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Convexidad en estructuras difusas
Convexity of Fuzzy Structures
Konvexita v usporiadaných štruktúrach

Tesis Doctoral

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RESUMEN (en español)

Esta tesis trata sobre la teoría de conjuntos difusos, los cuales fueron propuestos por Zadeh en 1965. Los conjuntos difusos permiten dar grados de pertenencia a un conjunto, con un valor en el intervalo [0, 1]. Abordar la imprecisión en problemas del mundo real es tal desafío que ha llevado a que se creasen varias extensiones de conjuntos difusos, como los conjuntos difusos *hesitant*, los conjuntos difusos intuicionistas y los conjuntos difusos intervalo-valorados. Estas extensiones pueden resultar útiles en situaciones donde las herramientas difusas clásicas no son tan eficientes, por ejemplo, cuando no hay un procedimiento preciso para determinar los grados de pertenencia. Debido a su potencial para diversas aplicaciones, estas extensiones han llamado la atención de muchos investigadores. En esta tesis, nos centramos en los conjuntos difusos del tipo *typical hesitant*, que son un caso particular de conjuntos difusos *hesitant* donde la función de pertenencia toma un número finito de valores, y en los conjuntos difusos intervalo-valorados, donde la función de pertenencia es un intervalo en cada valor.

Por otro lado, la convexidad es una herramienta matemática fundamental que resulta conveniente estudiar en diferentes escenarios, como los conjuntos clásicos o nítidos y los conjuntos difusos. Desde que Zadeh introdujo los conjuntos difusos, se han propuesto diferentes definiciones de convexidad en la literatura para tratar con este tipo de conjuntos y con sus extensiones.

De igual manera, la toma de decisiones ha sido un campo que ha captado la atención de muchos investigadores, como Bellman y Zadeh o Yager y Bason. Existen al menos tres factores cruciales a considerar en un proceso de toma de decisiones: 1) una colección de alternativas, 2) un conjunto de limitaciones en la elección dentro de múltiples alternativas, y 3) una función de utilidad que asocia la ganancia o pérdida resultante de elegir esa alternativa con cada elección.

En situaciones “reales” puede resultar muy desafiante especificar la función objetivo y las limitaciones con precisión. Según Czogala y Zimmermann, los conjuntos difusos pueden ser una herramienta muy útil para tratar con la imprecisión.

Teniendo estos comentarios en cuenta, esta tesis se centra en la convexidad de los conjuntos difusos y sus extensiones. Comenzamos con los conjuntos difusos *hesitant* y las funciones de agregación, que son funciones crecientes que combinan varias entradas para dar una salida y que cumplen que, si las entradas son todas cero, la salida es 0 y lo mismo con 1. Proponemos una definición adecuada de un conjunto difuso *hesitant* convexo basada en funciones de agregación y estudiamos cuándo la intersección de dos conjuntos convexos es también convexa. Obtenemos resultados positivos con el mínimo y el máximo. También hemos demostrado que existen funciones entre el mínimo y el máximo que verifican esa afirmación. Después de eso, señalamos que el uso de la función de agregación podría generar una falta de información, por lo que presentamos una definición adecuada de conjuntos difusos *hesitant* convexos basada en órdenes admisibles, que son órdenes totales que refinan el conocido orden reticular. Nuestra propuesta de convexidad es compatible con los alfa-cortes, es decir, si consideramos un conjunto difuso *hesitant* convexo, entonces sus alfa-cortes son conjuntos clásicos convexos. También es compatible con el soporte (*support*) y el núcleo (*core*) de un conjunto difuso *hesitant*. Para evitar el uso de las funciones de agregación, presentamos un estudio de la intersección de conjuntos difusos *hesitant* donde recuperamos el significado clásico de la intersección. Con esta definición de intersección y la definición de convexidad, obtenemos muy buenos resultados y proporcionamos una aplicación en la toma de decisiones con resultados interesantes al considerar objetivos y limitaciones difusas *hesitant* convexas.

En el caso de los conjuntos difusos intervalo-valorados, presentamos una definición de convexidad basada en órdenes de intervalos y estudiamos las propiedades de dicha convexidad. Obtenemos un comportamiento positivo con los alfa-cortes, el soporte y el núcleo de un conjunto difuso intervalo-valorado. También analizamos la definición de intersección para conjuntos difusos intervalo-valorados y proponemos otra que recupera el significado clásico de la intersección y es compatible con la convexidad definida. Es decir, la intersección de dos conjuntos convexos también es convexa. Finalmente, proporcionamos una aplicación para utilizar las teorías anteriores desarrolladas en procesos de toma de decisiones.

RESUMEN (en inglés)

This thesis discusses the concept of fuzzy sets, which were first proposed by Zadeh in 1965. Fuzzy sets allow for degrees of membership within a set, with the membership value being in the range of 0 to 1. Addressing imprecision in real-world problems has been a long-standing research challenge, leading to various extensions of fuzzy sets such as hesitant fuzzy sets, intuitionistic fuzzy sets, and interval-valued fuzzy sets. These extensions can prove useful in situations where classical fuzzy tools are not as efficient, for example when there is no objective procedure to determine crisp membership degrees. Due to their potential for various applications, these extensions have drawn the attention of many researchers. In this thesis, we focus on typical hesitant fuzzy sets, which are a particular case of hesitant fuzzy sets where the membership function takes a finite number of values, and interval-valued fuzzy sets, where the membership function is an interval for each value.

On the other hand, convexity is a fundamental mathematical technique that is useful in studying different scenarios, including crisp sets and fuzzy sets. Since Zadeh introduced fuzzy sets, different convexity types have been proposed in the literature to deal with this kind of sets and its extensions.

At the same time, decision-making has been a field that catches the attention of many researchers such as Bellman and Zadeh or Yager and Bason. There are at least three crucial factors to consider in a decision-making process 1) a collection of alternatives, 2) a set of limitations on the option within multiple alternatives, and 3) a utility function that associates the gain or loss resulting from choosing that alternative with each choice.
It is very challenging to specify the objective function and the limitations precisely in many real-world circumstances. According to Czogala and Zimmermann, fuzzy sets can be a very helpful tool for dealing with imprecision.

Bearing this in mind, this thesis is focused on the convexity of fuzzy sets and its extensions. We start with hesitant fuzzy sets and aggregation functions, which are increasing functions that combine various inputs in order to give one output and fulfill that if the inputs are all zero the output is 0 and the same with 1. We propose a proper definition of a convex hesitant fuzzy set based on aggregation functions and study when the intersection of two convex sets is also convex. We obtain positive results with the minimum and the maximum. We were also able to prove that there exist functions between the minimum and the maximum that verify that statement. After that, we point out that the use of the aggregation function could generate a lack of information, so we introduce an appropriate definition of convex hesitant fuzzy sets based on admissible orders, which are total orders that refine the well-known lattice order. Our proposal of convexity is compatible with the level sets, that is, if we consider a convex hesitant fuzzy set, then its level sets are convex crisp sets. It is also compatible with the support and core of a hesitant fuzzy set. In order to avoid aggregation functions, we present a study of the intersection of hesitant fuzzy sets where we recover the classical meaning of intersection. With this definition of intersection and the definition of convexity, we obtain very good results and provide an application in decision-making with interesting results when considering convex hesitant

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fuzzy goals and constraints.

In the case of interval-valued fuzzy sets, we introduce a definition of convexity based on interval orders and study the properties of these convex sets. We obtain a positive behaviour with the level sets, support and core of an interval-valued fuzzy set. We also analyse the definition of intersection for interval-valued fuzzy sets and propose another one that recovers the classical meaning of intersection and it is compatible with the convexity defined. That is, the intersection of two convex sets is also convex. Finally, we provide an application for using the previous theories developed in decision-making processes.

SR. PRESIDENTE DE LA COMISIÓN ACADÉMICA DEL PROGRAMA DE DOCTORADO En Matemáticas y Estadística

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Introduction

In 1965, Zadeh [91] proposed the idea of a fuzzy set. The degree of membership of any object to a set in fuzzy set theory is a value in the range $[0,1]$. This idea is a very useful instrument to explain human knowledge. Furthermore, Zadeh [92] suggested the first extensions of these sets in 1973, indicating the necessity to develop this theory.

One of the most used extensions is hesitant fuzzy sets, where the membership takes values on the power set of $[0,1]$. Numerous researchers rapidly became interested in these sets and presented a variety of extensions and operators to compute with these forms of information. Eventually, various applications were created [14, 50, 69].

Interval-valued fuzzy sets, where the values of the membership function are subintervals of the interval $[0,1]$, are another interesting and very used extension of fuzzy sets. There are lots of applications for this theory. For instance, it was used in the medical diagnosis of thyroid disease [70], image processing [36], approximate reasoning [10], interval-valued logic [63], medicine [4], clustering [67], among others. Atanassov's intuitionistic fuzzy sets are another frequently used extension of fuzzy sets and are equivalent to interval-valued fuzzy sets [20]. So working with interval-valued fuzzy sets and intuitionistic fuzzy sets is equivalent from a mathematical perspective, nevertheless, these sets are different from one another conceptually [74], and we use them depending on the situation.

All of these extensions can be included within the type-2 fuzzy sets [93]. In type-2 fuzzy sets, the value of the membership function is itself a fuzzy set. This extension catches the attention of several researchers, such as McCulloch and Wagner[58], Wu and Mendel[81, 82], Huidobro et al.[42], and others.

From another perspective, the idea of convexity is a fundamental mathematical technique that may be applied to study a large range of scenarios. Different convexity types have been discussed in the literature, initially to deal with crisp sets and later for fuzzy sets and intuitionistic fuzzy sets. These concepts enable us to work with any kind of set, whether it is crisp, fuzzy or intuitionistic [2, 21, 45, 90]. In order to do this to hesitant fuzzy sets, appropriate concepts for scenarios in which the universe is not always a vector space will be introduced [55] and also some methods to order the hesitant fuzzy elements will be presented. In the case of interval-valued fuzzy sets, we will also look for methods to order the intervals. Thus, we will do a parallel study about the convexity for the case of interval-valued fuzzy sets, where we will take into account the conclusions obtained for hesitant fuzzy sets. Of course, an appropriate adaptation to this new environment should be done and some different studies are required. It should be emphasized that convexity is one of the most crucial factors in the study of the geometric properties of both classical and fuzzy sets, as well as fuzzy multisets in particular, in addition to fuzzy sets in general. It has grown more potent due to its usage in a variety of fields, including optimization [53], image processing [75], robotics [51] or geometry [47], among others. One of the most relevant properties of convexity is that the intersection of two sets is also convex. Hence, we will deal with convexity and its preservation under the intersection.

Therefore, this thesis will deal with fuzzy sets, hesitant fuzzy sets and interval-valued fuzzy sets. We think that when the situation manages a discontinuous piece of information, hesitant fuzzy sets seem like a proper tool, while in the case of a continuum model we could use interval-valued fuzzy sets.

At the same time, decision-making has been a field that catches the attention of many researchers such as Bellman and Zadeh [8], Naz and Akram [60] or Yager and Bason [89]. There are at least three crucial factors to consider in a decision-making process:

1. a collection of alternatives
2. a set of limitations on the option within multiple alternatives
3. a utility function that associates the gain or loss resulting from choosing that

alternative with each choice

It is very challenging to specify the objective function and the limitations precisely in many real-world circumstances. Fuzzy sets can be a very helpful tool for dealing with imprecision, according to Czogala and Zimmermann [17].

Bearing this in mind, this work will be focused on convexity in fuzzy sets and its extensions. The main objectives are the following:

- Propose an appropriate definition of convex hesitant fuzzy sets and convex interval-valued fuzzy sets.
- Analyse the definition of intersection for hesitant fuzzy sets and interval-valued fuzzy sets.
- Study properties of convexity hesitant fuzzy sets and interval-valued fuzzy sets.
- Provide an application for using the previous theories in a decision making-problem.

In the first chapter, we will review the definition of fuzzy sets and the basic operations proposed by Zadeh in [91], and the ideas that correspond to the main generalizations of fuzzy sets. In addition, we will also introduce some basic notions about fuzzy convexity. In the second chapter, the main topic will be the convexity of hesitant fuzzy sets. Taking into account the state of the art on this topic, we propose two ways of defining convexity, one with aggregation functions and one without them. In order to end the chapter, two applications are shown. In Chapter 3, we study the convexity of interval-valued fuzzy sets and provide two applications, one for decision-making and one for ranking theory. Finally, we finish this document by showing its main conclusions.

Introducccción

En 1965, Zadeh [91] propuso la idea de un conjunto difuso o borroso. El grado de pertenencia de cualquier objeto a un conjunto, en la teoría de conjuntos difusos, es un valor dentro del rango $[0,1]$. Esta idea es un instrumento muy útil para explicar el razonamiento humano. Además, Zadeh [92] sugirió las primeras extensiones de estos conjuntos en 1973, indicando la necesidad de desarrollar esta teoría.

Una de las extensiones más utilizadas son los conjuntos difusos *hesitant*, donde la función de pertenencia toma valores en el conjunto partes de $[0,1]$. Muchos investigadores se interesaron rápidamente en estos conjuntos y presentaron una variedad de extensiones y operadores. También se crearon varias aplicaciones [14, 50, 69].

Los conjuntos difusos intervalo-valorados, donde los valores de la función de pertenencia son subintervalos del intervalo $[0,1]$, son otra extensión de los conjuntos difusos. Hay muchas aplicaciones con esta teoría. Por ejemplo, se usó en el diagnóstico médico de la enfermedad de la tiroides [70], procesamiento de imágenes [36], razonamiento aproximado [10], lógica de intervalos [63], medicina [4], etc. Los conjuntos difusos intuicionistas de Atanassov son otra extensión de los conjuntos difusos que se usa con frecuencia y son equivalentes a los conjuntos difusos intervalo-valorados [20]. Aunque trabajar con conjuntos difusos intervalo-valorados y con conjuntos difusos intuicionistas de Atanassov es equivalente desde una perspectiva matemática, estos conjuntos son diferentes entre sí conceptualmente [74], por lo que decidiremos con cual trabajar dependiendo del contexto en el que estemos.

Todas estas extensiones se pueden incluir dentro de los conjuntos difusos de tipo 2 [93]. En los conjuntos difusos de tipo 2, el valor de la función de pertenencia es en sí mismo un conjunto difuso. Esta extensión captó el interés de muchos investigadores como McCulloch y Wagner[58], Wu y Mendel[81, 82], Huidobro et al.[42], entre otros.

Desde otra perspectiva, la idea de convexidad es una técnica matemática fundamental que se puede aplicar en una amplia gama de escenarios. En la literatura se han discutido diferentes tipos de convexidad, inicialmente para tratar con conjuntos clásicos (o nítidos) y luego para conjuntos difusos y conjuntos difusos intuicionistas. Estos conceptos nos permiten trabajar con cualquier tipo de conjunto, ya sea nítido, difuso o intuicionístico [2, 21, 45]. Para hacer esto con los conjuntos difusos *hesitant*, se introducirán conceptos apropiados para escenarios en los que el universo no siempre es un espacio vectorial [55]. También se presentarán algunos métodos para ordenar los elementos difusos *hesitant*, lo cual nos permitirá manejar mejor la convexidad. En el caso de los conjuntos difusos intervalo-valorados, también buscaremos métodos para ordenar los intervalos. Haremos en este caso un estudio paralelo sobre la convexidad, donde tendremos en cuenta las conclusiones que ya se hayan establecido para los conjuntos difusos *hesitant*. Por supuesto, la adaptación a este nuevo entorno requerirá algunos estudios diferenciados. Cabe destacar que la convexidad es uno de los factores más cruciales en el estudio de las propiedades geométricas de los conjuntos tanto clásicos como difusos. Motivado por este hecho, su uso ha crecido sustancialmente en muchos campos, incluyendo la optimización [53], el procesamiento de imágenes [75], la robótica [51] o la geometría [47], entre otros. En general, una de las propiedades más relevantes es que la intersección de dos conjuntos convexos es convexa. Por lo tanto, también analizaremos el concepto de intersección y su preservación por convexidad.

Así, en esta tesis se hablará de conjuntos difusos, conjuntos difusos *hesitant* y conjuntos difusos intervalo-valorados. Creemos que cuando la situación maneja información discreta, los conjuntos difusos *hesitant* son una herramienta adecuada, mientras que en el caso de un modelo continuo sería más adecuado el uso de los conjuntos difusos intervalo-valorados.

Paralelamente, la toma de decisiones ha sido un campo que ha llamado la atención de muchos investigadores como Bellman y Zadeh [8], Naz y Akram [60] o Yager y Bason [89]. En un procedimiento de toma de decisiones hay al menos tres componentes importantes a tener en cuenta:

1. un conjunto de alternativas,
2. un conjunto de restricciones,
3. una función de utilidad que cuantifica la ganancia o pérdida que surge de la preferencia de esa alternativa con cada decisión.

En muchas situaciones reales, es extremadamente difícil describir con precisión la función objetivo y las restricciones. Para lidiar con la imprecisión, los conjuntos difusos pueden ser una herramienta muy útil [17].

Teniendo estos comentarios en cuenta, este trabajo se centrará en la convexidad de los conjuntos difusos y sus extensiones. Para ello, los objetivos principales son los siguientes:

- Proponer una definición apropiada de conjuntos difusos *hesitant* e intervalo-valorados convexos.
- Analizar la definición de intersección para conjuntos difusos *hesitant* e intervalo-valorados.
- Estudiar las propiedades de los conjuntos difusos *hesitant* e intervalo-valorados convexos.
- Proporcionar una aplicación para usar las teorías anteriores en un problema de toma de decisiones.

En el primer capítulo, revisaremos la definición de conjuntos difusos y las operaciones básicas propuestas por Zadeh en [91] y también las ideas que corresponden a las principales extensiones de los conjuntos difusos. Además, introduciremos algunas nociones básicas sobre la convexidad difusa. Para el segundo capítulo, el tema principal será la convexidad de los conjuntos difusos *hesitant*. Teniendo en cuenta lo que se ha hecho en la literatura, propondremos dos formas de definir la convexidad, una con funciones de agregación y otra sin ellas. Para finalizar el capítulo, se muestran dos aplicaciones. En el Capítulo 3, estudiamos la convexidad de los conjuntos difusos intervalo-valorados y proponemos una aplicación en la toma de decisiones y otra en rankings. Finalmente, cerraremos este trabajo presentando las principales conclusiones del mismo.

Úvod

V roku 1965 Zadeh [91] zaviedol pojem fuzzy množiny. Stupeň príslušnosti akéhokoľvek objektu k množine v teórii fuzzy množín je hodnota v intervale $[0,1]$. Táto myšlienka je veľmi užitočným nástrojom na popis niektorých aspektov ľudského poznania. Okrem toho Zadeh [92] navrhol prvé rozšírenia týchto množín v roku 1973, čo neskor rozvinuli ďalší autori.

Jedným z najpoužívanejších rozšírení sú hesitant fuzzy množiny, kde funkcia príslušnosti nadobúda hodnoty na potenčnej množine intervalu $[0,1]$. Mnohí autori sa pomerne rýchlo začali zaujímať o tieto zobrazenia a predstavili rôzne rozšírenia a operátory na manipuláciu s týmito formami informácií. Nakoniec vznikli aj rôzne aplikácie [14, 50, 69].

Ďalším rozšírením fuzzy množín sú intervalovo hodnotové fuzzy množiny, kde hodnoty funkcie príslušnosti sú podintervaly intervalu $[0,1]$. Existuje veľa aplikácií tejto teórie. Používa sa napríklad pri lekárskej diagnostike ochorenia štítnej žľazy [70], spracovaní obrazu [36], približnom odvodzovaní [10], intervalovej logike [63], medicíne vo všeobecnosti [4], atď. Atanassovove intuicionistické fuzzy množiny sú ďalším často používaným rozšírením fuzzy množín a sú ekvivalentom intervalovo-hodnotových fuzzy množín [20]. Štruktúry intervalovohodnotových fuzzy množín a Atanassovovych intuicionistických fuzzy množín sú teda z matematického hľadiska ekvivalentné, avšak tieto objekty sa navzájom koncepčne líšia [74], a teda ich používame v závislosti od kontextu.

Všetky tieto rozšírenia môžu byť zahrnuté do kategórie fuzzy množín typu 2 [93]. Vo fuzzy množinách typu 2 je samotná hodnota funkcie príslušnosti fuzzy množinou. Takýmito rozšíreniami sa zaoberajú autori ako napríklad McCulloch a Wagner [58], Wu a Mendel[81, 82], Huidobro a kol. [42][58] a iní.

Myšlienka konvexnosti je základnou matematickou technikou, ktorú možno použiť na štúdium širokej škály problémov. V literatúre sa diskutuje o rôznych typoch konvexnosti, najprv pre ostré množinami a neskôr fuzzy množiny a intuicionistické fuzzy množiny. Tieto koncepty nám umožňujú pracovať s akýmkoľvek druhom množiny, či už sú to ostré, neostré alebo intuicionistické [2, 21, 45, 90]. Aby to bolo možné urobiť pre hesitant fuzzy set, budú zavedené vhodné koncepty pre situácie, v ktorých základná množina nie je vždy vektorovým priestorom [55], v tejto práci tiež zavádzame niektoré metódy na usporiadanie hesitant fuzzy prvkov. V prípade IVFS sa tiež zaoberáme spôsobmi usporiadania intervalov. V tejto súvislosti sa zaoberáme konvexnosťou pre intervalovohodnotové zobrazenia, pričom využívame výsledky dosiahnuté pre hesitant fuzzy množiny. Pochopiteľne, isté úpravy a odlišné prístupy sú v tomto prípade nevyhnutné. Je potrebné zdôrazniť, že konvexnosť je jedným z najdôležitejších faktorov pri štúdiu geometrických vlastností klasických a fuzzy množín, ako aj fuzzy multimnožín. Tieto metódy získavajú na význame vďaka použitiu v rôznych oblastiach, vrátane optimalizácie [53], spracovania obrazu [75], robotiky [51] alebo geometrie [47]. Jednou z podstatných vlastností konvexnosti je jej zachovávanie pri prieniku. Preto sa v práci zaoberáme aj prienikmi a ich konvexnosťou.

Základnými skúmanými objektami tejto práce sú teda fuzzy množiny, hesitant fuzzy množiny a intervalovo hodnotové fuzzy množiny. Nazdávame sa, že keď si situácia vyžaduje diskrétny priestor hodnôt, hesitant fuzzy set sa javia ako správny nástroj, zatiaľ čo v prípade kontinua je vhodné použiť IVFS.

Rozhodovanie je oblasťou, ktorá priťahuje pozornosť mnohých výskumníkov ako Bellman a Zadeh [8], Naz a Akram [60], Yager a Bason [89]. Tri kľúčové faktory, ktoré je potrebné zvážiť v rozhodovacom procese, sú: 1) súbor alternatív, 2) súbor obmedzení možností v rámci viacerých alternatív a 3) funkcia užitočnosti, ktorá spája zisk alebo stratu vyplývajúcu z výberu tejto možnosti. Vo všeobecnosti je v reálnych situáciách pomerne náročné presne špecifikovať cieľovú funkciu a obmedzenia. Fuzzy množiny môžu byť podľa Czogala a Zimmermann [17] veľmi užitočným nástrojom na riešenie problémov, ktoré vo svojom popise obsahujú prvky neurčitosti.

Vzhľadom na to sa táto práca zameria na konvexnosť v oblasti fuzzy množín a ich rozšírení, stanovili sme si nasledujúce hlavné ciele:

- Navrhnúť vhodnú definíciu konvexnosti pre hesitant fuzzy množiny, ako aj pre intervalovohodnotové fuzzy množiny.
- Analyzovať existujúce definície prieniku pre hesitant fuzzy množiny a intervalovohodnotové fuzzy množiny.
- Skúmať vlastnosti konvexných hesitant fuzzy množín a intervalovohodnotových fuzzy množín.
- Prezentovať použitie predchádzajúcich teórií v rozhodovacom probléme.

V prvej kapitole zopakujeme definíciu fuzzy množiny a základné operácie pre ne navrhnuté Zadehom v [91] a myšlienky, ktoré zodpovedajú hlavným zovšeobecneniam fuzzy množín. Okrem toho zavedieme aj niektoré základné pojmy týkajúce sa fuzzy konvexnosti. V druhej kapitole je hlavnou témou konvexnosť hesitant fuzzy množín. Uvedieme prehľad existujúcich výsledkov a navrhneme spôsoby definovania konvexnosti, ako pomocou agregačných funkcií, tak aj bez ich použitia. Na záver kapitoly prezentujeme dve aplikácie. Nakoniec v kapitole 3 študujeme konvexnosť intervalovo hodnotových fuzzy množín a poskytujeme aplikáciu v rozhodovacom procese a v klasifikácii.

Chapter 1

Fuzzy sets and generalizations

The fundamental ideas for fuzzy sets, as well as their key extensions, will be covered in this chapter because they are crucial to understanding this work. Thus, the following sections are devoted to introducing the main concepts in this work and establishing the considered notation.

1.1 Fuzzy sets

In an effort to expand on the traditional set theory, L. A. Zadeh developed the idea of a fuzzy set [91]. Many authors have contributed to this notion since he first proposed it. As a result of all of this research, there are numerous definitions of a fuzzy set that are all equivalent in meaning. We first require an axiomatic reference set or universe, which we will designate by the symbol X.

In [91], the first definition was the following:

Definition 1.1 *A fuzzy set (class) A in X is characterized by a membership (characteristic) function $\mu_A(x)$ which associates with each point in X a real number in the interval $[0,1]$; the value of $\mu_A(x)$ at x represents the "grade of membership" of x in A.*

The motivation of Zadeh for this definition is that the closer $\mu_A(x)$ is to 1, the larger x belongs to the class A.

In this approach, several authors view the following as the typical definition of a fuzzy set:

Definition 1.2 *[21] Let X be a nonempty universe. A fuzzy set A on X is defined by means of a map* $\mu_A : X \to [0,1]$. *The map* μ_A *is said the membership function (or indicator) of A.*

In the theory of fuzzy sets, a fuzzy set A is frequently represented [25] as:

$$A = \{\langle x, \mu_A(x)\rangle : x \in X\}$$

where $\mu_A : X \to [0,1]$ represents the membership function of A.

The concept is really well described by the previous definitions. Thus, the set A and its membership function μ_A can both be used to represent a fuzzy set. Therefore, $\mu_A(x)$ or $A(x)$ indistinctly reflect the membership degree for a point x in X in the literature. However, in this thesis we are going to use just the membership representation. The fuzzy power set over X is the family of all fuzzy sets over X, and it is identified by the symbol $F(X)$ [68].

It should be emphasized that an ordinary set A, also known as a crisp set, can be thought of as a specific instance of a fuzzy set if its membership function is defined as

$$\mu_A(x) = \begin{cases} 1 & \text{if } x \in A \\ 0 & \text{if } x \notin A \end{cases}$$

Example 1.3 *Let us show an example of fuzzy sets.*

If X is the interval $[3,4]$, *the following sets are fuzzy sets:*

i) A is defined as $\mu_A(x) = \begin{cases} 1 & \text{if } x \in (3,4) \\ 0 & \text{if } x \notin (3,4) \end{cases}$

ii) B is defined as $\mu_B(x) = x - 3,\ \forall x \in [3,4]$

After recalling the fuzzy set description, we can move on to defining some key terms that are related to it, such as its fundamental operations. Although the membership function of a fuzzy set and a probability function when X is a countable set (or a probability density function when X is a continuum) are similar, it should be noted that there are significant differences between these concepts. These differences will become clearer once the rules for combining membership functions and their fundamental properties have been established. The idea of a fuzzy set is also entirely nonstatistical in nature [91].

Now we recall some interesting concepts related to fuzzy sets. Let us start with its support.

Definition 1.4 *[24, 91] Let A be a fuzzy set in X. The support of A, denoted by Supp(A), is defined as the crisp set*

$$Supp(A) = \{x \in X : \mu_A(x) \neq 0\}$$

The support could also be defined as $Supp(A) = \{x \in X : \mu_A(x) > 0\}$, since $\mu_A(x) \in [0,1]$, $\forall x \in X$.

It is easy to see that $Supp(A) = \emptyset$ if and only if $\mu_A(x) = 0\ \forall x \in X$.

Another interesting concept in fuzzy set theory is the core of a fuzzy set.

Definition 1.5 *[24, 91] Let A be a fuzzy set in X. The core of A, denoted by Core(A), is defined as the crisp set*

$$Core(A) = \{x \in X : \mu_A(x) = 1\}$$

It is also immediate that $Core(A) = \emptyset$ if and only if $\mu_A(x) = 0$, $\forall x \in X$.

We now describe the condition under which a fuzzy set is a content in another fuzzy set, i.e., whenever the first is a subset of the second.

Definition 1.6 *[91] Let A and B be two fuzzy sets in X. A is contained in B, which is denoted as $A \subseteq B$, if and only if $\mu_A(x) \leq \mu_B(x)$ for any $x \in X$. In symbols,*

$$A \subseteq B \Leftrightarrow \mu_A(x) \leq \mu_B(x), \quad \forall x \in X$$

When $A \subseteq B$ and $B \subseteq A$ we can consider that they are the same set. Thus, we obtain the following definition.

Definition 1.7 *[91] Two fuzzy sets A and B in X are equal, denoted by $A = B$, if and only if $\mu_A(x) = \mu_B(x)$, $\forall x \in X$.*

Before concluding this subsection, we introduce the fundamental operations between fuzzy sets.

Definition 1.8 *[68, 91] Let A be a fuzzy set in X.*

- *The standard complement of a A in X, which is denoted by A^c, is defined as the fuzzy set in X whose membership function is given by*

$$\mu_{A^c}(x) = 1 - \mu_A(x), \quad \forall x \in X$$

- *The standard intersection of A and B in X, which is denoted by $A \cap B$, is the fuzzy set of X defined by*

$$\mu_{A\cap B}(x) = \min\{\mu_A(x), \mu_B(x)\}, \quad \forall x \in X$$

- *The standard union A and B in X, which is denoted by $A \cup B$, is the fuzzy set of X defined by*

$$\mu_{A\cup B}(x) = \max\{\mu_A(x), \mu_B(x)\}, \quad \forall x \in X$$

Example 1.9 *Let X be the unit interval* $[3,4]$. *Let A and B be the fuzzy sets considered in Example 1.3.*

The complement of these fuzzy sets are:

i) A^c *defined as* $\mu_{A^c}(x) = \begin{cases} 0 & \text{if } x \in (3,4) \\ 1 & \text{if } x \notin (3,4) \end{cases}$

ii) B^c *is defined as* $\mu_{B^c}(x) = 1 - (x-3) = 4 - x$, $\forall x \in [3,4]$

Furthermore, it is clear that $A \not\subseteq B$ *and* $B \not\subseteq A$.

The intersection of A and B is the fuzzy set $A \cap B$ *defined as:*

$$\mu_{A\cap B}(x) = \begin{cases} 0 & \text{if } x = 4 \\ x & \text{if } x \neq 4 \end{cases}$$

It should be noted that the largest fuzzy set that is contained in both fuzzy sets is the intersection of the two fuzzy sets.

The union of A and B is the fuzzy set $A \cup B$ defined as:

$$\mu_{A\cup B}(x) = \begin{cases} 0 & \text{if } x = 3 \\ 1 & \text{if } x \neq 3 \end{cases}$$

With this classical interpretation, we can see that the smallest fuzzy set that contains both is the union of two fuzzy sets.

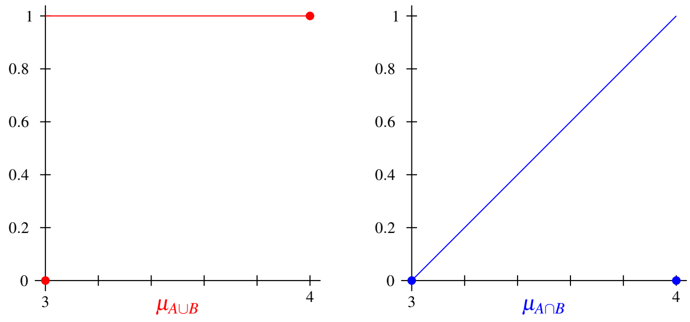

Figure 1.1: Union and intersection of A and B.

In Figure 1.1 the union of A and B is displayed in red, defined in Example 1.3, and the intersection in blue.

These fundamental operations can be found in the literature in a more generalized way. The goal is always to simply extend the fundamental operations for crisp sets. The main generalization is the one based on t-norms for the intersection and t-conorms for the union, which are particular cases of aggregation functions, concepts which will be used several times in this document. Thus, all these definitions are recalled here.

Definition 1.10 *[6, 59] Let $\mathscr{A} : \bigcup_{n\in\mathbb{N}}[0,1]^n \to [0,1]$ such that*

- $\mathscr{A}(0, \overset{(n)}{\ldots}, 0) = 0, \mathscr{A}(1, \overset{(n)}{\ldots}, 1) = 1$,
- $\mathscr{A}(a) = a$ *for all* $a \in [0,1]$,
- $\mathscr{A}$ *is increasing in each variable,*

then $\mathscr{A}$ is an aggregation function.

Important examples of aggregation functions are the arithmetic mean, the geometric mean, the median, the minimum, the maximum and the product. In general, the four main classes of aggregation functions are averaging, conjunctive, disjunctive and mixed (see [6]).

Definition 1.11 *An aggregation function $\mathscr{A}$ has averaging behaviour or is an averaging function if for every $(\alpha_1,\dots,\alpha_n) \in [0,1]^n$ it is bounded by* $\min\{\alpha_1,\dots,\alpha_n\} \leq \mathscr{A}(\alpha_1,\dots,\alpha_n) \leq \max\{\alpha_1,\dots,\alpha_n\}$.

Definition 1.12 *An aggregation function $\mathscr{A}$ has conjunctive behaviour or is a conjunctive aggregation function if for every $(\alpha_1,\dots,\alpha_n) \in [0,1]^n$ it is bounded by* $\mathscr{A}(\alpha_1,\dots,\alpha_n) \leq \min\{\alpha_1,\dots,\alpha_n\}$.

Definition 1.13 *An aggregation function $\mathscr{A}$ has disjunctive behaviour or is a disjunctive aggregation function if for every $(\alpha_1,\dots,\alpha_n) \in [0,1]^n$ it is bounded by* $\max\{\alpha_1,\dots,\alpha_n\} \leq \mathscr{A}(\alpha_1,\dots,\alpha_n)$.

Definition 1.14 *An aggregation function $\mathscr{A}$ is mixed if it does not belong to any of the above classes, i.e., it exhibits different types of behaviour on different parts of the domain.*

It is clear that the arithmetic mean, the geometric mean and the median are averaging functions, the minimum and the product are conjunctive and the maximum is disjunctive. An example of a mixed aggregation function (see [6]) could be:

$$\mathscr{A}(\alpha_1,\dots,\alpha_n) = \frac{\prod_{i=1}^{n}\alpha_i}{\prod_{i=1}^{n}\alpha_i + \prod_{i=1}^{n}(1-\alpha_i)} \text{ with the convention } \frac{0}{0} = 0.$$

When dealing with aggregation functions, some properties are very important for our purposes. In particular, we will focus on continuity and associativity.

Definition 1.15 *[34] Let $\mathscr{A}$ be an aggregation function. $\mathscr{A}$ is said to be:*

- *Continuous if the corresponding n-ary function* $\mathscr{A}|_{[0,1]^n}$ *is continuous, for any* $n \in \mathbb{N}$.
- *Associative if its associated two-argument function* $\mathscr{A}_2 : [0,1]^2 \to [0,1]$ *is associative.*

Consequently, the n-ary aggregation function can be constructed in a unique way by iteratively applying $\mathscr{A}|_{[0,1]^2}$ as

$$\mathscr{A}|_{[0,1]^n}(\alpha_1,\ldots,\alpha_n) = \mathscr{A}|_{[0,1]^2}(\mathscr{A}|_{[0,1]^2}(\ldots\mathscr{A}|_{[0,1]^2}(\alpha_1,\alpha_2),\alpha_3),\ldots,\alpha_n).$$

Thus bivariate associative aggregation functions univocally define extended aggregation functions. It is known that the product, minimum and maximum are associative aggregation functions while the arithmetic mean is not associative.

Triangular norms, or t-norms for short, and triangular conorms, or t-conorms, are two distinct families of aggregation functions since it is possible to extend them to $\bigcup\limits_{n\in\mathbb{N}}[0,1]^n$ by their associativity. They appear to be generalizations of minimum and maximum, which are used to define the intersection and union of two fuzzy sets, respectively.

Definition 1.16 *[48] A map* $T : [0,1]\times[0,1] \to [0,1]$ *is said to be a t-norm if it satisfies the following conditions:*

- *Associativity:* $T(T(\alpha_1,\alpha_2),\alpha_3) = T(\alpha_1,T(\alpha_2,\alpha_3))$ *for all* $\alpha_1,\alpha_2,\alpha_3 \in [0,1]$.
- *Commutativity:* $T(\alpha_1,\alpha_2) = T(\alpha_2,\alpha_1)$ *for all* $\alpha_1,\alpha_2 \in [0,1]$.
- *Monotonicity:* $T(\alpha_1,\alpha_3) \le T(\alpha_2,\alpha_3)$ *for all* $\alpha_1,\alpha_2,\alpha_3 \in [0,1]$ *with* $\alpha_1 \le \alpha_2$.
- *Boundary condition:* $T(\alpha,1) = \alpha$ *for all* $\alpha \in [0,1]$.

In a precisely analogous approach, a t-conorm is defined formally as follows:

Definition 1.17 *[48] A map* $S : [0,1]\times[0,1] \to [0,1]$ *is said to be a t-conorm if it is associative, commmutative, increasing, increasing on each argument and it fulfills the boundary condition:*

$$S(\alpha,0) = \alpha \text{ for all } \alpha \in [0,1].$$

Clearly, there is a way to connect these two ideas via duality. If T is a t-norm, it can be used to create a t-conorm $S_T : [0,1] \times [0,1] \to [0,1]$ as follows: $S_T(\alpha.\beta) = 1 - T(1-\alpha, 1-\beta)$, for every $\alpha.\beta \in [0,1]$. We say that T and S_T are dual or complementary.

Let us present several common t-norms and their dual t-conorms.

- The minimum t-norm and the maximum t-conorm:

$$T_M(\alpha.\beta) = \min\{\alpha.\beta\} \text{ and } S_M(\alpha.\beta) = \max\{\alpha.\beta\}$$

- The product t-norm and the probabilistic sum t-conorm:

$$T_P(\alpha.\beta) = \alpha \cdot \beta \text{ and } S_P(\alpha.\beta) = \alpha + \beta - \alpha \cdot \beta$$

- The Lukasiewicz t-norm and t-conorm:

$$T_L(\alpha.\beta) = \max\{\alpha + \beta - 1, 0\} \text{ and } S_L(\alpha.\beta) = \min\{\alpha + \beta, 1\}$$

- The drastic t-norm and t-conorm:

$$T_D(\alpha.\beta) = \begin{cases} \min\{\alpha.\beta\} & \text{if } \alpha = 1 \text{ or } \beta = 1 \\ 0 & \text{otherwise} \end{cases}$$

$$S_D(\alpha.\beta) = \begin{cases} \max\{\alpha.\beta\} & \text{if } \alpha = 0 \text{ or } \beta = 0 \\ 1 & \text{otherwise} \end{cases}$$

Moreover, it is known that for any t-norm T we have that $T(\alpha.\beta) \leq T_M(\alpha.\beta)$ and for any t-conorm S we have that $S(\alpha.\beta) \geq S_M(\alpha.\beta)$ for every $\alpha.\beta \in [0,1]$.

Thus, given any t-norm T and any t-conorm S, the general intersection and union of two fuzzy sets A and B could be defined, for any $x, y \in X$, by $\mu_{A \cap_T B}(x) = T(\mu_A(x), \mu_B(x))$ and $\mu_{A \cup_S B}(x) = S(\mu_A(x), \mu_B(x))$, respectively.

1.2 Generalizations of fuzzy sets

Fuzzy sets cover some uncertainty cases, however, membership grades are real numbers that cannot be always determined with complete accuracy. The use of exact real numbers as membership grades seems to go against the basic idea of fuzziness in many real-world situations where any assessment of membership grades is often an approximation that entails some degree of arbitrariness. Numerous generalized forms of fuzzy sets have been proposed in the literature as solutions to this issue.

