derivation [15] that contribute at the same order in fields (quartic) and derivatives (four), albeit they are relatively suppressed by a $1/\lambda_{\rm YM}^2$ factor.

Moving on to the on-shell approach, it has not really been applied to the WSS model, but to other AdS/QCD models with an IR cutoff in the holographic radial direction [95]. In this case the pion field is typically identified with the value of the gauge field at the cutoff, which together with a fixed asymptotic value at the boundary determines the solution for the gauge field. Then one proceeds in a similar way as in the off-shell derivation, evaluating the action on the solution and integrating over the radial direction to obtain the effective action. However, there are two main differences with the off-shell approach. The first one is that there is no expansion in Kaluza-Klein modes. Instead, solutions are found by fixing the value of the field at the cutoff, so even for the linearized equations they will typically consists of a superposition of the massless mode and the whole Kaluza-Klein tower. The second difference is that the full set of equations is solved, including equations with only field theory derivatives and non-linear terms. This can be done systematically using an expansion in derivatives and factors of the pion field, which is on-shell in this derivation (*i.e.*, terms proportional to the equations of motion of the pion field vanish).

It was also argued in [95], and shown for the AdS/QCD model introduced in [87], that the off-shell and on-shell derivations of the effective action should agree if the former is put on-shell which in the low momentum expansion requires integrating out all the massive vector bosons. In the WSS model one could in principle attempt a similar on-shell derivation contemplating the pion as the value of the field at an IR cutoff at $Z = 0$.

Our approach using scattering amplitudes have some similarities with the on-shell approach in that we will be solving the equations of motion in an expansion around the linearized solution, however, it will be the UV rather than the IR value of the gauge fields that will determine the expansion.

6.4 Expansion in a background axial gauge field

We will set $V_\mu^a = 0$ and $A_{5\,\mu}^a \sim O(\epsilon)$ with ϵ treated as a small parameter. The solutions for the gauge field on the D8-branes will be expanded in ϵ, that counts the number of factors of the source appearing in each term of the expansion

$$A_M = \epsilon A_M^{(1)} + \epsilon^2 A_M^{(2)} + \epsilon^3 A_M^{(3)} + \dots \ . \tag{6.4.1}$$

The $O(\epsilon)$ term contribution captures the two-point function of the current and the $O(\epsilon^3)$ contribution the four-point function. Our goal is to compute both in the following.

At each order we can arrange the equations of motion for the gauge field as follows

$$\begin{aligned} &\partial_Z\left[u(Z)^3 f_{Z\mu}^{(n)\,a}\right] + \frac{1}{u(Z)}\eta^{\alpha\beta}\partial_\alpha f_{\beta\mu}^{(n)\,a} = I_\mu^{(n)\,a} \\ &u(Z)^3\eta^{\alpha\beta}\partial_\alpha f_{\beta Z}^{(n)\,a} = I_Z^{(n)\,a} \ , \end{aligned} \tag{6.4.2}$$

where $I_\mu^{(1)\,a} = I_Z^{(1)\,a} = 0$. Following the holographic dictionary, we should impose boundary conditions such that

$$\lim_{Z\to\pm\infty} A_\mu^{(1)\,a}(x,Z) = \pm\hat{A}_{5\,\mu}^a(x) \ , \ \lim_{Z\to\pm\infty} A_\mu^{(n)\,a}(x,Z) = 0 \ , n > 1 \ . \tag{6.4.3}$$

We will work with Fourier transforms of the fields, and split the gauge potentials as before in transverse, longitudinal, radial, and gauge parts

$$A_\mu^{(n)\,a}(q) = A_\mu^{(n)\mu\perp a} + iq_\mu A^{(n)\,\|\,a} + iq_\mu C^{(n)\,a} \ , \ A_Z^{(n)\,a}(q) = B_Z^{(n)\,a} + \partial_Z C^{(n)\,a} \ . \tag{6.4.4}$$

The equations of motion for each component of the gauge field are

$$\begin{aligned}
&\partial_Z \left[u(Z)^3 \partial_Z A_\mu^{(n)\perp a} \right] - \frac{q^2}{u(Z)} A_\mu^{(n)\,\perp\,a} = I_\mu^{(n)\,\perp\,a} \\
&q^2 \partial_Z \left[u(Z)^3 \left(B_Z^{(n)\,a} - \partial_Z A^{(n)\|\,a} \right) \right] = iq^\alpha I_\alpha^{(n)\,a} \\
&q^2 u(Z)^3 \left(\partial_Z A^{(n)\,\|\,a} - B_Z^{(n)\,a} \right) = I_Z^{(n)\,a} \ .
\end{aligned} \tag{6.4.5}$$

Here and in the following indices will be raised and lowered with the flat Minkowski metric. At orders $n > 1$ we have to solve inhomogeneous equations. Since $iq^\alpha I_\alpha^{(n)\,a} + \partial_Z I_Z^{(n)\,a} = 0$ we can split

$$I_\mu^{(n)\,a} = iq_\mu \partial_Z J^{(n)\,a} + I_\mu^{(n)\,\perp\,a} \ , \ I_Z^{(n)\,a} = q^2 J^{(n)\,a} \ . \tag{6.4.6}$$

Then, the inhomogeneous solution for the longitudinal and radial parts is, up to gauge transformations,

$$A^{(n)\,\|\,a} = 0, \quad B_Z^{(n)\,a} = -\frac{1}{u(Z)^3} J^{(n)\,a} \ . \tag{6.4.7}$$

The field strengths are

$$\begin{aligned}
f_{Z\mu}^{(n)\,a} &= \partial_Z A_\mu^{(n)\,\perp\,a} + \frac{iq_\mu}{q^2} \frac{I_Z^{(n)\,a}}{u^3} \\
f_{\mu\nu}^{(n)\,a} &= i \left(q_\mu A_\nu^{(n)\,\perp\,a} - q_\nu A_\mu^{(n)\,\perp\,a} \right) \ .
\end{aligned} \tag{6.4.8}$$

For the transverse component of the gauge field, the inhomogeneous solution can be formally found using a Green's function

$$A^{(n)\,\perp\,a}(Z) = \int_{-\infty}^{\infty} dZ_1 \, G(Z, Z_1; q^2) I_\mu^{(n)\,\perp\,a}(Z_1) \ . \tag{6.4.9}$$

The Green's function is the solution to

$$\partial_Z \left[u(Z)^3 \partial_Z G(Z, Z_1; q^2) \right] - \frac{q^2}{u(Z)} G(Z, Z_1; q^2) = \delta(Z - Z_1) \tag{6.4.10}$$

with the boundary conditions

$$\lim_{Z \to \pm\infty} G(Z, Z_1; q^2) = 0 \ . \tag{6.4.11}$$

For general values of q^2 we have not been able to found a closed form analytic solution for the Green's function. However, for $q^2 = 0$ it takes a simple form

$$G(Z, Z_1; 0) = \pi \left\{ \begin{array}{lcl} \psi_0(Z)(\psi_0(Z_1) - 1) & , & Z < Z_1 \\ \psi_0(Z_1)(\psi_0(Z) - 1) & , & Z > Z_1 \end{array} \right. \ , \tag{6.4.12}$$

where $\psi_0(Z)$ was given in (6.3.11).[3] The asymptotic expansion is

$$G(Z, Z_1; 0) \underset{|Z| \to \infty}{\simeq} -\frac{1}{Z} \left\{ \begin{array}{lcl} \psi_0(Z_1) - 1 & , & Z \to -\infty \\ \psi_0(Z_1) & , & Z \to +\infty \end{array} \right. \ . \tag{6.4.15}$$

[3]Although not needed in our paper, one can obtain the finite momentum result using an analytic expansion of the Green's function around $q^2 = 0$:

$$G(Z, Z_1; q^2) = G^{(0)}(Z, Z_1) + q^2 G^{(1)}(Z, Z_1) + (q^2)^2 G^{(2)}(Z, Z_1) + \dots \ , \tag{6.4.13}$$

where $G^{(0)}(Z, Z_1) = G(Z, Z_1; 0)$ and where higher order terms in the expansion can be computed iteratively using

$$G^{(n)}(Z, Z_1) = \int_{-\infty}^{\infty} dZ_2 \, G(Z, Z_2; 0) \frac{G^{(n-1)}(Z_2, Z_1)}{u(Z_2)} \ . \tag{6.4.14}$$

6.5 Two-point function of the axial current

Before computing the scattering amplitude we need to compute the residue of the massless pole in the axial current two-point function. This determines the pion decay constant, that enters as well in the coefficients of the four-pion interaction, so it is necessary to know its value in order to compare with ChPT.

In order to compute the two-point function it is enough to find the solution for the D8-brane gauge field at $O(\epsilon)$, so the solutions to the linearized equations of motion suffice. Since we are interested in the massless pole, we can expand around $q^2 = 0$. Then, the solution is (6.2.8) plus small corrections

$$A_\mu^{(1)\,a} = \frac{2}{\pi}\arctan(Z)\hat{A}^a_{5\,\mu}(q) + O(q^2)\ ,\ A_Z^{(1)\,a} = -2\frac{iq^\alpha}{q^2}\hat{A}^a_{5\,\alpha}(q)\phi_0(Z) + O(q^2)\ , \tag{6.5.1}$$

Here we have taken into account (6.2.6) for $q^2 \neq 0$. The field strength at $O(\epsilon)$ is

$$f_{Z\mu}^{(1)\,a} = 2\phi_0(Z)\left(\delta_\mu^\nu - \frac{q_\mu q^\nu}{q^2}\right)\hat{A}^a_{5\,\nu}(q)\ . \tag{6.5.2}$$

The expectation value of the axial current is determined by the canonical momentum as in (6.0.1), and to $O(\epsilon)$ we only need the term originating from the $O(F^2)$ terms in the D8-brane action, $\Pi^{[2]}$ in (6.2.11). We find for the canonical momentum, using (6.1.9),

$$\pi_a^\mu \simeq -4\tilde{T}_8\left(\frac{\pi\alpha'}{L^2}\right)^2\left(\eta^{\mu\nu} - \frac{q^\mu q^\nu}{q^2}\right)\hat{A}^a_{5\,\nu}(q)u(Z)^3\phi_0(Z) = -\frac{N_c\lambda_{\rm YM}}{108\pi^4}\left(\eta^{\mu\nu} - \frac{q^\mu q^\nu}{q^2}\right)\hat{A}^a_{5\,\nu}(q)\ . \tag{6.5.3}$$

Restoring units, the expectation value of the axial current at this order reads

$$\langle J_5^{a\,\mu}(q)\rangle \simeq -\frac{N_c\lambda_{\rm YM}}{54\pi^4}M_{\rm KK}^2\left(\eta^{\mu\nu} - \frac{q^\mu q^\nu}{q^2}\right)A^a_{5\,\nu}(q) = -f_\pi^2\left(\eta^{\mu\nu} - \frac{q^\mu q^\nu}{q^2}\right)A^a_{5\,\nu}(q)\ . \tag{6.5.4}$$

This determines the susceptibilities of the axial current. Indeed, considering configurations constant in time,

$$\delta A^a_{5\,0}(q) = \delta\mu_5^a(2\pi)\delta(q_0)\ ,\ \delta\left\langle J_5^{a\,0}(q)\right\rangle = \delta\rho_5^a(2\pi)\delta(q_0)\ , \tag{6.5.5}$$

with μ_5^a the axial chemical potentials and ρ_5^a the axial charge densities, we obtain

$$\delta\rho_5^a \simeq f_\pi^2\delta^{ab}\delta\mu_5^b \quad\Rightarrow\quad \frac{\partial\rho_5^a}{\partial\mu_5^b} \simeq f_\pi^2\delta^{ab}\ . \tag{6.5.6}$$

The two-point function of the axial current is, at this order

$$\left\langle J_5^{a\,\mu}(-q)J_5^{b\,\nu}(q)\right\rangle \simeq if_\pi^2\left(\eta^{\mu\nu} - \frac{q^\mu q^\nu}{q^2}\right)\delta^{ab}\ . \tag{6.5.7}$$

The residue of the massless pole agrees with the expectation from the effective action. Note that the Ward identity for current conservation is satisfied

$$-iq_\mu\left\langle J_5^{a\,\mu}(-q)J_5^{b\,\nu}(q)\right\rangle = 0\ . \tag{6.5.8}$$

This mends the problem of the chiral effective theory Ward identity for the axial current that we mentioned in 2.3.3, one should had included the contact term proportional to the susceptibilities in (2.3.38).