Considering the two most significant generalizations of fuzzy sets – hesitant fuzzy sets and interval-valued fuzzy sets – is the next step. The first one appears in a natural way for instance in a realistic decision-making problem. When a group of experts is tasked with evaluating candidate alternatives, it is common for them to find some disagreements. Due to the differing opinions of the experts and the difficulty in persuading each other, achieving a consensus can be challenging. Instead, it is immediate for them to arrive at a set of possible values. The second one appears when the expert has a clear idea of a lower and an upper bound for the membership degree, although he or she does not know which is the exact value of that membership. It is clear that they are not as specific as fuzzy sets, but this lack of specificity makes them more realistic in some applications and therefore more credible. The following subsections illustrate the similarities and distinctions between these topics, as well as the main concepts related to them which are necessary for the remaining chapters.

1.2.1 Hesitant fuzzy sets

Hesitant fuzzy sets were first presented by Torra in 2010 [76]. Later, Xia and Xu created a number of aggregation operators for uncertain fuzzy information [83] and used them with multi-criteria discrete-valued data [62]. This large family of sets, the hesitant fuzzy sets, includes both intuitionistic fuzzy sets and interval-valued fuzzy sets. The concept of hesitant fuzzy sets was first suggested by Grattan-Guinnes in 1976 [35], although there have been a lot more studies on the subject since Torra proposed his work in 2010. They were known as set-valued fuzzy sets at that time.

A hesitant fuzzy set A is defined by Torra [76] in terms of a function h_A that, when applied to X, returns a subset of $[0,1]$.

We can find definitions like the following in the literature that are equivalent to Torra's original suggestion.

Definition 1.18 *[68] Let X be the reference universe. A hesitant fuzzy set A over X is characterised by a function $h_A : X \to \mathscr{P}([0,1])$, where $\mathscr{P}([0,1])$ denotes the power set of the interval* $[0,1]$.

It is clear that hesitant fuzzy sets are thus an extension of fuzzy sets. The corresponding membership function of a hesitant fuzzy set arrives in any subset of $[0,1]$, but the membership function of a fuzzy set arrives on a single value in $[0,1]$. Following Torra's definition of hesitant fuzzy sets, Xia and Xiu [83] added the mathematical description of a hesitant fuzzy set as follows:

$$A = \{\langle x, h_A(x)\rangle : x \in X\}$$

where $h_A(x)$ is a set of some values in $[0,1]$. The family of all hesitant fuzzy sets over the universe X will be denoted by $HFS(X)$.

A particular sort of hesitant fuzzy set is a typical hesitant fuzzy set. There are many definitions of this type of set in the literature. Here we present two definitions that are equivalent.

Definition 1.19 *[61] A typical hesitant fuzzy set A in the universe X is a hesitant fuzzy set where for each $x \in X$, $h_A(x)$ is a finite subset of* $[0,1]$.

Other similar definitions for this concept were introduced by different authors. Thus,

Definition 1.20 *[5, 69] Let $\mathbb{H} = \{S \subseteq [0,1] : S \text{ is finite and } S \neq \emptyset\}$. A typical hesitant fuzzy set A in the universe X is given by $A = \{\langle x, h_A(x)\rangle : x \in X\}$, where $h_A : X \longrightarrow \mathbb{H}$.*

We will denote the family of all typical hesitant fuzzy sets over the universe X by $THFS(X)$. In order to use hesitant fuzzy sets properly it is advised to take into account typical hesitant fuzzy sets [61, 69]. From now on, we will be using typical hesitant fuzzy set despite the fact we write just hesitant fuzzy set. Thus, a hesitant fuzzy set will be by default an element in $THFS(X)$. Each $h_A(x) \in \mathbb{H}$ is called a typical hesitant fuzzy element.

About the notation, we will follow Definition 1.20, so we will talk about a hesitant fuzzy set A using its membership function h_A. We will use $\#h_A(x)$ as the number of elements of the typical hesitant fuzzy element $h_A(x)$ for any x in X and we will denote by $\mathbf{0}_{\mathbb{H}}$ and $\mathbf{1}_{\mathbb{H}}$ the typical hesitant fuzzy elements $\{0\}$ and $\{1\}$, respectively.

We are also considering that typical hesitant fuzzy elements are ordered in an increasing way, i.e., if $h_A(x)$, $x \in X$, is a typical hesitant fuzzy element where $\#h_A(x) = n$, then $h_A(x)^1 \leq h_A(x)^2 \leq \cdots \leq h_A(x)^n$, where $h_A(x)^i$ is the ith component of $h_A(x)$. Some authors as Santos et al. [71] and Bedregal et al. [5] ignore this statement an apply a function σ which is a increasing permutation such that given $h_A(x)$ with $\#h_A(x) = n$, then $\sigma(h_A(x)) = \{\sigma(h_A(x))^1, \ldots, \sigma(h_A(x))^n\}$ with $\sigma(h_A(x))^1 \leq \sigma(h_A(x))^2 \cdots \leq \sigma(h_A(x))^n$.

The set of all unitary subsets of $\mathscr{P}([0,1])$ is called the set of diagonal or degenerate elements of $\mathbb{H}$ and is denoted by $\mathscr{E}_{\mathbb{H}} = \{h \in \mathbb{H} : \#h = 1\}$. With these sets we recover the idea of fuzzy values. We will denote by $\mathbb{H}^{(n)} = \{h \in \mathbb{H} : \#h = n\}$. Moreover, it is clear that $\mathbb{H} = \bigcup_{n \in \mathbb{N}} \mathbb{H}^{(n)}$ with

$$\mathbb{H}^{(n)} = \{(\alpha_1, \ldots, \alpha_n) \in [0,1]^n : \alpha_i \leq \alpha_j \text{ if } i < j\}.$$

Example 1.21 *Let $X = \{0, 0.5, 1\}$ be the referential. The following two sets are examples of a typical hesitant fuzzy sets:*

i) $A = \{\langle 0, \{0.25, 0.5\}\rangle, \langle 0.5, \{0\}\rangle, \langle 1, \{0.2, 0.4, 0.6, 0.8\}\rangle\}$

ii) $B = \{\langle x, h_B(x)\rangle : x \in X\}$ *where* $h_B(x) = \left\{\dfrac{e^x}{e}\right\}$

It is clear that B can be seen as a fuzzy set, but A is not.

A hesitant fuzzy set can also be obtained from a set of fuzzy sets:

Definition 1.22 *[69, 76] Let $M = \{\mu_1, \mu_2, ..., \mu_n\}$ be a set of n membership functions. The hesitant fuzzy set associated to M is the one given by the membership* *function* $h_M : X \to \mathbb{H}^{(n)}$ *defined as follows:*

$$h_M(x) = \bigcup_{\mu \in M} \{\mu(x)\}$$

It is interesting that this description fits decision-making so well when experts need to evaluate a range of options. In this situation, M stands for the expert opinions for each alternative, and h_M for the opinions of the group of experts.

Since the typical hesitant fuzzy elements may have different cardinals, one of the major problems of typical hesitant fuzzy elements is that there is no easy way to compare them. We are going to discuss potential solutions to this issue in this research. In order to avoid misunderstandings with orders on the real line or for intervals, we will use the symbol "$\trianglelefteq$" to denote when a typical hesitant fuzzy element is smaller than or equal to another one, and "$\triangleleft$" when there is no equality.

Preorders for typical hesitant fuzzy elements based on score functions

For typical hesitant fuzzy sets, the membership value at any point in the referential is a finite and nonempty subset of the interval $[0,1]$. Thus, we have several values associated with each point. A first approach for dealing with this multiple information is to summarize it in just a value. This idea is considered several times when we are working with hesitant fuzzy sets and it was done by means of the score functions. As we can see in [27], they are in fact aggregation functions applied to the elements in $\mathbb{H}$. Thus, one of the most usual score functions is the arithmetic mean, denoted as previously by $\mathscr{M}$.

Thus, a first approach for comparing typical hesitant fuzzy elements when given by Xia and Xu [83] by using this score function. More precisely.

Definition 1.23 *[83] Let A be a typical hesitant fuzzy set in X. The arithmetic mean score function for any typical hesitant fuzzy element* $h_A(x)$ *is defined as:*

$$s(h_A(x)) = \frac{1}{\#h_A(x)} \sum_{\gamma \in h_A(x)} \gamma, \forall x \in X$$

where $\#h_A(x)$ *denotes the cardinal of* $h_A(x)$.

If we consider $h_A(x) = \{h_A(x)^1, h_A(x)^2, \ldots, h_A(x)^n\}$, that is, if $h_A(x)^i$ denotes the component ith of $h_A(x)$, it is clear that s is just the consequence of applying the arithmetic mean to the components of $h_A(x)$. That is,

$$s(h_A(x)) = \mathscr{M}(h_A(x)) = \frac{1}{n}\sum_{i=1}^{n} h_A(x)^i$$

This is, in fact, the most typical score function. For this reason, sometimes it is just called the score function.

This is also the reason why Xia and Xu, and later Rashid and Beg, use it to define the following relations.

Definition 1.24 *[83] Let X be the referential, let A and B be two sets in $THFS(X)$ and let x be an element in X. It is said that the typical hesitant fuzzy element $h_A(x)$ is s-lower than or equal to the typical hesitant fuzzy element $h_B(x)$, and it is denoted by $h_A(x) \trianglelefteq h_B(x)$, if $s(h_A(x)) \leq s(h_B(x))$.*

It is immediate that $\trianglelefteq$ is a preorder (reflexive and transitive) in $\mathbb{H}$, but it is not an order, since it is not symmetric.

They used this relation between typical hesitant fuzzy elements to define a partial preorder for typical hesitant fuzzy sets.

Definition 1.25 *[83, 65] Let X be the referential set and let A and B be two sets in $THFS(X)$. A is said to be lower than or equal to B w.r.t. the score function s if*

$$s(h_A(x)) \leq s(h_B(x)), \textit{ for all } x \in X$$

Due to its importance, a lot of score functions were proposed in the literature, apart from the one considered in Definition 1.23. Thus, Fahardinia [26] proposed the score function:

$$S_{Nia}(h_A(x)) = \frac{\sum_{i=1}^{n} \delta(i) h_A(x)^i}{\sum_{i=1}^{n} \delta(i)}$$

such that $\{\delta(i)\}_1^n$ is an increasing positive-valued sequence of index i. It is common to set $\{\delta(i)\}_1^n = \{i\}_1^n$.

Xu and Xia [86] introduce a class of hesitant fuzzy element ranking functions based on the distance between the typical hesitant fuzzy element and the one with the same cardinality but all the values equal to 1. There are two typical ranking functions among others [28], and they are

- the hesitant normalized Hamming distance score function, defined as $S_{xux}^{-d_{hnh}}(h_A(x)) = \frac{1}{n}\sum_{i=1}^{n}\left|h_A(x)^i - 1\right|,$
- the hesitant normalized Euclidean distance score function, defined as $S_{xux}^{-d_{\text{hne}}}(h_A(x)) = \left(\frac{1}{n}\sum_{i=1}^{n}\left(h_A(x)^i - 1\right)^2\right)^{\frac{1}{2}}$

Considering the same ideas, Xu and Xia also proposed in [87] some other distance measures for ranking typical hesitant fuzzy elements such as:

- $S_{xux}^{-d_3}(h_A(x)) = \max_{1\le i\le n}\left\{\left|h_A(x)^i - 1\right|\right\}$
- $S_{xux}^{-d_4}(h_A(x)) = \max_{1\le i\le n}\left\{\left|h_A(x)^i - 1\right|^2\right\}$
- $S_{xux}^{-d_5}(h_A(x)) = \frac{1}{2}\left(\frac{1}{n}\sum_{i=1}^{n}\left|h_A(x)^i - 1\right| + \max_{1\le i\le n}\left\{\left|h_A(x)^i - 1\right|\right\}\right)$
- $S_{xux}^{-d_6}(h_A(x)) = \frac{1}{2}\left(\sqrt{\frac{1}{n}\sum_{i=1}^{n}\left|h_A(x)^i - 1\right|^2} + \max_{1\le i\le n}\left\{\left|h_A(x)^i - 1\right|^2\right\}\right)$

It should be pointed out that for all of these ranking functions, as they are based on distances, for two different typical hesitant fuzzy elements $h_A(x)$ and $h_B(x)$, if $S_{xux}^{-d}(h_A(x)) \geq S_{xux}^{-d}(h_B(x))$, then $h_A(x) \trianglelefteq h_B(x)$. In fact, we could consider that $1-d$ is measuring the same as the score function and we are using the same criterium.

The similarity measure, correlation measure, or relative closeness measure can be used in place of the distance measure to obtain the ranking order of typical hesitant fuzzy elements and appear to provide different sorts of ranking systems.

Although the score function introduced in Definition 1.23 is the most typical, Farhadinia [27] defined several score functions for rating typical hesitant fuzzy elements. Let us note that, in fact, they are just examples of the most usual aggregation function applied to the components of a typical hesitant fuzzy element. Thus, with these examples, we will also show examples of aggregation functions and fix the related notation about score functions.

- The smallest score function:

$$S_{\nabla}(h_A(x)) = \begin{cases} 1, & \text{if } h_A(x)^i = 1, \forall i \in \{1, \ldots, \#h_A(x)\} \\ 0, & \text{otherwise} \end{cases}$$

- The greatest score function:

$$S_{\triangle}(h_A(x)) = \begin{cases} 0, & \text{if } h_A(x)^i = 0, \forall i \in \{1, \ldots, \#h_A(x)\} \\ 1, & \text{otherwise} \end{cases}$$

- The geometric-mean score function:

$$S_{GM}(h_A(x)) = \left(\prod_{i=1}^{n} h_A(x)^i \right)^{\frac{1}{n}}$$

- The minimum score function:

$$S_{\text{Min}}\,(h_A(x)) = \min \left\{ h_A(x)^1, h_A(x)^2, \ldots, h_A(x)^n \right\}$$

- The maximum score function:

$$S_{\text{Max}}\,(h_A(x)) = \max \left\{ h_A(x)^1, h_A(x)^2, \ldots, h_A(x)^n \right\}$$

- The product score function:

$$S_P(h_A(x)) = \prod_{i=1}^{n} h_A(x)^i$$

- The bounded sum score function:

$$S_{BS}(h_A(x)) = \min\left\{1, \sum_{i=1}^{n} h_A(x)^i\right\}$$

- The fractional score function:

$$S_F(h_A(x)) = \frac{\prod_{i=1}^{n} h_A(x)^i}{\prod_{i=1}^{n} h_A(x)^i + \prod_{i=1}^{n} \left(1 - h_A(x)^i\right)}$$

with the convention $\frac{0}{0} = 0$.

In all the cases, we may determine the order of the typical hesitant fuzzy elements $h_A(x)$ and $h_B(x)$ by using any Farhadinia's scoring function, let us denote them S_*, thus if $S_*(h_A(x)) \leq S_*(h_B(x))$, then $h_A(x) \trianglelefteq h_B(x)$.

The problem that appears using score functions is that we can have two different objects $h_A(x)$ and $h_B(x)$ such that $h_A(x) \neq h_B(x)$, but satisfying $S(h_A(x)) = S(h_B(x))$, which is a bit counter-intuitive as we can see in the following example.

Example 1.26 *Let us consider two typical hesitant fuzzy elements $h_A(x)$ and $h_B(x)$ defined as $h_A(x) = \{0, 0.2, 0.8, 1\}$ and $h_B(x) = \{0, 0.1, 0.9, 1\}$. We have that,*

- $\mathscr{M}(h_A(x)) = 0.5 = \mathscr{M}(h_B(x))$
- $S_{xux}^{-d_{hnh}}(h_A(x)) = 0.5 = S_{xux}^{-d_{hnh}}(h_B(x))$
- $S_{xux}^{-d_3}(h_A(x)) = 0.5 = S_{xux}^{-d_3}(h_B(x))$
- $S_{\nabla}(h_A(x)) = 0 = S_{\nabla}(h_B(x))$
- $S_{\triangle}(h_A(x)) = 1 = S_{\triangle}(h_B(x))$
- $S_{min}(h_A(x)) = 0 = S_{min}(h_B(x))$
- $S_{max}(h_A(x)) = 1 = S_{max}(h_B(x))$

- $S_{BS}(h_A(x)) = 1 = S_{BS}(h_B(x))$
- $S_F(h_A(x)) = 0.5 = S_F(h_B(x))$

Thus, using the score function to summarize the values of the membership function is a procedure where a lot of information is lost. Therefore, it is not possible to obtain a symmetric relation based on these functions and so order on $\mathbb{H}$. To solve this problem, it is necessary to consider a new approach for ordering typical hesitant fuzzy elements.

Partial orders for typical hesitant fuzzy elements

First of all, we would like to introduce the more natural way to rank or compare typical hesitant fuzzy elements, which is the lattice order, also known as product order:

Definition 1.27 *Given $h_A(x)$ and $h_B(x)$ two elements in $\mathbb{H}^{(n)}$, $h_A(x)$ is said to be lower than or equal to $h_B(x)$ w.r.t. the lattice order, and it is denoted by $h_A(x) \trianglelefteq_{Lo} h_B(x)$, if and only if $h_A(x)^i \leq h_B(x)^i$ for any $i \in \{1, 2, \ldots, n\}$, where $h_A(x)^i$ is the component ith of $h_A(x)$.*

Although this order is a generalization of the usual order on $\mathbb{R}$, it presents at least two inconveniences. First, it is limited as the typical hesitant fuzzy elements must have the same cardinality, and second, it is not a total order.

Bedregal et al. [5] pointed out that two procedures could be implemented to compare two typical hesitant fuzzy elements with different cardinals:

- φ-normalization, remove elements of the set having more elements,
- ψ-normalization, add elements to the set having fewer elements.

Other authors such as Fahardinia [26], Xia [84] or Zhang [95] also considered the ψ-normalization.

Bedregal et al. [5] defined the φ-normalization by the function

$$\varphi(h,k) = \begin{cases} h_A(x) & \text{if } \#h_A(x) \leq k \\ h_A(x) \text{ without the first } \#h_A(x) - k \text{ elements} & \text{otherwise} \end{cases}$$

where $h_A(x)$ is a typical hesitant fuzzy element, $k \in \mathbb{N}$ and $\#h_A(x)$ denotes the cardinality of $h_A(x)$.

Example 1.28 *If we consider the typical hesitant fuzzy elements $h_A(x)$ and $h_B(x)$ given by $h_A(x) = \{0.1, 0.4, 0.8\}$ and $h_B(x) = \{0.2, 0.5, 0.6, 0.9\}$, we could apply the φ-normalization in the following way. We obtain that*

$$\varphi(h_A(x), \#h_B(x)) = \varphi(h_A(x), 4) = h_A(x)$$

$$\varphi(h_B(x), \#h_A(x)) = \varphi(h_B(x), 3) = \{0.5, 0.6, 0.9\},$$

so it is clear that $\varphi(h_A(x), \#h_B(x)) \trianglelefteq_{Lo} \varphi(h_B(x), \#h_A(x))$.

Thus, when we consider φ-normalization the comparison by means of the lattice order is not restricted to typical hesitant fuzzy elements with the same cardinality. However, the second problem remains, since not all the elements are comparable. For instance, if we consider $h_C(x) = \{0.1, 0.2, 0.3, 0.9\}$, we have that

$$\varphi(h_A(x), \#h_C(x)) = \{0.1, 0.4, 0.8\}$$

$$\varphi(h_C(x), \#h_A(x)) = \{0.2, 0.3, 0.9\}$$

and therefore we obtain that $\varphi(h_A(x), \#h_C(x)) \ntrianglelefteq_{Lo} \varphi(h_C(x), \#h_A(x))$ and also that $\varphi(h_C(x), \#h_A(x)) \ntrianglelefteq_{Lo} \varphi(h_A(x), \#h_C(x))$.

Bedregal et al. [5] only take into account this kind of φ-normalization, where $h_A(x)$ without $\#h_A(x) - k$ elements means we are considering $\{h_A(x)^{\#h_A(x)-k+1}, \ldots, h_A(x)^{\#h_A(x)}\}$. Nevertheless, it is clear that there are more possible ways such as removing the greatest elements instead of the lowest ones, but using the lattice order we will always have the drawback of being a partial order.

On the other hand, the ψ-normalization is defined by the function

$$\psi(h_A(x), k) = \begin{cases} h_A(x) & \text{if } \#h_A(x) \geq k \\ h_A(x) \text{ with } k - \#h_A(x) \text{ more elements} & \text{otherwise} \end{cases}$$

where $h_A(x)$ is a typical hesitant fuzzy element and $k \in \mathbb{N}$.

In the literature it is possible to find two common ψ-normalizations which are the following [30]:

Let us consider two typical hesitant fuzzy elements $h_A(x)$ and $h_B(x)$ with cardinalities l_1 and l_2, respectively, that is, $h_A(x) = \{h_A(x)^1, h_A(x)^2, \ldots, h_A(x)^{l_1}\}$ and $h_B(x) = \{h_B(x)^1, h_B(x)^2, \ldots, h_B(x)^{l_2}\}$.

- Optimistic: we add the maximum element, which means that we extend $h_A(x)$ to $h_A(x)_{max} = \{h_A(x)^1, h_A(x)^2, \ldots, \overbrace{h_A(x)^{l_1}, \ldots, h_A(x)^{l_1}}^{l_2-l_1+1 \text{ times}}\}$
- Pesimistic: we add the minimum element, which means that we extend $h_A(x)$ to $h_A(x)_{min} = \{\overbrace{h_A(x)^1, \ldots, h_A(x)^1}^{l_2-l_1+1 \text{ times}}, h_A(x)^2, \ldots, h_A(x)^{l_1}\}$

Based on this idea, we can find the Xu and Xia's order [86] defined as:

$$h_A(x) \trianglelefteq_{XX} h_B(x) \text{ if and only if } \begin{cases} h_A(x)_{min} \trianglelefteq_{Lo} h_B(x) & \text{if } l_1 \leq l_2 \\ h_A(x) \trianglelefteq_{Lo} h_B(x)_{min} & \text{otherwise} \end{cases}$$

Example 1.29 *If we consider $h_A(x)$ and $h_B(x)$ defined in Example 1.28, we could apply the ψ-normalization in the following ways.*

- *The optimistic ψ-normalization of $h_A(x)$ is $h_A(x)_{max} = \{0.1, 0.4, 0.8, 0.8\}$ and therefore we have to compare this element with $h_B(x) = \{0.2, 0.5, 0.6, 0.9\}$.*
- *If we consider the pessimistic ψ-normalization, we have to compare $h_A(x)_{min} = \{0.1, 0.1, 0.4, 0.8\}$ and $h_B(x) = \{0.2, 0.5, 0.6, 0.9\}$.*

It is clear, in both cases that these elements are not comparable.

Garmendia et al. [30] proposed a different way of normalization. First, we should introduce the following operator:

Definition 1.30 *Let $h_A(x) = \{h_A(x)^1, h_A(x)^2, \ldots, h_A(x)^l\}$ be a typical hesitant fuzzy element and $r \in \mathbb{N}$. We define*

$$h_A(x)_{(r)} = \{\overbrace{h_A(x)^1, \ldots, h_A(x)^1}^{r \text{ times}}, \ldots, \overbrace{h_A(x)^l, \ldots, h_A(x)^l}^{r \text{ times}}\}$$

Keeping this in mind, Garmendia et al. [30] defined the following order:

$$h_A(x) \trianglelefteq_G h_B(x) \text{ if and only if } h_A(x)_{\left(\frac{lcm(l_1,l_2)}{l_1}\right)} \trianglelefteq_{Lo} h_B(x)_{\left(\frac{lcm(l_1,l_2)}{l_2}\right)}$$

where $lcm(l_1, l_2)$ is the least common multiple of l_1 and l_2.

Example 1.31 *If we consider $h_A(x)$ and $h_B(x)$ defined in Example 1.28, we could try to compare them with the Garmendia et al. order. First, we should construct $h_A(x)_{\left(\frac{lcm(l_1,l_2)}{l_1}\right)}$ and $h_B(x)_{\left(\frac{lcm(l_1,l_2)}{l_2}\right)}$.*

- $h_A(x)_{(4)} = \{0.1, 0.1, 0.1, 0.1, 0.4, 0.4, 0.4, 0.4, 0.8, 0.8, 0.8, 0.8\}$
- $h_B(x)_{(3)} = \{0.2, 0.2, 0.2, 0.5, 0.5, 0.5, 0.6, 0.6, 0.6, 0.9, 0.9, 0.9\}$

And again we have the same drawback, these elements are not comparable with the lattice order, since $h_A(x)^1_{(4)} = 0.1 < 0.2 = h_B(x)^1_{(3)}$ and $h_A(x)^9_{(4)} = 0.8 > 0.6 = h_B(x)^9_{(3)}$.

Another approach also based on normalization was given by Zhang and Yang.

Definition 1.32 *[94] Let us consider two typical hesitant fuzzy elements $h_A(x) = \{h_A(x)^1, h_A(x)^2, \ldots, h_A(x)^{l_1}\}$ and $h_B(x) = \{h_B(x)^1, h_B(x)^2, \ldots, h_B(x)^{l_2}\}$. The order $\trianglelefteq_{ZY}$ is defined as*

$$h_A(x) \trianglelefteq_{ZY} h_B(x) \text{ if and only if } \begin{cases} h_A(x)^i \leq h_B(x)^i & i = 1, \ldots, , l_1 \quad \text{if } l_1 \leq l_2 \\ h_A(x)^{l_1 - l_2 + i} \leq h_B(x)^i & i = 1, \ldots, , l_2 \quad \text{otherwise} \end{cases}$$

Clearly, this order is a combination of the lattice order and the φ-normalization. In addition, this is also a partial order as we can see in the following example.

Example 1.33 *If we consider again the elements in Example 1.28, that is, $h_A(x) = \{0.1, 0.4, 0.8\}$ and $h_B(x) = \{0.2, 0.5, 0.6, 0.9\}$, then $l_1 = 3 < l_2 = 4$. Thus, $h_A(x) \not\trianglelefteq_{ZY} h_B(x)$ due to $0.8 \not\leq 0.6$ and $h_B(x) \not\trianglelefteq_{ZY} h_A(x)$ because $0.5 \not\leq 0.1$. Therefore, $h_A(x)$ and $h_B(x)$ are not comparable by means of the order $\trianglelefteq_{ZY}$.*

Normalization seems to be a nice possible way in order to compare two typical hesitant fuzzy elements. However, we could have some troubles as they is no way to define total orders. In fact, some drawbacks even arise when having a total order for same-cardinality elements. Thus, let us suppose that we have a total order $\trianglelefteq$ on $\mathbb{H}^{(n)}$, and that we have two typical hesitant fuzzy elements $h_A(x)$ and $h_B(x)$ with cardinalities $\#h_A(x) = m$ and $\#h_B(x) = n$. Let us suppose also that $m < n$. If we apply φ-normalization of $h_B(x)$, i.e., $\varphi(h_B(x), \#h_A(x))$, another problem arises, which is how to compare $h_B(x)$ and $\varphi(h_B(x), \#h_A(x))$ as if we compute the φ-normalization of $h_B(x)$ we would obtain that $h_B(x)$ and $\varphi(h_B(x), \#h_A(x))$ should be representing the same information.

Example 1.34 *For instance, if we consider again the elements in Example 1.28, we have that* $h_B(x) = \{0.2, 0.5, 0.6, 0.9\}$ *and its φ-normalization with respect to* $\#h_A(x)$ *is* $\varphi(h_B(x), \#h_A(x)) = \varphi(h_B(x), 3) = \{0.5, 0.6, 0.9\}$. *It is clear that these two elements are not the same. However,* $\varphi(h_B(x), \#h_A(x))$ *is representing* $h_B(x)$. *Thus, there is a loss of information which could be very significant.*

In a similar way, we have the same problem with the ψ-normalization and the method proposed by Garmendia et al. [30].

Total orders for typical hesitant fuzzy sets

When we consider total orders, we usually obtain better results as they are a particular case of partial orders. So as we add more restrictions, it is usually easier to obtain good theoretical results. Nevertheless, in practice, it is more difficult to obtain an order where for given any $h_A(x)$ and $h_B(x)$, we have to decide which one is larger. One of the most employed techniques for dealing with total orders is the use of admissible orders. The idea behind an admissible order is to consider a partial order and make it linear. The usual partial order considered is the lattice order, just taking into account that the only agreement among all of the proposed orders for typical hesitant fuzzy elements is that they all refine the lattice order, which is the natural order on $\mathbb{R}^n$. More precisely,

Definition 1.35 *[79] Let* $(\mathbb{H}^{(n)}, \trianglelefteq)$ *be an ordered set. The order* $\trianglelefteq$ *is called admissible if it is a linear order on* $\mathbb{H}^{(n)}$ *and if it refines the lattice order (if* $h_A(x) \trianglelefteq_{Lo} h_B(x)$

then if $h_A(x) \trianglelefteq h_B(x)$*).*

Appropriate mappings acting on the n elements of typical hesitant fuzzy elements can provide partial orders as well as admissible orders. Wang and Xu [79] proposed the following definition:

Definition 1.36 *[79] Let* $\trianglelefteq$ *be an admissible order on* $\mathbb{H}^{(n)}$*. The order* $\trianglelefteq$ *is called a generated admissible order if there exist n continuous functions* $f_i : \mathbb{H}^{(n)} \to [0,1], i = 1,2,\ldots,n$*, such that* $h_A(x) \trianglelefteq h_B(x)$ *if, and only if,*

$$\{f_1(h_A(x)), f_2(h_A(x)), \ldots, f_n(h_A(x))\} \trianglelefteq_{Lex} \{f_1(h_B(x)), f_2(h_B(x)), \ldots, f_n(h_B(x))\},$$

for all $h_A(x), h_B(x) \in \mathbb{H}^{(n)}$*, where* $h_A(x) \trianglelefteq_{Lex} h_B(x)$ *if* $(h_A(x) = h_B(x)) \vee [\exists m > 0 : \forall i < m, (h_A(x)^i = h_B(x)^i) \wedge (h_A(x)^m < h_B(x)^m)]$

In this sense, the n functions $f_1, f_2, \ldots, f_n$ are called a generating n-tuple of the order $\trianglelefteq$.

Wang and Xu not only proposed some methods to obtain admissible orders for hesitant fuzzy sets that have the same cardinal, but also characterised some results of admissible orders.

Theorem 1.37 *[79] Let* $\trianglelefteq$ *be an admissible order on* $\mathbb{H}^{(n)}$*. Then it cannot be induced by* $n-1$ *continuous functions* $f_i : [0,1]^n \to [0,1], i = 1,2,\ldots,n-1$*.*

Theorem 1.38 *[79] Let* $f_i : \mathbb{H}^{(n)} \to [0,1] (i = 1,2,\ldots,n)$ *be n continuous aggregation functions such that* $\forall h_A(x), h_B(x) \in \mathbb{H}^{(n)}, f_i(h_A(x)) = f_i(h_B(x)) (i = 1,2,\ldots,n)$ *hold if and only if* $h_A(x) = h_B(x)$*. Define the relation* $\trianglelefteq f_1, f_2, \ldots, f_n$ *on* $\mathbb{H}^{(n)}$ *by*

$$h_A(x) \trianglelefteq_{f_1,f_2,\ldots,f_n} h_B(x) \Leftrightarrow (h_A(x) = h_B(x)) \vee \left(h_A(x) \prec_{f_1,f_2,\ldots,f_n} h_B(x)\right),$$

where $h_A(x) \prec_{f_1,f_2,\ldots,f_n} h_B(x)$ *if and only if*

$$(\exists m > 0)(\forall i < m)(f_i(h_A(x)) = f_i(h_B(x))) \wedge (f_m(h_A(x)) < f_m(h_B(x))).$$

Then $\trianglelefteq_{f_1,f_2,\ldots,f_n}$ *is an admissible order on* $\mathbb{H}^{(n)}$*.*

Nevertheless, this method has the limitation that the cardinals of the typical hesitant fuzzy elements should be the same. Now, we will recall another perspective of admissible orders for typical hesitant fuzzy elements.

Definition 1.39 *[56] An order $\trianglelefteq$ on $\mathbb{H}$ is admissible if it refines the standard partial order $\trianglelefteq_{RH}$ defined by*

$$h_A(x) \trianglelefteq_{RH} h_B(x) \text{ if and only if } (h_A(x) = \mathbf{0}_{\mathbb{H}} \text{ or } (h_B(x) = \mathbf{1}_{\mathbb{H}}) \text{ or}$$

$$(\#h_A(x) = \#h_B(x) = n \text{ and } h_A(x)^{(i)} \leq h_B(x)^{(i)}, \forall i \in \mathbb{N}_n),$$

where $\mathbb{N}_k = \{1, 2, \ldots, k\}$ be the subset of natural numbers.

It should be noticed that the admissible orders in Definition 1.39 are different from the admissible orders for typical hesitant fuzzy elements proposed in Definition 1.35, as they considered a total order for typical hesitant fuzzy element when they are restricted to a cardinal n [57].

Now we present two admissible orders proposed by Matzenauer et al., which will be very important along this document.

Theorem 1.40 *[56] The relations $\trianglelefteq_{Lex1}$ and $\trianglelefteq_{Lex2}$ on $\mathbb{H}$, are given, respectively, as follows, for any typical hesitant fuzzy elements $h_A(x)$ and $h_B(x)$ with $m = \#h_A(x)$ and $n = \#h_B(x)$:*

- $h_A(x) \trianglelefteq_{Lex1} h_B(x)$ *iff*

$$\begin{cases} \exists i \in \mathbb{N}_{\min\{m,n\}} : h_A(x)^i < h_B(x)^i \text{ and } h_A(x)^j = h_B(x)^j, \forall j < i; \\ \text{or } m \leq n \text{ and } h_A(x)^j = h_B(x)^j, \forall j \in \mathbb{N}_m, \end{cases}$$

- $h_A(x) \trianglelefteq_{Lex2} h_B(x)$ *iff*

$$\begin{cases} \exists i \in \mathbb{N}_{\min\{m,n\}} : h_A(x)^i < h_B(x)^i \text{ and } h_A(x)^j = h_B(x)^j, \forall j > i; \\ \text{or } m \leq n \text{ and } h_A(x)^j = h_B(x)^j, \forall j \in \mathbb{N}_m; \end{cases}$$

are admissible orders.

Example 1.41 *Let us consider two typical hesitant fuzzy elements defined as* $h_A(x) = \{0, 0.2, 0.8, 1\}$ *and* $h_B(x) = \{0, 0.1, 0.9, 1\}$. *Then if we consider these orders, we obtain that*

- $h_B(x) \trianglelefteq_{Lex1} h_A(x)$ *as it exists* $2 \in \{1, 2, 3, 4\}$ *such that* $h_B(x)^2 = 0.1 < 0.2 = h_A(x)^2$ *and for all* $j < 2$, *i.e., for* $j = 1$, *we have that* $h_B(x)^1 = 0 = h_A(x)^1$.
- $h_A(x) \trianglelefteq_{Lex2} h_B(x)$ *as it exists* $3 \in \{1, 2, 3, 4\}$ *such that* $h_A(x)^3 = 0.8 < 0.9 = h_B(x)^3$ *and for all* $j > 3$, *i.e., for* $j = 4$, *we have that* $h_A(x)^4 = 1 = h_B(x)^4$.

Now we will discuss the case when we consider two typical hesitant fuzzy elements with different cardinalities. Let us show the following example:

Example 1.42 *Let us consider two typical hesitant fuzzy elements defined as* $h_A(x) = \{0, 0.2, 0.8\}$ *and* $h_B(x) = \{0, 0.2, 0.8, 1\}$. *Then if we consider these orders, we obtain that*

- $h_A(x) \trianglelefteq_{Lex1} h_B(x)$ *as* $\#h_A(x) = 3 \leq \#h_B(x) = 4$ *and* $h_A(x)^j = h_B(x)^j, \forall j \in \mathbb{N}_3$.
- $h_A(x) \trianglelefteq_{Lex2} h_B(x)$ *as* $\#h_A(x) = 3 \leq \#h_B(x) = 4$ *and* $h_A(x)^j = h_B(x)^j, \forall j \in \mathbb{N}_3$.

Once we have seen these two examples of admissible order, we will show a method to generate admissible orders from an increasing function.

Theorem 1.43 *[56] Let* $A^* : \mathbb{H} \to [0, 1]$ *be a function such that* $\mathscr{A}^*$ *is increasing w.r.t.* $\trianglelefteq_{RH}$, $\mathscr{A}^*(\mathbf{0}_{\mathbb{H}}) = 0$ *and* $\mathscr{A}^*(\mathbf{1}_{\mathbb{H}}) = 1$ *and* $f^* : \mathbb{H} \to \mathbb{R}$ *be a function such that:*

IC: If $f^*(X) = f^*(Y)$ *then* $\#X = \#Y$ *(injective-cardinality property)*

is satisfied. The relation defined by

$$X \trianglelefteq^{f^*}_{\mathscr{A}^*} Y \Leftrightarrow \begin{cases} X = Y, \text{ or} \\ \mathscr{A}^*(X) < \mathscr{A}^*(Y), \text{ or} \\ \mathscr{A}^*(X) = \mathscr{A}^*(Y) \text{ and } f^*(X) < f^*(Y), \end{cases}$$

is a total admissible order on $\mathbb{H}$ *if, for each* $n \in \mathbb{N}^+$, *where* $\mathscr{A}^*_n$, *which is the restriction of* $\mathscr{A}^*$ *to* $\mathbb{H}^{(n)}$, *is injective.*

De Miguel et al. proposed in [19] an algorithm for group decision-making using n-dimensional fuzzy sets, admissible orders and OWA operators. It is possible to use it in typical hesitant fuzzy sets but first we should normalize them.

Wang and Xu worked with linguistic term sets in [80] and proposed some total orders for extended hesitant fuzzy sets, which are hesitant fuzzy sets that have been normalized and have the same cardinal.

It is clear that any of the previous proposals for ordering typical hesitant fuzzy elements will allow us to define a subsethood relation in $THFS(X)$, as we will see in detail in Chapter 2. In the remaining parts of this subsubsection we will review other main operations for hesitant fuzzy sets.

Definition 1.44 *[69, 76] Let X be the universe and let A be a hesitant fuzzy set in X. The standard complement, or just complement for short, of A in X, which is denoted by A^c, is the hesitant fuzzy set defined by the following membership function:*

$$h_{A^c}(x) = \bigcup_{\gamma \in h_A} \{1 - \gamma(x)\}$$

We would like to remind the reader that we are considering typical hesitant fuzzy elements ordered in an increasing way, so after we apply Definition 1.44, we will order the components of the typical hesitant fuzzy elements.

Example 1.45 *Let X, A and B be the sets defined in Example 1.21. The complement of A and B are:*

$$A^c = \{\langle 0, \{0.5, 0.75\}\rangle, \langle 0.5, \{1\}\rangle, \langle 1, \{0.2, 0.4, 0.6, 0.8\}\rangle\}$$

and B^c with membership function:

$$h_{B^c}(x) = \left\{1 - \frac{e^x}{e}\right\}, \quad \forall x \in X$$

Logically, the intersection and union of two hesitant fuzzy sets were also defined in the literature. Now, we will review the most standard definitions for these concepts.

Definition 1.46 *[69, 76] Let A and B be two hesitant fuzzy sets in X. The standard intersection of A and B, which is denoted by $A \cap B$, is the hesitant fuzzy set in X defined by*

$$h_{A\cap B}(x) = \{h \in (h_A(x) \cup h_B(x)) : h \leq \min\{\max\{h_A(x)\}, \max\{h_B(x)\}\}\}$$

Example 1.47 *Let X, A and B be again the sets defined in Example 1.21. The standard intersection of A and B is obtained as follows:*

- *Since $h_A(0) = \{0.25, 0.5\}$ and $h_B(0) = \{1/e\}$, we have that $h_A(x) \cup h_B(0) = \{0.25, 0.5, 1/e\}$ and $\min\{\max\{h_A(0)\}, \max\{h_B(0)\}\} = \min\{0.5, 1/e\} = 1/e$. Thus,*

$$h_{A\cap B}(0) = \{h \in \{0.25, 0.5, 1/e\} : h \leq \min\{0.5, 1/e\}\} = \{0.25, 1/e\}.$$

- *Since $h_A(0.5) = \{0\}$ and $h_B(0.5) = \{1/\sqrt{e}\}$, then*

$$h_{A\cap B}(0.5) = \{h \in \{0, 1/\sqrt{e}\} : h \leq \min\{0, 1/\sqrt{e}\}\} = \{0\}$$

- *Since $h_A(1) = \{0.2, 0.4, 0.6, 0.8\}$ and $h_B(x) = \{1\}$, then*

$$h_{A\cap B}(1) = \{h \in \{0.2, 0.4, 0.6, 0.8, 1\} : h \leq \min\{0.8, 1\}\} = \{0.2, 0.4, 0.6, 0.8\}$$

Hence,

$$A \cap B = \{\langle 0, \{0.25, 1/e\}\rangle, \langle 0.5, \{0\}\rangle, \langle 1, \{0.2, 0.4, 0.6, 0.8\}\rangle\}$$

These three hesitant fuzzy sets A, B and $A \cap B$ are represented in Figure 1.2.

On the other hand, De Miguel et al. [18] proposed the concept of meet-convolution.

Definition 1.48 *[18] Let A and B be two hesitant fuzzy sets in X. The meet-convolution of A and B, which is denoted by $A \cap_{MC} B$, is the hesitant fuzzy set in X defined by*

$$h_{A\cap_{MC}B}(x) = sup\{\min\{h_A(u), h_B(v)\} : u, v \in X, \min\{u, v\} = x\}$$

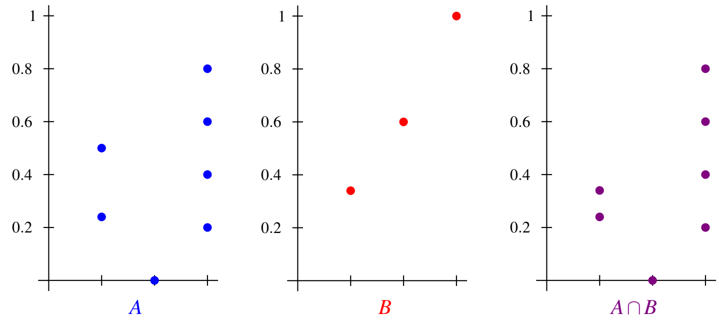

Figure 1.2: Intersection of A and B.

Example 1.49 *Let A, B and X be the sets defined in Example 1.21.*

The meet-convolution of A and B is obtained as follows:

- *For* $x = 0$*, we have that* $h_{A \cap_{MC} B}(0) = sup\{\min\{h_A(u), h_B(v)\} : \min\{u,v\} = 0\}$*. Thus, for instance, if* $u = 0$ *and* $v = 0$*, we have that* $\min\{u,v\} = 0$ *and we should do the minimum between* $h_A(u) = \{0.25, 0.5\}$ *and* $h_B(v) = \{1/e\}$*. For comparing these two sets, we will use the lexicographical order type 1 (see Theorem 1.40).*

  Thus, we obtain the following:

u	v	$\min\{u,v\}$	$A(u)$	$B(v)$	$\min\{A(u),B(v)\}$
0	0	0	$\{0.25,0.5\}$	$\{1/e\}$	$\{0.25,0.5\}$
0	0.5	0	$\{0.25,0.5\}$	$\{1/\sqrt{e}\}$	$\{0.25,0.5\}$
0	1	0	$\{0.25,0.5\}$	$\{1\}$	$\{0.25,0.5\}$
0.5	0	0	$\{0\}$	$\{1/e\}$	$\{0\}$
1	0	0	$\{0.2,0.4,0.6,0.8\}$	$\{1/e\}$	$\{0.2,0.4,0.6,0.8\}$

  and therefore $h_{A \cap_{MC} B}(0) = \{0.25, 0.5\}$.

- *For* $x = 0.5$ *we have that*

  and then $h_{A \cap_{MC} B}(0.5) = \{0.2, 0.4, 0.6, 0.8\}$.

- *Finally, for* $x = 1$ *we have that Thus,* $h_{A \cap_{MC} B}(1) = \{0.2, 0.4, 0.6, 0.8\}$.

u	v	$\min\{u,v\}$	$A(u)$	$B(v)$	$\min\{A(u),B(v)\}$
0.5	0.5	0.5	$\{0\}$	$\{1/\sqrt{e}\}$	$\{0\}$
0.5	1	0.5	$\{0\}$	$\{1\}$	$\{0\}$
1	0.5	0.5	$\{0.2,0.4,0.6,0.8\}$	$\{1/\sqrt{e}\}$	$\{0.2,0.4,0.6,0.8\}$

u	v	$\min\{u,v\}$	$A(u)$	$B(v)$	$\min\{A(u),B(v)\}$
1	1	1	$\{0.2,0.4,0.6,0.8\}$	$\{1\}$	$\{0.2,0.4,0.6,0.8\}$

Hence, using the lexicographical order type 1, we have obtained that the intersection of A and B, $A \cap_{MC} B$, is the hesitant fuzzy set:

$$\{\langle 0,\{0.25,0.5\}\rangle,\langle 0.5,\{0.2,0.4,0.6,0.8\}\rangle,\langle 1,\{0.2,0.4,0.6,0.8\}\rangle\}$$

We can find an illustration of this in Figure 1.3.

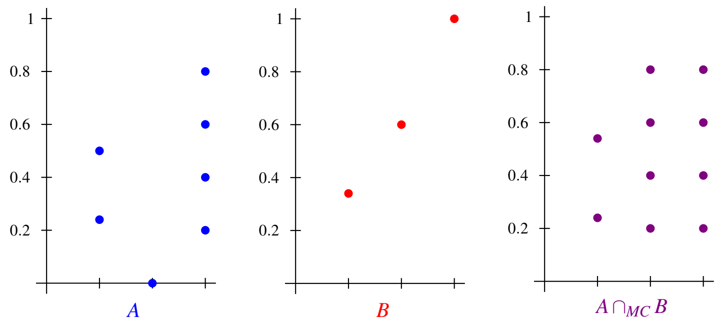

Figure 1.3: Meet-convolution of A and B.

As we saw in the last example, this type of intersection appears to be rather counter-intuitive. For instance, for the intermediate element of X, we have that $h_A(0.5) = \{0\}$, but the membership values for the intersection are clearly greater than zero. Maybe for this reason, both definitions for the intersection, Definitions 1.46 and 1.48, continue to be used. And the same happens for the union.

Definition 1.50 *[69, 76] Let X be the reference and let A and B be two hesitant fuzzy sets in X. The standard union of A and B, which is denoted by $A \cup B$, is the hesitant fuzzy set defined by*

$$h_{A \cup h_B}(x) = \{h \in (h_A(x) \cup h_B(x)) : h \geq \max\{\min\{h_A(x)\},\min\{h_B(x)\}\}\}$$

Example 1.51 *Let X, A and B be the sets defined in Example 1.21. The standard union of A and B is the hesitant fuzzy set obtained as follows:*

$$h_{A\cup B}(0) = \{h \in \{0.25, 0.5, 1/e\} : h \geq \max\{0.25, 1/e\}\} = \{1/e, 0.5\}$$

$$h_{A\cup B}(0.5) = \{h \in \{0, 1/\sqrt{e}\} : h \geq \max\{0, 1/\sqrt{e}\}\} = \{1/\sqrt{e}\}$$

$$h_{A\cup B}(1) = \{h \in \{0.2, 0.4, 0.6, 0.8, 1\} : h \geq \max\{0.2, 1\}\} = \{1\}$$

Consequently,

$$A \cup B = \{\langle 0, \{1/e, 0.5\}\rangle, \langle 0.5, \{1/\sqrt{e}\}\rangle, \langle 1, \{1\}\rangle\}$$

This can be illustrated in Figure 1.4, where we have A, B and their union.

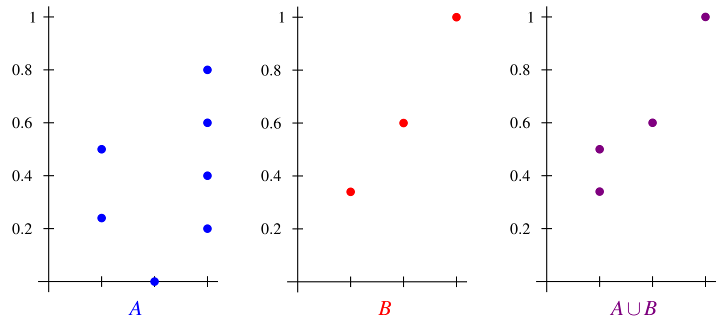

Figure 1.4: Union of A and B.

On the other hand, De Miguel et al. [18] proposed the concept of join-convolution.

Definition 1.52 *[18] Let A and B be two hesitant fuzzy sets in X. The join-convolution of A and B, which is denoted by $A \cup_{JC} B$, is the hesitant fuzzy set in X defined by*

$$h_{(A\cup_{JC}B}(x) = sup\{\min\{h_A(u), h_B(v)\} : u, v \in X, \max\{u, v\} = x\}$$

Example 1.53 *Let A, B and X be the sets defined in Example 1.21, and the join-convolution of A and B is obtained as follows, considering again the lexicographical order type 1.*

- *For* $x = 0$ *we have that* $h_{A\cup_{JC}B}(0) = sup\{\min\{A(u), B(v)\} : \max\{u, v\} = 0\}$. *Thus,*

u	v	$\max\{u,v\}$	$A(u)$	$B(v)$	$\min\{A(u),B(v)\}$
0	0	0	$\{0.25, 0.5\}$	$\{1/e\}$	$\{0.25, 0.5\}$

and then $h_{A\cup_{JC}B}(0) = \{0.25, 0.5\}$.

- *For* $x = 0.5$ *we obtain that*

u	v	$\max\{u,v\}$	$A(u)$	$B(v)$	$\min\{A(u),B(v)\}$
0	0.5	0.5	$\{0.25, 0.5\}$	$\{1/\sqrt{e}\}$	$\{0.25, 0.5\}$
0.5	0	0.5	$\{0\}$	$\{1/e\}$	$\{0\}$
0.5	0.5	0.5	$\{0\}$	$\{1/\sqrt{e}\}$	$\{0\}$

Then $(h_{A\cup_{JC}B}(0.5) = \{0.25, 0.5\}$.

- *For* $x = 1$ *we have that*

u	v	$\max\{u,v\}$	$A(u)$	$B(v)$	$\min\{A(u),B(v)\}$
0	1	1	$\{0.25, 0.5\}$	$\{1\}$	$\{0.25, 0.5\}$
0.5	1	1	$\{0\}$	$\{1\}$	$\{0\}$
1	0	1	$\{0.2, 0.4, 0.6, 0.8\}$	$\{1/e\}$	$\{0.2, 0.4, 0.6, 0.8\}$
1	0.5	1	$\{0.2, 0.4, 0.6, 0.8\}$	$\{1/\sqrt{e}\}$	$\{0.2, 0.4, 0.6, 0.8\}$
1	1	1	$\{0.2, 0.4, 0.6, 0.8\}$	$\{1\}$	$\{0.2, 0.4, 0.6, 0.8\}$

Thus , $h_{A\cup_{JC}B}(1) = \{0.25, 0.5\}$.

As a result, using the lexicographical order type 1, we obtain that

$$A \cup_{JC} B = \{\langle 0, \{0.25, 0.5\}\rangle, \langle 0.5, \{0.25, 0.5\}\rangle, \langle 1, \{0.25, 0.5\}\rangle\}$$

We can find an illustration of this union in Figure 1.5.

Here we have a similar situation as with the meet-convolution. As it is shown in the previous example, this type of union looks again a bit counter-intuitive as we can see for the membership function of the join-convolution at the point 1. For this reason, we will consider, by default, the standard intersection and union of hesitant fuzzy sets given at Definitions 1.46 and 1.50.

Finally, we will recall some definitions we will consider for the support and core of a hesitant fuzzy set and we will conclude with some operations for hesitant fuzzy elements which will be later needed.

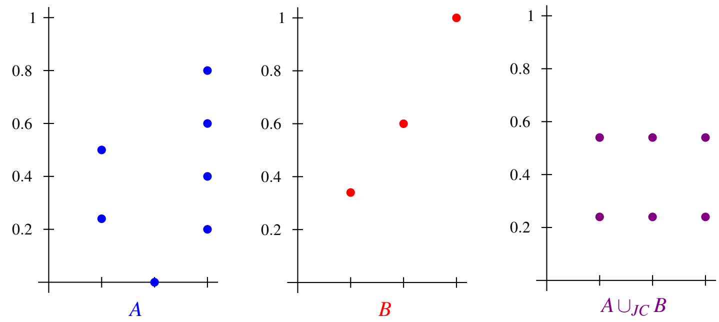

Figure 1.5: Meet-convolution of A and B.

Definition 1.54 *[65] Let A be a hesitant fuzzy set in X. The support of A, which is denoted by $Supp(A)$, is the crisp set*

$$Supp(A) = \{x \in X : \max\{h_A(x)\} \neq 0\}$$

And now we introduce the definition of core.

Definition 1.55 *[65] Let A be a hesitant fuzzy set in X. The core of A, denoted by $Core(A)$, is the crisp set*

$$Core(A) = \{x \in X : \max\{h_A(x)\} = 1\}$$

For concluding this introduction to hesitant fuzzy sets, we will introduce some concepts for hesitant fuzzy elements which are similar to the sum and the scalar product.

Definition 1.56 *[65, 83] Let $h_A(x)$ and $h_B(x)$ be two hesitant fuzzy elements.*

1. $k \odot h_A(x) = \cup_{\gamma \in h_A(x)} \{1 - (1-\gamma)^k\}$.

2. $h_A(x) \oplus h_B(x) = \cup_{\gamma_1 \in h_A(x), \gamma_2 \in h_B(x)} \{\gamma_1 + \gamma_2 - \gamma_1 \gamma_2\}$.

1.2.2 Interval-valued fuzzy sets

In the previous section we introduce hesitant fuzzy sets, which can handle the uncertainty provoked by several values. In the case of interval-valued fuzzy sets, we deal with the uncertainty generated by two values and all intermediate values. This happens, for instance, when we are able to obtain a lower and upper bound for the membership function, but we do have not precise information about the real value.

Interval-valued fuzzy sets were introduced independently by Zadeh [93], Grattan-Guiness [35], Jahn [44], and Sambuc [70] in the seventies. In cases where traditional fuzzy tools are unhelpful, such as when there is no objective method for choosing crisp membership degrees, interval-valued fuzzy set can be helpful. Many researchers have quickly been interested in these extensions as a result of their tremendous potential for a variety of applications. Thus, for instance, Sambuc [70] used them in medical diagnosis in thyrodian pathology, Bustince and Burillo [10] and Gozalczany [33] in approximate reasoning and Cornelis et al. [16] and Turksen and Zhong [77] in logic, among many others.

The definition of interval-valued fuzzy set we are considering is the following:

Definition 1.57 *[7] An interval-valued fuzzy set A on X is a mapping $A : X \to L([0,1])$ such that $A(x) = [\underline{A}(x), \overline{A}(x)]$, where $L([0,1])$ denotes the family of closed intervals included in the unit interval $[0,1]$.*

Thus, an interval-valued fuzzy set A is totally characterized by two mappings, $\underline{A}$ and $\overline{A}$, from X into $[0,1]$ such that $\underline{A} \leq \overline{A}$. It could be represented as $A = \{\langle x, [\underline{A}(x), \overline{A}(x)]\rangle : x \in X\}$, where $\underline{A}(x)$ and $\overline{A}(x)$ are the lower and upper bounds of the membership interval and they satisfy that $0 \leq \underline{A}(x) \leq \overline{A}(x) \leq 1, \forall x \in X$. The collection of all the interval-valued fuzzy set in X is denoted by $IVFS(X)$.

Naturally, a regular fuzzy set can be expressed as follows:

$$\{\langle x, [\mu_A(x), \mu_A(x)]\rangle : x \in X\}$$

and as a result, interval-valued fuzzy sets effectively generalize fuzzy sets.

Example 1.58 *Let X be the interval $[0,1]$. The following sets are examples of an interval-valued fuzzy set:*

i) $A = \{\langle x, [0.25, 0.5]\rangle : x \in X\}$

ii) $B = \{\langle x, [\underline{B}(x), \overline{B}(x)]\rangle : x \in X\}$ *where* $\underline{B}(x) = \dfrac{x}{e}$ *and* $\overline{B}(x) = \dfrac{e^x}{e}$*, for any* $x \in X$.

In 1986, Atanassov [3] proposed another extension of the notion of fuzzy set called intuitionistic fuzzy set (IFS). So, in contrast to a fuzzy set,an intuitionistic fuzzy set has associated two functions: a membership function and a non-membership function. A point's degree of set membership is represented by the first function and its degree of set non-membership by the second. Let us recall their definition.

Definition 1.59 *[7] An intuitionistic fuzzy set A in X is defined as* $A = \{\langle x, \mu_A(x), \nu_A(x)\rangle : x \in X\}$*, where* $\mu_A(x)$ *and* $\nu_A(x)$ *are the degrees of membership and nonmembership of x in A, respectively.* $\mu_A(x)$ *and* $\nu_A(x)$ *must satisfy that* $\mu_A(x), \nu_A(x) \in [0,1]$ *and* $0 \leq \mu_A(x) + \nu_A(x) \leq 1$.

We will denoted the family of all intuitionistic fuzzy sets on X by $IFS(X)$.

Atanassov [3] asserted that his intuitionistic fuzzy sets are equivalent to interval-valued fuzzy sets. Despite not solving the same issue, both are frequently utilized in the literature. The option that best suits the circumstances is typically the one that is picked [61]. Since they are mathematically equal, we will not deal with intuitionistic fuzzy sets and will use interval-valued fuzzy sets. However, from a mathematical point of view, we can consider this equivalence to use any interesting result for intuitionistic fuzzy sets.

By focusing on interval-valued fuzzy sets, we can take into consideration either the epistemic interpretation or the ontic interpretation. The first one will be the one that is chosen in our study. We, therefore, assume that inside the membership interval of potential membership degrees, there is only one actual, real-valued membership degree of an element, as it is shown in Figure 1.6.

Our proposal requires the preservation of convex interval-valued fuzzy sets under intersection, which makes it crucial to first define the concept of the intersection of two interval-valued fuzzy sets. However, to do this in a coherent manner, it is necessary to define the inclusion between two interval-valued fuzzy sets beforehand.

Let us consider the following two interval-valued fuzzy sets in Figure 1.7. It seems only obvious that in order to determine whether or not B is included in A, we must compare intervals.

In addition, we need a definition of the convexity coherent with the fuzzy set definition. We also need to define and analyze the union, which will depend on

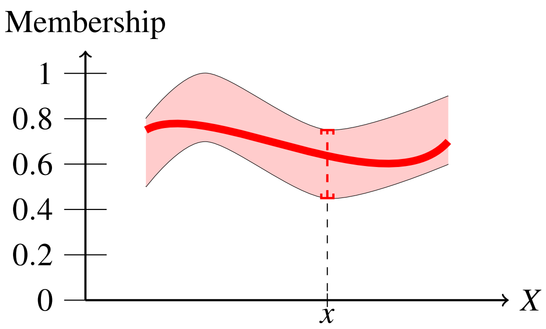

Figure 1.6: Epistemic interpretation.

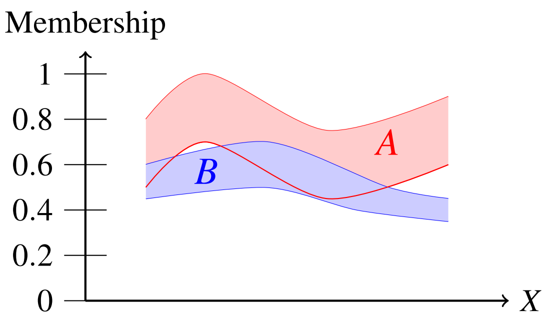

Figure 1.7: Is B "included" in A?

the order between intervals that is taken into consideration, in order to explore the features of the level sets of an interval-valued fuzzy set. To define the inclusion in $IVFS(X)$ and subsequently the union and intersection of interval-valued fuzzy set, we must first examine various orderings of real intervals.

Orders in $L([0,1])$

There are several ways to compare intervals and here are the most common relations presented in [37]. If $a = [\underline{a}, \overline{a}]$ and $b = [\underline{b}, \overline{b}]$ are two intervals in $L([0,1])$, we say that a is lower than or equal to b if:

- Interval dominance [29]: $a \preceq_{ID} b$ if $\overline{a} \leq \underline{b}$
- Lattice order [32]: $a \preceq_{Lo} b$ if $\underline{a} \leq \underline{b}$ and $\overline{a} \leq \overline{b}$, which is induced by the usual partial order in $\mathbb{R}^2$

- Lexicographical order type 1 [11]: $a \preceq_{Lex1} b$ if $\underline{a} < \underline{b}$ or ($\underline{a} = \underline{b}$ and $\overline{a} \leq \overline{b}$)
- Lexicographical order type 2 [11]: $a \preceq_{Lex2} b$ if $\overline{a} < \overline{b}$ or ($\overline{a} = \overline{b}$ and $\underline{a} \leq \underline{b}$)
- The Xu and Yager order [88]: $a \preceq_{XY} b$ if $\underline{a} + \overline{a} < \underline{b} + \overline{b}$ or ($\underline{a} + \overline{a} = \underline{b} + \overline{b}$ and $\overline{a} - \underline{a} \leq \overline{b} - \underline{b}$)
- Maximax order [72]: $a \preceq_{MM} b$ if $\overline{a} \leq \overline{b}$
- Maximin order [73, 78]: $a \preceq_{Mm} b$ if $\underline{a} \leq \underline{b}$
- Hurwicz order [43]: $a \preceq_{H(\alpha)} b$ if $\alpha \cdot \underline{a} + (1-\alpha) \cdot \overline{a} \leq \alpha \cdot \underline{b} + (1-\alpha) \cdot \overline{b}$ with $\alpha \in [0,1]$.
- Weak order [9]: $a \preceq_{wo} b$ if $\underline{a} \leq \overline{b}$

These relations are connected in some cases. It is fairly obvious that if an interval *a* w.r.t. the interval dominance is lower than or equal to *b*, then *a* is likewise lower than or equal to *b* w.r.t. the lattice order. All of these implications, along with a few others like them, are compiled in Figure 1.8.

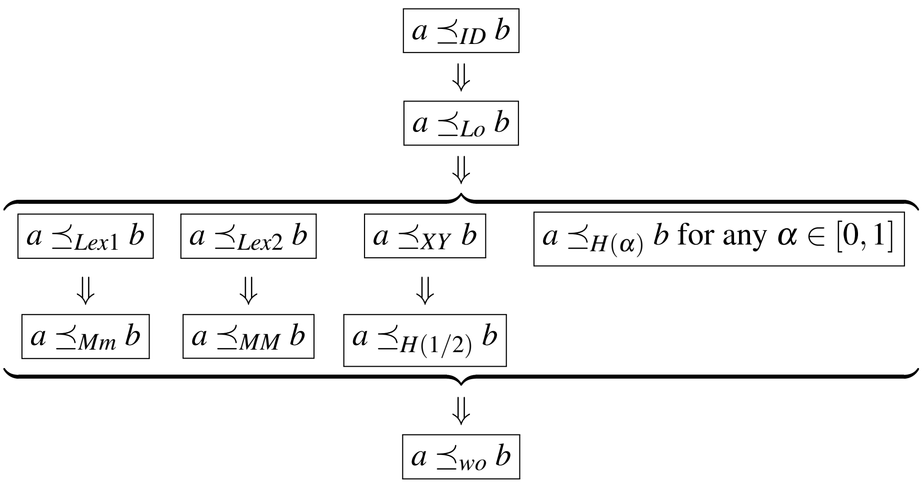

Figure 1.8: Relations between the interval relations.

At first sight, taking into account their names and the considered relations, the reader could think that these expressions are truly orders, but this is not true. As we can see in Table 1.1, some of these ways to compare intervals are not orders as they do not fulfill the order relation requirements (reflexivity, antisymmetry and transitivity). However, we will refer to all of them as orders since this is the usual name in the literature. In Table 1.1, we claim whether they are total orders or not and also we identify the case of some preorders which are not orders.

	Reflexive	Antisymmetric	Transitive	Preorder	Order	Total Order
ID	✗	✓	✓	✗	✗	✗
Lo	✓	✓	✓	✓	✓	✗
Lex_1	✓	✓	✓	✓	✓	✓
Lex_2	✓	✓	✓	✓	✓	✓
XY	✓	✓	✓	✓	✓	✓
Mm	✓	✗	✓	✓	✗	✗
MM	✓	✗	✓	✓	✗	✗
$H(\alpha)$	✓	✗	✓	✓	✗	✗
wo	✓	✗	✗	✗	✗	✗

Table 1.1: Properties of the different relations.

After doing this short analysis, we can confidently assert that only the lexicographical orders types 1 and 2 and the Xu and Yager order are total orders, being the lattice order just a partial order.

With respect to total orders in $L([0,1])$, in this work we are considering the so-called admissible orders, whose definition we review here.

Definition 1.60 *[11] An admissible order on $L([0,1])$, $\preceq_{ao}$, is a binary relation on $L([0,1])$ fulfilling:*

- *it is a total order*
- *it refines the lattice order, that is, for every $a,b \in L([0,1])$, if $a \preceq_{Lo} b$ then $a \preceq_{ao} b$.*

The ability to construct admissible orders using aggregation functions is an important aspect to take into account [11]. An aggregation function is defined on

$\bigcup_{n\in\mathbb{N}}[0,1]^n$. In particular, $[0,1]^2$ could be considered. There is a natural bijection between $L([0,1])$ and $K([0,1]) = \{(u,v) \in [0,1]^2 \mid u \leq v\}$ that links the interval $[\underline{a},\overline{a}]$ to the point made by its endpoints in $\mathbb{R}^2$. Consequently, we can add together the information presented as an interval using aggregation methods. Bustince et al. [11] develop the following procedure to create admissible orders based on this concept.

Proposition 1.61 *[11] Let $\mathscr{A},\mathscr{B}$ be continuous aggregation functions, such that for all $(u,v),(u',v') \in K([0,1])$, the equalities $\mathscr{A}(u,v) = \mathscr{A}(u',v')$ and $\mathscr{B}(u,v) = \mathscr{B}(u',v')$ can only hold if $(u,v) = (u',v')$. Define the relation $\preceq_{\mathscr{A},\mathscr{B}}$ on $L([0,1])$ by $a \preceq_{\mathscr{A},\mathscr{B}} b$ if and only if*

$$\mathscr{A}(\underline{a},\overline{a}) < \mathscr{A}(\underline{b},\overline{b})$$

or

$$\mathscr{A}(\underline{a},\overline{a}) = \mathscr{A}(\underline{b},\overline{b}) \text{ and } \mathscr{B}(\underline{a},\overline{a}) \leq \mathscr{B}(\underline{b},\overline{b})$$

Then $\preceq_{\mathscr{A},\mathscr{B}}$ is an admissible order on $L([0,1])$.

A possible procedure of building admissible orders on $L([0,1])$ is defining them using the weighted mean which is a particular case of continuous aggregation function (see [11]):

$$K_\alpha(u,v) = (1-\alpha)\cdot u + \alpha \cdot v, \text{ where } \alpha \in [0,1]$$

This mapping can be used to represent the α-quantile of a probability distribution that is evenly distributed over the range $[u,v]$. In order to derive the admissible order $\preceq_{K_\alpha,K_\beta}$, which is denoted, for convenience, as $\preceq_{\alpha,\beta}$, we can apply Proposition 1.61 to the aggregation functions K_α and K_β (see [37]).

The Xu and Yager order or the lexicographical orders type 1 and type 2 are examples of these admissible orders. More precisely, $\preceq_{Lex1}\equiv\preceq_{0,1}$, $\preceq_{Lex2}\equiv\preceq_{1,0}$ and $\preceq_{XY}\equiv\preceq_{1/2,\beta}$ for any $\beta \in (1/2,1]$ (see [11]).

Inclusion

According to the fuzzy set theory, A is said to be contained in B if and only if its membership function is smaller than or equal to that of B, where $A,B \in FS(X)$ (see

[91]). Once we have orders on intervals we can suggest the following definition of containment for interval-valued fuzzy sets, which goes further than the fuzzy set definition.

Definition 1.62 *[37] Let $(L([0,1]), \preceq_o)$ be the set of closed interval included in $[0,1]$ and $\preceq_o$ an order on $L([0,1])$. Let A and B be any sets in $IVFS(X)$, we say that A is o-included in B, which is denoted by $A \subseteq_o B$ if, and only if,*

$$A(x) \preceq_o B(x), \forall x \in X$$

It is obvious that if $\preceq_o$ is an order in $L([0,1])$, hence $\subseteq_o$ is an order in $IVFS(X)$. While $\preceq_o$ may be a total order, $\subseteq_o$ is only a partial order.

Example 1.63 *Consider the interval-valued fuzzy set A, B and C defined as in Figure 1.9.*

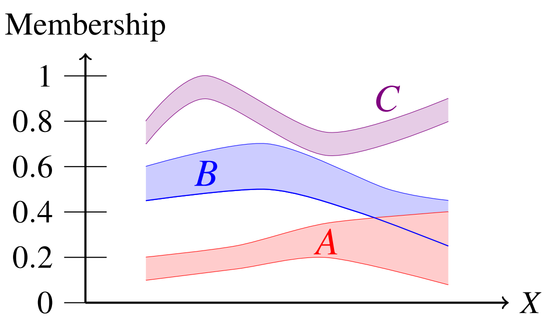

Figure 1.9: Membership functions of A, B and C.

It is clear that $A, B \subseteq_{ID} C$ and therefore they are ID-included in C w.r.t. any of the considered orders. We also have $A \subseteq_{Lo} B$, but $A \not\subseteq_{ID} B$. Thus, A is included in B for any considered order except for the interval dominance. Finally, we can say that B or C are not included in A for any order.

As we commented, the inherited relation in $IVFS(X)$ is not a total order even in case $\preceq_o$ is a total order. Thus, if we consider the lexicographical order type 1 and the interval-valued fuzzy set in Figure 1.10, we have that A and B are not comparable by means of this order.

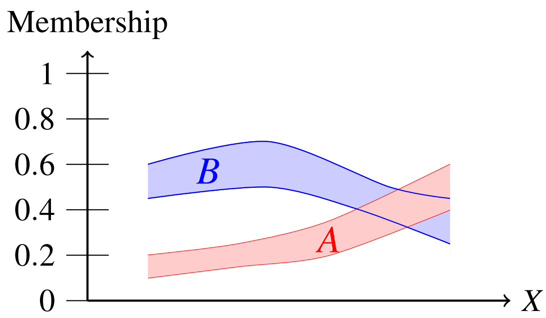

Figure 1.10: A and B are $\subseteq_{Lex1}$- incomparable.

Thus, the inclusion for interval-valued fuzzy sets is based on orders on $L([0,1])$. We will take this into account to define in Chapter 3 the union and intersection of elements in $IVFS(X)$. However, some operations, as the complement, are independent of the chosen order and we can consider them just now.

Complement

In the literature, a number of operations have been used for the idea of a complement set. Now let us focus on one of the most fundamental.

Definition 1.64 *[23] Let A be in $IVFS(X)$. The complement of A, denoted by A^c, is defined by $\underline{A^c}(x) = 1 - \overline{A}(x)$ and $\overline{A^c}(x) = 1 - \underline{A}(x)$ for any $x \in X$, that is,*

$$A^c(x) = [1 - \overline{A}(x), 1 - \underline{A}(x)]$$

This idea can be made more inclusive by using a negation.

Definition 1.65 *[31] A function $N : [0,1]^n \to [0,1]$ is a negation if for all $x \in [0,1]^n$ there is $N(0) = 1$ and $N(1) = 0$ and N is decreasing.*

Furthermore, N is a strong negation if $N(N(x)) = x$ for every $x \in [0,1]$. Note that every strong negation is strictly diminishing and continuous.

As a result, we define the complement with respect to N as follows:

$$A^{Nc}(x) = [N(\overline{A}(x)), N(\underline{A}(x))]$$

For all x in X, we will by default take into account the standard negation $N(x) = 1 - x$ for any $x \in X$.

Despite the existence of further fuzzy set extensions (see [61, 63, 68]), for this study we use only the ones that we have built in this chapter. In Figure 1.11, we show the connection between the different fuzzy set extensions. In this figure, we can see that interval-valued fuzzy sets and typical hesitant fuzzy sets are particular cases of hesitant fuzzy sets. Due to their importance, we will devote Chapter 3 to the first ones and Chapter 2 to the second ones. At this point, we should recall that, for simplicity, we are calling hesitant fuzzy sets for the typical hesitant fuzzy sets.

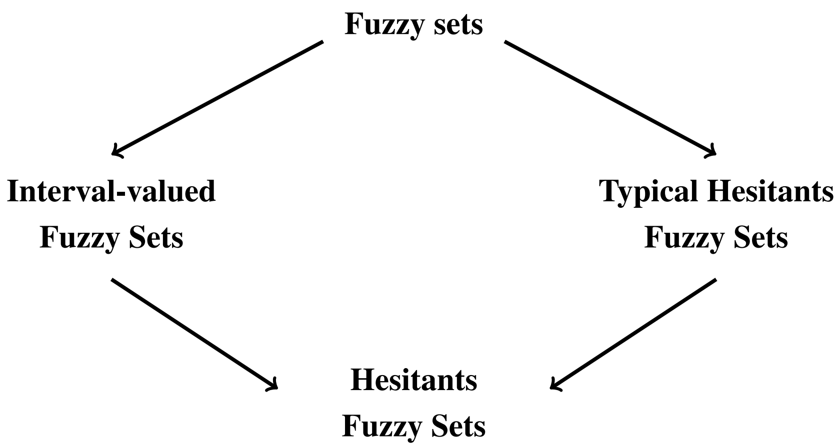

Figure 1.11: Extensions of fuzzy set.

Finally, as convexity is one of the main points of this thesis, we will recall some concepts and results of convexity in fuzzy sets in the next section.

1.3 Convexity of fuzzy sets

As we said in the introduction, convexity is a relevant concept in many areas of mathematics. In particular, convexity, as the fundamental theory in optimization research, has naturally formed one of the most important areas in fuzzy mathematics. This section introduces several concepts related to convexity.

As a starting point, we should review the traditional notion of convexity of ordinal sets. If X is a vector space, a crisp set $A \subseteq X$ is called convex if $\lambda x+(1-\lambda)y$ belongs to A for every $\lambda \in [0,1]$ and every $x, y \in A$ (see [21]). Obviously, this idea differs from the convex function concept. To put it another way, a function $f : X \to \mathbb{R}$ is called convex if and only if

$$f(\lambda x+(1-\lambda)y) \leq \lambda f(x)+(1-\lambda)f(y),\ \forall \lambda \in [0,1]$$

and it is said to be concave if $-f$ is convex.

A possible way to define convexity in fuzzy set theory is the following.

Definition 1.66 *[91] Let X be a vector space. A fuzzy set A defined on X is convex, if for each $x, y \in X$, $\lambda \in [0,1]$ there is*

$$\mu_A(\lambda x+(1-\lambda)y) \geq \lambda \mu_A(x)+(1-\lambda)\mu_A(y)$$

With this definition, the membership function is a concave function, but it is still called a convex set since it takes into account the ideas behind the classical convexity. Thus, now, the membership function at the point $\lambda x+(1-\lambda)y$ is at least a combination of the membership values on x and y. In fact, if x and y belong to A, that is, if $\mu_A(x) = \mu_A(y)$, then we have that $\mu_A(\lambda x+(1-\lambda)y) = 1$ and therefore $\lambda x+(1-\lambda)y$ also belong to A.

A really relevant concept in convexity is an α-set or a level set. Let us introduce its definition.

Definition 1.67 *[91] Let X be a referential and A a fuzzy set on X. Then the α-set of A, denoted by A_α, is the crisp set defined as*

$$A_\alpha = \{x \in X : \mu_A(x) \geq \alpha\}$$

for any $\alpha \in (0,1]$.

However, with this definition of α-set, Definition 1.66 of convexity has at least two drawbacks:

1. When the universe X we are working on is not a vector space, such as a lattice-valued fuzzy set, since the addition in the lattice is, in general, not defined, we could find some problems.

2. The behaviour of convexity is not as good as it should be. In other words, a fuzzy set for which all its level sets are convex, may not be convex [65], as we can see in Example 1.68.

Example 1.68 *[21] Let X be $\mathbb{R}$. Let A be the fuzzy set of X given by*

$$\mu_A(x) = \begin{cases} 0.2 & \text{if } x \leq 0 \\ 0.3 & \text{otherwise} \end{cases}$$

The level sets or α-cuts of A are defined by the crisp sets $A_\alpha = \{x \in X : \mu_A(x) \geq \alpha\}$ for any $\alpha \in (0,1]$. Thus,

$$A_\alpha = \begin{cases} \mathbb{R} & \text{if } \alpha \leq 0.2 \\ (0,\infty) & \text{if } 0.2 < \alpha \leq 0.3 \\ \emptyset & \text{if } \alpha > 0.3 \end{cases}$$

It is clear that all of them are convex subsets of the real line. However, the fuzzy set A fails to be convex:

If $\lambda = 0.5$, x=-4 and y=2, we may notice that $\lambda x + (1-\lambda)y = -1$, so that

$$\mu_A(\lambda x + (1-\lambda)y) = 0.2$$

whereas

$$\lambda \mu_A(x) + (1-\lambda)\mu_A(y) = 0.25.$$

Zadeh tried to apply a general notion of convexity to fuzzy sets, when he presented the following idea of convex fuzzy sets. Initially, he considered that the referential was the Euclidean n-space $\mathbb{R}^n$. However, we will consider the most general definition, for any vector space.

Definition 1.69 *[91] Let X be a vector space.A fuzzy set A on X is convex if and only if the sets A_α are convex for all α in the interval $(0,1]$.*

In addition, Zadeh put up another, simpler concept that is equivalent to the first one. Additionally, we can see that the traditional concept of convexity is still present here.

Definition 1.70 *[91] Let X be a vector space. A fuzzy set A is convex if and only if*

$$\mu_A(\lambda x_1 + (1-\lambda)x_2) \geq \min\{\mu_A(x_1), \mu_A(x_2)\}$$

for all $x_1, x_2 \in X$ *and all* $\lambda \in [0,1]$.

Later, in 1992 Ammar [2] proposed the idea of calling quasi-convex to the previous Zadeh's definition of convexity.

According to this definition, μ_A need not be a convex or concave function (see [91]), although it also means that the membership degree for any intermediate point is at least the membership degree for at the minimum one of the points x and y. Therefore, this is again the idea behind the convexity of a set. We actually analyze convexity as a convexity of sets, as we already mentioned. The fuzzy set on the left is convex (in the fuzzy sense), as shown in Figure 1.12, but its membership function is a convex function in some parts of the referential, but also a concave function in other parts. Moreover, there is an example of a non-convex fuzzy set on the right.

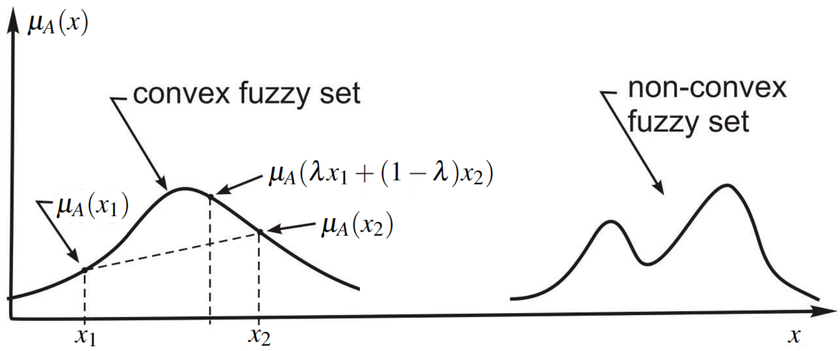

Figure 1.12: Convex and not convex fuzzy sets in $\mathbb{R}$ [91]

It is immediate to check that Definitions 1.69 and 1.70 are equivalent.