6.6 Four-point function of the axial current

Before computing the scattering amplitude we need to find the leading pole contributions to the four-point function of the current, from the $O(\epsilon^3)$ terms in the expansion of the D8-brane gauge field. This boils down to the calculation of the $O(\epsilon^2)$ and $O(\epsilon^3)$ inhomogeneous terms in (6.4.5), which are then introduced in (6.4.7) and (6.4.9) to get the solution for the gauge field.

The inhomogeneous terms at $O(\epsilon^3)$ can be split in two types of contributions, corresponding to different Witten diagrams. One contribution, $I_v^{(3)}$, corresponds to a four-point vertex, a diagram where four gauge field propagators join at a single point in the bulk. The other contribution, $I_e^{(3)}$, takes the form of an exchange diagram between two three-point vertex, the field propagates in the bulk between two points and there are two other propagators

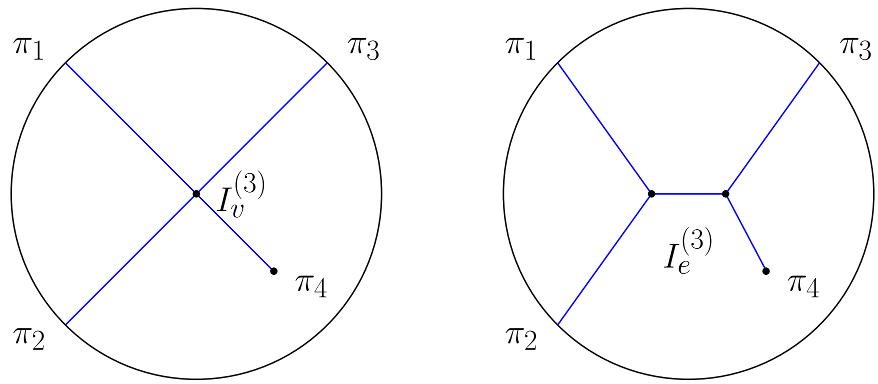


Figure 6.6.1: Four-point vertex (left) and exchange (right) Witten diagrams used to compute the one-point function of the axial current. Lines ending at the boundary represent bulk-to-boundary propagators and introduce factors proportional to the source, while lines connecting points in the interior represent bulk propagators.

at each point. We have sketched the two Witten diagrams in Fig. 6.6.1. For each type we are interested only in terms with massless poles that will be the only ones contributing to the pion scattering amplitude. The leading pole contribution to the scattering amplitude has a massless pole for each external leg. Within $O(\epsilon^3)$, the only leading pole contributions are those terms with two (three) massless pole factors in I_Z (I_μ). From (6.5.1) one can see that the only terms depending on the first order Abelian field strength that contain massless poles are those proportional to $A_Z^{(1)\,a}$ or $f_{Z\mu}^{(1)\,a}$. This fact will significantly reduce the number of terms we need to consider.

The details of the calculation have been relegated to the Appendix D. We identify three types of terms that can give a contribution to the four-point function:

- *Contributions from $O(F^4)$ terms in the canonical momentum.* The $O(\epsilon)$ solution to the gauge fields could give a direct contribution to the four-point function of the axial current through the $\sim F^3$ term in the canonical momentum, $\Pi^{[4]}$ in (6.2.11). However, it turns out that this does not give any contribution to the leading pole, so we can ignore it for the purpose of computing the scattering amplitude.

- *Contributions from $O(F^2)$ terms.* The non-Abelian terms in the field strength introduce a cubic coupling among gauge fields in the bulk. Joining two such vertices with a bulk gauge field propagator results in an exchange Witten diagram (cf. right panel of Fig. 6.6.1) that does give a contribution to the leading pole of the four-point function. The contribution is:

$$
\begin{aligned}
&\langle J_5^{\mu_1\,a_1}(p_1) J_5^{\mu_2\,a_2}(p_2) J_5^{\mu_3\,a_3}(p_3) J_5^{\mu_4\,a_4}(p_4)\rangle_e \\
&\simeq -2if_\pi^2 \left(\prod_{i=1}^4 \frac{p_i^{\mu_i}}{p_i^2}\right) \delta_{\sum_{i=1}^4 p_i} \Big[\Big\{ (p_1\cdot p_2) - \frac{4b}{\pi^3 M_{\rm KK}^2} \left[(p_1\cdot p_3)^2 + (p_1\cdot p_4)^2 - 2(p_1\cdot p_2)^2 \right] \Big\} \delta^{a_1a_2}\delta^{a_3a_4} \\
&+ (2\leftrightarrow 3) + (2\leftrightarrow 4) \Big] \,,
\end{aligned}
\tag{6.6.1}
$$

  where b is given in (6.3.12). Non-Abelian terms also introduce quartic couplings among gauge fields, but these do not contribute to the leading pole.

- *Contributions from $O(F^4)$ terms in the D8-brane action.* These terms introduce a quartic coupling between the Abelianized field strengths. This quartic coupling results in a vertex Witten diagram (cf. left panel of Fig. 6.6.1) that also contributes to the leading pole of the four-point function as follows:

$$
\begin{aligned}
&\langle J_5^{\mu_1\,a_1}(p_1) J_5^{\mu_2\,a_2}(p_2) J_5^{\mu_3\,a_3}(p_3) J_5^{\mu_4\,a_4}(p_4)\rangle \\
&\simeq i\frac{f_\pi^2}{M_{\rm KK}^2} \frac{3^5\Gamma\left(\frac{13}{6}\right)}{4\sqrt{\pi}\Gamma\left(\frac{8}{3}\right)\lambda_{\rm YM}^2} \left(\prod_{i=1}^4 \frac{p_i^{\mu_i}}{p_i^2}\right) \delta_{\sum_{i=1}^4 p_i} \left(\delta^{a_1a_2}\delta^{a_3a_4} + \delta^{a_1a_3}\delta^{a_2a_4} + \delta^{a_1a_4}\delta^{a_2a_3}\right) \\
&\qquad \times \left[(p_1\cdot p_2)(p_3\cdot p_4) + (2\leftrightarrow 3) + (2\leftrightarrow 4) \right] \,.
\end{aligned}
\tag{6.6.2}
$$

6.7 Scattering amplitude

We are now ready to extract the pion correlators and the scattering amplitude. Recall the expression in (2.3.39), where the current four-point function is given in terms of the pion four-point function. We can thus read off the pion four-point function from (6.6.1) and (6.6.2) by removing the $p_i^{\mu_i}$ factors and dividing by f_π^4. Furthermore, the scattering amplitude appears in the four-point function of the pions, recall (2.3.40), as the residue of the leading pole, once the delta function corresponding to the momentum conservation and an i factor have been factored out.

The resulting amplitude has the expected structure (2.3.34), with the exchange contribution (6.6.1) being

$$\begin{aligned} A_e(s,t,u) &= -\frac{2(p_1\cdot p_2)}{f_\pi^2}+\frac{8b}{\pi^3 f_\pi^2 M_{\rm KK}^2}\left[(p_1\cdot p_3)^2+(p_1\cdot p_4)^2-2(p_1\cdot p_2)^2\right] \\ &= \frac{s}{f_\pi^2}+\frac{2b}{\pi^3 f_\pi^2 M_{\rm KK}^2}\left[t^2+u^2-2s^2\right]. \end{aligned} \tag{6.7.1}$$

The first term in the exchange contribution agrees with the first term in Weinberg's amplitude (2.3.36). This therefore proves that the dual holographic derivation is indeed consistently capturing pion dynamics from spontaneous chiral symmetry breaking. The remaining terms take the form expected from vector meson exchange, giving a contribution to the $O(p^4)$ terms which agrees with the expressions in (6.3.12),

$$L_2^e=-L_1^e=\frac{N_c\lambda_{\rm YM}}{6^3\pi^7}b\ . \tag{6.7.2}$$

The vertex contribution (6.6.2) contains further terms that we can associate to pion self-interactions

$$\begin{aligned} A_v(s,t,u) &= \frac{1}{f_\pi^2 M_{KK}^2}\frac{3^5\Gamma\left(\frac{13}{6}\right)}{4\sqrt{\pi}\Gamma\left(\frac{8}{3}\right)\lambda_{YM}^2}\left[(p_1\cdot p_2)(p_3\cdot p_4)+(2\leftrightarrow 3)+(2\leftrightarrow 4)\right] \\ &= \frac{3^5\Gamma\left(\frac{13}{6}\right)}{2^4\sqrt{\pi}\Gamma\left(\frac{8}{3}\right)\lambda_{\rm YM}^2}\frac{s^2+t^2+u^2}{f_\pi^2 M_{KK}^2}. \end{aligned}$$

Comparing with Weinberg's amplitude (2.3.36), and using the expression for f_π^2 in (6.3.12), we can read off an additional contribution to the value of the coefficients in the pion effective action

$$L_2^v=2L_1^v=\frac{18N_c}{(4\pi)^4\lambda_{\rm YM}}\frac{\Gamma\left(\frac{13}{6}\right)}{\sqrt{\pi}\Gamma\left(\frac{8}{3}\right)}\ . \tag{6.7.3}$$

The numerical value of the constant factor with the gamma functions is approximately $\frac{\Gamma\left(\frac{13}{6}\right)}{\sqrt{\pi}\Gamma\left(\frac{8}{3}\right)}\approx 0.406$.

The full value of the LECs when the vector bosons are integrated out is the sum of the exchange and vertex contributions $\hat{L}_1=L_1^e+L_1^v$, $\hat{L}_2=L_2^e+L_2^v$. However, if the vector bosons are kept in the effective action, then only the vertex contributions produce non-vanishing LECs in the chiral Lagrangian $L_1=L_1^v$, $L_2=L_2^v$.