Proposition 1.71 *Let X be a vector space. A fuzzy set A is convex in the sense of Definition 1.70 if and only if the sets* A_α *are convex for all* α *in the interval* $(0,1]$.

Proof: Let us suppose that the α-cuts are convex. Let $x_1, x_2 \in X$. If $\alpha = \min\{\mu_A(x_1), \mu_A(x_2)\}$, $A_\alpha = \{y \in X : \mu_A(y) \geq \min\{\mu_A(x_1), \mu_A(x_2)\}\}$. It is obvious that $x_1, x_2 \in$

A_α. As the α-cuts are convex, $\lambda x_1 + (1-\lambda)x_2 \in A_\alpha$. Thus, $\mu_A(\lambda x_1 + (1-\lambda)x_2) \geq \alpha = \min\{\mu_A(x_1), \mu_A(x_2)\}$.

On the other hand, let A be a convex fuzzy set. For any $\alpha \in (0,1]$ and any $x_1, x_2 \in A_\alpha$, we have that $\mu_A(\lambda x_1 + (1-\lambda)x_2) \geq \min\{\mu_A(x_1), \mu_A(x_2)\} \geq \min\{\alpha, \alpha\} = \alpha$, since A is convex. Therefore, $\lambda x_1 + (1-\lambda)x_2 \in A_\alpha$ and then A_α is a crisp convex set. ■

Thus, from now on, when we deal with the convexity of fuzzy sets we will use Definition 1.70.

The fact that the intersection of any two convex sets is also convex is a crucial aspect of convexity.

Theorem 1.72 *[91] Let X be a vector space. Let A and B be two fuzzy sets on X. If A and B are convex, then $A \cap B$ is convex.*

Zadeh provided a thorough analysis of the preservation of convexity. Here, we shall review the findings that are most pertinent to our goals.

Definition 1.73 *[91] A fuzzy set A is bounded if and only if its α-sets A_α are bounded for all $\alpha \in (0,1]$.*

Lemma 1.74 *[91] Let A be a bounded fuzzy set and let $M = sup\{\mu_A(x) : x \in X\}$. M will be referred to as the maximal grade in A. Then there is at least one point x_0 at which M is essentially attained in the sense that, for each $\varepsilon > 0$, every spherical neighborhood of x_0 contains points in the set $Q(\varepsilon) = \{x : \mu_A(x) \geq M - \varepsilon\}$.*

Definition 1.75 *[91] Let X be a vector space. A fuzzy set A is strongly convex if and only if for any two points x_1 and x_2, $x_1 \neq x_2$, and any λ in the open interval (0,1)*

$$\mu_A(\lambda x_1 + (1-\lambda)x_2) > \min\{\mu_A(x_1), \mu_A(x_2)\}$$

Working with convex fuzzy sets, it should be noted that the intersection of two strongly convex sets is also strongly convex [91].

Some other properties of convexity are collected below.

Theorem 1.76 *[91] Let X be a vector space, and let A be a fuzzy set on X. If A is convex, then its core is a convex crisp set.*

Corollary 1.77 *[91] Let X be a vector space, and let A be a fuzzy set on X If. A is strongly convex, then the point at which M is essentially attained is unique.*

In 1987, Drewniak [22] worked with Zadeh's theory and achieved these results with $X = \mathbb{R}^n$.

Theorem 1.78 *[22] Let A with $\mu_A : \mathbb{R}^n \to [0,1]$ denote a fuzzy set in $\mathbb{R}^n$ for a given positive integer n.*

(a) If A is a convex fuzzy set, then $Supp(A)$ is a convex set.

(b) If A is a strongly convex fuzzy set, then $Supp(A) = \mathbb{R}^n$.

New definitions are required if the universe X is not a vector space. We can investigate the idea of a convex crisp set introduced by Llinares [55] in depth in order to think of additional potential approaches to define the convexity of a fuzzy set for any referential.

Definition 1.79 *[21, 55] Let X be a nonempty set. A convex structure on X is a map $H : X \times X \times [0,1] \to X$ that satisfies the following properties:*

(i) $H(x,y,\lambda) = H(y,x,1-\lambda)$, for every $x,y \in X$ and $\lambda \in [0,1]$.

(ii) $H(x,x,\lambda) = x$, for every $x \in X$ and $\lambda \in [0,1]$.

(iii) $H(x,y,1) = x$, for every $x,y \in X$.

Definition 1.80 *[21, 55] A subset A of X is said to be convex with respect to H, or H-convex for short, if $H(x,y,\lambda) \in A$, for all $x,y \in A$ and for all $\lambda \in [0,1]$.*

Due to Condition (ii) in Definition 1.79, any set with a single element is H-convex for any H.

If X is a vector space, it is immediate that any subset A of X is convex if and only if it is H-convex with $H(x,y,\lambda) = \lambda x + (1-\lambda)y$.

We will show in the following example that there are H-convex sets that are not convex.

Example 1.81 *[21] Let us consider the map $H : \mathbb{R} \times \mathbb{R} \times [0,1] \to \mathbb{R}$ defined by:*

$$H(x,y,\lambda) = \begin{cases} x & \text{if } \lambda = 1 \\ y & \text{if } \lambda = 0 \\ x & \text{if } \lambda \in (0,1) \text{ and } x = y \\ 0 & \text{if } \lambda \in (0,1) \text{ and } x \neq y \end{cases}$$

It is easy to check that H is a convex structure on $\mathbb{R}$.

Then, it is clear that the set $A = \{0, 0.3\}$ is H-convex, but it is not convex because for $\lambda = 0.5$ and $x = 0$ and $y = 0.3$ we have that

$$\lambda x + (1-\lambda)y = 0.15 \notin A$$

As a result, the idea of H-convex sets is a generalization of the idea of a convex set.

Taking this fact into consideration, it only makes sense to formulate a new definition for fuzzy convexity.

Definition 1.82 *[21] Let X be the universe. Assume that X is equipped with a convex structure through a map $H : X \times X \times [0,1] \to X$. Then, a fuzzy set A of X (whose membership function is $\mu_A : X \to [0,1]$) is said to be an H-convex fuzzy set if for every $x,y \in X$ and every $\lambda \in [0,1]$ it holds that $\mu_A(H(x,y,\lambda)) \geq \min\{\mu_A(x), \mu_A(y)\}$.*

Convex fuzzy sets are clearly expanded by this definition. Hence, if X is a vector space and $H(x,y,\lambda) = \lambda x + (1-\lambda)y$, we have Definition 1.70, moreover, there is a relationship between H-convex fuzzy sets and its α-cuts.

Proposition 1.83 *[21] Let X be the universe and let H be a convex structure on X. The following statements are equivalent:*

(i) A is an H-convex fuzzy set,

(ii) any α-cut of A is an H-convex crisp set.

According to this argument, the H-convexity of a fuzzy set's α-cuts defines any H-convex fuzzy sets.

Chapter 2

Convexity of hesitant fuzzy sets

In this chapter, we are going to show two different proposals for defining the convexity of a hesitant fuzzy set. The first one is based on aggregation functions [38], whereas a totally different approach is considered in the second one. Both of them are based on the revision done in the related literature, as can be seen in the following section.

2.1 Overview of convexity of hesitant fuzzy sets

Several different approaches to the idea of convexity of hesitant fuzzy sets have been considered in the literature. This chapter starts with the introduction of the most relevant since this will be the starting point for the new proposal given here.

The first approach was given by Rasihd and Beg [65]. An important step, in this case, was to deal with the uncertainty associated with any membership degree. Thus, they considered α-cuts as a good way to solve this problem. More precisely, they started by suggesting a definition for the α-cuts of a hesitant fuzzy set based on the score function which is, in fact, an aggregation function.

Definition 2.1 *[65] Let X be a universe, let A be a hesitant fuzzy set defined on X, let s be the score function in $THFS(X)$ and let α be a number in the interval* $(0,1]$. *The crisp subset of X defined by*

$$A_\alpha = \{x \in X : s(h_A(x)) \geq \alpha\}$$

is said to be the α-cut (level set) of the hesitant fuzzy set A.

Hence, they defined convexity using the score function given in Definition 1.23.

Definition 2.2 *[65] Let $(X,+,\cdot)$ be a vector space and let s be the score function in $THFS(X)$. A hesitant fuzzy set A on the universe X is said to be convex if it holds, for any $x,y \in X$ and any $\lambda \in [0,1]$, that*

$$s(h_A(\lambda x+(1-\lambda)y)) \geq s(\lambda \odot h_A(x) \oplus (1-\lambda) \odot h_A(y))$$

where $\oplus$ and $\odot$ are the operations considered in Definition 1.56 and $-$ is the usual subtraction in the real line.

Definition 2.2 has at least two drawbacks:

1. It cannot be considered when the universe X is not a vector space since the addition and scalar multiplication could be not defined.
2. It has not an appropriate behaviour with respect to the level sets, that is, it is not cut-consistent. The reason is that even in the case that all the level sets of a hesitant fuzzy set are convex, the set may be not convex.

Due to its importance, the second drawback is illustrated in a detailed way in the following example.

Example 2.3 *Let X be the real line with the usual addition and multiplication on $\mathbb{R}$. Let A be the hesitant fuzzy set on X given by*

$$h_A(x) = \begin{cases} \{0.2, 0.25, 0, 27\} & \text{if } x \leq 0 \\ \{0.3, 0.35, 0.4\} & \text{otherwise} \end{cases}$$

As $s(\{0.2,0.25,0,27\}) = 0.24$ and $s(\{0.3,0.35,0.4\}) = 0.35$, the α-cuts are:

- *If $0 < \alpha \leq 0.24$:*

$$A_\alpha = \{x \in X : s(h_A(x)) \geq \alpha\} = \mathbb{R}$$

- *If $0.24 < \alpha \leq 0.35$*

$$A_\alpha = \{x \in X : s(h_A(x)) \geq \alpha\} = (0,\infty)$$

- *If* $0.35 < \alpha \leq 1$

$$A_\alpha = \{x \in X : s(h_A(x)) \geq \alpha\} = \emptyset$$

Thus, it is clear that any level set of A is a convex set of the real line.

However, if we consider $\lambda = 0.5$, $x = -5$ *and* $y = 3$ *then we have that* $0.5x + (1-0.5)y = -1$. *Thus,*

$$h_A(0.5x + (1-0.5)y) = h(-1) = \{0.2, 0.25, 0, 27\}$$

and therefore

$$s(h_A(0.5x + (1-0.5)y)) = 0.24$$

On the other hand,

$$0.5 \odot h_A(x) = \{0.1056, 0.1340, 0.1456\}$$

and

$$(1-0.5) \odot h_A(y) = \{0.1633, 0.1938, 0.2254\}$$

Hence $0.5 \odot h_A(x) \oplus (1-0.5) \odot h_A(y) = \{0.2517, 0.2789, 0.3072,\ 0.2754,\ 0.3018,\ 0.3292,\ 0.2852, 0.3112, 0.3382\}$. *Thus,*

$$s(\lambda \odot h_A(x) \oplus (1-\lambda) \odot h_A(y)) = 0.2976$$

and therefore $s(h_A(0.5x + (1-0.5)y)) < s(\lambda \odot h_A(x) \oplus (1-\lambda) \odot h_A(y))$ *that is, A is not a convex hesitant fuzzy set with respect to Definition 2.2.*

As a result, the initial Rashid and Beg concept of convexity was changed by themselves.

Definition 2.4 *[65] Let* $(X, +, \cdot)$ *be a vector space and let s be the score function in* $THFS(X)$. *A hesitant fuzzy set A on the universe X is said to be quasi-convex if it holds, for any* $x, y \in X$ *and any* $\lambda \in [0, 1]$, *that*

$$s(h_A(\lambda x + (1-\lambda)y)) \geq \min\{s(h_A(x)), s(h_A(y))\}$$

The quasi-convexity is cut-consistent, that is, this concept has the cutworthy property, as follows from the following proposition.

Proposition 2.5 *[65] Let $(X,+,\cdot)$ be a vector space and let A be a hesitant fuzzy set on $THFS(X)$. The followings statements are equivalent:*

1. *A is a quasi-convex hesitant fuzzy set.*

2. *Any α-cut of A is a convex crisp set.*

Thus, the second drawback of the initial definition is now solved. Moreover, we can see that quasi-convexity is a generalization of convexity.

Theorem 2.6 *[65] Let $(X,+,\cdot)$ be a vector space and let A be a hesitant fuzzy set on $THFS(X)$. If A is a convex hesitant fuzzy set, then A is a quasi-convex hesitant fuzzy set.*

According to this theory, every convex hesitant fuzzy set is also a quasi-convex hesitant fuzzy set. The opposite, however, is not true. Due to Proposition 2.5, Example 2.3 shows that *A* is a quasi-convex hesitant fuzzy set since the level sets are convex sets of the real line. However, we could see in this example that it is not a convex hesitant fuzzy set.

The second weakness in Definition 2.2 may also be solved if a convex structure is used to extend the idea of convexity, even in the case it is not a vector space.

Definition 2.7 *[65] Let X be the universe. Let H be a convex structure on X. A hesitant fuzzy set A in X is said to be an H-convex hesitant fuzzy set if for all $x,y \in X$, and $\lambda \in [0,1]$ it holds that*

$$s(h_A(H(x,y,\lambda))) \geq \min\{s(h_A(x)), s(h_A(y))\}$$

This definition also fulfills the cutworthy approach, as it was proven in the next proposition.

Proposition 2.8 *[65] Let X be a universe and let H be a convex structure on X. Then the following statements are equivalent:*

1. *A is an H-convex hesitant fuzzy set.*

2. *Any α-cut of A is an H-convex crisp set.*

Rashid and Beg put up this theory in [65], but it can be investigated in various ways. We would get new results if we choose an aggregation function other than the score. We will analyze this subject in the section that follows.

2.2 Aggregation functions in convexity of hesitant fuzzy sets

In the previous section, the different values of the membership function at any point are summarized by means of the score function. This is, in fact, the arithmetic mean of these values. Thus, along this section, a more general approach is considered, based on aggregation functions. Of course, this proposal has to be coherent with the previous studies and, in particular, to avoid the two drawbacks of some of them: only vector spaces can be considered as referential and not cut-consistent.

Moreover, an extra property will be required. Thus, we are interested in a definition that guarantees the convexity of the intersection of two convex hesitant fuzzy sets. In this section, we will use Definition 1.46 proposed by Torra[76] for the intersection of hesitant fuzzy sets. It is defined as the membership values in either of the two sets for this point that are lower than or equal to the lowest of the two maximums. Some partial results on this topic were published in [38, 41].

According to the traditional concept of convexity, we should take into account that every intermediate point is actually a membership degree that must be at least as high as the membership degree at which we are confident it is at the extreme points. Nevertheless, as the membership value for hesitant fuzzy sets can be a collection of values, we will aggregate them using an aggregation function before checking the earlier requirement.

Definition 2.9 *Let X be an ordered set, let A be a hesitant fuzzy set on X and let $\mathscr{A}$ be an aggregation function. A is said to be $\mathscr{A}$-convex, if for each $x, y, z \in X$ with $x < y < z$ it follows that*

$$\mathscr{A}(h_A(y)) \geq \min\{\mathscr{A}(h_A(x)), \mathscr{A}(h_A(z))\}$$

It is clear that this definition only makes sense if the ordered set has at least three ordered elements, so this is the case we will consider by default in this section.

In order to verify the cut-consistency of this definition, we need to consider a concept of α-level set based on the same ideas.

Definition 2.10 *Let X be any referential, let A be a hesitant fuzzy set on X and let $\mathscr{A}$ be an aggregation function. For any $\alpha \in (0,1]$, we define the $\mathscr{A} - \alpha$-level set of A, or simply α-level set when there is no ambiguity, as follows:*

$$A_\alpha^{\mathscr{A}} = \{x \in X : \alpha \leq \mathscr{A}(h_A(x))\}$$

It is possible to find an equivalent between convex hesitant fuzzy sets and the crisp convexity of the α-level sets as we show in the following result.

Proposition 2.11 *Let X be a totally ordered set, let A be a hesitant fuzzy set on X and let $\mathscr{A}$ be an aggregation function. A is a $\mathscr{A}$-convex if and only if $A_\alpha^{\mathscr{A}}$ are convex crisp sets for all $\alpha \in (0,1]$.*

Proof: Let us consider $x, y, z \in X$ such that $x < y < z$.

If $x \in A_\alpha^{\mathscr{A}}$ and $z \in A_\alpha^{\mathscr{A}}$, then $\alpha \leq \mathscr{A}(h_A(x))$ and $\alpha \leq \mathscr{A}(h_A(z))$. Consequently, as A is convex, we have $\min\{\mathscr{A}(h_A(x)), \mathscr{A}(h_A(z))\} \leq \mathscr{A}(h_A(y))$. For this reason, $\alpha \leq \mathscr{A}(h_A(y))$ and so $y \in A_\alpha^{\mathscr{A}}$. Thus $A_\alpha^{\mathscr{A}}$ is a convex crisp set.

On the other hand, we can take $c = \min\{\mathscr{A}(h_A(x)), \mathscr{A}(h_A(z))\} \in [0,1]$. Then, $x, z \in A_c^{\mathscr{A}}$. As $A_c^{\mathscr{A}}$ is a convex crisp set, we have that $y \in A_c^{\mathscr{A}}$ and so $\min\{\mathscr{A}(h_A(x)), \mathscr{A}(h_A(z))\} \leq \mathscr{A}(h_A(y))$. ■

One of the advantages of α-cuts or level sets of fuzzy sets is that given some particular level sets, we can construct one unique set from them. This allows us to work with level sets instead of the original sets which could be easier depending on the context. The main drawback of using Proposition 2.11 is that given some α-level sets it is not possible to be sure about the set they came from. For instance, if we are using the arithmetic mean $\mathscr{M}$ as the aggregation function, $X = \{x\}$ and $A = \langle x, \{0,1\}\rangle$, $B = \langle x, \{0, 0.5, 1\}\rangle$ or $C = \langle x, \{0, 0.1, 0.2, 0.8, 0.9, 1\}\rangle$, then $A_\alpha^{\mathscr{M}} = B_\alpha^{\mathscr{M}} = C_\alpha^{\mathscr{M}}, \forall \alpha \in (0,1]$, although they are clearly different sets. In Section 2.3 we will define another kind of α-level set that would work properly with our purposes of reconstruction of the set, but the considered here is the most coherent proposal for the level set taking into account the ideas behind the concept of convexity managed in this section.

The major goal of this part of our research is to examine how the $\mathscr{A}$-convexity of the intersection is affected by the aggregation function $\mathscr{A}$. For the purposes of our study, we have divided it in some circumstances based on the characteristics of the aggregation function $\mathscr{A}$, as follows:

1. $\mathscr{A}$ is lower than the minimum at some point
2. $\mathscr{A}$ is greater than the maximum at some point
3. $\mathscr{A}$ is equal to the maximum or the minimum
4. $\mathscr{A}$ is between the maximum and the minimum but it is different from both

2.2.1 The case under the minimum

The first situation is when there exists at least one point $(\alpha,\beta) \in [0,1]^2$ that fulfills $\mathscr{A}(\alpha,\beta) < \min\{\alpha,\beta\}$.

It is well known that t-norms are aggregation functions such that $T(\alpha,\beta) \leq \min\{\alpha,\beta\}$, $\forall(\alpha,\beta) \in [0,1]^2$. Let us check the next case to verify if a concrete t-norm (product t-norm) satisfies that convexity is conserved with the intersection.

Example 2.12 *Let $X = \{x,y,z\}$ be the referential with $x < y < z$ and let T_P the product t-norm, that is, $T_P(\alpha,\beta) = \alpha \cdot \beta, \forall \alpha,\beta \in [0,1]$. Let A and B be two hesitant fuzzy set defined by:*

$$h_A(x) = \{0.5\}, \forall x \in X$$

$$h_B(x) = h_B(z) = \{0.1, 0.6\}, \quad h_B(y) = \{0.2, 0.3, 1\}$$

Then the intersection is given by:

$$h_{A\cap B}(x) = h_{A\cap B}(z) = \{0.1, 0.5\}$$

and

$$h_{A\cap B}(y) = \{0.2, 0.3, 0.5\}$$

In Figure 2.1 we can see a graphical representation of A, B and $A \cap B$.

Let us check if h_A and h_B are $T_P -$ convex:

$$T_P(h_A(x)) = T_P(h_A(y)) = T_P(h_A(z)) = 0.5$$

$$T_P(h_B(x)) = T_P(h_B(z)) = 0.1 \cdot 0.6 = 0.06$$

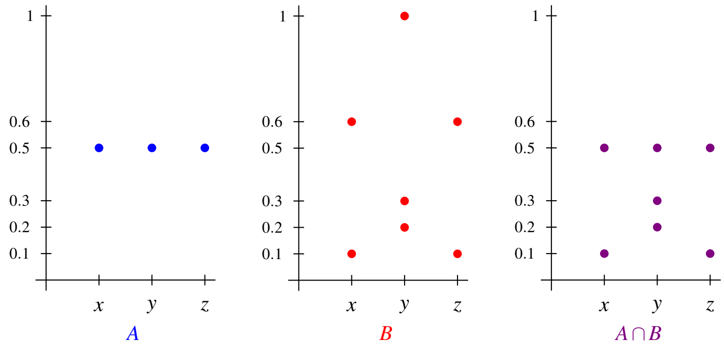

Figure 2.1: Graphical representation of the T_P-intersection.

and

$$T_P(h_B(y)) = 0.2 \cdot 0.3 \cdot 1 = 0.06$$

by applying the associativity of the t-norm.

So, it is clear that A and B are T_P-convex.

However,

$$T_P(h_{A\cap B}(x)) = T_P(h_{A\cap B}(z)) = 0.1 \cdot 0.5 = 0.05$$

and

$$T_P(h_{A\cap B}(y)) = 0.2 \cdot 0.3 \cdot 0.5 = 0.03$$

Thus,

$$T_P(h_{A\cap B}(y)) < \min\{T_P(h_{A\cap B}(x)), T_P(h_{A\cap B}(z))\}$$

that is, $A \cap B$ is not T_P-convex.

We have seen a negative behaviour for a specific t-norm and therefore we know that the convexity is not preserved in general for any t-norm or for any aggregation function when it is below the minimum. In fact, we can prove a general result for any aggregation function taking a value below the minimum.

Proposition 2.13 *Let X be an ordered set. If $\mathscr{A}$ is an aggregation function such that there is at least one pair of mutually distinct elements $(\alpha_1, \alpha_2) \in [0,1]^2$ for which $\mathscr{A}(\alpha_1, \alpha_2) < \min\{\alpha_1, \alpha_2\}$, then $\mathscr{A}$ does not preserve $\mathscr{A}$-convexity for the intersection.*

Proof: Let x, y, z be three elements in X such that $x < y < z$. If we denote by β the value of $\mathscr{A}(\alpha_1, \alpha_2)$ and we consider the hesitant fuzzy sets A and B defined by,

$$h_A(x) = h_A(z) = \{\alpha_1, \alpha_2\}, h_A(y) = \{\beta\}$$

$$h_B(x) = h_B(y) = h_B(z) = \{\min\{\alpha_1, \alpha_2\}\}$$

It is immediate to prove that both A and B are $\mathscr{A}$-convex.

On the other hand, their intersection is the hesitant fuzzy set defined by

$$h_{A\cap B}(x) = h_{A\cap B}(z) = \{\min\{\alpha_1, \alpha_2\}\}, h_{A\cap B}(y) = \{\beta\}$$

Then

$$\mathscr{A}(h_{A\cap B}(y)) = \beta < \min\{\mathscr{A}(h_{A\cap B}(x)), \mathscr{A}(h_{A\cap B}(z))\} =$$

$$\min\{\min\{\alpha_1, \alpha_2\}, \min\{\alpha_1, \alpha_2\}\} = \min\{\alpha_1, \alpha_2\}$$

Thus, the intersection is not $\mathscr{A}$-convex and therefore, it is possible to find a counterexample for any aggregation function lower than the minimum at least at one point. ∎

This proof could be illustrated in Figure 2.2, where we suppose that $\alpha_1 < \alpha_2$.

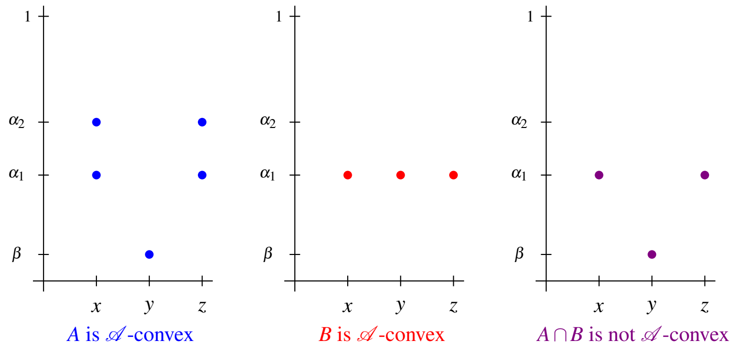

Figure 2.2: Graphical representation of the general counterexample considered at the proof of Proposition 2.13.

Corollary 2.14 *Let X be an ordered set. If $\mathscr{A}$ is a conjunctive aggregation different from the minimum, then $\mathscr{A}$ does not preserve $\mathscr{A}$-convexity for the intersection.*

Hence, any t-norm other than the minimum is not a suitable option for defining the convexity when using Definition 2.9.

2.2.2 The case over the maximum

Then, we examined what occurs when there exists a point $(\alpha,\beta) \in [0,1]^2$ such that $\mathscr{A}(\alpha,\beta) > \max\{\alpha,\beta\}$.

Once more, we begin with a particular example, to study if the convexity is properly preserved for the intersection, at least for a particular case of aggregation functions fulfilling this property. In particular, we are going to consider the case of t-conorms since any t-conorm S is known to be an aggregation function that satisfies the condition that $S(\alpha,\beta) \geq \max\{\alpha,\beta\}$, $\forall(\alpha,\beta) \in [0,1]^2$.

Example 2.15 *Let $X = \{x,y,z\}$ be the referential with $x < y < z$ and the Lukasiewicz t-conorm:*

$$S_L(\alpha,\beta) = \min\{\alpha+\beta,1\}, \quad \forall\alpha,\beta \in [0,1].$$

Let A and B be two hesitant fuzzy set defined as follows

$$h_A(x) = \{0.1,0.8\}, \forall x \in X$$

$$h_B(x) = \left\{0.8 - \frac{|x|}{5}, 0.9 - \frac{|x|}{5}\right\}, \forall x \in X$$

Thus the intersection is

$$h_{A\cap B}(x) = h_{A\cap B}(z) = \{0.1,0.6,0.7\}$$

and

$$h_{A\cap B}(y) = \{0.1,0.8\}$$

A, B and $A \cap B$ are graphically represented in Figure 2.3.

Thus,

$$S_L(h_A(x)) = S_L(h_A(y)) = S_L(h_A(z)) = 0.9$$

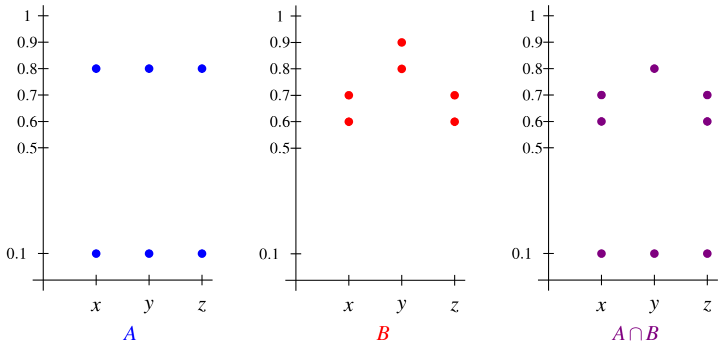

Figure 2.3: Two S_L-convex hesitant fuzzy sets whose intersection is not S_L-convex.

$$S_L(h_B(x)) = S_L(h_B(z)) = \min\{0.6 + 0.7, 1\} = 1$$

and

$$S_L(h_B(y)) = \min\{0.8 + 0.9, 1\} = 1$$

Then, it is clear that A and B are S_L-convex. Nevertheless, $A \cap B$ is not S_L-convex, since

$$S_L(h_{A\cap B}(x)) = S_L(h_{A\cap B}(z)) = \min\{\min\{0.1 + 0.6, 1\} + 0.7, 1\} = 1$$

and

$$S_L(h_{A\cap B}(y)) = \min\{0.8 + 0.1, 1\} = 0.9$$

We know now that there is at least an aggregation function assuming a value over the maximum such that the convexity is not preserved for the intersection. This is the case of the Lukasievicz t-conorm.

At this point, we also check the behaviour of another important t-conorm different from the maximum, which is called the product t-conorm or probabilistic sum.

Example 2.16 *Let us consider again that $X = \{x, y, z\}$ with $x < y < z$. Let A and B be two hesitant fuzzy sets considered in Example 2.15, where the intersection of A and B was also obtained. If we consider the product t-conorm*

$$S_P(\alpha, \beta) = \alpha + \beta - \alpha \cdot \beta, \quad \forall \alpha, \beta \in [0, 1]$$

we have that A and B are S_P − convex, since:

$$S_P(h_A(x)) = S_P(h_A(y)) = S_P(h_A(z)) = 0.8 + 0.1 - 0.8 \cdot 0.1 = 0.82$$

$$S_P(h_B(x)) = S_P(h_B(z)) = 0.6 + 0.7 - 0.6 \cdot 0.7 = 0.88$$

and

$$S_P(h_B(y)) = 0.8 + 0.9 - 0.8 \cdot 0.9 = 0.98$$

On the contrary, $A \cap B$ is not S_P − convex, since

$$S_P(h_{A\cap B}(x)) = S_P(0.1 + 0.6 - 0.1 \cdot 0.6, 0.7) = S_P(0.64, 0.7) = 0.892,$$

and

$$S_P(h_{A\cap B}(z)) = S_P(h_{A\cap B}(x)) = 0.892,$$

but

$$S_P(h_{A\cap B}(y)) = 0.1 + 0.8 - 0.1 \cdot 0.8 = 0.82.$$

These two counterexamples allow us to confirm that the required property is not fulfilled for the most important t-conorms different from the maximum. In fact, we will prove in the next result that the answer is negative for any aggregation function assuming at least one value over the maximum. For this purpose, we will generalize the previous counterexamples.

Proposition 2.17 *Let X be an ordered set. If $\mathscr{A}$ is an aggregation function such that there exists $(\alpha_1, \alpha_2) \in [0,1]^2$ such that $\mathscr{A}(\alpha_1, \alpha_2) > \max\{\alpha_1, \alpha_2\}$, then $\mathscr{A}$ does not preserve $\mathscr{A}$-convexity for the intersection.*

Proof: Let x, y, z be three elements in X such that $x < y < z$ and let us denote by β the value $\mathscr{A}(\alpha_1, \alpha_2)$. If we consider two hesitant fuzzy sets A and B defined as follows:

$$h_A(x) = h_A(z) = \{\alpha_1, \alpha_2\}, h_A(y) = \{\beta\}$$

$$h_B(x) = h_B(y) = h_B(z) = \{\max\{\alpha_1, \alpha_2\}\}$$

It is easy to check that both A and B are $\mathscr{A}$-convex. Their intersection is the hesitant fuzzy set determined by the membership function:

$$h_{A\cap B}(x) = h_{A\cap B}(z) = \{\alpha_1, \alpha_2\}, h_{A\cap B}(y) = \{\max\{\alpha_1, \alpha_2\}\}$$

A, B and their intersection are represented in Figure 2.4 for the case $\alpha_1 < \alpha_2$.

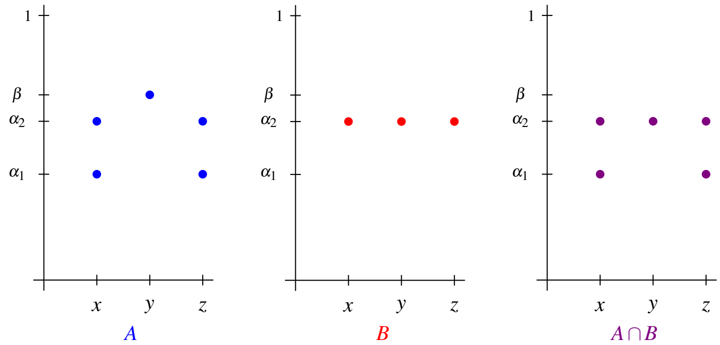

Figure 2.4: Graphical representation of A, B and $A \cap B$.

Then

$$\mathscr{A}(h_{A\cap B}(y)) = \mathscr{A}(\max\{\alpha_1, \alpha_2\}) = \max\{\alpha_1, \alpha_2\},$$

$$\mathscr{A}(h_{A\cap B}(x)) = \mathscr{A}(\alpha_1, \alpha_2) = \beta$$

and

$$\mathscr{A}(h_{A\cap B}(z)) = \mathscr{A}(h_{A\cap B}(x)) = \beta.$$

Thus, the intersection $A \cap B$ is not $\mathscr{A}$-convex as

$$\mathscr{A}((h_A \cap h_B)(y)) = \max\{\alpha_1, \alpha_2\} < \beta = \min\{\mathscr{A}(h_{A\cap B}(x)), \mathscr{A}(h_{A\cap B}(z))\}.$$

■

Corollary 2.18 *Let X be an ordered set. If $\mathscr{A}$ is a disjunctive aggregation different from the maximum, then $\mathscr{A}$ does not preserve $\mathscr{A}$-convexity for the intersection.*

Consequently, any t-conorm other than the maximum is not a suitable option for defining the convexity when using Definition 2.9.

2.2.3 The case of the maximum and minimum

From the previous subsections we know that the convexity is not preserved under intersections if it is based on an aggregation function assuming a value above the maximum or below the minimum. Thus, we will focus on the case of aggregation functions between these two maps, that is, the case of average functions. We will start by studying the two extreme cases. In both we will obtain a positive behaviour. In fact, the case of the maximum was already studied by Janis et al. [46] in 2018, being this paper the starting point for these general studies.

Proposition 2.19 *[46] Let X be an ordered set and let A and B be two hesitant fuzzy sets on X. If A and B are* max*-convex, then $A \cap B$ is also a* max*-convex hesitant fuzzy set.*

The remaining case of the minimum was not studied previously, but it is also possible to prove its good behaviour with respect to this property.

Proposition 2.20 *Let X be an ordered set and let A and B be two hesitant fuzzy sets on X. If A and B are* min*-convex, then $A \cap B$ is also a* min*-convex hesitant fuzzy set.*

Proof: Let A and B be two min-convex hesitant fuzzy sets on X. Let $x, y, z \in X$ such that $x \leq y \leq z$. Due to the definition of intersection it is clear that:

$$min\{(h_{A\cap B}(x)\} = min\{h_A(x), h_B(x)\}, \forall x \in X$$

As h_A is min-convex, we have that

$$\min\{h_A(y)\} \geq \min\{\min\{h_A(x)\}, \min\{h_A(z)\}\} = \min\{h_A(x), h_A(z)\}$$

and the same happens for B,

$$\min\{h_B(y)\} \geq \min\{\min\{h_B(x)\}, \min\{h_B(z)\}\} = \min\{h_B(x), h_B(z)\}$$

Thus, for $A \cap B$, we have that

$$\min\{(h_{A\cap B}(y)\} = min\{h_A(y), h_B(y)\} \geq$$

$$\min\{\min\{h_A(x), h_A(z)\}, \min\{h_B(x), h_B(z)\}\} =$$

$$\min\{\min\{h_A(x), h_B(x)\}, \min\{h_A(z), h_B(z)\}\} =$$

$$\min\{\min\{h_{A\cap B}(x)\}, \min\{h_{A\cap B}(z)\}\}$$

and therefore, we have proven that $A \cap B$ is min-convex. ∎

2.2.4 Averaging functions different from maximum and minimum

Until now we have completely studied and characterized the behaviour of any aggregation function which is conjunctive, disjunctive or mixed with respect to the preservation of the convexity under intersections. The remaining case is the case of the averaging functions different from maximum and minimum. Thus, we need to study the aggregation functions that are neither $\min\{x,y\}$ nor $\max\{x,y\}$ but fall between those two values.

We will divide our study into two parts. The case when the averaging function is strictly increasing and the case when it is just increasing. For the first one, we will obtain a general result. However, for the second one, there is no a common behaviour.

The strict increasing case

We start by assuming that the aggregation functions are strictly monotonic functions.

The arithmetic mean is a well-known example of a strictly increasing aggregation function. Using the arithmetic mean as our aggregation function, let us just look at an example.

Example 2.21 *Let us consider that our domain would be $\{x,y,z\}$ with $x < y < z$. Let $\mathscr{M}$ be the arithmetic mean. This, in the unit square,*

$$\mathscr{M}(\alpha,\beta) = \frac{\alpha+\beta}{2}, \forall \alpha,\beta \in [0,1]$$

Let A and B be two hesitant fuzzy sets on X defined as follows:

$$h_A(x) = h_A(z) = \{0.4\}, \quad h_A(y) = \{0.2, 0.6\}$$

$$h_B(x) = h_B(y) = h_B(z) = \{0.4\}$$

It is clear that the membership function of the intersection is

$$h_{A\cap B}(x) = h_{A\cap B}(z) = \{0.4\}$$

and

$$h_{A\cap B}(y) = \{0.2, 0.4\}$$

The three sets are represented in Figure 2.5.

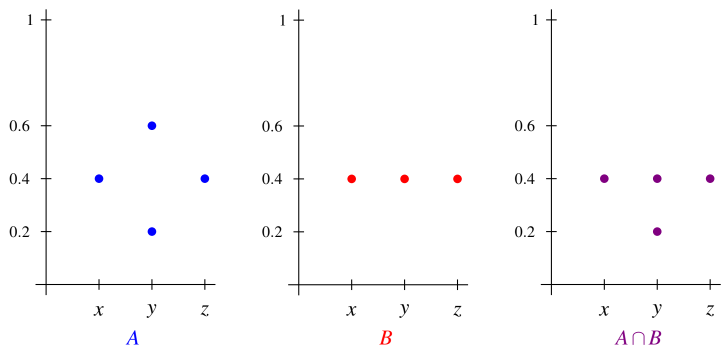

Figure 2.5: A, B and their intersection.

One the one hand, we have that

$$\mathscr{M}(h_A(x)) = \mathscr{M}(h_A(y)) = \mathscr{M}(h_A(z)) = 0.4$$

$$\mathscr{M}(h_B(x)) = \mathscr{M}(h_B(z)) = \mathscr{M}(h_B(y)) = 0.4$$

and therefore A and B are $\mathscr{M}$-convex. However,

$$\mathscr{M}(h_{A\cap B}(x)) = \mathscr{M}(h_{A\cap B}(z)) = 0.4$$

and

$$\mathscr{M}(h_{A\cap B}(y)) = \frac{0.4+0.2}{2} = 0.3$$

that is, $A\cap B$ is not $\mathscr{M}$-convex.

Therefore, we are aware that convexity is not being preserved for the intersection by the arithmetic mean. Moreover, we can prove that this negative answer can be obtained in general for any strictly increasing aggregation function.

Proposition 2.22 *Let X be an ordered set. If $\mathscr{A}$ is a strictly monotonic aggregation function, then $\mathscr{A}$ does not preserve $\mathscr{A}$-convexity under intersection.*

Proof: We are considering that $\min\{\alpha_1,\alpha_2\} \leq \mathscr{A}(\alpha_1,\alpha_2) \leq \max\{\alpha_1,\alpha_2\}, \forall \alpha_1,\alpha_2 \in [0,1]$. Since $\mathscr{A}$ is strictly, then $\mathscr{A}$ cannot be equal to the maximum. Then there exists $\alpha_1,\alpha_2 \in [0,1]$ such that $\mathscr{A}(\alpha_1,\alpha_2) < \max\{\alpha_1,\alpha_2\}$. This implies that $\alpha_1 \neq \alpha_2$ (otherwise the previous inequality is an equality since $\mathscr{A}$ is between minimum and maximum). If we consider $\beta = \mathscr{A}(\alpha_1,\alpha_2)$, then $\beta < \max\{\alpha_1,\alpha_2\}$. Thus, if we consider $x,y,z \in X$ with $x < y < z$ and we define the hesitant fuzzy sets A and B as follows:

$$h_A(x) = h_A(z) = \{\beta\}, h_A(y) = \{\alpha_1,\alpha_2\}$$

$$h_B(x) = h_B(y) = h_B(z) = \{\beta\}$$

we have that both A and B are $\mathscr{A}$-convex. Their intersection is the hesitant fuzzy set defined by

$$h_{A\cap B}(x) = h_{A\cap B}(z) = \{\beta\}, h_{A\cap B}(y) = \{\beta, \min\{\alpha_1,\alpha_2\}\}$$

These sets are illustrated in Figure 2.6, where we have supposed, without loss of generality, that $\alpha_1 < \alpha_2$.