6.8 Integrating out vector bosons and Hidden Local Symmetry

One might find it convenient to integrate out only the vector mesons above some mass threshold, in particular keeping only the lightest vector mode. In the scattering calculation the contribution from each massive vector mode can be identified using an eigenfunction expansion of the bulk propagator (6.4.10) entering in the exchange diagram

$$G(Z,Z_1;q)=-\sum_{n=1}^{\infty}\frac{\psi_n(Z)\psi_n(Z_1)}{q^2+m_n^2}\ , \tag{6.8.1}$$

where the eigenfunctions satisfy the following equations

$$\begin{aligned} &\partial_Z\left(u(Z)^3\partial_Z\psi_n(Z)\right)+\frac{m_n^2}{u(Z)}\psi_n(Z)=0\ ,\ \lim_{Z\to\pm\infty}\psi_n(Z)=0 \\ &\int_{-\infty}^{\infty}dZ\frac{\psi_n(Z)\psi_m(Z)}{u(Z)}=\delta_{nm}\ . \end{aligned} \tag{6.8.2}$$

It should be noted that the $O(p^2)$ contribution to the pion scattering amplitude is obtained from the $q^2 = 0$ value of the bulk propagator. When expressed in this form, the value of f_π obtained from the amplitude is determined by the exchange of an infinite tower of massive modes. Notice that a bulk vertex diagram or a bulk exchange diagram might not correspond necessarily to vertex or exchange processes in the field theory dual, although it seems natural to do this identification. Under this assumption, though, the vector bosons corresponding to mass eigenstates would not couple to the pions as Weinberg's ρ meson discussed in Sec. 2.3.4 and the effective action of the pion field (before integrating out the massive modes) do not have the $O(\partial^2\pi^4)$ terms expected in the chiral Lagrangian. Then, the effective action written in terms of these fields would not comply with HLS invariance in any obvious way.

The LECs obtained from integrating out all massive vector modes except the lightest one would naïvely be obtained by replacing the full bulk propagator by the truncated sum

$$G_{n>1}(Z, Z_1; q) = -\sum_{n=2}^{\infty} \frac{\psi_n(Z)\psi_n(Z_1)}{q^2 + m_n^2} \,. \tag{6.8.3}$$

In this case the effective action of the pion would have the expected $O(\partial^2\pi^4)$ terms of the chiral Lagrangian, with an effective value of f_π determined by the modes that have been integrated out. However, the right value of f_π measured in the full scattering amplitude would be recovered only after considering the tree level exchange by the lightest vector meson.

The issue with HLS invariance of mass eigenstates has been pointed out for instance in [89, 93, 96], where an alternative basis of radial functions has been proposed to construct an explicitly HLS invariant action. It would be interesting if the scattering amplitude calculation could be connected to the HLS covariant formalism in some way.

6.9 Discussion

In this paper we presented a computation of the pion scattering amplitude for two massless flavors in the WSS model [11, 15, 16]. Our main result is given in (6.7.3). These would be the coefficients of pion self-interactions in the effective action before vector mesons have been integrated out. It differs from the result that was extracted from the effective action (6.3.1) appearing in the seminal work [15, 16], where the coefficients can be removed by a field redefinition of the vector meson fields that puts the action in the standard form (6.3.5). The main qualitative difference between the previously quoted results and the actual value of the LECs can be summarized in the following table

previous result	scattering result
$L_2^{SU(2)} = -L_1^{SU(2)}$	$L_2^{SU(2)} = 2L_1^{SU(2)}$
$L_i \sim N_c \lambda_{YM}$	$L_i \sim N_c \lambda_{YM}^{-1}$

These relations hold before massive vector mesons, corresponding to mass eigenstates in the gravity dual, have been integrated out. At lower energies, vector meson exchange contributions modify the LECs, and we find that the relation $\hat{L}_2^{SU(2)} \simeq -\hat{L}_1^{SU(2)}$, with the values that were originally proposed, holds up to the $1/\lambda_{\rm YM}^2$ corrections in the strong coupling limit that we have computed.[4] In other cases where the massive vector bosons do not correspond to the mass eigenstates of the holographic dual (see, *e.g.*, [89, 93, 96]), the identification of the L_i coefficients is not as straightforward, but there will be a correction to the coefficients of the effective action $\sim 1/\lambda_{\rm YM}^2$ such that the low momentum amplitude reproduces our results.

We will elaborate more on the implications below. In addition to the WSS model, the relation $L_2^{SU(2)} = -L_1^{SU(2)}$ was also obtained in bottom-up models that followed essentially the same approach to derive the effective action [86, 87, 91, 92, 94, 95], with slight deviations from the classical value when other fields are integrated out [88]. The relation between L_1 and L_2 could possibly be modified already at the classical level if the LECs are extracted from the pion scattering amplitude following the method used in our work, depending on whether they are completely determined by vector meson exchange or not. However, in the bottom-up models there is an additional bilinear field dual to the chiral condensate whose effect in the scattering amplitude should be studied more carefully, so we cannot extrapolate directly the WSS results to those models. We also expect that the value of the LECs in the effective action depend generically on which fields have been integrated out.

The result for two flavors determines already the four-derivative terms of the chiral effective action for an arbitrary number of flavors N_f. The reason is that, in the large-N_c limit, the only contributions to the chiral

[4] The modified LECs are measured always at energies much below the vector meson masses, where the vector meson exchange contribution can be approximated by an effective local pion self-interaction, as in (2.3.49).

effective action with $O(N_c)$ coefficients are single trace terms (see, *e.g.*, [25])

$$\mathcal{L}_{p^4} = L_3^{SU(N_f)}\mathrm{Tr}\left(\partial_\mu\Sigma^\dagger\partial^\mu\Sigma\partial_\nu\Sigma^\dagger\partial^\nu\Sigma\right) + \tilde{L}_3^{SU(N_f)}\mathrm{Tr}\left(\partial_\mu\Sigma^\dagger\partial_\nu\Sigma\partial^\mu\Sigma^\dagger\partial^\nu\Sigma\right) \ . \tag{6.9.1}$$

But for $N_f = 3$ the last term can be rewritten as combination of the term proportional to $L_3^{SU(N_f)}$ and double-trace terms, giving

$$L_1^{SU(3)} = \frac{1}{2}\tilde{L}_3^{SU(N_f)} \ , \ L_2^{SU(3)} = \tilde{L}_3^{SU(N_f)} \ , \ L_3^{SU(3)} = L_3^{SU(N_f)} - 2\tilde{L}_3^{SU(N_f)} \ . \tag{6.9.2}$$

Then, for $N_f = 2$, the two independent terms that are left have coefficients

$$L_1^{SU(2)} = \frac{L_3^{SU(N_f)} - \tilde{L}_3^{SU(N_f)}}{2} \ , \ L_2^{SU(2)} = \tilde{L}_3^{SU(N_f)} \ . \tag{6.9.3}$$

Therefore for an arbitrary number of flavors we claim that the result is simply $\tilde{L}_3^{SU(N_f)} = 2L_3^{SU(N_f)} = L_2^{SU(2)}$ with $L_2^{SU(2)}$ determined by the equation (6.7.3).

We have limited our calculation to the leading contributions to the pion scattering amplitude at low momentum. Higher order corrections in momentum can be computed systematically using the expressions for the bulk propagator introduced in (6.4.13) and (6.4.14). For momentum of the order or larger than the confinement scale the full form of the bulk propagator would be necessary. At very large momentum, though, stringy α' corrections become relevant, and the scattering amplitude might be approximated by integrating a flat space string amplitude along the holographic coordinate [67,98,100–103], an approach that has been applied in [104] to holographic duals of confining theories. However, as discussed there, in the high energy regime the WSS model is dual to a six-dimensional theory, so not suitable for a comparison with high-momentum scattering in QCD.

Chapter 7

Summary and general conclusions/ Outlook of the work

The AdS/CFT or holographic duality is one of the key tools of theoretical physics. Its appearance changed the understanding of gauge theories, gravitation and their relationship. Also it is a realization of the holographic principle which establishes that the degrees of freedom of a theory of gravity in a volume must be encoded in its boundary. The basic feature of the AdS/CFT correspondence is that it establishes an equivalence between a gravitational theory in a hyperbolic spacetime and a field theory living on the boundary of it. The duality emerges from a String Theory framework starting from the study of fundamental objects of the theory called D-branes. Those D-branes can be seen as hypersurfaces where open strings can end giving a quantum field theory living on the brane or as solitonic objects which are the solutions to the supergravity equations of motion, so those both perspectives gave birth to the holographic duality.

The formulation of this conjecture is a point in favor of string theories, which would therefore have phenomenological interest even in areas for which they were not designed, through the connection of the gravitational problem with the corresponding dual quantum field theory. In this way, string theories play the role of a tool to study quantum field theories, used as a framework to model certain physical systems. Therefore, the interest of the gauge/gravity duality is also phenomenological. The main advantage of this correspondence lies in the fact that it relates opposite regimes of both theories. When the gravitational problem in String Theory is tractable, corresponding to the limit of classical gravity, the regime in the field theory is the one of a large coupling constant in the planar limit. Given that there are generally no ways to study the theories with a large coupling constant, since the perturbation expansion in Feynman diagrams would not be applicable, the gauge/gravity duality would be the only theoretical tool with which to deal with strong coupling beyond numerical simulations. The main limitation of this duality is the fact that gravitational duals of the main phenomenologically interesting theories, such as QCD, is are available, but only the duals of theories with larger symmetry. However, the development and generalization of this correspondence has allowed to find the dual of a large number of field theories, some of them showing similar characteristics to the main theories of phenomenological interest. In this way, the holographic duality is not a precision tool that provides quantitatively good results, but rather qualitative results. It is expected to be able to capture characteristics and behaviors that are common to different theories and try to establish results that are universal. Therefore, one can understand the duality applied to a field theory that models with greater or lesser accuracy a given physical system as a tool that allows to make predictions in such theory through the corresponding calculation in the gravitational dual. In that sense, finding the duals to interesting theories and understanding the properties of these duals is of great relevance.

So the gauge/gravity duality has proven to be a very useful tool in the understanding of QFT's outside the perturbative regime. In particular, holography has been able to shed light not only on generic mechanisms of strongly coupled theories, but also on processes potentially occurring in experimental set-ups, such as the heavy ion collisions. Despite the success it is important to bear in mind that holography provides computational tools for some models rather than for QCD itself

In the following, we summarize the main findings of our original work illustrated in chapters 4, 5 and 6 and point out some possible future directions.

In chapters 4 and 5 we presented a method to compute the scattering length of the effective interaction between states created by a scalar operator of fixed dimension Δ in a large-N strongly coupled gauge theory with a holographic dual. Then we performed explicitly the calculation of the main contribution to the scattering length of a contact interaction in the two-to-two elastic scattering of spin-0 particle in three different models to show the power of our method. Those three models are dubbed confining in the sense that the spectrum should be discrete and have a mass gap. The models are: the hard wall model that consists of a AdS_5 geometry with a sharp cutoff, and other two confining models with different geometry, like the Witten-Yang-Mills model, [11] where it ends smoothly, or at a singularity, like in the dual to $\mathcal{N} = 1^*$ super Yang-Mills (SYM) theory, the GPPZ solution [14]. The purpose was to compute and compare the scattering lengths and address the question if these differences could be reflected in the scattering amplitudes and lead to qualitatively different results.

First we found, in the hard wall model with a quartic potential, a constraint on the scaling dimension of a scalar operator $\Delta > d/4$. For $d < 4$ this is more stringent than the unitarity constraint and may be applicable to an extended family of large-N theories with a discrete spectrum of massive states. We also argue that for scalar potentials with polynomial terms of order K, a constraint more restrictive than the unitarity bound will appear for $d < 2K/(K-2)$.