Although A and B are $\mathscr{A}$-convex, we have that

$$\mathscr{A}(h_{A\cap B}(y)) = \mathscr{A}(\min\{\alpha_1,\alpha_2\},\beta) < \mathscr{A}(\min\{\alpha_1,\alpha_2\},\max\{\alpha_1,\alpha_2\})$$

since $\mathscr{A}$ is strictly increasing and $\beta < \max\{\alpha_1,\alpha_2\}$.

Thus,

$$\mathscr{A}(h_{A\cap B}(y)) < \mathscr{A}(\alpha_1,\alpha_2) = \beta = \min\{\mathscr{A}h_{A\cap B}(x)),\mathscr{A}(h_{A\cap B}(z)\}$$

so, the intersection $A \cap B$ is not $\mathscr{A}$-convex. ■

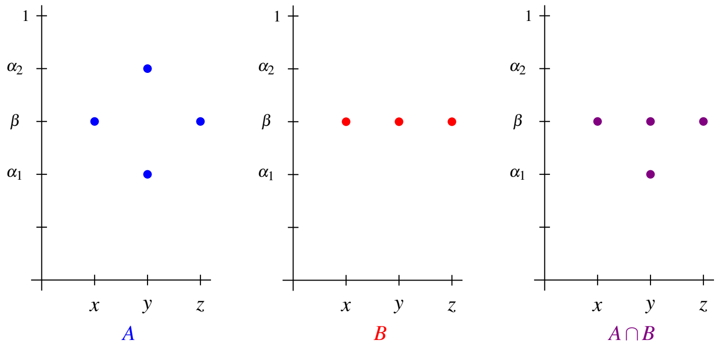

Figure 2.6: Graphical representation of A, B and their intersection.

The non-strict increasing case

Until now, we have been able to obtain general results for all the cases of aggregation functions considered. Thus, in most of them the convexity is not preserved for the intersection, but for any conjunctive, disjunctive, mixed, strictly increasing averaging aggregation function, the maximum and the minimum, we know when convexity is preserved and when it is not. In the remaining case, the case of averaging functions different from maximum and minimum which are not strictly monotonic, there is no a general behaviour. Thus, we can obtain functions preserving the convexity and functions that do not preserve it.

In order to obtain these examples, we will use the following result, which will allow us to define the averaging function from a map on $[0,1]^2$ and extend it by its associativity.

Proposition 2.23 *If $\mathscr{A}$ is an associative averaging function, then there exists $\alpha_1, \alpha_2 \in [0,1]$ with $\alpha_1 \neq \alpha_2$ such that $\mathscr{A}(\alpha_1, \alpha_2) = \min\{\alpha_1, \alpha_2\}$ or $\mathscr{A}(\alpha_1, \alpha_2) = \max\{\alpha_1, \alpha_2\}$.*

Proof: If there exists $\alpha_1, \alpha_2 \in [0,1]$ with $\alpha_1 \neq \alpha_2$ such that $\mathscr{A}(\alpha_1, \alpha_2) = \min\{\alpha_1, \alpha_2\}$ the proof is finished. Otherwise, it has to be $\mathscr{A}(\alpha_1, \alpha_2) > \min\{\alpha_1, \alpha_2\}$ for any $\alpha_1, \alpha_2 \in [0,1]$ with $\alpha_1 \neq \alpha_2$. Let's suppose $\alpha_1 < \alpha_2$, if we denote by $\beta = \mathscr{A}(\alpha_1, \alpha_2)$, we have that $\alpha_1 < \beta$ and

$$\mathscr{A}(\alpha_1, \beta) = \mathscr{A}(\alpha_1, \mathscr{A}(\alpha_1, \alpha_2)) = \mathscr{A}(\mathscr{A}(\alpha_1, \alpha_1), \beta) = \mathscr{A}(\alpha_1, \alpha_2) = \beta$$

Thus, $\mathscr{A}(\alpha_1, \beta) = \max\{\alpha_1, \beta\}$ and therefore the proof is concluded. ■

Thus, if $\mathscr{A}$ is associative and it is between minimum and maximum, it has to be equal to max or min in at least one point, that is, it is not possible that $\min\{\alpha, \beta\} < \mathscr{A}(\alpha, \beta) < \max\{\alpha, \beta\}, \forall \alpha, \beta \in [0,1]$.

We will use the previous result to obtain more examples where the convexity is not preserved the convexity for the intersection, apart from the arithmetic mean.

Example 2.24 *For the aggregation function generated by $\mathscr{A}^1$, which is a combination between* $\min\{x,y\}$ *and* $\max\{x,y\}$*, where:*

$$\mathscr{A}^1(\alpha, \beta) = \begin{cases} \max\{\alpha, \beta\} & \text{if } \alpha + \beta > 1 \\ \min\{\alpha, \beta\} & \text{if } \alpha + \beta \leq 1 \end{cases}$$

if we consider $X = \{x, y, z\}$ with $x < y < z$ and the hesitant fuzzy set A and B defined as:

$$h_A(x) = h_A(z) = \{0.4\}, \quad h_A(y) = \{0.2, 1\}$$

$$h_B(x) = h_B(y) = h_B(z) = \{0.4\}$$

then the intersection is

$$h_{A \cap B}(x) = h_{A \cap B}(z) = \{0.4\}$$

and

$$h_{A \cap B}(y) = \{0.2, 0.4\}$$

We can see A, B and $A \cap B$ in Figure 2.7.

Since

$$\mathscr{A}^1(h_A(x)) = \mathscr{A}^1(h_A(z)) = 0.4, \quad \mathscr{A}^1(h_A(y)) = 1$$

$$\mathscr{A}^1(h_B(x)) = \mathscr{A}^1(h_B(z)) = \mathscr{A}^1(h_B(y)) = 0.4$$

we have that A and B are $\mathscr{A}^1$-convex. However, $A \cap B$ is not $\mathscr{A}^1$-convex, since

$$\mathscr{A}^1(h_{A \cap B}(x)) = \mathscr{A}^1(h_{A \cap B}(z)) = 0.4$$

and

$$\mathscr{A}^1(h_{A \cap B}(y)) = \min\{0.2, 0.4\} = 0.2$$

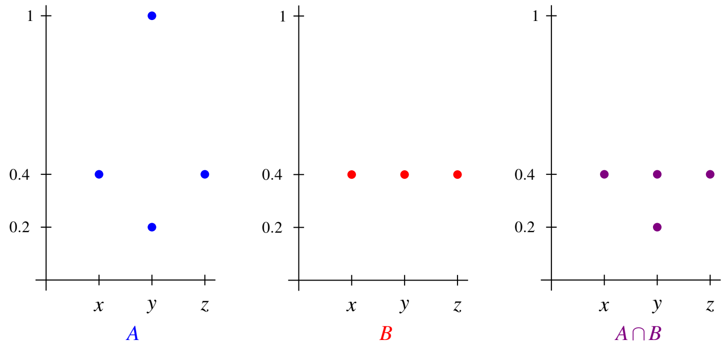

Figure 2.7: Counterexample for $\mathscr{A}^1$.

Thus, the averaging function generated by $\mathscr{A}^1$ is not strictly increasing and it does not preserve the convexity for the intersection.

Example 2.25 *If we consider the aggregation function generated by:*

$$\mathscr{A}^2(\alpha,\beta) = \begin{cases} 1 & \text{if } \alpha+\beta = 2 \\ 0.5 & \text{if } 1 < \alpha+\beta < 2 \\ \min\{\alpha,\beta\} & \text{if } \alpha+\beta \leq 1 \end{cases}$$

we have that the hesitant fuzzy sets A and B considered at the previous example are also $\mathscr{A}^2$-convex:

$$\mathscr{A}^2(h_A(x)) = \mathscr{A}^2(h_A(z)) = 0.4, \quad \mathscr{A}^2(h_A(y)) = 0.5$$

$$\mathscr{A}^2(h_B(x)) = \mathscr{A}^2(h_B(z)) = \mathscr{A}^2(h_B(y)) = 0.4$$

but again it does not preserve the convexity for the intersection, since $A \cap B$ is not $\mathscr{A}^2$-convex:

$$\mathscr{A}^2(h_{A\cap B}(x)) = \mathscr{A}^2(h_{A\cap B}(z)) = 0.4$$

and

$$\mathscr{A}^2(h_{A\cap B}(y)) = \min\{0.2, 0.4\} = 0.2$$

Example 2.26 *Finally, if we consider:*

$$\mathscr{A}^3(\alpha,\beta) = \begin{cases} \min\{\alpha,\beta\} & \textit{if } \alpha,\beta \leq 0.5 \textit{ or } \alpha,\beta \geq 0.5 \\ 0.5 & \textit{otherwise} \end{cases}$$

the aggregation function generated by $\mathscr{A}^3$ *is an averaging function non-strictly increasing. Moreover, if we consider* $X = \{x,y,z\}$ *with* $x < y < z$ *and the two typical hesitant fuzzy set A and B over X, defined as follows:*

$$h_A(x) = h_A(z) = \{0.5\}, \quad h_A(y) = \{0.3, 0.7\}$$

$$h_B(x) = h_B(y) = h_B(z) = \{0.5\}$$

we obtain that the intersection is given by

$$h_{A\cap B}(x) = h_{A\cap B}(z) = \{0.5\}$$

and

$$h_{A\cap B}(y) = \{0.3, 0.5\}$$

A, B and $A \cap B$ *are illustrated in Figure 2.8.*

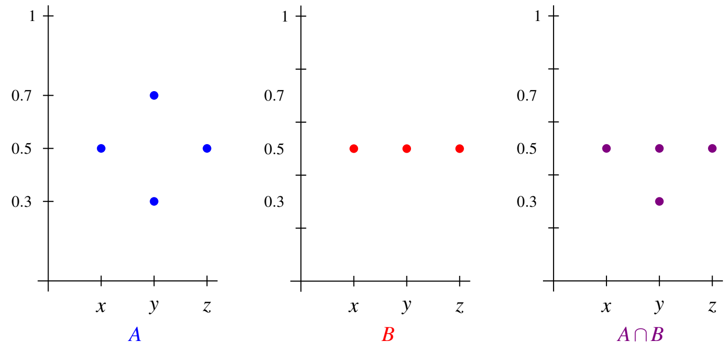

Figure 2.8: Counterexample for $\mathscr{A}^3(x,y)$.

On the one hand, we have that

$$\mathscr{A}^3(h_A(x)) = \mathscr{A}^3(h_A(z)) = 0.5, \quad \mathscr{A}^3(h_A(y)) = 0.5$$

$$\mathscr{A}^3(h_B(x)) = \mathscr{A}^3(h_B(z)) = \mathscr{A}^3(h_B(y)) = 0.5$$

so that A and B are $\mathscr{A}^3$-convex. However, $A \cap B$ is not $\mathscr{A}^3$-convex, since

$$\mathscr{A}^3(h_{A\cap B}(x)) = \mathscr{A}^3(h_{A\cap B}(z)) = 0.5$$

and

$$\mathscr{A}^3(h_{A\cap B}(y)) = \min\{0.3, 0.5\} = 0.3$$

And again, it does not preserve the convexity of the intersection.

At this point, after all these negative examples, we could think that it is possible to prove a general result about the no-preservation of the convexity for this kind of aggregation function. However, this is not true, since there are functions in this family preserving the convexity for the intersection. One of them is shown in the following result.

Proposition 2.27 *Let X be an ordered set. If we consider the map*

$$\mathscr{A}^4(\alpha,\beta) = \begin{cases} \max\{\alpha,\beta\} & \text{if } \alpha,\beta \in [0,0.5] \\ \min\{\alpha,\beta\} & \text{if } \alpha,\beta \in (0.5,1] \\ 0.5 & \text{otherwise} \end{cases}$$

then it is possible to generate from it an averaging function different from the maximum and the minimum, non-strictly increasing and such the intersection of any two $\mathscr{A}^4$-convex hesitant fuzzy sets is a $\mathscr{A}^4$-convex hesitant fuzzy set.

Proof: The map $\mathscr{A}^4$ is illustrated at Figure 2.9.

It is clear from this representation that $\mathscr{A}^4$ is increasing but not strictly. Apart from that, from its definition, we have that $\mathscr{A}^4(0,0) = \max\{0,0\}$ and $\mathscr{A}^4(1,1) = \min\{1,1\}$. Moreover, this mapping is a nullnorm (see [12]) and it is known that nullnorms are associative. So, it can be in a natural way extended to a mapping $\mathscr{A}^4 : \cup_n[0,1]^n \to [0,1]$, which is also an aggregation function.

By definition it is clear that $\min\{\mathbf{x}\} \le \mathscr{A}^4(\mathbf{x}) \le \max\{\mathbf{x}\}$ for any $\mathbf{x} \in \cup_n[0,1]^n$. Furthermore, $\mathscr{A}^4(0.7, 0.2) = 0.5$, so it is also clear that $\mathscr{A}^4$ is different from the minimum or the maximum.

Finally, we will show that $\mathscr{A}^4$-convexity is preserved by intersections.

Let us suppose that X is an ordered set. Let A and B be $\mathscr{A}^4$-convex hesitant fuzzy sets. Let $x, y, z \in X$ such that $x < y < z$.

The proof will be divided into three cases:

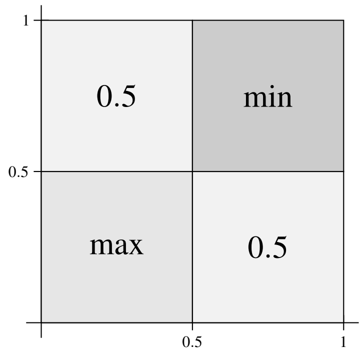

Figure 2.9: Graphical representation of $\mathscr{A}^4$.

1. The case $\mathscr{A}^4(h_{A\cap B}(y)) > 0.5$.

   (a) If $\mathscr{A}^4(h_{A\cap B}(x)) \leq 0.5$ or $\mathscr{A}^4(h_{A\cap B}(z)) \leq 0.5$ then

   $$\min\{\mathscr{A}^4(h_{A\cap B}(x)), \mathscr{A}^4(h_{A\cap B}(z))\} \leq 0.5 < \mathscr{A}^4(h_{A\cap B}(y))$$

   and therefore the condition to be $A \cap B$ $\mathscr{A}^4$-convex is fulfilled in this case.

   (b) If $\mathscr{A}^4(h_{A\cap B}(x)) > 0.5$ and $\mathscr{A}^4(h_{A\cap B}(z)) > 0.5$ then, by the definition of $\mathscr{A}^4$, $\mathscr{A}^4(h_{A\cap B}(x)) = \min\{h_{A\cap B}(x)\}$ and $\mathscr{A}^4(h_{A\cap B}(z)) = \min\{h_{A\cap B}(z)\}$ and considering the definition of the intersection (Definition 1.46), we have that $0.5 < \mathscr{A}^4(h_{A\cap B}(x)) = \min\{h_{A\cap B}(x)\} = \min\{h_A(x), h_B(x)\}$ and that $0.5 < \mathscr{A}^4(h_{A\cap B}(z)) = \min\{h_{A\cap B}(z)\} = \min\{h_A(z), h_B(z)\}$.
   Then,

   $$\min\{\mathscr{A}^4(h_{A\cap B}(x)), \mathscr{A}^4(h_{A\cap B}(z))\} = \min\{h_A(x), h_B(x), h_A(z), h_B(z)\} =$$

   $$\min\{\min\{\min\{h_A(x)\}, \min\{h_A(z)\}\}, \min\{\min\{h_B(x)\}, \min\{h_B(z)\}\}\}.$$

   As we noticed that $0.5 < \min\{h_A(x), h_B(x)\}$ and $0.5 < \min\{h_A(z), h_B(z)\}$, by definition of $\mathscr{A}^4$, we have that

   $$\mathscr{A}^4(h_A(x)) = \min\{h_A(x)\} \quad \mathscr{A}^4(h_A(z)) = \min\{h_A(z)\}$$
   $$\mathscr{A}^4(h_B(x)) = \min\{h_B(x)\} \quad \mathscr{A}^4(h_B(z)) = \min\{h_B(z)\}$$

Then,

$$\min\{\mathscr{A}^4(h_{A\cap B}(x)), \mathscr{A}^4(h_{A\cap B}(z))\} =$$

$$\min\{\min\{\mathscr{A}^4(h_A(x)), \mathscr{A}^4(h_A(z))\}, \min\{\mathscr{A}^4(h_B(x)), \mathscr{A}^4(h_B(z))\}\}.$$

But, by the $\mathscr{A}^4$-convexity of A and B we have that

$$\min\{\mathscr{A}^4(h_{A\cap B}(x)), \mathscr{A}^4(h_{A\cap B}(z))\} \le \min\{\mathscr{A}^4(h_A(y)), \mathscr{A}^4(h_B(y))\}$$

On the other hand, $\mathscr{A}^4(h_{A\cap B}(y)) > 0.5$, we also have that $\mathscr{A}^4(h_{A\cap B}(y)) = \min\{h_{A\cap B}(y)\} = \min\{h_A(y), h_B(y)\} = \min\{\min\{h_A(y)\}, \min\{h_B(y)\}\} = \min\{\mathscr{A}^4(h_A(y)), \mathscr{A}^4(h_B(y))\}$.

Thus, we have proven that

$$\min\{\mathscr{A}^4(h_{A\cap B}(x)), \mathscr{A}^4(h_{A\cap B}(z))\} \le \mathscr{A}^4(h_{A\cap B}(y)).$$

2. If $\mathscr{A}^4(h_{A\cap B}(y)) < 0.5$, then $\mathscr{A}^4(h_{A\cap B}(y)) = \max\{h_{A\cap B}(y)\}$.

   By the definition of the intersection, we have $\max\{h_{A\cap B}(y)\} = \max\{h_A(y)\}$ or $\max\{h_{A\cap B}(y)\} = \max\{h_B(y)\}$. Suppose we have the first case (the proof for the second case it totally analogous). Since $\max\{h_A(y)\} < 0.5$, then $\mathscr{A}^4(h_A(y)) = \max\{h_A(y)\}$. By applying that A is a $\mathscr{A}^4$-convex hesitant fuzzy set, we have $\mathscr{A}^4(h_A(x)) \le \mathscr{A}^4(h_A(y))$ or $\mathscr{A}^4(h_A(z)) \le \mathscr{A}^4(h_A(y))$. Let us consider that we have the first case (again the second case is analogous). Thus, $\mathscr{A}^4(h_A(x)) < 0.5$ and then $\mathscr{A}^4(h_A(x)) = \max\{h_A(x)\} < 0.5$. By considering again the definition of the intersection, we see that $\max\{h_{A\cap B}(x)\} \le \max\{h_A(x)\} < 0.5$ and therefore $\mathscr{A}^4(h_{A\cap B}(x)) = \max\{h_{A\cap B}(x)\}$. Now, if we join the above inequalities and equalities, we have:

$$\mathscr{A}^4(h_{A\cap B}(x)) = \max\{h_{A\cap B}(x)\} \le \max\{h_A(x)\} = \mathscr{A}^4(h_A(x)) \le$$

$$\mathscr{A}^4(h_A(y)) = \max\{h_{A\cap B}(y)\} = \mathscr{A}^4(h_{A\cap B}(y))$$

   and then

$$\mathscr{A}^4(h_{A\cap B}(y)) \ge \min\{\mathscr{A}^4(h_{A\cap B}(x)), \mathscr{A}^4(h_{A\cap B}(z))\}.$$

3. The case $\mathscr{A}^4(h_{A\cap B}(y)) = 0.5$.

   If $\mathscr{A}^4(h_{A\cap B}(x)) \leq 0.5$ or $\mathscr{A}^4(h_{A\cap B}(z)) \leq 0.5$, then the proof is trivial. Thus, we will consider that $\mathscr{A}^4(h_{A\cap B}(x)) > 0.5$ and $\mathscr{A}^4(h_{A\cap B}(z)) > 0.5$. In that case,

   $$\mathscr{A}^4(h_{A\cap B}(x)) = \min\{h_{A\cap B}(x)\} = \min\{h_A(x), h_B(x)\} > 0.5$$

   and

   $$\mathscr{A}^4(h_{A\cap B}(z)) = \min\{h_{A\cap B}(z)\} = \min\{h_A(z), h_B(z)\} > 0.5$$

   Then

   $$\min\{h_A(x)\}, \min\{h_A(z)\}, \min\{h_B(x)\}, \min\{h_B(z)\} > 0.5$$

   and therefore $\mathscr{A}^4(h_A(x)) = \min\{h_A(x)\} > 0.5$ and similarly we prove that

   $$\mathscr{A}^4(h_A(z)), \mathscr{A}^4(h_B(x)), \mathscr{A}^4(h_B(z)) > 0.5.$$

   As A and B are $\mathscr{A}^4$-convex, then $\mathscr{A}^4(h_A(y)) > 0.5$ and $\mathscr{A}^4(h_B(y)) > 0.5$. Then, $\min\{h_A(y), h_B(y)\} > 0.5$ and therefore $\mathscr{A}^4(h_{A\cap B}(y)) = \min\{h_{A\cap B}(y)\} > 0.5$ which is a contradiction, so we can assure that $\mathscr{A}^4(h_{A\cap B}(x)) \leq 0.5$ or $\mathscr{A}^4(h_{A\cap B}(z)) \leq 0.5$ and therefore

   $$\mathscr{A}^4(h_{A\cap B}(y)) = 0.5 \geq \min\{\mathscr{A}^4(h_{A\cap B}(x)), \mathscr{A}^4(h_{A\cap B}(z))\}.$$

Thus, we have proven that $A \cap B$ is $\mathscr{A}^4$-convex. ■

Note that in fact any nullnorm and its extensions could be used in the previous demonstration.

As all four cases are studied, we know the behaviour of the different aggregation functions with respect to the preservation of convexity under intersections. Now we can say that only minimum, maximum and some specific aggregation functions between them are appropriate to define convexity for hesitant fuzzy sets.

2.3 Convexity of hesitant fuzzy sets without using aggregation functions

In this section we show the second approach of convexity for hesitant fuzzy sets. The problem when using aggregation functions is that one can find one result coming from different inputs. This is a clear summarization of the information which allows identifying any membership value with a number. Thus, any element in $\bigcup_{n\in\mathbb{N}}[0,1]^n$ is mapped into an element in $[0,1]$ and we have to deal with numbers in the real line, where we have a lot of known good properties. However, as we commented previously, this is an important lack of information. We will try to solve this problem by considering a different approach for the convexity of hesitant fuzzy sets. Now we will deal with the original membership function and no fusion of the values will be done. Thus, we have to manage elements in $\bigcup_{n\in\mathbb{N}}[0,1]^n$ and we have to order them since it is an essential step when working with convexity. Therefore, the orders in $\mathbb{H}$ considered in Section 1.2.1 will be essential in this approach.

2.3.1 Operations for hesitant fuzzy sets based on orders in $\mathbb{H}$

It is clear that we are now considering a different point of view for dealing with hesitant fuzzy sets. Taking into account this, we can use this approach as well as the considered orders to redefine the main operations. More precisely, we will propose a new definition of the intersection and union of hesitant fuzzy sets and define the idea of level sets. For the study of convexity, the intersection and level sets of hesitant fuzzy sets are crucial concepts, and the union is required to understand the idea of a level set.

Intersection

First, we would like to recall the classical notion of intersection. For crisp sets, the intersection of two sets A and B is the largest set contained in A and B. Thus, the intersection is closely related to the content between sets and a definition for this operation for hesitant fuzzy sets is required. Taking into account that if μ_A and μ_B

are fuzzy sets then μ_A is contained in μ_B if and only if $\mu_A(x) \leq \mu_B(x)$ for all $x \in X$, it is clear that we can follow a similar reasoning for the case of hesitant fuzzy sets based on orders for typical hesitant fuzzy elements. Thus,

Definition 2.28 *Let A and B be two hesitant fuzzy sets in X. Let us consider an order $\trianglelefteq_o$ in $\mathbb{H}$. It is said that A is o-content in B, and it is denoted by $A \subseteq_o B$, if and only if $A(x) \trianglelefteq_o B(x)$ for any $x \in X$.*

It is immediate to prove that $\subseteq_o$ is an order in $THFS(X)$, which is not a total order, even in the case $\trianglelefteq_o$ so is. It is also clear that Definition 1.46 proposed by Torra does not fulfill, in general, that it is the largest set contained in A and B. Then we propose the following definition.

Definition 2.29 *Let A and B be two hesitant fuzzy sets in X. Let us consider an order $\trianglelefteq_o$ in $\mathbb{H}$. The o-intersection of A and B, which is denoted by $A \cap_o B$, is the largest hesitant fuzzy sets o-contained in A and B.*

Under this definition, we tried to collect the main ideas for the intersection of classical sets. An equivalent definition is obtained from the following proposition in the case the order is total. It is in general more useful for practical cases.

Proposition 2.30 *Let $\trianglelefteq_o$ be a total order on $\mathbb{H}$. For any $A, B \in THFS(X)$, the o-intersection of A and B is the hesitant fuzzy set whose membership function assumes the value $\min_o\{h_A(x), h_B(x)\}$, for any $x \in X$, where $\min_o$ denotes the minimum w.r.t. the order $\trianglelefteq_o$.*

Proof: If we denote by H the hesitant fuzzy set with this membership function, it is clear that

$$h_H(x) = \begin{cases} h_A(x) & \text{if } h_A(x) \trianglelefteq_o h_B(x), \\ h_B(x) & \text{if } h_B(x) \trianglelefteq_o h_A(x). \end{cases}$$

This set is obviously a hesitant fuzzy set. Thus, we know that $h_H(x) = h_A(x)$ if $h_A(x) \trianglelefteq_o h_B(x)$ and $h_H(x) = h_B(x)$ if $h_B(x) \trianglelefteq_o h_A(x)$. The fact that $h_H(x) \trianglelefteq_o h_A(x)$ and $h_H(x) \trianglelefteq_o h_B(x)$ for each $x \in X$ is true since $\trianglelefteq_o$ is transitive. Therefore $H \subseteq_o A$ and $H \subseteq_o B$.

Additionally, if we take into account a set $C \in THFS(X)$ with $C \subseteq_o A$ and $C \subseteq_o B$, thus for each $x \in X$, since there are no incomparable items under $\trianglelefteq_o$, we get two possibilities:

a) If $h_A(x) \trianglelefteq_o h_B(x)$, then $h_H(x) = h_A(x)$. But, $h_C(x) \trianglelefteq_o h_A(x) = h_H(x)$.

b) If $h_B(x) \trianglelefteq_o h_A(x)$, then $h_H(x) = h_B(x)$. But, $h_C(x) \trianglelefteq_o h_B(x) = h_H(x)$.

So, for both cases, we arrive at the conclusion that $h_C(x) \trianglelefteq_o h_H(x)$ and thus $C \subseteq_o H$.

As a result, H is the largest hesitant fuzzy set which is o-included in A and B and that implies it is their o-intersection. ■

Example 2.31 *Let A, B and X be the sets defined in Example 1.21 and let us consider the lexicographical order type 1.*

The $\trianglelefteq_{Lex1}$-intersection of A and B is obtained as follows:

- *For $x = 0$, we have that $h_{A\cap_{Lex1}B}(0) = \min_{Lex1}\{h_A(0), h_B(0)\} = \min_{Lex1}\{\{0.25, 0.5\}, \{1/e\}\} = \{0.25, 0.5\}$ as it exists $i \in \{1\}$ such that $h_A(0)^1 < h_B(0)^1$.*
- *For $x = 0.5$, we have that $h_{A\cap_{Lex1}B}(0.5) = \min_{Lex1}\{h_A(0.5), h_B(0.5)\} = \min\{\{0\}, \{1/\sqrt{e}\}\} = \{0\}$.*
- *For $x = 1$, we have that $h_{A\cap_{Lex1}B}(1) = \min_{Lex1}\{h_A(1), h_B(1)\} = \min_{Lex1}\{\{0.2, 0.4, 0.6, 0.8\}, \{1\}\} = \{0.2, 0.4, 0.6, 0.8\}$.*

Hence,

$$A \cap_{Lex1} B = \{\langle 0, \{0.25, 0.5\}\rangle, \langle 0.5, \{0\}\rangle, \langle 1, \{0.2, 0.4, 0.6, 0.8\}\rangle\}$$

It is represented in Figure 2.10 and it is clear that it is different that the intersections obtained in Examples 1.47 and 1.49.

Lexicographical orders are particular examples of admissible orders, which are total orders refining the lattice order. It should be noticed that we ask for a total order because if we use a partial order, the intersection can not be properly defined as we can see in the following example.

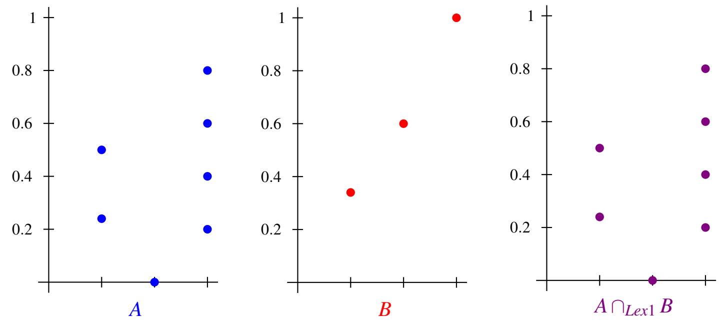

Figure 2.10: $Lex1$-intersection of A and B.

Example 2.32 *Let us consider that $X = \{x\}$ and the hesitant fuzzy sets A and B in X are defined by $A = \{\langle x, \{0, 0.2, 0.8, 1\}\rangle\}$ and $B = \{\langle x, \{0, 0.1, 0.9, 1\}\rangle\}$.*

If we use the minimum score function, then $S_{min}(h_A(0)) = 0 = S_{min}(h_B(0))$ and therefore $h_A(x) \trianglelefteq_{min} h_B(x)$ and $h_B(x) \trianglelefteq_{min} h_A(x)$, but $h_A(x) \neq h_B(x)$. Thus, as the relation $\trianglelefteq_{min}$ is just a preorder, and therefore it is not symmetric, we can not decide which one is bigger so the intersection could be any of them and it is not properly defined.

In fact, the problem could remain when the relation is an order, but it is not a total order. For instance, if we consider the lattice order $\trianglelefteq_{Lo}$, taking into account the definition of intersection given in Definition 2.29, we know that, in this example, $h_A(x)$ and $h_B(x)$ are not comparable and therefore, we cannot obtain the minimum of both elements. It could be possible to consider the infimum instead of the minimum, in this case it would be $h_C(x) = \{0, 0.1, 0.8, 1\}$, however, we can not always assure the existence of the infimum as we need a lower bounded set to guarantee it. This happens in this case, since the lattice order generates a lattice. Really this is the reason for the name given to this order.

Union

If the lowest set that includes both sets is the definition of the union of two sets, then there is again a different interpretation of the union for each order we use in $THFS(X)$. As the union would be a useful tool for the following item, we can thus do a research similar to the one given for the intersection. It is also clear that

Definition 1.50 proposed by Torra does not fulfill in general that it is the smallest set containing A and B. Then we propose the following definition of the union.

Definition 2.33 *Let $A, B \in THFS(X)$ and let $\trianglelefteq_o$ be an order in $\mathbb{H}$. We define the o-union of A and B, denoted by $A \cup_o B$, as the smallest hesitant fuzzy set such that A and B are o-contained in it.*

In an analogous way to the intersection, we will only consider total orders.

Proposition 2.34 *Let $\trianglelefteq_o$ be a total order on $\mathbb{H}$. For any $A, B \in THFS(X)$, the o-union of A and B is the hesitant fuzzy set whose membership function, at any point $x \in X$, is $\max_o\{h_A(x), h_B(x)\}$, for any $x \in X$, where $\max_o$ denotes the maximum w.r.t. the order $\trianglelefteq_o$.*

Proof: If we consider

$$h_H(x) = \max_o\{h_A(x), h_B(x)\} = \begin{cases} h_B(x) & \text{if } h_A(x) \trianglelefteq_o h_B(x), \\ h_A(x) & \text{if } h_B(x) \trianglelefteq_o h_A(x). \end{cases}$$

it is clear that this is the membership function of a hesitant fuzzy set H, since $\trianglelefteq_o$ is a total order and it is clear that H is well-defined, since $h_A(x) \trianglelefteq_o h_B(x)$ or $h_B(x) \trianglelefteq_o h_A(x)$, for all $x \in X$.

By definition, it is obvious that $A \subseteq_o H$ and $B \subseteq_o H$.

Lastly, if we assume that there exists a hesitant fuzzy set $C \in THFS(X)$ such that $A \subseteq_o C$ and $B \subseteq_o C$, then $h_A(x) \trianglelefteq_o h_C(x)$ and $h_B(x) \trianglelefteq_o h_C(x)$, for all $x \in X$. By the transitivity of $\trianglelefteq_o$, it is immediate that $h_H(x) \trianglelefteq_o h_C(x)$ and therefore $H \subseteq_o C$. Thus, H is the smallest hesitant fuzzy set o-containing A and B. ■

Example 2.35 *In Figure 2.11, it is possible to find the $Lex1$-union of the hesitant fuzzy sets considered in Example 1.21.*

Level sets of hesitant fuzzy sets

An α-cut or a level set is one of the most crucial ideas in fuzzy sets, according to Klir [49]. In this section, we provide an appropriate definition of a level set for hesitant fuzzy set taking into account the criterion of avoiding the loss of information

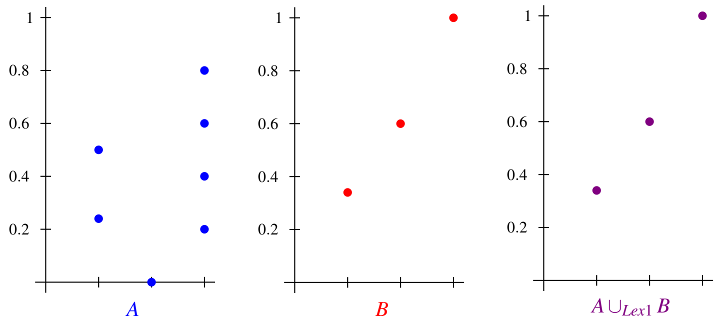

Figure 2.11: $Lex1$-union of A and B.

by fusing the values of the membership function, since the ones considered in the previous chapter were based on that idea and, therefore, are not appropriate in this case.

Definition 2.36 *Let $\trianglelefteq_o$ be an order on $\mathbb{H}$. For any $A \in THFS(X)$ and for any $\alpha \in \mathbb{H}$, we define the α-level sets of A w.r.t. the order $\trianglelefteq_o$ as follows:*

$$A^o_\alpha = \{x \in X : \alpha \trianglelefteq_o h_A(x)\}$$

It is indeed remarkable that if we employ a different order, we would get different level sets because the definition varies depending on the order we choose.

Example 2.37 *Let $X = \{x, y, z\}$. Let us consider $A \in HFS(X)$ defined as:*

$$A = \{\langle x, \{0.3\}\rangle, \langle y, \{0.5, 0.6\}\rangle, \langle z, \{0.4, 0.7, 0.8\}\rangle\}$$

We have calculated some level sets of this set for different orders in Table 2.1.

Order \ Level (α)	$\{0.3\}$	$\{0.5,0.6\}$	$\{0.4,0.7,0.8\}$	$\{0.4,0.6\}$
Lexicographical type 1	$\{x,y,z\}$	$\{y\}$	$\{y,z\}$	$\{y,z\}$
Lexicographical type 2	$\{x,y,z\}$	$\{y,z\}$	$\{z\}$	$\{y,z\}$

Table 2.1: Level sets for different orders.

Based on this example, we can notice there could be some relations between level sets.

Proposition 2.38 *If $\trianglelefteq_1$ and $\trianglelefteq_2$ are orders in $\mathbb{H}$ such that $a \trianglelefteq_1 b$ implies $a \trianglelefteq_2 b$, then for any $A \in THFS(X)$ and any $\alpha \in \mathbb{H}$ we have that $A^1_\alpha \subseteq A^2_\alpha$.*

Proof: Since $A^1_\alpha = \{x \in X : \alpha \trianglelefteq_1 h_A(x)\}$ and $A^2_\alpha = \{x \in X : \alpha \trianglelefteq_2 h_A(x)\}$, it is immediate that $A^1_\alpha \subseteq A^2_\alpha$. ■

Let us take a quick look at some of the characteristics that these level sets satisfy for a fixed order.

Proposition 2.39 *Let $\trianglelefteq_o$ be an order on $\mathbb{H}$. For any $A, B \in THFS(X)$ and any $\alpha, \beta \in \mathbb{H}$, we have that:*

i) If $\alpha \trianglelefteq_o \beta$, then $A^o_\beta \subseteq A^o_\alpha$.

ii) $A \subseteq_o B \Leftrightarrow A^o_\alpha \subseteq B^o_\alpha$ for any $\alpha \in \mathbb{H}$.

iii) $(A \cap_o B)^o_\alpha \subseteq A^o_\alpha \cap B^o_\alpha$. Moreover, if $\trianglelefteq_o$ is a total order, then $(A \cap_o B)^o_\alpha = A^o_\alpha \cap B^o_\alpha$.

iv) $A^o_\alpha \cup B^o_\alpha \subseteq_o (A \cup_o B)^o_\alpha$. Therfore, if $\trianglelefteq_o$ is a total order, then $A^o_\alpha \cup B^o_\alpha = (A \cup_o B)^o_\alpha$.

Proof: Let us consider $A, B \in THFS(X)$ and $\alpha, \beta \in \mathbb{H}$.

i) If $\alpha \trianglelefteq_o \beta$, then it is immediate by definition that $A^o_\beta \subseteq A^o_\alpha$, since $\trianglelefteq_o$ is transitive.

ii) If $A \subseteq_o B$ then $h_A(x) \trianglelefteq_o h_B(x), \forall x \in X$. Thus, if $\alpha \trianglelefteq_o h_A(x)$, since $\trianglelefteq_o$ is transitive, then $\alpha \trianglelefteq_o h_B(x)$ and so $A^o_\alpha = \{x \in X : \alpha \trianglelefteq_o h_A(x)\} \subseteq \{x \in X : \alpha \trianglelefteq_o h_B(x)\} = B^o_\alpha$.

On the other hand, for any $x \in X$, if we use the inclusion for the level sets, we obtain that $x \in A^o_{h_A(x)}$ since $\trianglelefteq_o$ is reflexive, and therefore $x \in B^o_{h_A(x)}$. This is equivalent to saying that $h_A(x) \trianglelefteq_o h_B(x)$. As we have this result for all $x \in X$, this means that $A \subseteq_o B$.

iii) Since $A \cap_o B \subseteq_o A$ and $A \cap_o B \subseteq_o B$, by applying ii), we have that $(A \cap_o B)^o_\alpha \subseteq A^o_\alpha$ and $(A \cap_o B)^o_\alpha \subseteq B^o_\alpha$ and then $(A \cap_o B)^o_\alpha \subseteq A^o_\alpha \cap B^o_\alpha$.

On the other hand, if $x \in A^o_\alpha \cap B^o_\alpha$, then $\alpha \trianglelefteq_o h_A(x)$ and $\alpha \trianglelefteq_o h_B(x)$. Thus, if we consider a total order, from Proposition 2.30 we obtain that $h_{A \cap B}(x) = h_A(x)$ or $h_{A \cap B}(x) = h_B(x)$ and so $\alpha \trianglelefteq_o h_{A \cap B}(x)$.

iv) As $A \subseteq_o A \cup_o B$ and $B \subseteq_o A \cup_o B$, by applying ii), we obtain that $A^o_\alpha \subseteq_o (A \cup_o B)^o_\alpha$ and $B^o_\alpha \subseteq_o (A \cup_o B)^o_\alpha$. Thus, $A^o_\alpha \cup B^o_\alpha \subseteq_o (A \cup_o B)^o_\alpha$.

On the other hand, for any $x \in X$ we obtain that $h_{A \cup_o B}(x) = h_B(x)$ or $h_{A \cup_o B}(x) = h_B(x)$, by applying Proposition 2.34, as $\trianglelefteq_o$ is a total order. Therefore, if $x \in (A \cup_o B)^o_\alpha$, then $\alpha \trianglelefteq_o h_{A \cup B(x)}$ and then $\alpha \trianglelefteq_o h_A(x)$ or $\alpha \trianglelefteq_o h_B(x)$. Then, $x \in A^o_\alpha \cup B^o_\alpha$.

■

In fuzzy sets theory, Decomposition Theorems [49] are known for allowing to represent a fuzzy set through its α-cuts, so we would like to adapt this for hesitant fuzzy sets. Before giving a general result, we will present it in an example.

Example 2.40 *Let $X = \{x, y, z\}$. We will consider the hesitant fuzzy set A defined in Example 3.23 and the lexicographical order type 1, where the level sets were $A^{Lex1}_{\{0.3\}} = \{x, y, z\}$, $A^{Lex1}_{\{0.4,0.7,0.8\}} = \{y, z\}$ and and $A^{Lex1}_{\{0.5,0.6\}} = \{z\}$.*

If we choose the proper elements, the level sets of the hesitant fuzzy set can be used to represent it. Now, we will obtain a hesitant fuzzy set ${}^{Lex1}_{\alpha}A$ based on these level sets whose membership function at any $x \in X$ is:

$$h_{{}^{Lex1}_{\alpha}A}(x) = \begin{cases} \alpha & \text{if } x \in A^{Lex1}_\alpha, \\ \mathbf{0}_{\mathbb{H}} & \text{otherwise.} \end{cases}$$

With this procedure, we are hesitant fuzzifying the level sets, that is, we begin with level sets (crisp sets) and then we obtain hesitant fuzzy sets.

Thus,

$$h_{{}^{Lex1}_{\{0.3\}}A}(t) = \{0.3\}, \forall t \in X$$

$$h_{{}^{Lex1}_{\{0.4,0.7,0.8\}}A}(t) = \begin{cases} \{0.4,0.7,0.8\} & \text{if } t \in \{y,z\}, \\ \mathbf{0}_{\mathbb{H}} & \text{if } t = x, \end{cases}$$

and

$$h_{{}^{Lex1}_{\{0.5,0.6\}}A}(t) = \begin{cases} \{0.5,0.6\} & \text{if } t = z, \\ \mathbf{0}_{\mathbb{H}} & \text{if } t \in \{x,y\}. \end{cases}$$

It is immediate that the $Lex1-$union of these hesitant fuzzy sets is the original set A. That is,

$$A = {}^{Lex1}_{\{0.3\}}A \cup_{Lex1} {}^{Lex1}_{\{0.4,0.7,0.8\}}A \cup_{Lex1} {}^{Lex1}_{\{0.5,0.6\}}A$$

Based on this idea, we propose the following theorem:

Theorem 2.41 (Decomposition Theorem) *Let $\trianglelefteq_o$ be a total order in $\mathbb{H}$ with least element $\mathbf{0}^o_{\mathbb{H}}$. For every $A \in THFS(X)$, we have that*

$$A = \bigcup_{\alpha\in\mathbb{H}}{}_o \; {}_{\alpha}{}^{o}A$$

where $\cup_o$ denotes the $o-$union and ${}_{\alpha}{}^{o}A(x) = \alpha$ if $x \in A^o_\alpha$, ${}_{\alpha}{}^{o}A(x) = \mathbf{0}^o_{\mathbb{H}}$ if $x \notin A^o_\alpha$.

Proof: Let A be any set in $THFS(X)$. For any $x \in X$, we have that $h_A(x) = \beta \in \mathbb{H}$. Then, $h_A(x) = {}_{\beta}{}^{o}A(x)$ and therefore $h_A(x) \trianglelefteq_o \bigcup_{\alpha\in\mathbb{H}}{}_o \; {}_{\alpha}{}^{o}A(x)$, by the definition of $\cup_o$.

On the other hand, as $\trianglelefteq_o$ is a total order, for any x in X, there exists a $\beta_x \in \mathbb{H}$ such that $\bigcup_{\alpha\in\mathbb{H}}{}_o \; {}_{\alpha}{}^{o}A(x) = {}_{\beta_x}{}^{o}A(x)$.

By the definition of ${}_{\beta_x}{}^{o}A(x)$, we get two possible cases:

- If $x \notin A^o_{\beta_x}$, then ${}_{\beta_o}{}^{o}A(x) = \mathbf{0}^o_{\mathbb{H}} \trianglelefteq_o h_A(x)$.
- If $x \in A^o_{\beta_x}$, then $\beta_x \trianglelefteq_o h_A(x)$ and so ${}_{\beta_x}{}^{o}A(x) = \beta_x \trianglelefteq_o h_A(x)$.

So by the symmetry of the order $\trianglelefteq_o$, we obtain that $h_A(x) = \bigcup_{\alpha\in\mathbb{H}}{}_o \; {}_{\alpha}{}^{o}A(x)$. ■

Instead of using the hesitant fuzzy set, this theorem enables us to operate with level sets.

We will consider it in the next corollary because it is remarkable that multiple elements could produce the same level set. If we take into account that $\Lambda(A)$ is the set of all elements that indicate various level sets of A, then there is an equivalent relation in X. As a result, the following result is a simplified version of the preceding one in which just one element is taken from each class in $\Lambda(A)$. It means that instead of taking $\mathbb{H}$, we consider $\Lambda(A)$. For instance, in Example 3.23, $\Lambda(A) = \{\{0.3\},\{0.4,0.7,0.8\},\{0.5,0.6\}\}$.

Corollary 2.42 *Let $\trianglelefteq_o$ be a total order in $\mathbb{H}$ with least element $\mathbf{0}^o_{\mathbb{H}}$. For every $A \in THFS(X)$, we have that*

$$A = \bigcup_{\alpha\in\Lambda(A)}{}_o \; {}_{\alpha}{}^{o}A$$

This is how a hesitant fuzzy set is represented without using the same level set twice. The proof is straight from the Decomposition Theorem.

2.3.2 Convexity of hesitant fuzzy sets

In previous section we considered that a hesitant fuzzy set A on X is $\mathscr{A}$-convex if for each $x < y < z$ in X there is $\mathscr{A}(h_A(y)) \geq \min\{\mathscr{A}(h_A(x)), \mathscr{A}(h_A(z))\}$ where $\mathscr{A}$ is an aggregation function. With this definition Huidobro et al. [38] were able to characterize the cases when the convexity of two typical hesitant fuzzy sets through intersections is preserved. However, as two different typical hesitant fuzzy elements could have the same value for a given aggregation function while being different sets, we think this definition is not proper and we propose the following one that can achieve better results and we do not reduce the information about the membership values of the hesitant fuzzy sets.

Definition 2.43 *Let X be an ordered set and let $\trianglelefteq_o$ be a total order on $\mathbb{H}$. A hesitant fuzzy set A is o-convex if $\min_o\{h_A(x), h_A(z)\} \trianglelefteq_o h_A(y)$ for any $x,y,z \in X$ such that $x < y < z$.*

In a similar way, we can also define strict convexity.

Definition 2.44 *Let X be an ordered set and let $\trianglelefteq_o$ be a total order on $\mathbb{H}$. A hesitant fuzzy set strictly A is o-convex if $\min_o\{h_A(x), h_A(z)\} \triangleleft_o h_A(y)$, for any $x,y,z \in X$ such that $x < y < z$.*

Definition 2.43 is accurate, because, as the following result proves, there is an equivalence between the convexity for a given hesitant fuzzy set and the convexity of its level sets.

Proposition 2.45 *Let X be an ordered vector space, let $A \in THFS(X)$ and let $\trianglelefteq_o$ be a total order in $\mathbb{H}$. A is an o-convex typical hesitant fuzzy set if and only if A^o_α are convex crisp sets for all $\alpha \in \mathbb{H}$.*

Proof: Let us consider $x, y, z \in X$ such that $x \leq y \leq z$.

For any $\alpha \in \mathbb{H}$, if $x \in A^o_\alpha$ and $z \in A^o_\alpha$, thus $\alpha \trianglelefteq_o h_A(x)$ and $\alpha \trianglelefteq_o h_A(z)$. Therefore, since A is convex, we have $\min_o\{h_A(x), h_A(z)\} \trianglelefteq_o h_A(y)$. By the transitivity of $\trianglelefteq_o$, $\alpha \trianglelefteq_o h_A(y)$ or $\alpha \trianglelefteq_o h_A(y)$ and so $y \in A^o_\alpha$. This is true, in particular, for the case $y = \lambda x + (1-\lambda)z$ with $\lambda \in [0,1]$. Thus A^o_α is a convex crisp set.

On the other hand, since $\trianglelefteq_o$ is a total order, we can consider $\alpha = \min_o\{h_A(x), h_A(z)\} \in \mathbb{H}$. Then, $x, z \in A^o_\alpha$. As A^o_α is a convex crisp set, we have that $y \in A^o_\alpha$ and so $\min_o\{h_A(x), h_A(z)\} \trianglelefteq_o h_A(y)$. ∎

As we saw in Subsection 1.2.1, admissible orders are a particular case of total orders, so they could be a good option for dealing with convexity.

Admissible orders will also be very important for the preservation of the convexity for the support and the core of a hesitant fuzzy set. For the support, we could consider the usual proposal given in Definition 1.54. But also we could think on a natural way to define the support of a hesitant fuzzy set, which is also coherent with the ideas for support for fuzzy sets (Definition 1.4). In fact, we will prove that both definitions are equivalent when we manage admissible orders in $\mathbb{H}$. More precisely,

Proposition 2.46 *Let A be a hesitant fuzzy set in X and $\trianglelefteq_o$ an admissible order on $\mathbb{H}$. We have that*

$$Supp(A) = \{x \in X : h_A(x) \neq \mathbf{0}_{\mathbb{H}}\}$$

Proof: For any x in X, we have that $x \in Supp(A)$ if, and only if, $\max\{h_A(x)\} \neq 0$. As $\mathbf{0}_{\mathbb{H}} = \{0\}$ and any element in $\mathbb{H}$ is a finite subset of $[0,1]$, this is equivalent to say that $h_A(x) \neq \mathbf{0}_{\mathbb{H}}$. ∎

Thus, we can generalize the definition of support to any order with a least element, as follows:

Definition 2.47 *Let A be a hesitant fuzzy set in X and $\trianglelefteq_o$ an order on $\mathbb{H}$ with least element $\mathbf{0}^o_{\mathbb{H}}$. The support of A is defined as*

$$Supp^o(A) = \{x \in X : h_A(x) \neq \mathbf{0}^o_{\mathbb{H}}\}$$

In the case of admissible orders on $\mathbb{H}$, we have that the least element is $\mathbf{0}_{\mathbb{H}}$ and therefore Definition 1.54 and Definition 2.47 are equivalent, as we have proven in Proposition 2.46.

Also, the definition of convexity fits well with this general idea of support of a hesitant fuzzy set as we can see in the following result.

Proposition 2.48 *Let us consider the ordered vector space X, let $\trianglelefteq_o$ be a total order on $\mathbb{H}$ with least element $\mathbf{0}^o_{\mathbb{H}}$ and $A \in THFS(X)$. If A is an o-convex hesitant fuzzy set, then the support of A is a convex crisp set.*

Proof: Let A be an o-convex hesitant fuzzy set. For any $x, z \in Supp^o(A) = \{x \in X : h_A(x) \neq \mathbf{0}^o_{\mathbb{H}}\}$ and any $\lambda \in [0,1]$, we have the following cases:

- if $x = z$, then $\lambda x + (1-\lambda)z = x$ and it is then clear that $\lambda x + (1-\lambda)z \in Supp^o(A)$.
- if $x < z$, then $x \leq \lambda x + (1-\lambda)z \leq z$. Moreover, as A is an o-convex hesitant fuzzy set, we have that

  $$\min_o\{h_A(x), h_A(z)\} \trianglelefteq_o h_A(\lambda x + (1-\lambda)z)$$

  and as $\mathbf{0}^o_{\mathbb{H}}$ is the least element, $h_A(x) \neq \mathbf{0}^o_{\mathbb{H}}$ and $h_A(z) \neq \mathbf{0}^o_{\mathbb{H}}$, we obtain that $h_A(\lambda x + (1-\lambda)z) \neq \mathbf{0}^o_{\mathbb{H}}$, that is, $\lambda x + (1-\lambda)z \in Supp^o(A)$.
- the case $z < x$ is totally analogous to the previous case.

Therefore, $Supp^o(A)$ is a crisp convex set. ■

Not only with the support but also convexity works with the core of a hesitant fuzzy set. Let us start by considering an appropriate definition of core for hesitant fuzzy sets. Again, we will consider the original ideas of core of a fuzzy set (Definition 1.5). Thus,

Definition 2.49 *Let A be a hesitant fuzzy set in X and an order $\preceq_o$ on $\mathbb{H}$ with greatest element $\mathbf{1}^o_{\mathbb{H}}$. The core of A, which is denoted by $Core^o(A)$, is the crisp set*

$$Core^o(A) = \{x \in X : h_A(x) = \mathbf{1}^{\mathbf{o}}_{\mathbb{H}}\}$$

Nevertheless, even in the case of admissible orders, this definition is not equivalent to the one given at Definition 1.55, as we show in the next example.

Example 2.50 *Let us consider $X = \{x\}$, any admissible order on $\mathbb{H}$ and the hesitant fuzzy set A defined by $h_A(x) = \{0.5, 1\}$. If we consider Definition 1.55, the core of A is X, since $\max\{h_A(x)\} = 1$. However, for Definition 2.49, the core of A is the empty set, since $\{0.5, 1\} \neq \{1\}$ and $\{1\}$ is the greatest element of the order.*

In the following result we show that the core is compatible with the definition of convexity.

Proposition 2.51 *Let us consider the ordered vector space X, the total order $\trianglelefteq_o$ on $\mathbb{H}$ with greatest element $\mathbf{1}^o_{\mathbb{H}}$ and $A \in THFS(X)$. If A is an o-convex hesitant fuzzy set, then the core of A w.r.t. this order is a convex crisp set.*

Proof: Let us suppose that $Core^o(A)$ is not a convex crisp set. That is, there exists $y = \lambda x + (1-\lambda)z \in X$ with $\lambda \in (0,1)$ such that $y \notin Core^o(A)$ for $x, z \in Core^o(A)$. Then $h_A(y) \neq \mathbf{1}^o_{\mathbb{H}}$. As $\mathbf{1}^o_{\mathbb{H}}$ is the greatest element, $h_A(y) \lhd_o \mathbf{1}^o_{\mathbb{H}} = \min_o\{h_A(x), h_A(z)\}$, which is a contradiction, since A is an o-convex hesitant fuzzy set. Thus, $Core^o(A)$ is a crisp convex set. ∎

An interesting property of convexity is being preserved when intersections, i.e., the intersection of two convex hesitant fuzzy sets is also convex.

Proposition 2.52 *Let X be an ordered set and let $\trianglelefteq_o$ a total order on $\mathbb{H}$. If $A, B \in THFS(X)$ are o-convex (resp. strictly o-convex) and $A \cap_o B \neq \emptyset$, then $A \cap_o B$ is also o-convex (resp. strictly o-convex).*

Proof: Let x, y, z in X with $x < y < z$.

If $h_A(y) \trianglelefteq_o h_B(y)$, by Proposition 2.30 we have that $h_{A \cap_o B}(y) = h_A(y)$. Since A is o-convex (resp. strictly o-convex), $h_A(x) \trianglelefteq_o h_A(y)$ (resp. $h_A(x) \lhd_o h_A(y)$) or

$h_A(z) \trianglelefteq_o h_A(y)$ (resp. $h_A(z) \triangleleft_o h_A(y)$). But by the definition of the intersection for this order we have that $h_{A\cap_o B}(x) \trianglelefteq_o h_A(x)$ and $h_{A\cap_o B}(z) \trianglelefteq_o h_B(z)$. By the transitivity, $h_{A\cap_o B}(x) \trianglelefteq_o h_A(y) = h_{A\cap_o B}(y)$ (resp. $h_{A\cap_o B}(x) \triangleleft_o h_A(y) = h_{A\cap_o B}(y)$) or $h_{A\cap_o B}(z) \trianglelefteq_o h_A(y) = h_{A\cap_o B}(y)$ (resp. $h_{A\cap_o B}(z) \triangleleft_o h_A(y) = h_{A\cap_o B}(y)$).

The case $h_B(y) \trianglelefteq_o h_A(y)$ is totally analogous. Therefore, $A \cap_o B$ is o-convex (resp. strictly o-convex). ■

If we consider the objectives and restrictions in a decision-making process as typical hesitant fuzzy sets, this theorem is crucial. They will also be convex at their intersection if they are convex. When the choice is a convex hesitant fuzzy set, we can use the following theorem to get some interesting optimization results.

Theorem 2.53 *Let X be an infinite ordered set. Let $\trianglelefteq_o$ be a total order on $\mathbb{H}$ with a least element.*

i) If A is an o-convex hesitant fuzzy set over X and $x^* \in Supp^o(A)$ is a strict local maximizer of h_A, then it is also a global maximizer of h_A over $Supp^o(A)$.

ii) If A is a strictly o-convex hesitant fuzzy set over X and $x^* \in Supp^o(A)$ is a local maximizer of h_A, then it is also a global maximizer of h_A over $Supp^o(A)$.

Proof: Suppose that $x^* \in Supp^o(A)$.

If x^* a strict local maximizer of h_A, this means that there exists a neighborhood Y such that for all $x \in Y$, we have that $h_A(x) \triangleleft_o h_A(x^*)$.

Let us suppose that there exists $x' \in Supp^o(A)$, different from x^*, such that $h_A(x^*) \trianglelefteq_o h_A(x')$. It is clear that $x' \notin Y$ because otherwise x^* would not be a strict local maximizer.

Let us consider $y \in Y$ such that $x' < y < x^*$ or $x^* < y < x'$. If we suppose that A is o-convex, we have that $h_A(x') \trianglelefteq_o h_A(y)$ or $h_A(x^*) \trianglelefteq_o h_A(y)$. Then, if we take y close enough to x^*, that is, $y \in Y$ and $y \neq x^*$, that contradicts $h_A(y) \triangleleft_o h_A(x^*)$.

On the other hand, if x^* is just a local maximizer of h_A, then there is a neighborhood Y where $h_A(y) \trianglelefteq_o h_A(x^*), \forall y \in Y$. Let us assume that there exists x' a global maximizer such that $h_A(x^*) \trianglelefteq_o h_A(x')$. If we consider that A is strictly o-convex, we get $h_A(x^*) \trianglelefteq_o h_A(x') \triangleleft_o h_A(y)$ or $h_A(x^*) \triangleleft_o h_A(y)$, so $h_A(x^*) \triangleleft_o h_A(y)$ for all the points y between x' and x^*. If we pick y close enough to x^*, that is $y \in Y$, there is a contradiction. ∎

Finally, we will study the set of the points at which the membership function attains its maximum.

Theorem 2.54 *Let X be an ordered set. Let $\trianglelefteq_o$ be a total order on $\mathbb{H}$ with least element. Let A be an o-convex hesitant fuzzy set over the ordered set X.*

i) The set of points at which h_A attains its global maximum over its support is a convex crisp set.

ii) If A is strictly o-convex, h_A attains its global maximum over $Supp^o(A)$ at no more than one point if X is uncountable or no more than two points if X is finite or countable.

Proof: We consider that A is an o-convex hesitant fuzzy set over X

i) Let us suppose that α is the typical hesitant fuzzy element where h_A attains its maximum value. If we construct the level set of A associated to α w.r.t. the order $\trianglelefteq_o$, it is a convex crisp set by Proposition 2.39 as A is an o-convex hesitant fuzzy set, and it coincides with the set of points at which h_A attains its global maximum over $Supp^o(A)$.

ii) Let us assume that $x^*, x' \in Supp^o(A)$ are two different global maximizers, that is, $h_A(x) \trianglelefteq_o h_A(x^*) = h_A(x')$ for all $x \in X$. Let us suppose that $x‘ < x^*$ (the case $x^* < x'$ is totally analogous).

   If there exists $y \in X$ such that $x' < y < x^*$ (this always happens in the uncountable case), since A is strictly convex, then $h_A(x^*) = h_A(x') \triangleleft_o h_A(y)$, and that contradicts the fact that there are two global maximizers and so the of points at which h_A attains its global maximum over $Supp^o(A)$ has no more than one point.

   If X is countable or finite and the maximizers are consecutive (otherwise we are at the previous case), h_A only has, at most, two global maximizers. Otherwise, if it has three consecutive maximizers, which are denoted by x^*, x'

and x'', with $x^* < x' < x''$, then $h_A(x^*) = h_A(x'') \lhd_o h_A(x')$ by the strictly o-convexity of A, but this is a contradiction with the fact that x^* and x'' are global maximizers. ■

Thus, in the finite case, we could have two consecutive global maximizers, since we could have two points such that there is no other point between them, as we can see at the following example.

Example 2.55 *It should be noticed that if we consider* $X = \{x_1, x_2, x_3, x_4\}$ *with* $x_1 < x_2 < x_3 < x_4\}$*, the lexicographical order type 1 on* $\mathbb{H}$ *and the hesitant fuzzy set* $A = \{\langle x_1, \{0.3\}\rangle, \langle x_2, \{0.6\}\rangle, \langle x_3, \{0.6\}\rangle, \langle x_4, \{0.3\}\rangle\}$*, A is strictly convex but it has two maximum points that are together. If instead of two, there are three points,* $B = \{\langle x_1, \{0.6\}\rangle, \langle x_2, \{0.6\}\rangle, \langle x_3, \{0.6\}\rangle, \langle x_4, \{0.3\}\rangle\}$*, then B is not strictly convex as* $h_B(x_2) \not\lhd_{Lex1} \min\{h_B(x_1), h_B(x_3)\}$.

Theorems 2.53 and 2.54 can be applied to any admissible order $\unlhd_o$, since it refines the standard partial order $\unlhd_{RH}$ and therefore $(\mathbb{H}, \unlhd_o)$ is a bounded lattice.

2.3.3 Decision-making based on hesitant fuzzy sets

Theories regarding using fuzzy sets in decision-making can be found in the literature. Bellman and Zadeh [8], for instance, worked to demonstrate how a choice can be thought of as a collection of objectives and restrictions with symmetry between them. Using this method, we can treat objectives and restrictions as if they were symmetrically related notions joined together by the conjunction “and”.

It is assumed in fuzzy set theory that each element’s level of membership in a set is known. Unfortunately, there are instances in real life where the membership function is not completely understood. This happens very frequently in decision-making, when any expert can provide a value for the membership function. With this in mind, hesitant fuzzy sets are a strong and useful tool for expressing uncertain information, as it allows the membership degree of an element to a set represented by multiple alternative values in [0,1] (see [60]).

In this work, we will employ the Bellman and Zadeh methodology [8], which states that the choice D would be the intersection of all the hesitant fuzzy constraints and goals if we were to treat the constraints and the goals as hesitant fuzzy set over the set of alternatives, X.

A decision, in the sense of Yager and Basson [89], is the place at which all objectives and restrictions meet. So we could reach the following definition by using this concept as a guide.

Definition 2.56 *Let $X = \{x_1, \dots, x_n\}$ be the set of alternatives, $G_1, \dots, G_p$ be the set of goals that can be expressed as hesitant fuzzy sets on the space of alternatives, and $C_1, \dots, C_m$ be the set of constraints that can also be expressed as hesitant fuzzy sets on the space of alternatives. Let $\trianglelefteq_o$ be an order on $\mathbb{H}$. The goals and constraints then combine to form a decision D_o, which is a hesitant fuzzy set resulting from the intersection of the goals and the constraints, that is,*

$$D_o = G_1 \cap_o, \dots \cap_o G_p \cap_o C_1 \cap_o \dots \cap_o C_m.$$

For any $x \in X$, the interpretation of $D(x)$ might be the extent to which the alternative x satisfies the objectives and constraints. We must choose the best option before the decision has been made.

Since the intersection is actually an o-intersection, it follows immediately from this definition that D directly depends on the chosen order $\trianglelefteq_o$ in $\mathbb{H}$. As a result, depending on the sequence we use, the decision D, built as the intersection of the goals and restrictions, would alter.

In order to a better understanding, let us show the following example.

Example 2.57 *A big company has to decide a new country to expand their facilities between one of three locations x_1, x_2 and x_3. They would like to choose a place that reduces the cost of real estate, G, and is near stores, C_1. Let $X = \{x_1, x_2, x_3\}$. In this case, there is a committee of experts that evaluates several aspects, so hesitant fuzzy set could be a good option to model it. Let us assume that the membership grades of the hesitant fuzzy goal G is*

$$G = \{\langle x_1, \{0.4, 0.8, 0.8\}\rangle, \langle x_2, \{0.6, 0.8, 1\}\rangle, \langle x_3, \{0.6, 0.7, 0.8\}\rangle\}$$

where they measure the electricity, the rent and the water prize; and the membership function of the hesitant fuzzy constraint C_1 is

$$C_1 = \{\langle x_1, \{0.5, 0.6, 0.7\}\rangle, \langle x_2, \{0.9, 0.6, 1\}\rangle, \langle x_3, \{0.6, 0.9, 0.9\}\rangle\}$$

where they take into account the difficulty of getting workforce, the machinery and the salaries. In this case, we have to reorder this constraint as

$$C_1 = \{\langle x_1, \{0.5, 0.6, 0.7\}\rangle, \langle x_2, \{0.6, 0.9, 1\}\rangle, \langle x_3, \{0.6, 0.9, 0.9\}\rangle\}$$

If we consider lexicographical order type 1, the membership values of the hesitant fuzzy decision D_{Lex1} are:

$$D_{Lex1} = \{\langle x_1, \{0.4, 0.8, 0.8\}\rangle, \langle x_2, \{0.6, 0.8, 1\}\rangle, \langle x_3, \{0.6, 0.7, 0.8\}\rangle\}$$

and the optimal decision would be x_2, due to the fact that it is the alternative with the highest possible value of D_{Lex1} in terms of the lexicographical order type 1. Nevertheless, if we employ lexical order type 2, the membership degrees of the hesitant fuzzy decision D_{Lex2} are:

$$D_{Lex2} = \{\langle x_1, \{0.5, 0.6, 0.7\}\rangle, \langle x_2, \{0.6, 0.8, 1\}\rangle, \langle x_3, \{0.6, 0.7, 0.8\}\rangle\}$$

but in this case the optimal decision does not change and it is still to x_2.

Once we have seen this easy example, we would like to point out how important is the chosen order on $\mathbb{H}$.

In a hesitant fuzzy decision, similar to the one above, all the objectives and restrictions are hesitant fuzzy sets over exactly the same set of alternatives, however this can occasionally change. We can prevent this circumstance by employing the extension principle.

Definition 2.58 (Extension principle) *Let $(\mathbb{H}, \trianglelefteq_o)$ be a complete lattice. Any function $f : X \to Y$ produces a functions $\tilde{f} : THFS(X) \to THFS(Y)$ such that, for any $A \in THFS(X)$, $\tilde{f}(A)$ is the hesitant fuzzy set in Y whose membership function is:*

$$h_{\tilde{f}(A)}(y) = \sup_{o \atop x|y=f(x)} h_A(x), \forall y \in Y$$

where $\sup_o$ means the supremum w.r.t. the order $\trianglelefteq_o$

When there is not ambiguity, $\tilde{f}$ could also be denoted as f.

With this technique, the situation where the hesitant fuzzy constraints or goals are defined in different spaces can be mapped into exactly the same universe.

Let us show in the following example how this methodology works.

Example 2.59 *Continuing Example 2.57, but there is now another space Y denoting the set of previous works done in that countries, $Y = \{y_1, y_2, y_3, y_4\}$. The information about these former works is the following: y_1 and y_2 were made in x_1, y_3 was done in x_2 and y_4 was performed jointly in x_2 and x_3.*

Using this data, we create the following function:

$$f : Y \to X$$

characterized by $f(y_1) = x_1$, $f(y_2) = x_1$, $f(y_3) = x_2$ and $f(y_4) = x_3$.

Additionally, we are aware of a fuzzy restriction over Y that evaluates the significance of each work described by the cost to carry it out and the success they had: $C_2^Y = \{\langle y_1, \{0.6, 0.8\}\rangle, \langle y_2, \{0.7, 0.9\}\rangle, \langle y_3, \{0.75, 0.8\}\rangle, \langle y_4, \{0.5, 0.9\}\rangle\}$. It is represented as C_2^Y to emphasize that it is a hesitant fuzzy set over the space Y. Applying the extension principle now will allow us to have all of the objectives and constraints represented as hesitant fuzzy sets over the same domain. In this situation, we will employ lexicographical order type 1. For x_1, $h_{f(C_2^Y)}(x_1) = \sup_{y|x=f(y)} h_{C_2^Y}(y) = \sup\{h_{C_2^Y}(y_1), h_{C_2^Y}(y_2)\} = \sup\{\{0.6, 0.8\}, \{0.7, 0.9\}\} = \{0.7, 0.9\}$. In a similar way, $h_{f(C_2^Y)}(x_2) = \{0.75, 0.8\}$ and $h_{f(C_2^Y)}(x_3) = \{0.5, 0.9\}$.

As a consequence,

$$f(C_2^Y) = \{\langle x_1, \{0.7, 0.9\}\rangle, \langle x_2, \{0.75, 0.8\}\rangle, \langle x_3, \{0.5, 0.9\}\rangle\}$$

Eventually, the decision is

$$D'_{Lex1} = G \cap_{Lex1} C_1 \cap_{Lex1} f(C_2^Y)$$

where the membership values are now:

$$D'_{Lex1} = \{\langle x_1, \{0.4, 0.8, 0.8\}\rangle, \langle x_2, \{0.6, 0.8, 1\}\rangle, \langle x_3, \{0.5, 0.9\}\rangle\}$$

Thus, the optimal decision is x_2.

There are circumstances where a decision parameter is dependent on another domain. Yager and Basson [89] proposed the idea of fuzzy conditional sets to be capable of dealing with such circumstances. With these concepts in mind, we arrive at the subsequent definition:

Definition 2.60 *Let X and Y be two crisp sets and let $(\mathbb{H}, \trianglelefteq_o)$ be a complete lattice. If we have a family of hesitant fuzzy sets on X given by $\{A|y \in THFS(X) : y \in Y\}$ and $B \in THFS(Y)$, we can obtain a new hesitant fuzzy set on X by combining the information given by $A|y$ for any $y \in Y$ and B. This set will be denoted by $A|_B$ and its membership function is:*

$$h_{A|_B}(x) = \sup_{y\in Y}{}_o \min{}_o\{h_B(y), h_{A|y}(x)\}$$

We can see how to apply these ideas in our practical example.

Example 2.61 *Let us add more details to Example 2.59. The company is required to minimize the cost of moving experienced workers and basic machinery to the new office. They would focus on the distance from the main office to the new one. Let $Z = \{Near(N), Med(M), Far(F)\}$. This restriction is set by the hesitant fuzzy set $B^Z = \{\langle N, \{0.8\}\rangle, \langle M, \{0.5\}\rangle, \langle F, \{0.3\}\rangle\}$. The following conditioned typical hesitant fuzzy sets describe how the options and distance to the main office are related:*

$$C_3^X|N = \{\langle x_1, \{0.6, 0.6\}\rangle, \langle x_2, \{0.6, 0.7\}\rangle, \langle x_3, \{0.5, 0.8\}\rangle\}$$
$$C_3^X|M = \{\langle x_1, \{0.6, 0.7\}\rangle, \langle x_2, \{0.4, 0.7\}\rangle, \langle x_3, \{0.7, 0.9\}\rangle\}$$
$$C_3^X|F = \{\langle x_1, \{0.3, 0.5\}\rangle, \langle x_2, \{0.4, 0.4\}\rangle, \langle x_3, \{0.3, 0.4\}\rangle\}$$

Then, we can construct the typical hesitant fuzzy sets $C_3^X|_{B^Z}$. Thus, for x_1 we obtain that

$$h_{C_3^X|_{B^Z}}(x_1) = \sup_{z\in Z}{}_{Lex1} \min{}_{Lex1}\{h_{B^z}(z), h_{C_3^X|z}(x_1)\}.$$

As

$$\min{}_{Lex1}\{h_{B^Z}(N), h_{C_3^X|N}(x_1)\} = \min{}_{Lex1}\{\{0.8\}, \{0.6, 0.6\}\} = \{0.6, 0.6\}$$
$$\min{}_{Lex1}\{h_{B^Z}(M), h_{C_3^X|M}(x_1)\} = \min{}_{Lex1}\{\{0.5\}, \{0.6, 0.7\}\} = \{0.5\}$$
$$\min{}_{Lex1}\{h_{B^Z}(F), h_{C_3^X|F}(x_1)\} = \min{}_{Lex1}\{\{0.3\}, \{0.3, 0.5\}\} = \{0.3\}$$

we obtain that

$$h_{C_3^X|_{B^Z}}(x_1) = \sup_{Lex1}\{\{0.6, 0.6\}, \{0.5\}, \{0.3\}\} = \{0.6, 0.6\}$$

With an analogous process, we can calculate the constraint for x_2 and x_3 so that we get the hesitant fuzzy set $C_3^X|_{B^Z}$:

$$C_3^X|_{B^Z} = \{\langle x_1, \{0.6, 0.6\}\rangle, \langle x_2, \{0.6, 0.7\}\rangle, \langle x_3, \{0.5, 0.8\}\rangle\}$$

As a result, the decision is the hesitant fuzzy set $D''_{Lex1} = G \cap_{Lex1} C_1 \cap_{Lex1} f(C_2^Y) \cap_{Lex1} C_3^X|_{B^Z}$ described by:

$$D''_{Lex1} = \{\langle x_1, \{0.4, 0.8, 0.8\}\rangle, \langle x_2, \{0.6, 0.7\}\rangle, \langle x_3, \{0.5, 0.8\}\rangle\}$$

Thus, x_2 is again the optimal decision.

If we now combine Proposition 2.52 and Theorem 2.54 with the decision-making theory, we could achieve some good results:

Corollary 2.62 *Let X be an ordered set. Let $\trianglelefteq_o$ be an order on $\mathbb{H}$, let $G_1, \ldots, G_p$ be the hesitant fuzzy goals, $C_1, \ldots, C_m$ the hesitant fuzzy constraints, and $D = G_1 \cap, \ldots \cap G_p \cap C_1 \cap, \ldots \cap C_m$ be the resulting decision.*

- *If the hesitant fuzzy goals and the hesitant fuzzy constraints are o-convex hesitant fuzzy set, then the resulting decision D is an o-convex hesitant fuzzy set and the set of maximizing decisions of the hesitant fuzzy set D is a convex crisp set.*

- *If the hesitant fuzzy goals and the hesitant fuzzy constraints are strictly o-convex hesitant fuzzy set, then the resulting decision D is a strictly o-convex hesitant fuzzy set and the cardinality of the set of maximizing decisions is at most two.*

Let us sum up the decision-making problem of Example 2.61 in the following example.

Example 2.63 *We will summarize the hesitant goal and constraints of the previous examples:*

$$G = \{\langle x_1, \{0.4, 0.8, 0.8\}\rangle, \langle x_2, \{0.6, 0.8, 1\}\rangle, \langle x_3, \{0.6, 0.7, 0.8\}\rangle\}$$

$$C_1 = \{\langle x_1, \{0.5, 0.6, 0.7\}\rangle, \langle x_2, \{0.6, 0.9, 1\}\rangle, \langle x_3, \{0.6, 0.9, 0.9\}\rangle\}$$

$$f(C_2^Y) = \{\langle x_1, \{0.7, 0.9\}\rangle, \langle x_2, \{0.75, 0.8\}\rangle, \langle x_3, \{0.5, 0.9\}\rangle\}$$

$$C_3^X|_{B^Z} = \{\langle x_1, \{0.6, 0.6\}\rangle, \langle x_2, \{0.6, 0.7\}\rangle, \langle x_3, \{0.5, 0.8\}\rangle\}$$

If we assume that $x_1 < x_2 < x_3$ and consider the lexicographical order type 1, then we obtain that G, C_1, $f(C_2^Y)$ and $C_3^X|_{B^Z}$ are strictly convex hesitant fuzzy set, so the decision D_{Lex1} is also a convex hesitant fuzzy set w.r.t. the same order. It is immediate to check it as

$$D''_{Lex1} = \{\langle x_1, \{0.4, 0.8, 0.8\}\rangle, \langle x_2, \{0.6, 0.7\}\rangle, \langle x_3, \{0.5, 0.8\}\rangle\}$$

Then we can affirm that x_2 is a global maximizer, following the ideas considered in Corollary 2.62.

It is clear that the choice of the order on $\mathbb{H}$ is an essential step for this method. This is clearly shown at the following example.

Example 2.64 *Considering the same hesitant fuzzy set for the goal and constraints from the Examples 2.57, 2.59 and 2.61, we have obtained that the chosen optimal decision is x_2, since*

$$D''_{Lex1} = \{\langle x_1, \{0.4, 0.8, 0.8\}\rangle, \langle x_2, \{0.6, 0.7\}\rangle, \langle x_3, \{0.5, 0.8\}\rangle\}$$

It is clear from the notation used for the decision set that it depends on the order considered on $\mathbb{H}$. In that case, the considered order was the lexicographical order type 1. This is also important for obtaining the sets $f(C_2^Y)$ and $C_3^X|_{B^Z}$, since the supremum and minimum are considered. Thus, if we consider the lexicographical order type 2, we obtain that

$$D''_{Lex2} = \{\langle x_1, \{0.5, 0.6, 0.7\}\rangle, \langle x_2, \{0.6, 0.7\}\rangle, \langle x_3, \{0.6, 0.7, 0.8\}\rangle\}.$$

Thus, D_{Lex2} is not only convex but also strictly convex, so it is possible to assure that the unique optimal decision is x_3.

2.3.4 Ranking method based on hesitant fuzzy sets

In this section we propose a ranking method for hesitant fuzzy sets. The first step is to model it like the decision-making procedure explained in the previous section. Once we have constructed the decision D, we have to order their components, i.e., we should decide which hesitant fuzzy element is the largest. That element would be the first one, then we delete it and we do the same until we have no more elements. When two or more elements have the same value we have to decide on one randomly.

In order to achieve a better understanding, we are going to show some practical cases. Let us present this example where the data was extracted from [52].

Example 2.65 *In Taiwan, marketing has faced a hard task because of the high-speed railroad development. More airlines are making an effort to attract customers by lowering prices, particularly since the worldwide economic crisis in 2008. However, they soon realised that this is a lose-lose situation and that the one essential component to surviving in this extremely competitive home market is quality of service. The Civil Aviation Administration of Taiwan (CAAT) is interested in finding out which national airline in Taiwan provides the best customer service. In order to study the four main national airlines, UNI Air, Transasia, Mandarin and Daily Air; the CAAT sets up a committee. Four key criteria are provided to assess these four domestic airlines. These four criteria are the booking and ticket service (C_1), the check-in and boarding process (C_2), the cabin service (C_3) and the responsiveness of the company (C_4). The data is presented in Table 2.2.*

	C_1	C_2	C_3	C_4
UNI Air	{0.6,0.7,0.9}	{0.6,0.8}	{0.3,0.6,0.9}	{0.4,0.5,0.9}
Transasia	{0.7,0.8,0.9}	{0.5,0.8,0.9}	{0.4,0.8}	{0.5,0.6,0.7}
Mandarin	{0.5,0.6,0.8}	{0.6,0.7,0.9}	{0.3,0.5,0.7}	{0.5,0.7}
Daily Air	{0.6,0.9}	{0.7,0.9}	{0.2,0.4,0.7}	{0.4,0.5}

Table 2.2: Hesitant fuzzy decision matrix.