Then after the comparison of the results of the three models we found a similar behavior of the scattering length across different models that suggests that this could be a good observable for further study via holographic duality. The scattering lengths have similar functional dependences on the masses of the particles and on the conformal dimension of the operators that create them. Assuming these similarities hold more generally, they could be used to constrain the effective description of gapped strongly coupled theories beyond symmetry considerations. Available lattice data [105,106] can be used to check or calibrate holographic QCD models. In addition, comparison with the experimental data is possible and would be an interesting extension of the present work.

A reason why we find this kind of analysis useful, in particular in the context of applications of the duality to QCD or condensed matter systems, is that it can help to understand the effect of parameters in the gravitational action on the properties on the field theory side. The scattering length is one such physical observable and crucially depends on the coupling constants in the bulk scalar potential. One can therefore learn how varying the parameter values in the scalar potential determine the strength of the effective interactions and whether they are attractive or repulsive. This makes the gravitational action less of a 'black box' and helps in determining what kind of an action mimics the desired physics, given the field content. In addition, the scattering length calculations will be useful to further constrain the specific gravitational actions of models aspired to give reliable descriptions of QCD.

A less direct application, but one where holography duality really stands out, is to use the information from the confining phase to make predictions about the properties of the finite temperature and non-zero density deconfined phases. The gravitational action describes both the hadronic phase with confinement and a deconfined phase, where quarks and gluons should be the dynamical degrees of freedom, although they cannot be directly observed as they are not gauge-invariant fields. That this is possible at all stems from the fact that the same gravitational action describes both the hadronic phase with confinement and a deconfined phase, where quarks and gluons should be the dynamical degrees of freedom, although they cannot be directly observed as they are not gauge-invariant fields. So physical quantities, like the equation of state, critical temperatures of phase transitions, and transport coefficients in the deconfined phase depend on the same parameters of the gravitational action as the scattering lengths. For example, the coefficient of the quartic term in the bulk potential, v_4, is directly connected with the stiffness of the underlying equation of state for dense systems [107, 108]. Therefore, our analysis establishes a direct link between hadronic physics and the physics of quarks and gluons at low energies, something that seems out of reach using ordinary field theory methods.

Our analysis has been limited to a quartic contact interaction among scalars; see [17] for a discussion on generalizations to higher order polynomial Lagrangian. It would be interesting to extend our framework to include the coupling to the metric, as well as to introduce other fields of different spin, in particular gauge fields. In general, there will be a non-trivial momentum dependence that would also be interesting to investigate.
In chapter 6 we computed the pion decay constant and coefficients of fourth derivative terms in the chiral Lagrangian for massless quarks in the Witten-Sakai-Sugimoto model. Identification of the low energy coefficients in the chiral action is subtle as their values are shifted when the tower of massive vector bosons are integrated out. By a direct comparison with the existing standard procedure of constructing the chiral action with radial modes in the gravity dual, we explicitly show that there are finite 't Hooft coupling corrections that have been missed. This suggests that past derivations of effective actions from holographic models may have to be revisited and future derivations more carefully considered.

A further important extension of our work would be to study pion scattering with nonzero quark masses. Within the WSS model this requires considering non-antipodal embeddings and in the presence of an additional "tachyon" field [109] that is dual to a quark bilinear. In bottom-up models, such as the original AdS/QCD hard wall model [13], or V-QCD [12], the tachyon field is already included, and in addition the V-QCD model also has quartic terms in the action of the gauge fields dual to the flavor currents. The quark mass explicitly breaks the axial flavor symmetry and gives a mass to the pions. ChPT can still be used, but with additional terms in the effective action.

When quarks are massive, there is a finite scattering length determined by the pion mass that can be obtained from the pion scattering amplitude at leading order in low momentum (see, *e.g.*, [1]). Higher momentum corrections to the scattering amplitude can also be computed and used to constrain the LECs appearing in the chiral effective action. It would be interesting to extract the scattering amplitude in holographic models and compare with recent large-N lattice results [110].

The calculation of scattering amplitudes should be applied also to other setups. As our example on the WSS model shows, this might be a requisite step to correctly identify the low energy effective theory that captures the dynamics encoded by the holographic dual.

As a final remark and after presenting our work we can say that the method to compute scattering amplitudes developed in this thesis within the framework of holography is a useful tool to be applied in different models to obtain new insights from them and to give rise to some new interesting questions that need to be adressed in the future.

Appendix A

Normal modes and $\Delta < 2$ in the $\mathcal{N} = 1^*$ SYM dual

In this appendix we will fill in some gaps in the calculations presented in Sec. 5.2 in the main text.

A.1 Normal modes

The set of normalizable modes (5.2.10) maps to a set of Jacobi polynomials through the relation

$$P_n^{(\alpha,\beta)}(x) = \frac{(\alpha+1)^2}{n!}\, {}_2F_1\left(-n, 1+\alpha+\beta+n; \alpha+1; \frac{1}{2}(1-x)\right), \tag{A.1.1}$$

where $(\alpha+1)_n = \Gamma(\alpha+n+1)/\Gamma(\alpha+1)$ is the Pochhammer's symbol. Then,

$$\phi_{M_n}(u) = \frac{1}{n+1}(1-u)^{\frac{2+\nu}{2}} P_n^{(1,\nu)}(1-2u)\,. \tag{A.1.2}$$

Jacobi polynomials satisfy the orthogonality condition

$$\int_{-1}^{1} dx\,(1-x)(1+x)^\nu P_n^{(1,\nu)}(x) P_m^{(1,\nu)}(x) = \frac{2^{2+\nu}(n+1)}{(\nu+2n+2)(\nu+n+1)}\delta_{nm}. \tag{A.1.3}$$

Changing variables to $x = 1-2u$, $0 \le u \le 1$, the integral becomes $(\rho(u) = u/(2(1-u)^2))$

$$\int_{-1}^{1} dx\,(1-x)(1+x)^\nu P_n^{(1,\nu)}(x) P_m^{(1,\nu)}(x)$$

$$= 2^{2+\nu}\int_0^1 du\, u(1-u)^\nu P_n^{(1,\nu)}(1-2u) P_m^{(1,\nu)}(1-2u) = \frac{2^{3+\nu}(n+1)2}{\alpha_n^2}\int_0^1 du\,\rho(u)\varphi_n(u)\varphi_m(u). \tag{A.1.4}$$

Therefore, in order to satisfy the orthonormality condition, the coefficients α_n have to be fixed to

$$\alpha_n^2 = 2(n+1)(\nu+n+1)(\nu+2n+2). \tag{A.1.5}$$

A.2 Alternative quantization

In order to compute the two- and four-point functions using the approach in [17] we first compute the one-point function as a function of the sources using the regularized canonical momentum associated to the radial evolution of the scalar

$$\pi_\phi^R = \sqrt{-g}g^{zz}\partial_z\phi + \frac{\delta S_{c.t.}}{\delta\phi}, \tag{A.2.1}$$

where $S_{c.t.}$ is the counterterm action defined on a radial slice. For this discussion we will set the AdS radius to unity $L = 1$. The asymptotic expansion of the scalar field dual to an operator of dimension $\Delta_+ = \frac{d}{2} + \nu$ takes the form

$$\phi \underset{z \to 0}{\longrightarrow} z^{\frac{d}{2}-\nu} + C_\nu z^{\frac{d}{2}+\nu}. \tag{A.2.2}$$

In this case, the one-point function of the dual operator is

$$\mathcal{N}^{-1} \langle \mathcal{O} \rangle_{\Delta+} = \lim_{z \to 0} z^{\frac{d}{2}-\nu} \pi_\phi^R = 2\nu C_\nu. \tag{A.2.3}$$

For $0 < \nu < 1$ there is an alternative quantization where the dual operator has dimension $\Delta_- = \frac{d}{2} - \nu$. In this case the asymptotic expansion of the scalar is

$$\phi \underset{z \to 0}{\longrightarrow} z^{\frac{d}{2}+\nu} + D_\nu z^{\frac{d}{2}-\nu}. \tag{A.2.4}$$

The leading contribution to the regularized momentum close to the boundary is

$$\pi_\phi^R \underset{z \to 0}{\longrightarrow} 2\nu z^{\nu - \frac{d}{2}}, \tag{A.2.5}$$

and from this expression it follows that the expectation value of the dual operator is [111]

$$\mathcal{N}^{-1} \langle \mathcal{O} \rangle_{\Delta-} = \lim_{z \to 0} \left(-2\nu z^{\nu - \frac{d}{2}} \phi \right) = -2\nu D_\nu. \tag{A.2.6}$$

Since the equation of motion for the scalar (5.2.4) does not depend on the sign of ν, symmetry under a sign flip $\nu \to -\nu$ is expected. Indeed, using Euler's relation

$$ {}_2F_1(a, b; c; u) = (1-u)^{c-a-b} \, {}_2F_1(c-a, c-b; c; u) = (1-u)^{c-a-b} \, {}_2F_1(c-b, c-a; c; u), \tag{A.2.7}$$

the solution for the scalar can also be written as

$$\phi_M(u) = (1-u)^{\frac{2-\nu}{2}} \, {}_2F_1\left(1 - \frac{\nu}{2} + \frac{1}{2}\sqrt{\nu^2 + M^2 - 4}, 1 - \frac{\nu}{2} - \frac{1}{2}\sqrt{\nu^2 + M^2 - 4}; 2; u\right). \tag{A.2.8}$$

The bulk-to-boundary propagator is then the same as in (5.2.7) with the replacement $\nu \to -\nu$. This implies that the coefficient in (A.2.4) is $D_\nu = C_{-\nu}$ and then

$$\langle \mathcal{O} \rangle_{\Delta-} = \langle \mathcal{O} \rangle_{\Delta+} \Big|_{\nu \to -\nu}. \tag{A.2.9}$$

Since the two- and four-point functions and their residues are all derived from the one-point function, it follows that the scattering length and amplitude of states created by operators of dimension $\Delta_- = 2 - \nu$, $0 < \nu < 1$ can be obtained simply by continuing the results for $\Delta_+ = 2 + \nu$ to negative values $-1 < \nu < 0$.