Now we should transform this into a decision-making problem and then compute the intersection of the criteria. If we consider lexicographical order type 1, *then*

	UNI Air	*Transasia*	*Mandarin*	*Daily Air*
$C_1 \cap C_2 \cap C_3 \cap C_4$	*{0.3,0.6,0.9}*	*{0.4,0.8}*	*{0.3,0.5,0.7}*	*{0.2,0.4,0.7}*

So therefore, our ranking with lexicographical order type 1 is

$$Transasia > UNIAir > Mandarin > DailyAir$$

as

$$\{0.2,0.4,0.7\} \lhd_{Lex1} \{0.3,0.5,0.7\} \lhd_{Lex1} \{0.3,0.6,0.9\} \lhd_{Lex1} \{0.4,0.8\}.$$

On the other hand, if we consider the lexicographical order type 2, we obtain that

	UNI Air	*Transasia*	*Mandarin*	*Daily Air*
$C_1 \cap C_2 \cap C_3 \cap C_4$	*{0.4,0.5,0.9}*	*{0.5,0.6,0.7}*	*{0.3,0.5,0.7}*	*{0.2,0.4,0.7}*

and the ranking is UNI Air > Transasia > Mandarin > Daily Air.

Finally, let us show another example where the data was extracted from [1].

Example 2.66 *In the academic world, there are several methods to provide metarankings of universities. In this example, various data given by the Academic Ranking of World Universities (Shanghai Ranking, henceforth Sh) by [15], QS World University Rankings (henceforth QS) by [54], and Times Higher Education World University Rankings (henceforth THE) by [66]. These organizations use a unique scoring system, with a maximum score of 100 assigned to each university. Although the experts are well renowned for their overall rankings of top institutions worldwide, they also provide rankings of universities by specialty. Alcantud et al. [1] develop five fields that are appropriate to the classification methods used by the three experts which are Arts and Humanities (AH), Life Sciences and Medicine (LM), Engineering and Technology (ET), Natural Science and Mathematics (SCI) and Social Sciences (SOC).*

The data is presented in Table 2.3.

Now we should transform this data into hesitant fuzzy sets dividing by 100 and grouping by fields. This is in Table 2.4.

Then we have to obtain the intersection using lexicographical order type 1.

University	AH(THE)	AH(QS)	LM(THE)	LM(SH)	LM(QS)	ET(THE)	ET(SH)	ET(QS)	SCI(THE)	SCI(SH)	SCI(QS)	SOC(THE)	SOC(SH)	SOC(QS)
Stanford	87.1	86.8	87.6	69.4	91.2	91.9	92.1	93.3	89.9	91.4	92.5	93.6	80.1	89.2
Harvard	86.1	89.7	91.3	100	98.2	-	65.1	85.7	90.2	100	92.3	91.9	100	96.3
Oxford	84.4	99.1	91.1	60.9	92.3	87.6	64.4	86.1	87.3	72.3	90.4	93.5	59.9	94.2
Cambridge	83.9	93.5	88.5	75.6	91.8	88.8	74.8	90.5	88.8	92.2	97.0	87.5	59.4	91.2
California, Berkeley	81.4	87.2	81.6	58.0	85.6	90.6	86.8	90.2	89.9	96.3	93.4	86.9	79.6	87.3
Princeton	81.2	86.5	42.5	24.8	74.1	89.5	71.1	81.6	91.0	93.7	89.2	91.1	76.4	84.4
Yale	81.2	89.0	83.7	62.4	88.6	-	49.1	75.2	83.6	65.2	84.3	90.0	72.8	87.4

Table 2.3: Collected data.

University	AH	LM	ET	SCI	SOC
Stanford	{0.868, 0.871}	{0.694,0.876, 0.912}	{0.919, 0.921, 0.933}	{0.899, 0.914, 0.925}	{0.801, 0.892, 0.936}
Harvard	{0.861 , 0.897}	{0.913, 0.982, 1}	{0.651, 0.857}	{0.902, 0.923, 1}	{0.919, 0.963, 1}
Oxford	{0.844, 0.991}	{0.609, 0.911 , 0.923}	{0.644, 0.861, 0.876}	{0.723, 0.873, 0.904}	{0.599, 0.935, 0.942}
Cambridge	{0.839, 0.935}	{0.756, 0.885, 0.918}	{0.748, 0.888, 0.905}	{0.888, 0.922, 0.97}	{0.594, 0.875, 0.912}
California, Berkeley	{0.814 , 0.872}	{0.580, 0.816, 0.856}	{0.868, 0.902,0.906}	{0.899, 0.934, 0.963}	{0.796, 0.869, 0.873}
Princeton	{0.812, 0.865}	{0.248, 0.425, 0.741}	{0.711, 0.816, 0.895}	{0.892, 0.910, 0.937}	{0.764, 0.844, 0.911}
Yale	{0.812, 0.89}	{0.624, 0.837, 0.886}	{0.491, 0.752}	{0.652, 0.836, 0.843}	{0.728, 0.874, 0.900}

Table 2.4: Hesitant Fuzzy Data.

University	$AH \cap_{Lex1} LM \cap_{Lex1} ET \cap_{Lex1} SCI \cap_{Lex1} SOC$
Stanford	*{0.694,0.876, 0.912}*
Harvard	*{0.651, 0.857}*
Oxford	*{0.599, 0.935, 0.942}*
Cambridge	*{0.594, 0.875, 0.912}*
California, Berkeley	*{0.580, 0.816, 0.856}*
Princeton	*{0.248, 0.425, 0.741}*
Yale	*{0.491, 0.752}*

So therefore, our ranking with lexicographical order type 1 is:

Stanford > Harvard > Oxford > Cambridge > California, Berkeley > Yale > Princeton.

On the other hand, if we consider the lexicographical order type 2, we obtain that

University	$AH \cap_{Lex2} LM \cap_{Lex2} ET \cap_{Lex2} SCI \cap_{Lex2} SOC$
Stanford	*{0.868, 0.871}*
Harvard	*{0.651, 0.857}*
Oxford	*{0.644, 0.861, 0.876}*
Cambridge	*{0.748,0.888,0.905}*
California, Berkeley	*{0.580, 0.816, 0.856}*
Princeton	*{0.248, 0.425, 0.741}*
Yale	*{0.491, 0.752}*

And the ranking w.r.t. the lexicographical order type 2 is:

Cambridge > Stanford > Oxford > Harvard > California, Berkeley > Yale >Princeton.

Chapter 3

Convexity of interval-valued fuzzy sets

The other extension of fuzzy sets where we are interested in studying convexity is the case of the interval-valued fuzzy sets. Unfortunately, we could not find reasonable results about this topic in the literature. However, there are several interesting papers devoted to the study of the convexity of intuitionistic fuzzy sets. Thus, we will start this chapter by providing a review of them, since they could be the starting point for our purposes, taking into account the mathematical equivalence between interval-valued fuzzy sets and intuitionistic fuzzy sets, which was already commented in Chapter 1. Taking these studies into account, we will later present our proposal, which was published in [37, 39, 40].

3.1 Overview of convexity of intuitionistic fuzzy sets

For dealing with the convexity of intuitionistic fuzzy sets, it is essential to manage the concept of α-cut. Thus, we will start by recalling this definition.

Definition 3.1 *[21] Let $A = \{\langle x, \mu_A(x), \nu_A(x)\rangle : x \in X\}$ be an intuitionistic fuzzy set on a referential X and let α be a real number in the interval $(0,1]$. The α-cut of A is the crisp set denoted by A_α and defined by*

$$A_\alpha = \{x \in X : \mu_A(x) \geq \alpha \text{ and } \nu_A(x) \leq 1-\alpha\}$$

As we can see in the following proposition, there is a redundant condition in the previous definition.

Proposition 3.2 *[21] Let $A = \{\langle x, \mu_A(x), \nu_A(x)\rangle : x \in X\}$ be an intuitionistic fuzzy set on a referential X and let α be a real number in the interval $(0,1]$. The α-cut of A is equal to the α-cut of the fuzzy set whose membership function is μ_A, that is,*

$$A_\alpha = \{x \in X : \mu_A(x) \geq \alpha\}$$

This fuzzy set defined by means of the membership function of an intuitionistic fuzzy set A will be denoted by A^μ. It is clear that it is also possible to associate another fuzzy set A^ν such that $A^\nu(x) = 1 - \nu_A(x), \forall x \in X$.

Now that α-cuts have been established, we may utilize them to determine when an intuitionistic fuzzy set is convex.

Definition 3.3 *[21] Let X be a vector space. An intuitionistic fuzzy set A on X is said to be convex if its cuts A_α are convex subsets of X for all $\alpha \in (0,1]$.*

Since the convexity of the α-cuts is equivalent to the convexity of the sets, it is obvious that the idea of convexity in fuzzy sets is extended.

Corollary 3.4 *[21] Let A be an intuitionistic fuzzy set on the vector space X. The following statements are equivalent:*

(i) A is a convex intuitionistic fuzzy set,

(ii) A^μ is a quasi-convex fuzzy set.

Thus, the convexity of an intuitionistic fuzzy set is independent of the non-membership function associated with it. This is a natural consequence of Definition 3.3 and Proposition 3.2.

This corollary makes it easier to prove that the intersection of two convex intuitionistic fuzzy sets is convex too.

Proposition 3.5 *Let us consider the vector space X. If A and B are convex intuitionistic fuzzy sets on X, then $A \cap B$ is also convex, where*

$$A \cap B = \{\langle x, \min\{\mu_A(x), \mu_B(x)\}, \max\{\nu_A(x), \nu_B(x)\}\rangle : x \in X\}.$$

Proof: Let us consider $A, B \in IFS(X)$. If we suppose that A and B are convex intuitionistic fuzzy sets, then μ_A and μ_B are the membership functions of two quasi-convex fuzzy sets. If we denote them by A^μ and B^μ, respectively, we have that $A^\mu \cap B^\mu$. The membership function of $A^\mu \cap B^\mu$ coincides with $\mu_{A\cap B}$ by definition of the intersection of two intuitionistic fuzzy sets. Thus, $\mu_{A\cap B}$ is the membership function of a quasi-convex fuzzy set. Hence, $A \cap B$ is a convex intuitionistic fuzzy set. ■

The concept of quasi-convex intuitionistic fuzzy set was first described in [85].

Definition 3.6 *[21, 85] Let X be a vector space. An intuitionistic fuzzy set A is said to be quasi-convex if*

$$\mu_A(\lambda x + (1-\lambda)y) \geq \min\{\mu_A(x), \mu_A(y)\}$$

$$\nu_A(\lambda x + (1-\lambda)y) \leq \max\{\nu_A(x), \nu_A(y)\}$$

for all $x, y \in X$ and $\lambda \in [0,1]$.

In [21] the following result was demonstrated:

Proposition 3.7 *[21] Let X be a vector space and let A be an intuitionistic fuzzy set on X. The following statements are equivalent:*

(i) A is a quasi-convex intuitionistic fuzzy set.

(ii) The α-cuts of the fuzzy sets A^μ and A^ν are convex, for any $\alpha \in (0,1]$.

Thus, we have obtained that A is a quasi-convex intuitionistic fuzzy set if, and only if, the associated fuzzy sets A^μ and A^ν are quasi-convex.

It is also clear that convexity and quasi-convexity are related, but they are not equivalent. The next proposition demonstrates how one implies the other.

Proposition 3.8 *[21] Let X be a vector space. Let A be an intuitionistic fuzzy set on X. If A is quasi-convex, then it is also convex. The converse is not true in general.*

To show that the opposite is not true in general, it is possible to consider the example below.

Example 3.9 *[21] If we consider the intuitionistic fuzzy set A on $\mathbb{R}$ with the following membership and non-membership functions:*

$$\mu_A(x) = \nu_A(x) = \begin{cases} 0.5 & \text{if } x \in [1,2] \\ 0 & \text{otherwise} \end{cases}$$

we have that

$$A_\alpha = \{x : \mu_A(x) \geq \alpha \text{ and } \nu_A(x) \leq 1-\alpha\} = \begin{cases} [1,2] & \text{if } \alpha \in (0,0.5] \\ \emptyset & \text{if } \alpha \in (0.5,1] \end{cases}$$

It is clear that each A_α is convex for any $\alpha \in (0,1]$. Therefore A is a convex intuitionistic fuzzy set.

However, the 0.7-cut of A^ν is not a convex crisp set, since $(A^\nu)_{0.7} = (-\infty,1) \cup (2,\infty)$, and hence, A is not a quasi-convex intuitionistic fuzzy set.

Additional generalizations of convexity of intuitionistic fuzzy set can be introduced when the universe X is not a vector space, as shown below.

Definition 3.10 *[21] Let A be an intuitionistic fuzzy set defined on a universe X. Let $H : X \times X \times [0,1] \to X$ be a convex structure on X.*

(i) The intuitionistic fuzzy set A is said to inherit a convex structure from H, if for every $\alpha \in (0,1]$ the restriction of H to $A_\alpha \times A_\alpha \times [0,1]$ takes values in the A_α. That is, each α-cut A_α has also a convex structure induced by H.

(ii) The intuitionistic fuzzy set A is said to be convex with respect to H, if

$$\mu_A(H(x,y,\lambda)) \geq \min\{\mu_A(x), \mu_A(y)\} \text{ and } \nu_A(H(x,y,\lambda)) \leq \max\{\nu_A(x), \nu_A(y)\}$$

hold for all $x, y \in X$ and $\lambda \in [0,1]$.

Additionally, the relationship between an intuitionistic fuzzy set that is convex with regard to H and its α-cuts can be generalized.

Proposition 3.11 *[21] Let A be an intuitionistic fuzzy set defined on a universe X. Let $H : X \times X \times [0,1] \to X$ define a convex structure on X. The following statements are equivalent:*

(i) The intuitionistic fuzzy set A is convex with respect to H.

(ii) The restriction of H to each of the α-cuts of the fuzzy sets A^μ and A^ν defines a convex structure on the corresponding α-cut, for any $\alpha \in (0,1]$.

A summary of these results can be seen in Figure 3.1.

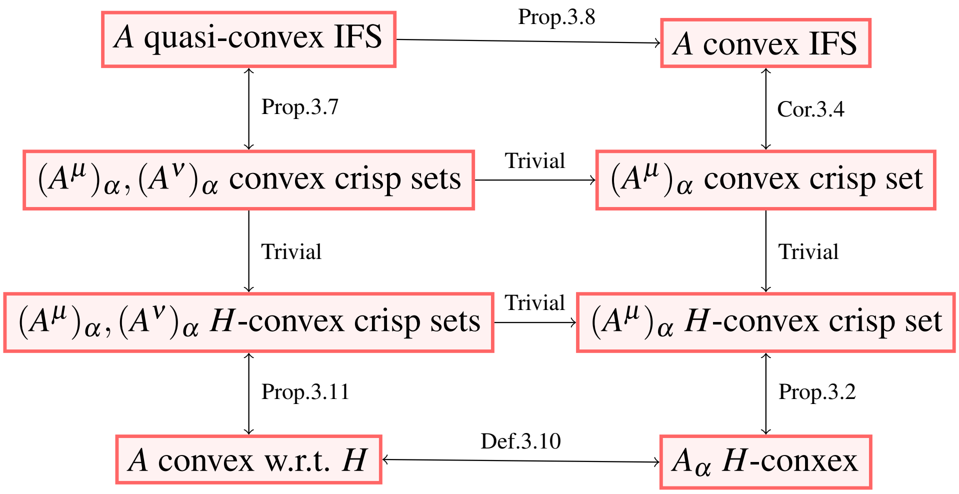

Figure 3.1: Relationships between convexity for intuitionistic fuzzy sets (IFSs) and the associated fuzzy sets.

3.2 Operations for interval-valued fuzzy sets

We will consider the previous studies about intuitionistic fuzzy sets and their correspondence with the interval-valued fuzzy sets as the starting point to analyze the convexity in the second case. Taking also into account that interval-valued fuzzy sets and (typical) hesitant fuzzy sets could be seen as two examples of the same family of generalized fuzzy sets, we will also consider the obtained results in Chapter 2, as well as the conclusions obtained there.

Thus, to study the convexity of interval-valued fuzzy sets, we will begin by analyzing the intersection and union of interval-valued fuzzy sets and defining level sets for interval-valued fuzzy sets.

3.2.1 Intersection

In the literature for classical sets, the intersection of two sets is described as the largest set that is contained in both sets. We will use again this idea to describe the intersection of two interval-valued fuzzy sets, so we have a different definition of intersection for each of the considered orders because, as we have seen, the order that is selected is important.

Definition 3.12 *[37, 39, 40] Let A, B be interval-valued fuzzy sets in X and let* $(L([0,1]), \preceq_o)$ *be a lattice. We define the o-intersection of A and B, denoted by* $A \cap_o B$*, as the greatest interval-valued fuzzy set such that* $A \cap_o B \subseteq_o A$ *and* $A \cap_o B \subseteq_o B$.

It is clear from this definition that the selection of the order totally determines the concept of intersection. Only in some particular cases, the obtained sets are related, as we can see in the following proposition.

Proposition 3.13 *Let* $(L([0,1]), \preceq_{o_1})$ *and* $(L([0,1]), \preceq_{o_2})$ *be two lattice. If* $a \preceq_{o_1} b$ *implies that* $a \preceq_{o_2} b$*, for all* $a, b \in L([0,1])$*, then we have that* $A \cap_{o_1} B \subseteq_{o_2} A \cap_{o_2} B$ *for any* $A, B \in IVFS(X)$.

Proof: By definition $A \cap_{o_1} B \subseteq_{o_1} A$ and $A \cap_{o_1} B \subseteq_{o_1} B$. By the relation between the orders and Definition 1.62, we can assure that $A \cap_{o_1} B \subseteq_{o_2} A$ and $A \cap_{o_1} B \subseteq_{o_2} B$. Finally, by definition $A \cap_{o_2} B$ is the greatest interval-valued fuzzy set o_2-included in A and B and therefore $A \cap_{o_1} B \subseteq_{o_2} A \cap_{o_2} B$. ∎

Thus, we can use the previous result to take into account the relations among the different orders given in Figure 1.8. At that moment, we were considering relations usually called orders in the literature, but some of them were not really orders. However, they were frequently used to compare two intervals and the previous definition of intersection could be extended for those cases. Hence, first of all, we will analyze these cases and we will classify them in accordance with their behaviour with respect to the intersection. For the first group (interval dominance and lattice order), partial relations are taken into consideration, which define the intersection as a unique set; for the second group (admissible orders, particularly the lexicographical orders and the Xu and Yager order), the intersection will once again be defined uniquely, since these are total orders; and finally, for the third group (maximin, maximax, Hurwicz, and weak orders), the intersection is not properly defined, since it is not a unique interval-valued fuzzy set.

Let us start by expressing the intersection using lattice order (*Lo*) or interval dominance (*ID*).

Proposition 3.14 *[37, 40] Let A, B be sets in $IVFS(X)$. Then, for any $x \in X$, we have that*

$$A \cap_{ID} B(x) = \min\{\underline{A(x)}, \underline{B(x)}\}$$
$$A \cap_{Lo} B(x) = [\min\{\underline{A(x)}, \underline{B(x)}\}, \min\{\overline{A(x)}, \overline{B(x)}\}]$$

Proof:

We start with the case of the interval dominance:

For any value x in X, it is clear that $\min\{\underline{A(x)}, \underline{B(x)}\}$ is a number in $[0,1]$ and therefore an element in $L([0,1])$. Thus, if we consider the fuzzy set I defined as $I(x) = \min\{\underline{A(x)}, \underline{B(x)}\}$ for any $x \in X$, or equivalently the interval-valued fuzzy set defined as $I(x) = [\min\{\underline{A(x)}, \underline{B(x)}\}, \min\{\underline{A(x)}, \underline{B(x)}\}]$ for any $x \in X$, we have that $\overline{I(x)} = \min\{\underline{A(x)}, \underline{B(x)}\} \leq \underline{A(x)}$ and $\overline{I(x)} \leq \underline{B(x)}$. Thus, $I(x) \preceq_{ID} A(x)$ and $I(x) \preceq_{ID} B(x)$ for any $x \in X$ and therefore $I \subseteq_{ID} A$ and $I \subseteq_{ID} B$.

Apart from that, if we consider a set $C \in IVFS(X)$ such that $C \subseteq_{ID} A$ and $C \subseteq_{ID} B$, then $\overline{C(x)} \leq \underline{A(x)}$ and $\overline{C(x)} \leq \underline{B(x)}$. So, $\overline{C(x)} \leq \min\{\underline{A(x)}, \underline{B(x)}\} = \underline{I(x)}$, that is, $C \subseteq_{ID} I$.

Thus, the fuzzy set I is the greatest interval-valued set that is *ID*-included in both sets and therefore it is the intersection of them.

Now, for the lattice order:

It is immediate that $[\min\{\underline{A(x)}, \underline{B(x)}\}, \min\{\overline{A(x)}, \overline{B(x)}\}] \in L([0,1])$ for any $x \in X$. Thus, we can define an interval-valued fuzzy set I as follows: $I(x) = [\min\{\underline{A(x)}, \underline{B(x)}\}, \min\{\overline{A(x)}, \overline{B(x)}\}]$ for all $x \in X$. Then, we have that $\underline{I(x)} = \min\{\underline{A(x)}, \underline{B(x)}\} \leq \underline{A(x)}$ and $\overline{I(x)} = \min\{\overline{A(x)}, \overline{B(x)}\} \leq \overline{A(x)}$. Thus, $I(x) \preceq_{Lo} A(x)$ and therefore $I \subseteq_{Lo} A$. Similarly, we can prove that $I \subseteq_{Lo} B$.

Finally, if we consider a set $C \in IVFS(X)$ such that $C \subseteq_{Lo} A$ and $C \subseteq_{Lo} B$, then $\underline{C(x)} \leq \underline{A(x)}$ and $\underline{C(x)} \leq \underline{B(x)}$. Therefore, $\underline{C(x)} \leq \min\{\underline{A(x)}, \underline{B(x)}\} = \underline{I(x)}$. It is analogous to prove that $\overline{C(x)} \leq \min\{\overline{A(x)}, \overline{B(x)}\} = \overline{I(x)}$ and then $C \subseteq_{Lo} I$.

Thus, the interval-valued fuzzy set I is the greatest, w.r.t. the lattice order, interval-valued such that it is Lo-included in A and B and therefore, by definition, I is the Lo-intersection of them. ∎

The reader may be cognizant now that the expression obtained for the lattice order is the intersection that is taken into account the most frequently in the literature, whereas the intersection of two interval-valued fuzzy sets using interval dominance is just a fuzzy set.

The intersection is found as a result of the following general result for total orders in the case of the lexicographical orders and the Xu and Yager order, which are particular cases of admissible order.

Proposition 3.15 *[37, 40] Let $\preceq_o$ be a total order on $L([0,1])$. For any $A, B \in IVFS(X)$, the o-intersection of A and B is the interval-valued fuzzy set defined by:*

$$A \cap_o B(x) = \begin{cases} A(x) & \text{if } A(x) \preceq_o B(x), \\ B(x) & \text{if } B(x) \preceq_o A(x). \end{cases}$$

Proof: It is clear that the set defined in the statement is an interval-valued fuzzy set. Let us denote it by I. We have that $I(x) = A(x)$ if $A(x) \preceq_o B(x)$ and $I(x) = B(x)$ if $B(x) \preceq_o A(x)$. Since $\preceq_o$ is transitive, we have that $I(x) \preceq_o A(x)$ and $I(x) \preceq_o B(x)$ for any $x \in X$. Thus, $I \subseteq_o A$ and $I \subseteq_o B$.

Moreover, if we consider a set $C \in IVFS(X)$ such that $C \subseteq_o A$ and $C \subseteq_o B$, then for any $x \in X$, as under $\preceq_o$ there are no incomparable elements, we have two cases:

a) If $A(x) \preceq_o B(x)$, then $I(x) = A(x)$. But, $C(x) \preceq_o A(x) = I(x)$.

b) If $B(x) \preceq_o A(x)$, then $I(x) = B(x)$. But, $C(x) \preceq_o B(x) = I(x)$.

Thus, in both cases we obtain that $C(x) \preceq_o I(x)$ and therefore $C \subseteq_o I$.

Therefore I is the greatest interval-valued fuzzy set o-included in A and B and, by definition, it is their o-intersection. ∎

This result is fulfilled for any total order, so for any admissible order. In particular, if we consider the admissible order $\preceq_{\mathscr{A},\mathscr{B}}$ obtained in Proposition 1.61, we have that

$$A \cap_{\mathscr{A},\mathscr{B}} B(x) = \begin{cases} A(x) & \text{if } A(x) \preceq_{\mathscr{A},\mathscr{B}} B(x), \\ B(x) & \text{if } B(x) \preceq_{\mathscr{A},\mathscr{B}} A(x), \end{cases}$$

Thus, we can use it in the particular cases of the lexicographical orders and the Xu and Yager order, where now we know that:

- Lexicographical order type 1:

$$A \cap_{Lex1} B(x) = \begin{cases} A(x) & \text{if } A(x) \preceq_{Lex1} B(x), \\ B(x) & \text{if } B(x) \preceq_{Lex1} A(x). \end{cases}$$

- Lexicographical order type 2:

$$A \cap_{Lex2} B(x) = \begin{cases} A(x) & \text{if } A(x) \preceq_{Lex2} B(x), \\ B(x) & \text{if } B(x) \preceq_{Lex2} A(x). \end{cases}$$

- Xu and Yager order:

$$A \cap_{XY} B(x) = \begin{cases} A(x) & \text{if } A(x) \preceq_{XY} B(x), \\ B(x) & \text{if } B(x) \preceq_{XY} A(x). \end{cases}$$

As we can see, a unique intersection can be obtained using interval dominance, lattice order, or any of the admissible orders. Unfortunately, as we can see from the results below, not all of the relations taken into consideration in Subsection 1.2.2 have the same behaviour.

Proposition 3.16 *[37, 39] Let A, B be sets in $IVFS(X)$. Then, for any $x \in X$, we have that*

- *Maximin order:* $A \cap_{Mm} B(x) = [\min\{\underline{A(x)}, \underline{B(x)}\}, v]$ *where v can be any number in the interval* $[\min\{\underline{A(x)}, \underline{B(x)}\}, 1]$.
- *Maximax order:* $A \cap_{MM} B(x) = [u, \min\{\overline{A(x)}, \overline{B(x)}\}]$ *where u can be any number in the interval* $[0, \min\{\overline{A(x)}, \overline{B(x)}\}]$.
- *Hurwicz order:* $A \cap_{H(\alpha)} B(x) = \left[u, \frac{k-\alpha \cdot u}{1-\alpha}\right]$ *where* $k = \min\{\alpha \cdot \underline{A(x)} + (1-\alpha) \cdot \overline{A(x)},\ \alpha \cdot \underline{B(x)} + (1-\alpha) \cdot \overline{B(x)}\}$ *and u is any number in the interval* $\left[\max\{0, \frac{k-(1-\alpha)}{\alpha}\}, k\right]$.

- *Weak order:* $A \cap_{wo} B(x) = [u,v]$ *where* u *and* v *could be any number in the interval* $[0, \min\{\overline{A(x)}, \overline{B(x)}\}]$ *and* v *can be any number in the interval* $[\min\{\overline{A(x)}, \overline{B(x)}\}, 1]$.

Proof: It is obvious to check that this set is a part of A and B in the first two cases (Mm and MM), and it is the largest element of $IVFS(X)$ that satisfies this property.

For the Hurwicz order, the intersection is well-defined, since $u \geq 0$, $u \leq \frac{k-\alpha \cdot u}{1-\alpha}$ if and only if $u \leq k$ and this is true by definition and, finally, $\frac{k-\alpha \cdot u}{1-\alpha} \leq 1$ if and only if $u \geq \frac{k-(1-\alpha)}{\alpha}$ and this is true by definition too. Moreover, since $\alpha u + (1-\alpha)\frac{k-\alpha \cdot u}{1-\alpha} = k$, we have that the defined set is $H(\alpha)$-included in A and B. Furthermore, if we take any other set C such that $C \subseteq_{H(\alpha)} A$ and $C \subseteq_{H(\alpha)} B$, thus, for any $x \in X$, $\alpha \underline{C(x)} + (1-\alpha)\overline{C(x)} \leq k = \alpha u + (1-\alpha)\frac{k-\alpha \cdot u}{1-\alpha}$. Hence, there is not an interval-valued fuzzy sets $H(\alpha)$-included in A and B which is not contained in the set defined in the statement.

For the weak order, since $u \leq \min\{\overline{A(x)}, \overline{B(x)}\}$, then $[u,v] \preceq_{wo} A(x)$ and $[u,v] \preceq_{wo} B(x)$ and any other set C such that $C \subseteq_{wo} A$ and $C \subseteq_{wo} B$ fulfils that $\underline{C(x)} \leq \overline{A(x)}$ and $\underline{C(x)} \leq \overline{B(x)}$, that is, $\underline{C(x)} \leq \min\{\overline{A(x)}, \overline{B(x)}\} \leq v$. ■

Hence, from Proposition 3.16, it is clear that the intersection is not always clearly defined since there is not a unique set fulfilling the required conditions, but an infinite collection of sets. Apart from that, it is also remarkable that the intersection of two interval-valued fuzzy sets for interval dominance is merely a fuzzy set (see Table 3.1).

By using the following examples, we may explain the earlier remarks.

Example 3.17 *Let us consider the case* $X = \{x\}$ *and the interval-valued fuzzy set* A *and* B *defined by* $A(x) = [0.4, 0.8]$ *and* $B(x) = [0.2, 0.9]$. *Then, the intersection for the last four relations in Table 3.1 is shown in Table 3.2 and illustrated in Figure 3.2, where non-uniqueness is clearly evidenced.*

If we consider the relations where the intersection is unique, we obtain the results in Table 3.3.

We can see again in this table that the intersection is just a fuzzy set for the case of the interval dominance. These examples are graphically represented in Figure 3.3.

Interval order	Is the intersection unique?	Is the intersection a proper IVFS?
Interval dominance	✓	✗
Lattice order	✓	✓
Lex. order type 1	✓	✓
Lex. order type 2	✓	✓
Xu and Yager order	✓	✓
Maximin order	✗	
Maximax order	✗	
Hurwicz order	✗	
Weak order	✗	

Table 3.1: Behaviour of the intersection.

$A\cap_{MM}B(x)$	$A\cap_{Mm}B(x)$	$A\cap_{H(1/2)}B(x)$	$A\cap_{wo}B(x)$
$[u,0.8]$	$[0.2,v]$	$[u,1.1-u]$	$[u,v]$
$u\in[0,0.8]$	$v\in[0.2,1]$	$u\in[0.1,0.55]$	$u\in[0,0.8]$ $v\in[0.8,1]$

Table 3.2: Intersection for the maximin, maximax, Hurwicz and weak relations.

Although this is not always the case, in this example both the Xu and Yager order and the type 1 lexicographical order provide the same intersection. For instance, if we consider C an interval-valued fuzzy set such that $C(x)=[0.4,0.5]$, we have that $B\preceq_{Lex1}C$ and $C\preceq_{XY}B$ and therefore $B\cap_{Lex1}C=B\neq B\cap_{XY}C=C$.

This example also makes it very evident that the intersection depends on the order that is taken into account. From now on, we will consider the relations where the intersection is uniquely specified.

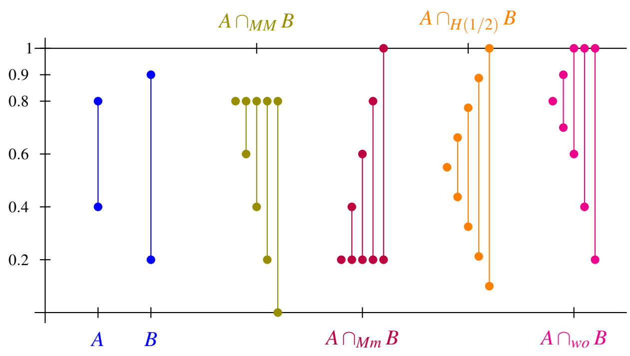

Figure 3.2: Not unique intersections of A and B.

$A \cap_{ID} B$	$A \cap_{Lo} B$	$A \cap_{Lex1} B$	$A \cap_{Lex2} B$	$A \cap_{XY} B$
0.2	$[0.2, 0.8]$	$[0.2, 0.9]$	$[0.4, 0.8]$	$[0.2, 0.9]$

Table 3.3: Intersection for interval dominance, lattice order and some well-known admissible orders.

3.2.2 Union

If the smallest set that includes two sets is an essential characteristic of the union of two sets, then there is also a different definition of union for each order that we are considering in $L([0,1])$.Since the union would be a useful tool for the following section, we can do a study similar to the one for the intersection.

Definition 3.18 *[39] Let A, B be sets in $IVFS(X)$ and let $(L([0,1]), \preceq_o)$ be a lattice. We define the o-union of A and B, denoted by $A \cup_o B$, as the smallest interval-valued fuzzy set such that $A \subseteq_o A \cup_o B$ and $B \subseteq_o A \cup_o B$.*

As a result of the previous studies, we will only take into account the relations where the intersection is unique and we will study the union using a similar scheme.

Proposition 3.19 *[39] Let A, B be sets in $IVFS(X)$. Then, for any $x \in X$ we have:*

- $A \cup_{ID} B(x) = \max\{\overline{A(x)}, \overline{B(x)}\}$.
- $A \cup_{Lo} B(x) = [\max\{\underline{A(x)}, \underline{B(x)}\}, \max\{\overline{A(x)}, \overline{B(x)}\}]$.

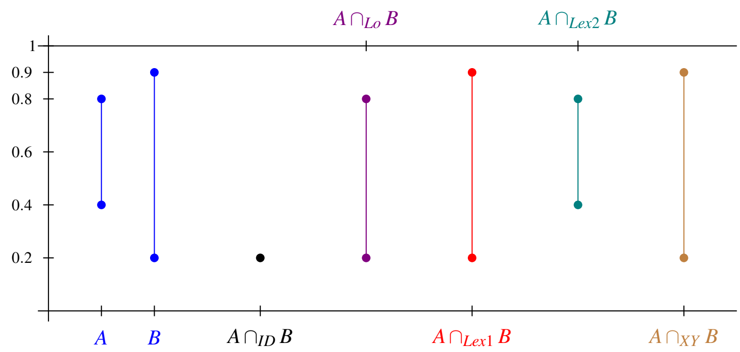

Figure 3.3: Intersection of A and B by different relations.

Proof: Interval dominance (ID): we should prove that $A \subseteq_{ID} A \cup_{ID} B$ and $B \subseteq_{ID} A \cup_{ID} B$ and that if there is another interval-valued fuzzy set containing both of them, then the union is contained in it. It is immediate that $A \subseteq_{ID} A \cup_{ID} B$ and $B \subseteq_{ID} A \cup_{ID} B$ by definition. Let us suppose there is an interval-valued fuzzy set C fulfilling $A \subseteq_{ID} C$ and $B \subseteq_{ID} C$. If $A \subseteq_{ID} C$, then $\overline{A(x)} \leq \underline{C(x)}$. If $B \subseteq_{ID} C$, thus $\overline{B(x)} \leq \underline{C(x)}$. If $A \cup_{ID} B(x) = \max\{\overline{A(x)}, \overline{B(x)}\}$, then $\max\{\overline{A(x)}, \overline{B(x)}\} \leq \underline{C(x)}$ and $A \cup_{ID} B \subseteq_{ID} C$.

Lattice order (Lo): let us check that this union is well defined. It is immediate from the definition that, $A \subseteq_{Lo} A \cup_{Lo} B$ and $B \subseteq_{Lo} A \cup_{Lo} B$. If we suppose that there exists an interval-valued fuzzy set $C \in IVFS(X)$, $C(x) = [\underline{C(x)}, \overline{C(x)}]$, such that $A \subseteq_{Lo} C$ and $B \subseteq_{Lo} C$. If $A \subseteq_{Lo} C$, then $\underline{A(x)} \leq \underline{C(x)}$ and $\overline{A(x)} \leq \overline{C(x)}$. If $B \subseteq_{Lo} C$, thus $\underline{B(x)} \leq \underline{C(x)}$ and $\overline{B(x)} \leq \overline{C(x)}$. Then $A \cup_{Lo} B \subseteq_{Lo} C$. ■

As with the intersection, the interval dominance is just a fuzzy set because the membership function assumes only one point at any element of the referential. Once again, the lattice order provides the union that is typically taken into account in the literature.

When it comes to total orders, we have the following:

Proposition 3.20 *[39] Let $\preceq_o$ be a total order on $L([0,1])$. For any $A, B \in IVFS(X)$,* *the o-union of A and B is the interval-valued fuzzy set defined by:*

$$A \cup_o B(x) = \begin{cases} B(x) & \text{if } A(x) \preceq_o B(x), \\ A(x) & \text{if } B(x) \preceq_o A(x). \end{cases}$$

Proof: Let us make sure that this union is well-defined. It follows that $\preceq_o$ is a total order, it is evident that $A \cup_o B$ can be defined for any $x \in X$, as $A(x) \preceq_o B(x)$ or $B(x) \preceq_o A(x)$.

By definition, it is immediate that $A \subseteq A \cup_o B$ and $B \subseteq A \cup_o B$.

We assume that there is an interval-valued fuzzy set $C \in IVFS(X)$ such that $A \subseteq_o C$ and $B \subseteq_o C$, thus, by the transitivity of $\preceq_o$, $A(x) \preceq_o C(x)$ and $B(x) \preceq_o C(x)$, for any $x \in X$. Hence, by definition, it is clear that $A \cup_o B(x) \preceq_o C(x)$ and therefore $A \cup_o B \subseteq_o C$. ■

Therefore, for the admissible order considered in Proposition 1.61, we obtain that

$$A \cup_{\mathscr{A},\mathscr{B}} B(x) = \begin{cases} B(x) & \text{if } A(x) \preceq_{\mathscr{A},\mathscr{B}} B(x), \\ A(x) & \text{if } B(x) \preceq_{\mathscr{A},\mathscr{B}} A(x), \end{cases}$$

and, in particular:

- Lexicographical order type 1:

$$A \cup_{Lex1} B(x) = \begin{cases} B(x) & \text{if } A(x) \preceq_{Lex1} B(x), \\ A(x) & \text{if } B(x) \preceq_{Lex1} A(x). \end{cases}$$

- Lexicographical order type 2:

$$A \cup_{Lex2} B(x) = \begin{cases} B(x) & \text{if } A(x) \preceq_{Lex2} B(x), \\ A(x) & \text{if } B(x) \preceq_{Lex2} A(x). \end{cases}$$

- Xu and Yager order:

$$A \cup_{XY} B(x) = \begin{cases} B(x) & \text{if } A(x) \preceq_{XY} B(x), \\ A(x) & \text{if } B(x) \preceq_{XY} A(x). \end{cases}$$

For a better understanding of this operation, we provide an example.

$A \cup_{ID} B$	$A \cup_{Lo} B$	$A \cup_{Lex1} B$	$A \cup_{Lex2} B$	$A \cup_{XY} B$
0.9	$[0.4, 0.9]$	$[0.4, 0.8]$	$[0.2, 0.9]$	$[0.4, 0.8]$

Table 3.4: Union for the lattice order, the lexicographical orders and the Xu and Yager order.

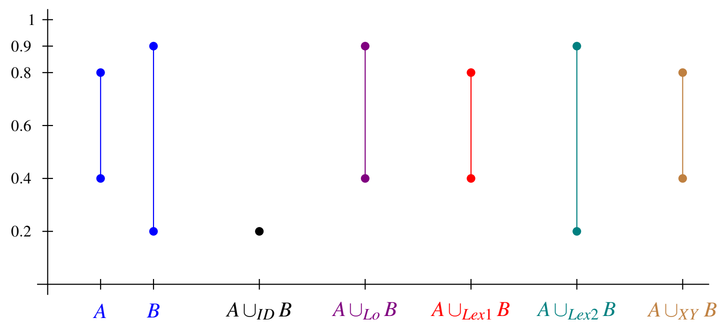

Figure 3.4: Union of A and B by different orders.