Appendix B

Numerical solutions in the AdS_6 soliton

In this appendix we will detail some steps for interested reader to reproduce the numerical data. In order to find the solution we use numerical shooting, starting with a regular solution at the origin $z = 1$ that is expanded in a power series $\chi_t(z)$ with a fixed value $\chi_t(1) = 1$ at the tip of the AdS soliton

$$\chi_t(z) = 1 + \sum_{n=1}^{N_t} a_n (1-z)^n. \tag{B.0.1}$$

Here we will consider $N_t = 8$, which is large enough to have reliable results. The value of the coefficients for the first terms in the series are

$$a1 \quad = \quad \tfrac{1}{20}\left((5-2\nu)2 - 4M_2\right) \tag{B.0.2}$$

$$a2 \quad = \quad \tfrac{1}{1600}\left((5-2\nu)2(4(\nu-15)\nu+65) + 16M_4 - 8(4(\nu-10)\nu+75)M_2\right) \tag{B.0.3}$$

$$\begin{aligned} a3 \quad = \quad & \tfrac{1}{288000}\left((5-2\nu)2(8\nu(\nu(2(\nu-40)\nu+815)-2100)+7425) - 64M_6\right. \\ & \left. +16(12(\nu-15)\nu+455)M_4 - 4(24\nu(\nu(2(\nu-30)\nu+485)-1425)+30175)M_2\right). \end{aligned} \tag{B.0.4}$$

This series is used to give boundary conditions to the numerical solution $\chi_N(z)$ at $z = 1-\epsilon$, where we take $\epsilon = 10^{-6}$. So we fix

$$\chi_N(1-\epsilon) = \chi_t(1-\epsilon), \quad \chi'_N(1-\epsilon) = \chi'_t(1-\epsilon). \tag{B.0.5}$$

The numerical solution and its derivative up to order 2ν can be found, *e.g.*, using NDSolve in Mathematica with these boundary conditions, in the interval $z \in [\epsilon, 1-\epsilon]$. The boundary value is taken to be approximately the value of the numerical solution at $z = \epsilon$:

$$\chi_M(0) \approx \chi_N(\epsilon). \tag{B.0.6}$$

The numerical value of the derivative at the boundary is also approximated by the value of the numerical solution at $z = \epsilon$:

$$\partial_z^{2\nu}\chi(0) \approx \partial_z^{2\nu}\chi_N(\epsilon). \tag{B.0.7}$$

If ν is not half-integer, then we need to be more cautious. In this case, we first do a change of variables in the radial coordinate $z = u^{1/(2\nu)}$ and proceed with the same shooting method. The equation (5.3.8) is replaced by

$$k_n = \lim_{M\to M_n} (M_n^2 - M^2) \times \frac{\partial_u \chi_M(u)}{\chi_M(0)}\Big|_{u=0}. \tag{B.0.8}$$

Numerically, this is evaluated using

$$\partial_u \chi_M \chi(0) \approx \partial_u \chi_M(u = \epsilon^{2\nu}). \tag{B.0.9}$$

ν	1/2	1	3/2	2	5/2
M_0	2.02	2.54	3.05	3.56	4.06
M_{WKB}	1.77	2.51	3.07	3.54	3.96
α_0	1.36	1.57	1.76	1.93	2.09
k_0	6.06	20.2	47.2	94.2	171
$\kappa_{0,0,0,0}$	0.85	1.16	1.48	1.89	2.11

Table B.0.1: Numerical values obtained from shooting for lowest modes $n = 0$. Notice that the WKB approximation detailed in Appendix C gives very accurate results.

n	0	1	2	3	4	5	6	7	8	9
M_n	2.02	4.45	6.93	9.43	11.93	13.43	16.93	19.43	21.94	24.44
M_{WKB}	1.77	4.34	6.86	9.38	11.89	14.39	16.90	19.41	21.92	24.42
α_n	1.36	2.10	2.63	3.07	3.45	3.80	4.12	4.41	4.69	4.95
k_n	6.06	31.1	76.2	141.	226.	332.	457.	602.	767.	953.

Table B.0.2: Numerical values obtained from shooting for excited modes for fixed $\nu = 1/2$. Notice that the WKB approximation detailed in Appendix C gives very accurate results and becomes increasingly better for higher modes as expected.

What remains to be done in order to find the spectrum of normal modes is to solve for $\chi_M(0) = 0$ for fixed M. The value M is tuned until a zero is found, an analysis that can be effectively performed using Newton's method. To fix the coefficients α_n we evaluate numerically the following integrals

$$\alpha_n^{-2} \approx \int_{\epsilon}^{1-\epsilon} dz\, z^{1-2\nu} \chi_M(z)^2 \Big|_{M=M_n}. \tag{B.0.10}$$

Finally, we compute the overlaps in a similar way, by performing the following integral numerically

$$\frac{\kappa_{n_1,n_2,n_3,n_4}}{\alpha_{n_1}\alpha_{n_2}\alpha_{n_3}\alpha_{n_4}} \approx \int_{\epsilon}^{1-\epsilon} dz\, z^{4-4\nu} \prod_{i=1}^{4} \chi_N(z) \Big|_{M=M_{n_i}}. \tag{B.0.11}$$

We have collected some numerical values in Table B.0.1 for the lowest mode and in Table B.0.2 for excited states for fixed ν that we obtain from numerical calculation.

Appendix C

WKB approximation

Following the method developed in [112] we will compute the masses of normal modes using the WKB approximation. First, we will write the equations of motion in the following form

$$\partial_x(f(x)y(x)) + \left(M^2 h(x) + p(x)\right) y(x) = 0. \tag{C.0.1}$$

The value of the masses can be obtained from the behavior of these functions close to the origin $x \to 1$ and the boundary $x \to \infty$. Close to the origin $x \to 1$,

$$f \approx f_1(x-1)^{s_1}, \quad h \approx h^1(x-1)^{s_2}, \quad p \approx p_1(x-1)^{s_3}, \tag{C.0.2}$$

whereas close to the boundary $x \to \infty$,

$$f \approx f_2 x^{r_1}, h \approx h_2 x^{r_2}, p \approx p_2 x^{r_3}. \tag{C.0.3}$$

The WKB approximation of the masses is given by the formula

$$M_n^2 = \frac{\pi^2}{\xi^2}(n+1)\left(n + \frac{\alpha_2}{\alpha_1} + \frac{\beta_2}{\beta_1}\right) + O(n_0), n \geq 0. \tag{C.0.4}$$

The quantities appearing in this expression are, an integral setting the scale

$$\xi = \int_1^\infty dx \sqrt{\frac{h}{f}}, \tag{C.0.5}$$

and the following combination of exponents and coefficients of the asymptotic expansions

$$\alpha_1 = s_2 - s_1 + 2, \quad \beta_1 = r_1 - r_2 - 2, \tag{C.0.6}$$

and

$$\alpha_2 = \begin{cases} |s_1 - 1| & s_3 - s_1 + 2 \neq 0 \\ \sqrt{(s_1-1)^2 - 4\frac{p_1}{f_1}} & s_3 - s_1 + 2 = 0 \end{cases}, \qquad \beta_2 = \begin{cases} |r_1 - 1| & r_1 - r_3 - 2 \neq 0 \\ \sqrt{(r_1-1)^2 - 4\frac{p_2}{f_2}} & r_1 - r_3 - 2 = 0 \end{cases}. \tag{C.0.7}$$

C.1 $\mathcal{N} = 1^*$ SYM

Let us now specify to the first case studied in this thesis in Sec. 5.2. While we were able to obtain the mass spectrum analytically, it is interesting to compare how close the WKB approximation is with the exact values. By comparing with the exact result, we will have an estimate of the error in the approximation for cases where the exact result is unknown. Starting with (5.2.4), we make a change variables $u = 1 - 1/x$ as well as a field redefinition

$$\phi(x) = x^{-1/2}(x-1)^{-1}y(x). \tag{C.1.1}$$

Then, the equation for $y(x)$ has the form (C.0.1) with

$$f(x)=1, \quad h(x)=\frac{1}{4x^2(x-1)}, p(x)=\frac{x-5+\nu^2(1-x)}{4x^2(x-1)}. \tag{C.1.2}$$

The asymptotic form of these functions close to the origin $x\to 1$ is as in (C.0.2) with

$$f_1=1, \quad s_1=0; \quad h_1=\frac{1}{4}, \quad s_2=-1; \quad p_1=-1, \quad s_3=-1. \tag{C.1.3}$$

Close to the boundary $x\to\infty$ the expansion is as in (C.0.3) with

$$f_2=1, \quad r_1=0; \quad h_2=\frac{1}{4}, \quad r_2=-3; \quad p_2=\frac{1-\nu^2}{4}, \quad r_3=-2. \tag{C.1.4}$$

We will plug these expressions in (C.0.5), (C.0.6), and (C.0.7), taking into account that $r_1-r_3-2=0$. This yields

$$\xi=\frac{\pi}{2}, \quad \alpha_1=1, \quad \beta_1=1, \quad \alpha_2=1, \quad \beta_2=\nu. \tag{C.1.5}$$

Using these values in the mass formula (C.0.4), the WKB approximation gives

$$M_n^2=4(n^2+(2+\nu)n+(1+\nu))+O(n^0), \quad n\geq 0. \tag{C.1.6}$$

Comparing with the exact formula (5.2.9) we see that the $O(n^0)$ correction is a ν-independent term

$$(M_n^2)_{\text{exact}}-(M_n^2)_{\text{WKB}}=4. \tag{C.1.7}$$

C.2 Non-supersymmetric theory

Let us now proceed with comparing the mass spectra for the non-supersymmetric case studied numerically in Sec.5.3. In this case we do not have an analytic expression for the masses, but we can compare the results from the WKB approximation with those from the numerical computation. Starting with the equation of motion (5.3.2), we make a change of variables $z=1/x$ and simple rename the field $\phi(x)=y(x)$. This results in an equation of the form (C.0.1) with

$$f(x)=x^6-x, \quad h(x)=x^2, \quad p(x)=\left(\frac{25}{4}-\nu^2\right)x^4. \tag{C.2.1}$$

The asymptotic form of these functions close to the origin $x\to 1$ is as in (C.0.2) with

$$f_1=5, \quad s_1=1; \quad h_1=1, \quad s_2=0; \quad p_1=\frac{25}{4}-\nu^2, \quad s_3=0. \tag{C.2.2}$$

Close to the boundary $x\to\infty$ the expansion is as in (C.0.3) with

$$f_2=1, \quad r_1=6; \quad h_2=1, \quad r_2=2; \quad p_2=\frac{25}{4}-\nu^2, \quad r_3=4. \tag{C.2.3}$$

We will plug these expressions in (C.0.5), (C.0.6), and (C.0.7), taking into account that $r_1-r_3-2=0$. This yields

$$\xi=\frac{\sqrt{\pi}\Gamma\left(6/5\right)}{\Gamma\left(7/10\right)}, \quad \alpha_1=1, \quad \beta_1=2, \quad \alpha_2=0, \quad \beta_2=2\nu. \tag{C.2.4}$$

Using these values in the mass formula (C.0.4), the WKB approximation gives

$$M_n^2=\frac{\pi^2}{\xi^2}(n+1)(n+\nu)+O(n^0), \quad n\geq 0. \tag{C.2.5}$$

We note that the WKB approximation compares really well with the numerical values, even for lowest lying modes.

Appendix D

Calculation of the four point function

In this Appendix we spell out the details of the calculation of the axial current four-point function following the procedure described in the main text. As listed in section 6.6, there are three possible contributions we have to study: $O(F^4)$ terms in the action and the associated terms in the canonical momentum as well as $O(F^2)$ terms in the action.

D.1 Contributions from $O(F^4)$ terms in the canonical momentum: contact terms

The $O(\epsilon)$ solution to the gauge fields could give a direct contribution to the four-point function of the axial current through the $\sim F^3$ term in the canonical momentum, $\Pi^{[4]}$ in (6.2.11). The leading pole contribution involves just the $f^{(1)}_{Z\mu}$ components

$$\begin{aligned}\Pi^{[4]} &\simeq \frac{1}{6}\left[4f^b_{Z\nu}\left(f^{a\,\mu Z}f^{b\,Z\nu}+f^{b\,\mu Z}f^{a\,Z\nu}\right)+4f^a_{Z\nu}f^{b\,\mu Z}f^{b\,Z\nu}\right.\\ &\quad\left.+2\left(f^{a\,Z\mu}f^b_{Z\nu}f^{b\,Z\nu}+2f^{b\,Z\mu}f^{b\,Z\nu}f^a_{Z\nu}\right)\right]\\ &= -\frac{1}{3}\left[f^b_{Z\nu}f^{b\,Z\nu}f^{a\,Z\mu}+2f^b_{Z\nu}f^{a\,Z\nu}f^{b\,Z\mu}\right].\end{aligned} \tag{D.1.1}$$

However, this contribution to the canonical momentum vanishes when $|Z|\to\infty$, since from (6.5.2),

$$\pi^\mu_a \sim u^2\Pi^{[4]\,\mu}_a \sim u^4\phi_0(Z)^3 \underset{|Z|\to\infty}{\sim} \frac{1}{|Z|^{10/3}} \longrightarrow 0\,. \tag{D.1.2}$$

Therefore, there is no contribution to the expectation value of the axial current or the four-point function from these terms.

D.2 Contributions from $O(F^2)$ terms: exchange diagram

Let us discuss the contributions coming from terms quadratic in the field strength. The $O(A^4)$ non-Abelian quartic term in the action introduces $O(\epsilon^3)$ non-linear terms in the equations of motion, proportional to

$$\begin{aligned}I^{[2]\,a}_Z &\sim \epsilon^{abc}\epsilon^{dec}\eta^{\mu\nu}A^{(1)\,b}_\mu A^{(1)\,d}_\nu A^{(1)\,e}_Z\\ I^{[2]\,a}_\mu &\sim \epsilon^{abc}\epsilon^{dec}A^{(1)\,b}_Z A^{(1)\,d}_Z A^{(1)\,e}_\mu\ ,\ \epsilon^{abc}\epsilon^{dec}\eta^{\alpha\beta}A^{(1)\,b}_\alpha A^{(1)\,d}_\beta A^{(1)\,e}_\mu.\end{aligned} \tag{D.2.1}$$

The antisymmetry of the structure constants guarantees that there are no $\sim (A^{(1)}_Z)^3$ terms. But these would be the only terms contributing to the leading pole. As they are absent, we can neglect the contributions coming from the quartic terms in the gauge potentials.

The non-linear terms in the equations of the bulk gauge field originating from the $O(A^3)$ terms in the action are

$$\begin{aligned}I^{[2]\,a}_Z &= -u^3\epsilon^{abc}\left[\eta^{\alpha\beta}\partial_\alpha\left(A^b_\beta A^c_Z\right)+\eta^{\alpha\beta}A^b_\alpha F^c_{\beta Z}\right]\\ I^{[2]\,a}_\mu &= -\epsilon^{abc}\left[\partial_Z\left(u^3A^b_ZA^c_\mu\right)+u^3A^b_ZF^c_{Z\mu}+\frac{1}{u}\eta^{\alpha\beta}\partial_\alpha\left(A^b_\beta A^c_\mu\right)+\frac{1}{u}\eta^{\alpha\beta}A^b_\alpha F^c_{\beta\mu}\right]\ .\end{aligned} \tag{D.2.2}$$

Let us first identify the vertex contributions, they are those with three field factors

$$\begin{aligned} I^{[2]\,(3)\,a}_{Z\,v} &= -u^3\epsilon^{abc}\epsilon^{cde}\eta^{\alpha\beta}A^{(1)\,b}_{\alpha}A^{(1)\,d}_{\beta}A^{(1)\,e}_{Z} \\ I^{[2]\,(3)\,a}_{\mu\,v} &= -\epsilon^{abc}\epsilon^{cde}\left[u^3A^{(1)\,b}_{Z}A^{(1)\,d}_{Z}A^{(1)\,e}_{\mu}+\frac{1}{u}\eta^{\alpha\beta}A^{(1)\,b}_{\alpha}A^{(1)\,d}_{\beta}A^{(1)\,e}_{\mu}\right]\,. \end{aligned} \tag{D.2.3}$$

Since all the terms have less massless pole factors than those required to give a contribution to the leading pole term, we can neglect these contributions in the correlator.

Let us now move on to the exchange contributions, they are those with two field factors

$$I^{[2]\,(3)\,a}_{Z\,e} = -u^3\epsilon^{abc}\left[\eta^{\alpha\beta}\partial\alpha\left(A^b_\beta A^c_Z\right)+\eta^{\alpha\beta}A^b_\alpha f^c_{\beta Z}\right]^{(1),(2)} \tag{D.2.4}$$

$$I^{[2]\,(3)\,a}_{\mu\,e} = -\epsilon^{abc}\left[\partial_Z\left(u^3A^B_ZA^c_\mu\right)+u^3A^b_Zf^c_{Z\mu}+\frac{1}{u}\eta^{\alpha\beta}\partial_\alpha\left(A^b_\beta A^c_\mu\right)+\frac{1}{u}\eta^{\alpha\beta}A^b_\alpha f^c_{\beta\mu}\right]^{(1),(2)}\,. \tag{D.2.5}$$

The superscript notation means that from the two factors in each term in the brackets, one should be $O(\epsilon)$ and the other $O(\epsilon^2)$, and we must consider all possibilities. In order to compute this contribution we will need the $O(\epsilon^2)$ inhomogeneous terms as well

$$\begin{aligned} I^{[2]\,(2)\,a}_{Z} &= -u^3\epsilon^{abc}\left[\eta^{\alpha\beta}\partial_\alpha\left(A^{(1)\,b}_{\beta}A^{(1)\,c}_{Z}\right)+\eta^{\alpha\beta}A^{(1)\,b}_{\alpha}f^{(1)\,c}_{\beta Z}\right] \\ I^{[2]\,(2)\,a}_{\mu} &= -\epsilon^{abc}\left[\partial_Z\left(u^3A^{(1)\,b}_{Z}A^{(1)\,c}_{\mu}\right)+u^3A^{(1)\,b}_{Z}f^{(1)\,c}_{Z\mu}+\tfrac{1}{u}\eta^{\alpha\beta}\partial_\alpha\left(A^{(1)\,b}_{\beta}A^{(1)\,c}_{\mu}\right)+\tfrac{1}{u}\eta^{\alpha\beta}A^{[2]\,(1)\,b}_{\alpha}f^{(1)\,c}_{\beta\mu}\right]. \end{aligned} \tag{D.2.6}$$

In this case there is a contribution to the leading pole term of the correlator from terms in $I^{(2)}_M$ with two massless pole factors, corresponding to two external legs of the exchange Witten diagram joining in a vertex with the internal leg. This leaves only one term that needs to be considered

$$I^{[2]\,(2)\,a}_{Z} \simeq 0 \tag{D.2.7}$$

$$I^{[2]\,(2)\,a}_{\mu} \simeq -\epsilon^{abc}u^3A^{(1)\,b}_{Z}f^{(1)\,c}_{Z\mu}. \tag{D.2.8}$$

Taking this into account, we can set $A^{(2)\,a}_Z\simeq 0$ to compute the leading pole contribution. This leaves

$$I^{[2]\,(3)\,a}_{Z} \simeq -u^3\epsilon^{abc}\left[\eta^{\alpha\beta}\partial_\alpha\left(A^{(2)\,b}_{\beta}A^{(1)\,c}_{Z}\right)+\eta^{\alpha\beta}A^{(2)\,b}_{\alpha}f^{(1)\,c}_{\beta Z}\right] \tag{D.2.9}$$

$$I^{[2]\,(3)\,a}_{\mu} = -\epsilon^{abc}\left[\partial_Z\left(u^3A^{(1)\,b}_{Z}A^{(2)\,c}_{\mu}\right)+u^3A^{(1)\,b}_{Z}f^{(2)\,c}_{Z\mu}\right]. \tag{D.2.10}$$

Finally, there could had been an $O(\epsilon^3)$ exchange contribution where the internal leg of the Witten diagram is the Abelian component of the D8-brane gauge field and the vertices are determined by the Wess-Zumino action (6.1.13). To compute this contribution one should first find the $O(\epsilon^2)$ solution for the Abelian field. The inhomogeneous terms in the Abelian field equation are proportional to

$$I^{[WZ]}_Z\sim\epsilon^{Z\mu\nu\alpha\beta}f^{(1)\,a}_{\mu\nu}f^{(1)\,a}_{\alpha\beta}\ ,\ I^{[WZ]}_\mu\sim\epsilon_\mu{}^{\nu Z\alpha\beta}f^{(1)\,a}_{\nu Z}f^{(1)\,a}_{\alpha\beta}\ . \tag{D.2.11}$$

But none of these terms has two massless pole factors, so they do not contribute to the leading pole term in the correlator of the axial current.

Moving on to the calculation of the solution to the gauge field, given the $O(\epsilon)$ solutions (6.5.1) with momenta p_i, p_j, the $O(\epsilon^2)$ inhomogeneous term contributing to the leading pole would be

$$I^{[2]\,(2)\,a}_{\mu}(p_i,p_j;Z,q)\simeq\frac{4}{\pi}\phi_0(Z)\int_{p_i}\int_{p_j}\delta_{q-p_i-p_j}\frac{ip_i^\alpha}{p_i^2}\left(\delta_\mu{}^\beta-\frac{p_{j\,\mu}p_j^\beta}{p_j^2}\right)\epsilon^{abc}\hat{A}^b_{5\,\alpha}(p_i)\hat{A}^c_{5\,\beta}(p_j)\ , \tag{D.2.12}$$

where we are using as shorthand notation

$$\int_p\equiv\int\frac{d^4p}{(2\pi)^4}\quad \delta_p=(2\pi)^4\delta^{(4)}(p)\ . \tag{D.2.13}$$

One can check using the symmetries of the integrand that $q^\mu I^{(2)}_\mu = 0$, so this is a transverse term and in addition it is independent of the radial coordinate Z. We can further simplify this expression by keeping only the leading pole term

$$I^{[2]\,(2)\,a}_\mu(p_i, p_j; Z, q) \simeq i^{(2)\,a}_\mu(p_i, p_j; q)\phi_0(Z) \ , \tag{D.2.14}$$

where, in order to make expressions more manageable we have defined

$$i^{(2)\,a}_\mu(p_i, p_j; q) = -\frac{4}{\pi}\int_{p_i}\int_{p_j}\delta_{q-p_i-p_j}\frac{ip_i^\alpha}{p_i^2}\frac{p_{j\,\mu}p_j^\beta}{p_j^2}\epsilon^{abc}\hat{A}^b_{5\,\alpha}(p_i)\hat{A}^c_{5\,\beta}(p_j) \ . \tag{D.2.15}$$

Then, from (6.4.7), the $O(\epsilon^2)$ gauge field solution is

$$A^{(2)\,a}_\mu(Z, q) \simeq \int dZ_1 G(Z, Z_1; q^2) I^{[2]\,(2)\,a}_\mu(p_i, p_j; Z_1, q) \simeq i^{(2)\,a}_\mu(p_i, p_j; q)\left[\Phi^{(2)}(Z) + q^2\widetilde{\Phi}^{(2)}(Z)\right] + O(q^4) \ , \tag{D.2.16}$$

where

$$\begin{aligned}\Phi^{(2)}(Z) &= \int dZ_1 G(Z, Z_1; 0)\phi_0(Z_1) = \frac{\pi}{2}\psi_0(Z)(\psi_0(Z) - 1)\\ \widetilde{\Phi}^{(2)}(Z) &= \int dZ_1 G^{(1)}(Z, Z_1)\phi_0(Z_1) \ .\end{aligned} \tag{D.2.17}$$

Here we are introducing an additional approximation, not only $p_i^2 \simeq 0$, $p_j^2 \simeq 0$ are close to lightlike values, but also we assume $|(p_i + p_j)^2| \ll 1$, i.e., low energy and momentum for the external pions.

Next, we compute the $O(\epsilon^3)$ inhomogeneous terms

$$\begin{array}{rcl} I^{[2]\,(3)\,a}_Z & \simeq & -2u^3\epsilon^{abc}\eta^{\alpha\beta}A^{(2)\,b}_\alpha\partial_\beta A^{(1)\,c}_Z \\ I^{[2]\,(3)\,a}_\mu & \simeq & -2\epsilon^{abc}u^3 A^{(1)\,b}_Z\partial_Z A^{(2)\,c}_\mu, \end{array} \tag{D.2.18}$$

where we have used $\partial_Z(u^3 A^{(1)}_Z) = 0$, $\eta^{\alpha\beta}\partial_\alpha A^{(2)\,a}_\beta = 0$ and kept the leading pole terms only. Assigning momentum p_k to the $O(\epsilon)$ factors

$$\begin{array}{rcl} I^{[2]\,(3)\,a}_Z(Z, p_l) & \simeq & -\frac{4}{\pi}\left[\Phi^{(2)}(Z) + q^2\widetilde{\Phi}^{(2)}(Z)\right]\int_{p_k}\int_q \delta_{p_l-p_k-q}\epsilon^{abc}i^{(2)\,b}_\alpha(p_i, p_j; q)\frac{p_k^\alpha p_k^\beta}{pk2}\hat{A}^c_{5\,\beta}(p_k) \\ I^{[2]\,(3)\,a}_\mu & \simeq & -\frac{4}{\pi}\left[\partial_Z\Phi^{(2)}(Z) + q^2\partial_Z\widetilde{\Phi}^{(2)}(Z)\right]\int_{p_k}\int_q \delta_{p_l-p_k-q}\epsilon^{abc}i^{(2)\,b}_\mu(p_i, p_j; q)\frac{ip_k^\alpha}{p_k^2}\hat{A}^c_{5\,\alpha}(p_k). \end{array} \tag{D.2.19}$$

Let us define

$$i^{(3)\,a}_\mu(p_i, p_j, p_k, p_l) = -\frac{4}{\pi}\int_{p_k}\int_q \delta_{p_l-p_k-q}\epsilon^{abc}i^{(2)\,b}_\mu(p_i, p_j; q)\frac{ip_k^\alpha}{p_k^2}\hat{A}^c_{5\,\alpha}(p_k) \ , \tag{D.2.20}$$

then

$$\begin{array}{rcl} I^{[2]\,(3)\,a}_Z(Z, p_l) & \simeq & -\left[\Phi^{(2)}(Z) + q^2\widetilde{\Phi}^{(2)}(Z)\right] ip_l^\mu i^{(3)\,a}_\mu(p_i, p_j, p_k, p_l) \\ I^{[2]\,(3)\,a}_\mu & \simeq & \left[\Phi^{(2)\prime}(Z) + q^2\widetilde{\Phi}^{(2)\prime}(Z)\right] i^{(3)\,a}_\mu(p_i, p_j, p_k, p_l). \end{array}$$

Following (6.4.7) and (6.4.9), the $O(\epsilon^3)$ solution for the gauge potential is

$$\begin{aligned} A^{[2]\,(3)\,a}_\mu(Z, p_l) &\simeq \left(\delta_\mu^{\ \nu} - \frac{p_{l\,\mu}p_l^\alpha}{p_l^2}\right) i^{(3)\,a}_\alpha(p_i, p_j, p_k, p_l)\left[\Phi^{(3)}(Z) + q^2\widetilde{\Phi}^{(3)}(Z)\right] \\ A^{[2]\,(3)\,a}_Z(Z, p_l) &= \frac{\Phi^{(2)}(Z) + q^2\widetilde{\Phi}^{(2)}(Z)}{1 + Z^2}\frac{ip_l^\alpha}{p_l^2} i^{(3)\,a}_\alpha(p_i, p_j, p_k, p_l) \ , \end{aligned} \tag{D.2.21}$$

where

$$\begin{aligned} \Phi^{(3)}(Z) &= \int dZ_1 G(Z, Z_1; 0)\Phi^{(2)\prime}(Z_1) = \frac{\pi}{6}\arctan(Z)\psi_0(Z)(\psi_0(Z) - 1) \\ \widetilde{\Phi}^{(3)}(Z) &= \int dZ_1 G(Z, Z_1; 0)\widetilde{\Phi}^{(2)\prime}(Z_1) \ . \end{aligned} \tag{D.2.22}$$

The $O(\epsilon^3)$ field strength is proportional to the pion mode solution (6.2.9). First note that

$$f^{(3)\,a}_{Z\mu}(Z,p_l) = \partial_Z A^{[2]\,(3)\,a}_{\mu}(Z,p_l) - ip_{l\,\mu} A^{[2]\,(3)\,a}_{Z} \propto \partial_Z \left(\Phi^{(3)}(Z) + q^2\widetilde{\Phi}^{(3)}(Z)\right) - \frac{\Phi^{(2)}(Z) + q^2\widetilde{\Phi}^{(2)}(Z)}{1+Z^2}. \tag{D.2.23}$$

On the other hand, using the definition of the Green's function in (D.2.22)

$$\begin{aligned}\partial_Z\left[u^3(Z)\partial_Z\Phi^{(3)}\right] &= \Phi^{(2)\prime}(Z)\\ \partial_Z\left[u^3(Z)\partial_Z\widetilde{\Phi}^{(3)}\right] &= \widetilde{\Phi}^{(2)\prime}(Z)\ .\end{aligned} \tag{D.2.24}$$

We can integrate once each equation and, since $u(Z)^3 = 1+Z^2 = 1/(\pi\phi_0(Z))$, it follows that

$$\begin{aligned}\partial_Z\Phi^{(3)} &= \frac{\Phi^{(2)}(Z)}{1+Z^2} + c\pi\phi_0(Z)\ ,\\ \partial_Z\widetilde{\Phi}^{(3)} &= \frac{\widetilde{\Phi}^{(2)}(Z)}{1+Z^2} + \tilde{c}\pi\phi_0(Z)\ .\end{aligned} \tag{D.2.25}$$

In the limit $Z\to\infty$ the terms proportional to $\Phi^{(2)}$, $\widetilde{\Phi}^{(2)}$ in the equations above are subleading, while the leading terms have the asymptotic form

$$\partial_Z\Phi^{(3)} \sim c\pi\phi_0(Z) \sim \frac{c}{Z^2}\ ,\ \partial_Z\widetilde{\Phi}^{(3)} \sim \tilde{c}\pi\phi_0(Z) \sim \frac{\tilde{c}}{Z^2}\ . \tag{D.2.26}$$

Using the expansion in (6.4.15), the coefficients of the asymptotic terms are determined by the following integrals

$$\begin{aligned}c &= \lim_{Z\to\infty} Z^2\partial_Z\Phi^{(3)} = \int_{-\infty}^{\infty} dZ_1\,\psi_0(Z_1)\Phi^{(2)\prime}(Z_1) = \frac{\pi}{12}\\ \tilde{c} &= \lim_{Z\to\infty} Z^2\partial_Z\widetilde{\Phi}^{(3)} = \int_{-\infty}^{\infty} dZ_1\,\psi_0(Z_1)\widetilde{\Phi}^{(2)\prime}(Z_1)\ .\end{aligned} \tag{D.2.27}$$

The first integral can easily be done taking into account that $\phi_0(Z) = \psi_0'(Z)$, so the integrand turns out to be a total derivative. The second integral can be manipulated to show it is equal to

$$\tilde{c} = \int_{-\infty}^{\infty} dZ_1\,\psi_0(Z_1)\widetilde{\Phi}^{(2)\prime}(Z_1) = -\int_{-\infty}^{\infty} d\tilde{Z}\frac{\left(\Phi^{(2)}(\tilde{Z})\right)^2}{u(\tilde{Z})} = -\frac{b}{(2\pi)^2}\ , \tag{D.2.28}$$

where b is given in (6.3.12).

The leading pole contribution in the field strength can be identified as

$$f^{(3)\,a}_{Z\mu}(Z,p_l) \simeq -\left[\frac{\pi^2}{12} - \frac{b}{4\pi}q^2\right]\phi_0(Z)\frac{p_{l\,\mu}p_l^{\alpha}}{p_l^2} i^{(3)\,a}_{\alpha}(p_i,p_j,p_k,p_l)\ . \tag{D.2.29}$$

Plugging in (D.2.15) and (D.2.20), and integrating over q results in

$$f^{(3)\,a}_{Z\mu}(Z,p_l) \simeq \frac{4}{3}\phi_0(Z)\int_{p_i}\int_{p_j}\int_{p_k}\delta_{p_l-p_k-p_i-p_j}\frac{p_{l\,\mu}p_i^{\alpha}p_j^{\beta}p_k^{\gamma}}{p_l^2p_i^2p_j^2p_k^2}(p_l\cdot p_j)\left[1-\frac{12b}{\pi^3}(p_i\cdot p_j)\right]\epsilon^{abc}\epsilon^{bde}\hat{A}^d_{5\,\alpha}(p_i)\hat{A}^e_{5\,\beta}(p_j)\hat{A}^c_{5\,\gamma}(p_k)\ . \tag{D.2.30}$$

Note that the radial dependence is the same as for the pion mode solution (6.2.9). Then, the calculation of the canonical momentum and expectation value of the current will proceed along similar steps, resulting in an exchange contribution to the axial current

$$\langle J_5^{\mu\,a}(p_l)\rangle_e \simeq \frac{2}{3}f_\pi^2\int_{p_i}\int_{p_j}\int_{p_k}\delta_{p_l-p_k-p_i-p_j}\frac{p_l^{\mu}p_i^{\alpha}p_j^{\beta}p_k^{\gamma}}{p_l^2p_i^2p_j^2p_k^2}(p_l\cdot p_j)\left[1-\frac{12b}{\pi^3}(p_i\cdot p_j)\right]\epsilon^{abc}\epsilon^{bde}\hat{A}^d_{5\,\alpha}(p_i)\hat{A}^e_{5\,\beta}(p_j)\hat{A}^c_{5\,\gamma}(p_k)\ . \tag{D.2.31}$$

Restoring units, the exchange contribution to the leading pole in the four-point function of the current is

$$\begin{aligned}&\langle J_5^{\mu_1\,a_1}(p_1)J_5^{\mu_2\,a_2}(p_2)J_5^{\mu_3\,a_3}(p_3)J_5^{\mu_4\,a_4}(p_4)\rangle_e\\ &\qquad\simeq -2if_\pi^2\left(\prod_{i=1}^{4}\frac{p_i^{\mu_i}}{p_i^2}\right)\delta_{\sum_{i=1}^4 p_i}\\ &\times\left[\left\{(p_1\cdot p_2) - \frac{4b}{\pi^3M_{KK}^2}\left[(p_1\cdot p_2)(p_2\cdot(p_3+p_4)) - (p_1\cdot p_4)(p_2\cdot p_4) - (p_1\cdot p_3)(p_2\cdot p_3)\right]\right\}\delta^{a_1a_2}\delta^{a_3a_4}\right.\\ &\left.+(2\leftrightarrow 3)+(2\leftrightarrow 4)\right]\ .\end{aligned} \tag{D.2.32}$$

Or, using momentum conservation

$$\begin{aligned}
&\langle J_5^{\mu_1\,a_1}(p_1)J_5^{\mu_2\,a_2}(p_2)J_5^{\mu_3\,a_3}(p_3)J_5^{\mu_4\,a_4}(p_4)\rangle_e \\
&\qquad\qquad \simeq -2if_\pi^2\left(\prod_{i=1}^4\frac{p_i^{\mu_i}}{p_i^2}\right)\delta_{\sum_{i=1}^4 p_i} \\
&\times\Big[\Big\{(p_1\cdot p_2)-\frac{4b}{\pi^3}\left[(p_1\cdot p_3)^2+(p_1\cdot p_4)^2-2(p_1\cdot p_2)^2\right]\Big\}\delta^{a_1a_2}\delta^{a_3a_4} \\
&+(2\leftrightarrow 3)+(2\leftrightarrow 4)\Big]\,.
\end{aligned} \tag{D.2.33}$$

D.3 Contributions from $O(F^4)$ terms: vertex diagram

The last possible contribution we have to study is originating from the $O(F^4)$ terms in the D8-brane action, the one that would introduce the non-linear terms in the equations

$$\begin{aligned}
I_Z^{[4]\,a} &= \frac{1}{2}\left(\frac{\pi\alpha'}{L^2}\right)^2 u^2\left(\partial_\alpha\Pi_a^{[4]\,\alpha}+\frac{\delta\mathcal{L}_{\rm DBI}^{[4]}}{\delta A_Z^a}\right) \\
I_\mu^{[4]\,a} &= -\frac{1}{2}\left(\frac{\pi\alpha'}{L^2}\right)^2\left[\left(\partial_Z\left(u^2\Pi_a^{[4]\,\mu}\right)-u^2\frac{\delta\mathcal{L}_{\rm DBI}^{[4]}}{\delta A_\mu^a}\right)+u^2\left(\partial_\alpha\left(\frac{\delta\mathcal{L}_{\rm DBI}^{[4]}}{\delta\partial_\alpha A_\mu^a}\right)-\frac{\delta\mathcal{L}_{\rm DBI}^{[4]}}{\delta A_\mu^a}\right)\right]\,.
\end{aligned} \tag{D.3.1}$$

At $O(\epsilon^3)$ we need to keep only terms that are at most cubic in the fields, so terms $\sim\frac{\delta\mathcal{L}_{\rm DBI}^{[4]}}{\delta A_M^a}$ can be dropped, and only terms involving three factors of the Abelianized field strengths $f_{MN}^{(1)\,a}$ remain. Among these, the leading pole contributions must come from terms with three factors of the $f_{Z\mu}^{(1)}$ components. One can check using (6.1.10) that $\frac{\delta\mathcal{L}_{\rm DBI}^{[4]}}{\delta\partial_\alpha A_\mu^a}$ does not introduce any such terms. Hence, using (D.1.1), all leading pole contributions are the following

$$\begin{aligned}
I_Z^{[4]\,(3)\,a} &\simeq -\frac{1}{6}\left(\frac{\pi\alpha'}{L^2}\right)^2 u^4\eta^{\gamma\lambda}\eta^{\alpha\beta}\partial_\alpha\left(f_{Z\gamma}^{(1)\,b}f_{Z\lambda}^{(1)\,b}f_{Z\beta}^{(1)\,a}+2f_{Z\gamma}^{(1)\,b}f_{Z\lambda}^{(1)\,a}f_{Z\beta}^{(1)\,b}\right) \\
I_\mu^{[4]\,(3)\,a} &\simeq \frac{1}{6}\left(\frac{\pi\alpha'}{L^2}\right)^2\eta^{\gamma\lambda}\partial_Z\left(u^4\left(f_{Z\gamma}^{(1)\,b}f_{Z\lambda}^{(1)\,b}f_{Z\mu}^{(1)\,a}+2f_{Z\gamma}^{(1)\,b}f_{Z\lambda}^{(1)\,a}f_{Z\mu}^{(1)\,b}\right)\right)\,.
\end{aligned} \tag{D.3.2}$$

Going to momentum space, and using (6.5.2) and (6.1.9), yields

$$\begin{aligned}
I_Z^{[4]\,(3)\,a}(Z,q) &\simeq -u(Z)\phi_0(Z)^2 iq^\alpha j_\alpha^{(3)\,a}(p_i,p_j,p_k,q) \\
I_\mu^{[4]\,(3)\,a}(Z,q) &\simeq \partial_Z(u(Z)\phi_0(Z)^2)j_\mu^{(3)\,a}(p_i,p_j,p_k,q)\,,
\end{aligned} \tag{D.3.3}$$

where the leading pole factor is

$$\begin{aligned}
j_\mu^{(3)\,a}(p_i,p_j,p_k,q) \simeq -\frac{3^5\pi}{4\lambda_{\rm YM}^2}\int_{p_i}\int_{p_j}\int_{p_k}\delta_{q-p_i-p_j-p_k}\frac{p_{k\,\mu}p_k^\nu p_i^\sigma p_j^\rho}{p_i^2p_j^2p_k^2}(p_i\cdot p_j) \\
\times\left(\delta^{a_ia_j}\delta^{aa_k}+\delta^{a_ia_k}\delta^{aa_j}+\delta^{a_ja_k}\delta^{aa_i}\right)\hat{A}_{5\,\sigma}^{a_i}(p_i)\hat{A}_{5\,\rho}^{a_j}(p_j)\hat{A}_{5\,\nu}^{a_k}(p_k)\,.
\end{aligned} \tag{D.3.4}$$

Following (6.4.7) and (6.4.9), the $O(\epsilon^3)$ solution for the gauge potential is

$$\begin{aligned}
A_\mu^{[4]\,(3)\,a}(Z,p_l) &\simeq \Psi^{(3)}(Z)\left(\delta_\mu^{\ \nu}-\frac{p_{l\,\mu}p_l^\alpha}{p_l^2}\right)j_\mu^{(3)\,a}(p_i,p_j,p_k,p_l) \\
A_Z^{[4]\,(3)\,a}(Z,p_l) &= \frac{u(Z)\phi_0(Z)^2}{1+Z^2}\frac{ip_l^\alpha}{p_l^2}j_\alpha^{(3)\,a}(p_i,p_j,p_k,p_l)\,,
\end{aligned} \tag{D.3.5}$$

where

$$\Psi^{(3)}(Z) \quad = \quad = \int dZ_1 G(Z,Z_1;0)\partial_{Z_1}\left(u(Z_1)\phi_0(Z_1)^2\right) = \frac{7Z\,{}_2F_1\left(\frac{1}{2},\frac{2}{3};\frac{3}{2};-Z^2\right)}{40\pi^2}+\frac{3Z\left(7Z^2+11\right)}{40\pi^2(Z^2+1)^2}-\frac{\Gamma\left(\frac{13}{6}\right)\arctan(Z)}{\pi^2\Gamma\left(\frac{8}{3}\right)}. \tag{D.3.6}$$

The leading pole contribution in the field strength can be identified as

$$f^{(3)\,a}_{Z\mu}(Z,p_l) \simeq -\frac{\Gamma\left(\frac{13}{6}\right)}{\pi^{3/2}\Gamma\left(\frac{8}{3}\right)}\phi_0(Z)\frac{p_{l\,\mu}p_l^\alpha}{p_l^2}j^{(3)\,a}_\alpha(p_i,p_j,p_k,p_l)\ . \tag{D.3.7}$$

Plugging in (D.3.4) results in

$$\begin{aligned} f^{(3)\,a}_{Z\mu}(Z,p_l) \simeq & \phi_0(Z)\frac{3^5\Gamma\left(\frac{13}{6}\right)}{4\sqrt{\pi}\Gamma\left(\frac{8}{3}\right)\lambda^2_{\text{YM}}}\int_{p_i}\int_{p_j}\int_{p_k}\delta_{p_l-p_i-p_j-p_k}\frac{p_{l\,\mu}p_k^\nu p_i^\sigma p_j^\rho}{p_i^2p_j^2p_k^2p_l^2}(p_i\cdot p_j)(p_k\cdot p_l) \\ & \times\left(\delta^{a_ia_j}\delta^{aa_k}+\delta^{a_ia_k}\delta^{aa_j}+\delta^{a_ja_k}\delta^{aa_i}\right)\hat{A}^{a_i}_{5\,\sigma}(p_i)\hat{A}^{a_j}_{5\,\rho}(p_j)\hat{A}^{a_k}_{5\,\nu}(p_k)\ . \end{aligned} \tag{D.3.8}$$

Note that the radial dependence is, once more, the same as for the pion mode solution (6.2.9). Then, the calculation of the canonical momentum and expectation value of the current will proceed along similar steps, resulting in a vertex contribution to the axial current

$$\begin{aligned} \langle J^{\mu\,a}_5(p_l)\rangle_v \simeq & f_\pi^2\frac{3^5\Gamma\left(\frac{13}{6}\right)}{8\sqrt{\pi}\Gamma\left(\frac{8}{3}\right)\lambda^2_{\text{YM}}}\int_{p_i}\int_{p_j}\int_{p_k}\delta_{p_l-p_i-p_j-p_k}\frac{p_l^\mu p_k^\nu p_i^\sigma p_j^\rho}{p_i^2p_j^2p_k^2p_l^2}(p_i\cdot p_j)(p_k\cdot p_l) \\ & \times\left(\delta^{a_ia_j}\delta^{aa_k}+\delta^{a_ia_k}\delta^{aa_j}+\delta^{a_ja_k}\delta^{aa_i}\right)\hat{A}^{a_i}_{5\,\sigma}(p_i)\hat{A}^{a_j}_{5\,\rho}(p_j)\hat{A}^{a_k}_{5\,\nu}(p_k)\ . \end{aligned} \tag{D.3.9}$$

Restoring units, the vertex contribution to the leading pole in the four-point function of the current is

$$\begin{aligned} & \langle J^{\mu_1\,a_1}_5(p_1)J^{\mu_2\,a_2}_5(p_2)J^{\mu_3\,a_3}_5(p_3)J^{\mu_4\,a_4}_5(p_4)\rangle_v \\ & \simeq i\frac{f_\pi^2}{M^2_{\text{KK}}}\frac{3^5\Gamma\left(\frac{13}{6}\right)}{4\sqrt{\pi}\Gamma\left(\frac{8}{3}\right)\lambda^2_{\text{YM}}}\left(\prod_{i=1}^4\frac{p_i^{\mu_i}}{p_i^2}\right)\delta_{\sum_{i=1}^4 p_i}\left(\delta^{a_1a_2}\delta^{a_3a_4}+\delta^{a_1a_3}\delta^{a_2a_4}+\delta^{a_1a_4}\delta^{a_2a_3}\right) \\ & \qquad\times\left[(p_1\cdot p_2)(p_3\cdot p_4)+(2\leftrightarrow 3)+(2\leftrightarrow 4)\right]\ . \end{aligned} \tag{D.3.10}$$

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