Example 3.21 *Under the same conditions of Example 3.17, the union for the different orders is calculated in Table 3.4.*

This is depicted graphically in Figure3.4.

In this instance, for the lattice order, A and B are not comparable, nevertheless, $B \subseteq_{Lex1} A$, $A \subseteq_{Lex2} B$ *and* $B \subseteq_{XY} A$. *Thus, it is logical that* $A \cup_{Lo} B \neq A$ *and* $A \cup_{Lo} B \neq B$, $A \cup_{Lex1} B = A$, $A \cup_{Lex2} B = B$ *and* $A \cup_{XY} B = A$. *It is once again clear that the order used to specify the inclusion operation in* $IVFS(X)$ *had a significant impact. It would seem that the union is closely tied to this idea.*

As in the intersection, if we use the lexicographical order type 1 or the Xu and Yager order we obtain the same interval-valued fuzzy set but, in general, this is not true. For instance, if we consider C an interval-valued fuzzy set such that $C(x) = [0.5, 0.6]$, *we have that* $A \preceq_{Lex1} C$ *and* $C \preceq_{XY} A$ *and therefore* $A \cup_{Lex1} C = C \neq A \cup_{XY} C = A$.

3.2.3 Level sets

An α-cut or a level set is one of the most crucial ideas in fuzzy sets, according to Klir [49]. We provide a reasonable definition of a level set for interval-valued fuzzy sets in this section.

Definition 3.22 *[40] Let $\preceq_o$ be an order on $L([0,1])$. For any $A \in IVFS(X)$ and for any $[\alpha,\beta] \in L([0,1])$, we define the $[\alpha,\beta]$-level sets of A w.r.t. the order $\preceq_o$ as follows:*

$$A^o_{[\alpha,\beta]} = \{x \in X : [\alpha,\beta] \preceq_o A(x)\}$$

In [64], Ramik and Vlach have considered the definition for level sets of intuitionistic fuzzy sets given in Definition 3.1. The mathematical connection between interval-valued fuzzy sets and intuitionistic fuzzy sets leads to equivalence to the previous proposal, for the particular case of the lattice order and $\beta = 1-\alpha$. Thus, we can consider that Definition 3.22 is in some sense a generalization of Definition 3.1.

It is evident that we would get various level sets if we used different orders, because the definition depends on the order we use.

Example 3.23 *Let $X = \{x,y,z\}$ be the referential. If we consider the interval-valued fuzzy sets A on X defined as $A = \{\langle x,[0.1,0.7]\rangle, \langle y,[0.2,0.8]\rangle, \langle z,[0.4,0.5]\rangle\}$, we have computed some level sets for different orders in Table 3.5.*

Order	$A_{[0.1,0.7]}$	$A_{[0.2,0.8]}$	$A_{[0.4,0.5]}$	$A_{[0.3,0.6]}$
Lattice order	$\{x,y\}$	$\{y\}$	$\{z\}$	$\emptyset$
Lexicographical type 1	$\{x,y,z\}$	$\{y,z\}$	$\{z\}$	$\{z\}$
Lexicographical type 2	$\{x,y\}$	$\{y\}$	$\{x,y,z\}$	$\{x,y\}$
Xu and Yager	$\{x,y,z\}$	$\{y\}$	$\{y,z\}$	$\{y\}$

Table 3.5: Level sets for different orders and levels.

We can notice from this example that some level sets are included in others. This is a direct consequence of the relationship between these orders.

Proposition 3.24 *[40] If $\preceq_1$ and $\preceq_2$ are orders in $L([0,1])$ such that $a \preceq_1 b$ implies $a \preceq_2 b$, then for any $A \in IVFS(X)$ and any $[\alpha,\beta] \in L([0,1])$ we have that $A^1_{[\alpha,\beta]} \subseteq A^2_{[\alpha,\beta]}$.*

Proof: By definition, $A^1_{[\alpha,\beta]} = \{x \in X : [\alpha,\beta] \preceq_1 A(x)\} \subseteq \{x \in X : [\alpha,\beta] \preceq_2 A(x)\} = A^2_{[\alpha,\beta]}$. ∎

The level sets of A obtained using lattice order are included in the level sets acquired using the lexicographical order type 1, type 2, or the Xu and Yager order, as in Example 3.23.

Let us just take a quick look at some of the characteristics that these level sets satisfy.

Proposition 3.25 *[40] Let $\preceq_o$ be an order on $L([0,1])$. For any $A,B \in IVFS(X)$ and any $[\alpha,\beta],[\gamma,\delta] \in L([0,1])$, we have that:*

i) If $[\alpha,\beta] \preceq_o [\gamma,\delta]$, then $A^o_{[\gamma,\delta]} \subseteq A^o_{[\alpha,\beta]}$.

ii) $A \subseteq_o B \Leftrightarrow A^o_{[\alpha,\beta]} \subseteq B^o_{[\alpha,\beta]}$ for any $[\alpha,\beta] \in L([0,1])$.

iii) $(A \cap_o B)^o_{[\alpha,\beta]} \subseteq A^o_{[\alpha,\beta]} \cap B^o_{[\alpha,\beta]}$. If $\preceq_o$ is a total order, then $(A \cap_o B)^o_{[\alpha,\beta]} = A^o_{[\alpha,\beta]} \cap B^o_{[\alpha,\beta]}$.

iv) $A^o_{[\alpha,\beta]} \cup B^o_{[\alpha,\beta]} \subseteq_o (A \cup_o B)^o_{[\alpha,\beta]}$. If $\preceq_o$ is a total order, then $A^o_{[\alpha,\beta]} \cup B^o_{[\alpha,\beta]} = (A \cup_o B)^o_{[\alpha,\beta]}$.

Proof: Let us consider $A,B \in IVFS(X)$ and $[\alpha,\beta],[\gamma,\delta] \in L([0,1])$.

i) If $[\alpha,\beta] \preceq_o [\gamma,\delta]$, then it is immediate by definition that $A^o_{[\gamma,\delta]} \subseteq A^o_{[\alpha,\beta]}$, since $\preceq_o$ is transitive.

ii) If $A \subseteq_o B$ then $A(x) \preceq_o B(x), \forall x \in X$. Thus, if $[\alpha,\beta] \preceq_o A(x)$, since $\preceq_o$ is transitive, then $[\alpha,\beta] \preceq_o B(x)$ and so $A^o_{[\alpha,\beta]} = \{x \in X : [\alpha,\beta] \preceq_o A(x)\} \subseteq \{x \in X : [\alpha,\beta] \preceq_o B(x)\} = B^o_{[\alpha,\beta]}$.

On the contrary, for any $x \in X$, if we apply the inclusion for the $A(x)$-level sets, we have that $x \in A^0_{A(x)}$ since $\preceq_o$ is reflexive, and therefore $x \in B^0_{A(x)}$. This is equivalent to say that $A(x) \preceq_o B(x)$. As we have proven it for any $x \in X$ we have that $A \subseteq_o B$.

iii) Since $A \cap_o B \subseteq_o A$ and $A \cap_o B \subseteq_o B$, by applying ii), we have that $(A \cap_o B)^o_{[\alpha,\beta]} \subseteq A^o_{[\alpha,\beta]}$ and $(A \cap_o B)^o_{[\alpha,\beta]} \subseteq B^o_{[\alpha,\beta]}$ and therefore $(A \cap_o B)^o_{[\alpha,\beta]} \subseteq A^o_{[\alpha,\beta]} \cap B^o_{[\alpha,\beta]}$.
On the other hand, if $x \in A^o_{[\alpha,\beta]} \cap B^o_{[\alpha,\beta]}$, then $[\alpha,\beta] \preceq_o A(x)$ and $[\alpha,\beta] \preceq_o B(x)$. As we are using a total order, from Proposition 3.15 we have that $A \cap_o B(x) = A(x)$ or $A \cap_o B(x) = B(x)$ and so $[\alpha,\beta] \preceq_o A \cap_o B(x)$.

iv) Since $A \subseteq_o A \cup_o B$ and $B \subseteq_o A \cup_o B$, by applying ii), we have that $A^o_{[\alpha,\beta]} \subseteq_o (A \cup_o B)^o_{[\alpha,\beta]}$ and $B^o_{[\alpha,\beta]} \subseteq_o (A \cup_o B)^o_{[\alpha,\beta]}$. Then, $A^o_{[\alpha,\beta]} \cup B^o_{[\alpha,\beta]} \subseteq_o (A \cup_o B)^o_{[\alpha,\beta]}$.
Conversely, for any $x \in X$ we have that $A \cup_o B(x) = B(x)$ or $A \cup_o B(x) = B(x)$, by applying Proposition 3.20, since $\preceq_o$ is a total order. Thus, if $x \in (A \cup_o B)^o_{[\alpha,\beta]}$, then $[\alpha,\beta] \preceq_o A \cup_o B(x)$ and therefore $[\alpha,\beta] \preceq_o A(x)$ or $[\alpha,\beta] \preceq_o B(x)$. Then, $x \in A^o_{[\alpha,\beta]} \cup B^o_{[\alpha,\beta]}$.

■

In fuzzy sets theory, we can represent a fuzzy set by its α-cuts through the Decompositions Theorems (see [49]), so the next task we consider is adapting these results of fuzzy sets into interval-valued fuzzy sets. Thus, we will try to identify an interval-valued fuzzy set through its level sets. First of all, we will do it in an example, where we can explain in detail the considered notation and then we will prove a general result.

Example 3.26 *Let $X = \{x,y,z\}$ be the referential. If we consider the interval-valued fuzzy set A defined in Example 3.23 and the lexicographical order type 1. Then the level sets are*

$$A^{Lex1}_{[0.1,0.7]} = \{x,y,z\}, \quad A^{Lex1}_{[0.2,0.8]} = \{y,z\} \quad \text{and} \quad A^{Lex1}_{[0.4,0.5]} = \{z\}$$

If we choose proper intervals, the interval-valued fuzzy set can be represented by its level sets. Let us use the following characteristic functions to define the level sets:

$$A^{Lex1}_{[0.1,0.7]} = 1 \cdot \{x\} + 1 \cdot \{y\} + 1 \cdot \{z\} = \{x,y,z\}$$

$$A^{Lex1}_{[0.2,0.8]} = 0 \cdot \{x\} + 1 \cdot \{y\} + 1 \cdot \{z\} = \{y,z\}$$

and

$$A^{Lex1}_{[0.4,0.5]} = 0 \cdot \{x\} + 0 \cdot \{y\} + 1 \cdot \{z\} = \{z\}$$

Now, we are going to obtain interval-valued fuzzy set based on these level sets defined as follows:

$$^{Lex1}_{[\alpha,\beta]}A = [\alpha,\beta] \cdot A^{Lex1}_{[\alpha,\beta]} = \begin{cases} [\alpha,\beta] & \text{if } x \in A^{Lex1}_{[\alpha,\beta]}, \\ [0,0] & \text{otherwise}. \end{cases}$$

With this operation, we are interval-valued fuzzifying the level sets, that is, we start from level sets (crisp sets) and we get interval-valued fuzzy set.

Then,

$$^{Lex1}_{[0.1,0.7]}A(t) = [0.1,0.7], \forall t \in X$$

$$^{Lex1}_{[0.2,0.8]}A(t) = \begin{cases} [0.2,0.8] & \text{if } t \in \{y,z\}, \\ [0,0] & \text{if } t = x, \end{cases}$$

and

$$^{Lex1}_{[0.4,0.5]}A(t) = \begin{cases} [0.4,0.5] & \text{if } t = z, \\ [0,0] & \text{if } t \in \{x,y\}. \end{cases}$$

In accordance with the previous notation, if we find a fixed interval in any of the level set functions, it indicates that the element belongs to that level set, as shown in the following example:

$$^{Lex1}_{[0.1,0.7]}A(x) = [0.1,0.7], \quad ^{Lex1}_{[0.1,0.7]}A(y) = [0.1,0.7] \quad \text{and} \quad ^{Lex1}_{[0.1,0.7]}A(z) = [0.1,0.7]$$

$$^{Lex1}_{[0.2,0.8]}A(y) = [0.2,0.8] \quad \text{and} \quad ^{Lex1}_{[0.2,0.8]}A(z) = [0.2,0.8]$$

$$^{Lex1}_{[0.4,0.5]}A(z) = [0.4,0.5]$$

It is clear that the $Lex1-$union of these interval-valued fuzzy set is the original set A. That is,

$$A = {}^{Lex1}_{[0.1,0.7]}A \cup_{Lex1} {}^{Lex1}_{[0.2,0.8]}A \cup_{Lex1} {}^{Lex1}_{[0.4,0.5]}A$$

On the basis of this concept, we put forward the following theorem:

Theorem 3.27 (Decomposition Theorem) *[40] Let $\preceq_o$ be a total order in $L([0,1])$ with least element $\mathbf{0}_o$. For every $A \in IVFS(X)$, we have that*

$$A = \underset{[\alpha,\beta]\in L([0,1])}{\cup_o} {}_{[\alpha,\beta]}^{\ o}A$$

where $\cup_o$ denotes the $o-$union and ${}_{[\alpha,\beta]}^{\ o}A(x) = [\alpha,\beta]$ if $x \in A^o_{[\alpha,\beta]}$ and $\mathbf{0}_o$ otherwise.

Proof: Let A be any set in $IVFS(X)$. For any $x \in X$, we have that $A(x) = [\gamma,\delta] \in L([0,1])$. Thus, $A(x) = {}_{[\gamma,\delta]}^{\ o}A(x)$ and therefore $A(x) \preceq_o \underset{[\alpha,\beta]\in L([0,1])}{\cup_o} {}_{[\alpha,\beta]}^{\ o}A(x)$, by the definition of $\cup_o$.

Conversely, since $\preceq_o$ is a total order, we have that $\underset{[\alpha,\beta]\in L([0,1])}{\cup_o} {}_{[\alpha,\beta]}^{\ o}A(x) = {}_{[\varepsilon,\zeta]}^{\ o}A(x)$ for some $[\varepsilon,\zeta] \in L([0,1])$.

By the definition of ${}_{[\varepsilon,\zeta]}^{\ o}A(x)$, we have two cases:

- If $x \notin A^o_{[\varepsilon,\zeta]}$, then ${}_{[\varepsilon,\zeta]}^{\ o}A(x) = \mathbf{0}_o \preceq_o A(x)$.
- If $x \in A^o_{[\varepsilon,\zeta]}$, then $[\varepsilon,\delta] \preceq_o A(x)$ and so ${}_{[\varepsilon,\zeta]}^{\ o}A(x) = [\varepsilon,\zeta] \preceq_o A(x)$.

Then, by the symmetry of $\preceq_o$, we have that $A(x) = \underset{[\alpha,\beta]\in L([0,1])}{\cup_o} {}_{[\alpha,\beta]}^{\ o}A(x)$. ∎

This theorem allows us to work with level sets instead of the interval-valued fuzzy set, but not all the operations hold.

For example, the standard complement for interval-valued fuzzy set is not cut-worthy, that is,

$$(A^c)^o_{[\alpha,\beta]} \neq (A^o_{[\alpha,\beta]})^c$$

as we can see in the following example.

Example 3.28 *Under the same conditions of Example 3.23 and considering Definition 1.64, we have that*

$$A = \{\langle x,[0.1,0.7]\rangle, \langle y,[0.2,0.8]\langle, \langle z,[0.4,0.5]\rangle\}$$

$$A^c = \{\langle x,[0.3,0.9]\rangle, \langle y,[0.2,0.8]\rangle, \langle z,[0.5,0.6]\rangle\}$$

If we consider again the lexicographical order type 1 and the level $[0.3,0.9]$, we obtain that $(A^c)^{Lex1}_{[0.3,0.9]} = \{t \in X : [0.3,0.9] \preceq_{Lex1} A^c(t)\} = \{x,z\}$.

On the other hand, $A^{Lex1}_{[0.3,0.9]} = \{z\}$ *and then*

$$(A^{Lex1}_{[0.3,0.9]})^c = \{x,y\} \neq (A^c)^{Lex1}_{[0.3,0.9]}$$

It is interesting that different intervals could generate the same level set, so we are going to take it into account in the next corollary. If we consider $\Lambda(A) = \{A(x) : x \in X\}$, there is an equivalent relation in $L([0,1])$ because $\Lambda(A)$ is the set of all intervals that represent different level sets of A. So the next result is a version of the first one where we only take one interval from each equivalent class in $\Lambda(A)$. That is, instead of considering $L([0,1])$, we would use $\Lambda(A)$. In Example 3.23, $\Lambda(A) = \{[0.1,0.7],[0.2,0.8],[0.4,0.5]\}$.

Corollary 3.29 *[40] Let* $\preceq_o$ *be a total order in* $L([0,1])$ *with least element* $\mathbf{0}_o$*. For every* $A \in IVFS(X)$*,*

$$A = \bigcup_{[\alpha,\beta]\in\Lambda(A)}^{o} {}_{[\alpha,\beta]}\overset{o}{A}$$

This is how interval-valued fuzzy sets are represented without using the same level set twice. The proof is a consequence of the previous Decomposition Theorem.

It is clear that we can apply these results for admissible orders, since they are total orders and as they refine the lattice order, we have that $[0,0]$ is the least element. Really, by the same reason, we also know that $[1,1]$ is the greatest element.

3.2.4 Convexity of interval-valued fuzzy sets

Taking into consideration the comments from the previous section, we do not consider all the relations introduced in Subsection 1.2.2. As we could see, the intersection based on the maximax, the maximin, the Hurwicz or the weak order does not work well and the interval dominance is not really an order. For the remaining intersections, we will determine whether the intersection of two convex interval-valued fuzzy set is a convex set as well for any order in $L([0,1])$.

In the literature, there are some approaches to convex interval-valued functions as [13]. Nevertheless, Cao [13] is not dealing with interval-valued fuzzy set, so we will consider the following definition of convexity that does not have the problem of defining the addition for interval-valued fuzzy set. We have also taking into account the problems on summarizing the membership values shown in Section 2.2. Thus, this proposal is, in some way, the result of all the previous studies.

Definition 3.30 *[37] Let X be an ordered set and let $\preceq_o$ be an order in $L([0,1])$. An interval-valued fuzzy set A on X is said to be o-convex, if for each $x < y < z$ in X the following inequalities are fulfilled:*

$$A(x) \preceq_o A(y) \text{ or } A(z) \preceq_o A(y)$$

This definition is based on the usual idea of convexity. It is easy to prove that if we consider a convex fuzzy set as an interval-valued fuzzy set, it is convex w.r.t. the previous definition. In addition, this definition has as particular cases the usual definition of convexity of crisp sets and fuzzy sets.

When X is a totally ordered set, the previous definition of convexity is equivalent to check

$$\min\{A(x),A(z)\} \preceq_o A(y)$$

If we work with partial orders, it may happen that $A(x)$ is not related to $A(z)$, so this is the reason for considering $A(x) \preceq_o A(y)$ or $A(z) \preceq_o A(y)$ in previous definition.

Basis on this idea, Huidobro et al. [39, 40] introduce the concept of strictly convex interval-valued fuzzy set.

Definition 3.31 *Let X be an ordered set and let $\preceq_o$ be an order on $L([0,1])$. An interval-valued fuzzy set A on X is said to be strictly o-convex, if for each $x < y < z$ in X the following inequalities are fulfilled:*

$$A(x) \prec_o A(y) \text{ or } A(z) \prec_o A(y)$$

which means that

$$A(x) \preceq_o A(y) \text{ and } A(x) \neq A(y) \quad \text{or} \quad A(z) \preceq_o A(y) \text{ and } A(z) \neq A(y)$$

Definition 3.30 is accurate because, as the following result proves, there is an equal relationship between convexity and the convexity of the level sets.

Proposition 3.32 *[39, 40] Let X be a ordered set and let $\preceq_o$ be an order in $L([0,1])$. Let A be an interval-valued fuzzy set on X. If A is o-convex, then $A^o_{[\alpha,\beta]}$ are convex crisp sets for all $[\alpha,\beta] \in L([0,1])$. The converse is true if $\preceq_0$ is a total order.*

Proof: Let us consider $x, y, z \in X$ such that $x < y < z$.

If $x \in A^o_{[\alpha,\beta]}$ and $z \in A^o_{[\alpha,\beta]}$, then $[\alpha,\beta] \preceq_o A(x)$ and $[\alpha,\beta] \preceq_o A(z)$. Moreover, as A is convex, we have $A(x) \preceq_o A(y)$ or $A(z) \preceq_o A(y)$. By the transitivity of $\preceq_o$, $[\alpha,\beta] \preceq_o A(y)$ or $[\alpha,\beta] \preceq_o A(y)$ and so $y \in A^o_{[\alpha,\beta]}$. Thus $A^o_{[\alpha,\beta]}$ is a convex crisp set.

Conversely, since $\preceq_o$ is a total order, we can consider $c = \min_o\{A(x), A(z)\} \in L([0,1])$. Then, $x, z \in A^o_c$. Since A^o_c is a convex crisp set, then $y \in A^o_c$ and so $\min_o\{A(x), A(z)\} \preceq_o A(y)$. ∎

Crisp sets and interval-valued fuzzy set are related to the idea of a level set. If we deal with the specific orders that were taken into consideration in the previous sections, we find that *Lo*-convexity implies *Lex*1-convexity, *Lex*2-convexity, and *XY*-convexity.

As in the case of hesitant fuzzy sets, we show in the next result that the definition of convexity also fits well with the definition of support of an interval-valued fuzzy set. First of all, we will propose a definition for the support of an interval-valued fuzzy set, based on how the support of fuzzy sets is defined.

Definition 3.33 *Let A be an interval-valued fuzzy set in X and let $\preceq_o$ be an order on $(L([0,1])$ with least element $\mathbf{0}_o$. The o-support of A, which is denoted by $Supp^o(A)$, is the crisp set*

$$Supp^o(A) = \{x \in X : A(x) \neq \mathbf{0}_o\}$$

Proposition 3.34 *Let X be a vector space and an order $\preceq_o$ in $L([0,1])$ with least element $\mathbf{0}_o$. If A is an o-convex interval-valued fuzzy set on X, then the o-support of A is a convex crisp set.*

Proof: Let A be a o-convex interval-valued fuzzy set. For any $x, z \in Supp^o(A)$ and any $\lambda \in (0,1)$, if we consider $y = \lambda x + (1-\lambda)z$, by the o-convexity of A, we have that $A(x) \preceq_o A(y)$ or $A(z) \preceq_o A(y)$. On the other hand, as $x, z \in Supp^o(A)$, we have that $A(x) \neq \mathbf{0}_o$ and $A(z) \neq \mathbf{0}_o$. Since $\mathbf{0}_o$ is the least element, then $A(y) \neq \mathbf{0}_o$ and therefore $y \in Supp^o(A)$, so $Supp^o(A)$ is a crisp convex set. ∎

In a similar way, we will also introduce the idea of core and prove that it works well with convexity.

Definition 3.35 *Let A be an interval-valued fuzzy set in X and let $\preceq_o$ be an order on $L([0,1])$ with greatest element $\mathbf{1}_o$. The core of A, which is denoted by $Core^o(A)$, is the crisp set*

$$Core^o(A) = \{x \in X : A(x) = \mathbf{1}_o\}$$

Proposition 3.36 *Let us consider the universe X, an order $\preceq_o$ on $L([0,1])$ with greatest element $\mathbf{1}_o$ and $A \in IVFS(X)$. If A is an o-convex interval-valued fuzzy set, then the core of A is a convex crisp set.*

Proof: If $x,z \in Core^o(A)$, then $A(x) = \mathbf{1}_o$ and $A(z) = \mathbf{1}_o$. Thus, for any $\lambda \in [0,1]$, we have that $y = \lambda x + (1-\lambda)z \in X$ and $A(x) \preceq_o A(y)$ or $A(z) \preceq_o A(y)$. Since $\mathbf{1}_o$ is the greatest element, we have that $A(y) = \mathbf{1}_o$, that is, $\lambda x + (1-\lambda)z \in Core^o(A)$ and therefore, $Core^o(A)$ is a crisp convex set. ■

Finally, we would like to study the significant attribute of convexity preservation under intersections. Unfortunately, as we will see at the following example, it is not possible to obtain a general result for any order.

Example 3.37 *Let $X = \{x,y,z\}$ with $x < y < z$. If we consider the interval-valued fuzzy sets A and B defined as follows:*

$$A = \{\langle x,[0.1,0.7]\rangle, \langle y,[0.2,0.8]\rangle, \langle z,[0.3,0.5]\rangle\}$$

$$B = \{\langle x,[0.1,0.7]\rangle, \langle y,[0.4,0.6]\rangle, \langle z,[0.3,0.5]\rangle\}$$

and we consider the lattice order, we obtain that,

$$A \cap_{Lo} B = \{\langle x,[0.1,0.7]\rangle, \langle y,[0.2,0.6]\rangle, \langle z,[0.3,0.5]\rangle\}$$

Then A is Lo-convex, since $[0.1,0.7] \preceq_{Lo} [0.2,0.8]$ and B is Lo-convex since $[0.3,0.5] \preceq_{Lo} [0.4,0.6]$. However, $A \cap_{Lo} B$ is not Lo-convex since $[0.2,0.6]$ is not related with $[0.1,0.7]$ or $[0.3,0.5]$ by means of the order relation $\preceq_{Lo}$.

We have reached a general, favorable outcome for total orders, which is reflected in the statement below.

Proposition 3.38 *[39, 40] Let X be an ordered set and let $\preceq_o$ a total order on $L([0,1])$. If $A, B \in IVFS(X)$ are o-convex (resp. strictly o-convex) , then $A \cap_o B$ is also o-convex (resp. strictly o-convex).*

Proof: Let x, y, z be three elements in X with $x < y < z$.

If $A(y) \preceq_o B(y)$, by Proposition 3.15 we have that $A \cap_o B(y) = A(y)$. Since A is o-convex (resp. strictly o-convex), $A(x) \preceq_o A(y)$ (resp. $A(x) \prec_o A(y)$) or $A(z) \preceq_o A(y)$ (resp. $A(z) \prec_o A(y)$). But by the definition of the intersection for this order we have that $A \cap_o B(x) \preceq_o A(x)$ and $A \cap_o B(z) \preceq_o B(z)$. By the transitivity, $A \cap_o B(x) \preceq_o A(y) = A \cap_o B(y)$ (resp. $A \cap_o B(x) \prec_o A(y) = A \cap_o B(y)$) or $A \cap_o B(z) \preceq_o A(y) = A \cap_o B(y)$ (resp. $A \cap_o B(z) \prec_o A(y) = A \cap_o B(y)$).

The case $B(y) \preceq_o A(y)$ is totally analogous. ∎

To continue, we arrive at the following conclusion by applying Proposition 3.38 to the situation of admissible order, and in particular the lexicographical orders and the Xu and Yager order.

Corollary 3.39 *[39, 40] If $\preceq_o$ is an admissible order, then o-convexity (resp. strictly o-convexity) is preserved under intersections.*

Finally, we will give some optimization-related conclusions that will be helpful for the following section.

Theorem 3.40 *[39, 40] Let $\preceq_o$ be an order on $L([0,1])$ with least element. Let A be an interval-valued fuzzy set over an ordered set X. Let x^* be an element in $Supp^o(A)$. If*

i) A is o-convex and x^* is a strict local maximizer of the membership function of A

or

ii) A is strictly o-convex and x^* is a local maximizer of the membership function of A

then x^* is also a global maximizer of the membership function of A over $Supp^o(A)$.

Proof: Suppose that $x^* \in Supp^o(A)$ is a strict local maximizer. It means that there exists a neighborhood Y such that for all $x \in Y$, there is $A(x) \prec_o A(x^*)$.

Let us suppose that there exists $x' \in Supp^o(A)$, different from x^*, such that $A(x^*) \preceq_o A(x')$.

By convexity, we have that $A(x') \preceq_o A(y)$ or $A(x^*) \preceq_o A(y)$, for all $y \in Supp^o(A)$ such that $x' < y < x^*$ or $x^* < y < x'$. Then, if we take y close enough to x^*, that is, $y \in Y$ and $y \neq x^*$, that contradicts $A(y) \prec_o A(x^*)$.

If A is strictly o-convex, but x^* is just a local maximizer, this means that there exists a neighborhood Y where $A(x) \preceq_o A(x^*)$ for any $x \in Y$. Let us suppose that x^* is not a global maximizer, then there exists $x' \in Supp^o(A)$ such that $A(x^*) \prec_o A(x')$. By the strictly o-convexity of A, we have that $A(x') \prec_o A(y)$ or $A(x^*) \prec_o A(y)$ for any element y between x^* and x'. If we choose y close enough to x^*, that is $y \in Y$, there is a contradiction since x^* is a local maximizer. ■

We also analyse the set where the membership function attains its maximum.

Theorem 3.41 *Let $\preceq_o$ be an order on $L([0,1])$ with a least element. Let A be an o-convex interval-valued fuzzy set over an ordered set X.*

i) The set of points at which A attains its global maximum over its support is a convex crisp set.

ii) If A is strictly o-convex, A attains its global maximum over $Supp^o(A)$ at no more than one point if X is uncountable or no more than two points if X is finite or countable.

Proof: Let's suppose that A is an o-convex interval-valued fuzzy set over an ordered set X.

i) Let us suppose that $[\alpha, \beta]$ is the maximum value for the membership function of A. If we build the level set associated with $[\alpha, \beta]$ following Proposition 3.32, it is a convex crisp set as A is a o-convex interval-valued fuzzy set. This level set is, in fact, the set of point at which A attains its global maximum.

ii) Let us suppose that $x^*, x' \in Supp^o(A)$ are two global maximizers, that is $A(x) \preceq_o A(x^*) = A(x')$ for all $x \in X$. We will assume that $x' < x^*$. The other case is totally analogous.

If X is countable, there exists $y \in X$ such that $x' < y < x^*$. Since A is strictly o-convex, then $A(x') \prec_o A(y)$ or $A(x^*) \prec_o A(y)$, and that contradicts the fact that they are global maximizers and so the set of points at which the membership function of A attains its global maximum over $Supp^o(A)$ has no more than one point.

If X is countable or finite and we have three global maximizers, x^*, x' and x'', they have to be consecutive. Otherwise, we are in a case similar to the previous one and a contradiction arises. Let us suppose that $x^* < x' < x''$, then $A(x^*) = A(x'') \prec_o A(x')$ by the strictly o-convexity of A, but this is a contradiction with the fact that x^* and x'' are global maximizers. ■

Previous results can be applied to any admissible order, since $(L([0,1]), \preceq_o)$ is then a bounded lattice with least element $\mathbf{0}_o = [0,0]$ and greatest element $\mathbf{1}_o = [1,1]$, taking into account that any admissible order refines the lattice order.

3.2.5 Decision-making based on interval-valued fuzzy sets

In this section, we offer a solution to a problem involving decision-making. This proposal has been introduced by Huidobro et al. in [39]. We also take into account here the comments given at the beginning of the Subsection 2.3.3. Again we have an approximate knowledge of the membership function. However, in this case, the available information about it is an interval where the value is included.

Utilizing Bellman and Zadeh method [8], the decision D would now be the intersection of the interval-valued fuzzy constraints and objectives if the constraints and the goals were thought of as interval-valued fuzzy set over the set of alternatives, X.

So according to Yager and Basson [89], a choice is created by intersecting all the goals and constraints. In light of this concept, Huidobro et al. suggested the definition below.

Definition 3.42 *[39, 40] Let $X = \{x_1, , \ldots, x_n\}$ be the set of alternatives, $G_1, , \ldots G_p$ be the set of goals that can be expressed as interval-valued fuzzy sets on the space of alternatives, and $C_1, \ldots, C_m$ be the set of constraints that can also be expressed as interval-valued fuzzy sets on the space of alternatives. Let $\preceq_o$ be an order on $L([0,1])$. The goals and constraints then combine to form a decision D_o, which is an interval-valued fuzzy set resulting from the intersection of the goals and the constraints. Thus, $D_o = G_1 \cap_o, \ldots \cap_o G_p \cap_o C_1 \cap_o \ldots \cap_o C_m$.*

For any $x \in X$, the meaning of $D_o(x)$ could refer to how well the alternative x validates the objectives and restrictions. After making a choice, we must choose the best alternative.

Since the intersection is actually an o-intersection, it follows that D_o depends on the chosen order $\preceq_o$ in $L([0,1])$ right away. As a result, depending on the sequence we are examining, the decision D_o, which is the intersection of all the goals and constraints, would vary. When there is no ambiguity, D_o could be denoted just as D.

Let us show an example.

Example 3.43 *A person has to choose to locate a new plant in one of three locations x_1, x_2 and x_3. He wants to select a location that minimizes real estate cost, G, and is located near supplies, C_1. Let $X = \{x_1, x_2, x_3\}$. This is a similar case as the one considered in Example 2.57, but in this case the information about any membership value is just an lower and an upper bound for his value. Thus, the appropriate sets to deal with this problem would be interval-valued fuzzy sets. Let's suppose that the membership functions of the goal G is*

$$G = \{\langle x_1, [0.2, 0.7]\rangle, \langle x_2, [0.6, 0.7]\rangle, \langle x_3, [0.4, 0.8]\rangle\}$$

and the membership function of the interval-valued fuzzy constraint C_1 is

$$C_1 = \{\langle x_1, [0.5, 0.6]\rangle, \langle x_2, [0.5, 0.9]\rangle, \langle x_3, [0.3, 0.9]\rangle\}.$$

If we consider lexicographical order type 1, we emphasize the lower endpoint of the interval. Then the membership functions of the interval-valued fuzzy decision D_{Lex1} is:

$$D_{Lex1} = \{\langle x_1, [0.2, 0.7]\rangle, \langle x_2, [0.5, 0.9]\rangle, \langle x_3, [0.3, 0.9]\rangle\}$$

and the optimal decision would be x_2, since it is the alternative with a maximum value of D_{Lex1} with respect to the lexicographical order type 1.

However, if we use lexicographical order type 2, then he membership functions of the interval-valued fuzzy decision D_{Lex2} is:

$$D_{Lex2} = \{\langle x_1, [0.5, 0.6]\rangle, \langle x_2, [0.6, 0.7]\rangle, \langle x_3, [0.4, 0.8]\rangle\}$$

and the optimal decision changes to x_3.

Following this simple illustration, it is demonstrated how crucial it is to choose the order on $L([0,1])$ correctly. It is clear that this correctness depends on any particular problem.

There are instances where the goals and constraints are defined in a different set of alternatives than they are in the previous example, which uses interval-valued fuzzy sets over the same collection of alternatives. We can keep clear of this circumstance if we apply the extension principle.

Definition 3.44 (Extension principle) *[39, 40] Let $(L([0,1]), \preceq_o)$ be a complete lattice. Any given function $f : X \to Y$ induces two functions, $f : IVFS(X) \to IVFS(Y)$ and $f^{-1} : IVFS(Y) \to IVFS(X)$, which are defined by*

$$[f(A)](y) = \sup_{o \atop x|y=f(x)} A(x)$$

for all $A \in IVFS(X)$, where sup_o denotes the supremum using the order $\preceq_o$ and $[f^{-1}(B)](x) = B(f(x))$ for all $B \in IVFS(Y)$.

With this procedure, when the interval-valued fuzzy constraints or goals are defined in different spaces, they can be mapped into the same space. When we have an n-ary function which maps $X_1 \times X_2 \times \cdots \times X_n$ to Y, we would assume that if $A \in IVFS(X_1 \times X_2 \times \cdots \times X_n)$, then $A(x_1, x_2, \ldots, x_n) = A(x_1) \cap_o A(x_2) \cap_o \cdots \cap_o A(x_n)$.

Let us show it by the following example.

Example 3.45 *Suppose the same conditions as in Example 3.43, but now there is another space Y meaning a set of former works developed by the potential financial* *directors, $Y = \{y_1, y_2, y_3, y_4\}$. We have some information about these former works: y_1 and y_2 were made by x_1, y_3 was supervised by x_2 and y_4 was produced by x_3.*

With this information we construct the following mapping:

$$f : Y \to X$$

defined by $f(y_1) = x_1$, $f(y_2) = x_1$, $f(y_3) = x_2$ and $f(y_4) = x_3$.

We also know a fuzzy constraint over Y that measures the impact of each one of works defined by: $C_2^Y = \{\langle y_1, [0.4, 0.65]\rangle, \langle y_2, [0.7, 0.9]\rangle, \langle y_3, [0.7, 1]\rangle, \langle y_4, [0.6, 0.9]\rangle\}$. It is denoted as C_2^Y in order to point out that it is an interval-valued fuzzy set over the space Y. Now we should apply the extension principle to have all the goals and constraints as interval-valued fuzzy sets over the same space. To apply the extension principle we should first decide which order are we taking into account, in this case, we would use lexicographical order type 1. Thus, for x_1,

$$[f(C_2^Y)](x_1) = \sup_{o \atop y|x_1 = f(y)} C_2^Y(y) = \sup_o \{C_2^Y(y_1), C_2^Y(y_2)\} = [0.7, 0.9].$$

Analogously, $[f(C_2)](x_2) = [0.7, 1]$ and $[f(C_2)](x_3) = [0.6, 0.9]$.

Consequently, $f(C_2^Y) = \{\langle x_1, [0.7, 0.9]\rangle, \langle x_2, [0.7, 1]\rangle, \langle x_3, [0.6, 0.9]\rangle\}$.

Finally, the decision is $D'_{Lex1} = G \cap_{Lex1} C_1 \cap_{Lex1} f(C_2^Y)$, that is, the membership degrees for the different alternatives in D'_{Lex1} are:

$$D'_{Lex1} = \{\langle x_1, [0.2, 0.7]\rangle, \langle x_2, [0.5, 0.9]\rangle, \langle x_3, [0.3, 0.9]\rangle\}$$

Thus, the optimal decision is still x_2.

Taking into account again the concept of conditional set, introduced in the fuzzy case by Yager and Basson [89], we provide the following definition, which was first considered in [39, 40].

Definition 3.46 *Let X and Y be two crisp sets and let $(L([0,1]), \preceq_o)$ be a complete lattice. If we have a family of interval-valued fuzzy sets on X given by $\{A|_y \in IVFS(X) : y \in Y\}$ and $B \in IVFS(Y)$, we can obtain a new interval-valued fuzzy set on X by combining the information given by $A|_y$ for any $y \in Y$ and B. This set will be denoted by $A|_B$ and its membership function is:*

$$A|_B(x) = \sup_{o \atop y \in Y} \min_o \{B(y), A|_y(x)\}$$

We can see how this conditional information is considered by means of the previous example.

Example 3.47 *Suppose the same conditions of Example 3.45. The company is forced to minimize the facility of employing workers. They would concentrate on the distance to the main office. Let $Z = \{Near(N), Med(M), Far(F)\}$. This constrain is given by the interval-valued fuzzy set*

$$B^Z = \{\langle N, [0.85, 1]\rangle, \langle M, [0.4, 0.7]\rangle, \langle F, [0.15, 0.35]\rangle\}.$$

The relation between the alternatives and the proximity to the main office is given by the following conditioned interval-valued fuzzy sets:

$$C_3^X|_N = \{\langle x_1, [0.7, 0.8]\rangle, \langle x_2, [0.5, 0.6]\rangle, \langle x_3, [0.3, 0.9]\rangle\},$$

$$C_3^X|_M = \{\langle x_1, [0.5, 0.6]\rangle, \langle x_2, [0.55, 0.7]\rangle, \langle x_3, [0.6, 0.9]\rangle\}$$
$$C_3^X|_F = \{\langle x_1, [0.35, 0.7]\rangle, \langle x_2, [0.4, 0.65]\rangle, \langle x_3, [0.35, 0.7]\rangle\}.$$

Thus, we can construct the interval-valued fuzzy set facility of hiring workers. For x_1, $C_3^X|_{B^Z}(x_1) = \sup_{Lex1}\limits_{z\in Z} \min_{Lex1}\{B^Z(z), C_3^X|_z(x_1)\} = \sup_{Lex1}\{\min_{Lex1}\{B^Z(N), C_3^X|_N(x_1)\}, \{\min_{Lex1}\{B^Z(M), C_3^X|_M(x_1)\}, \{\min_{Lex1}\{B^Z(F), C_3^X|_F(x_1)\}\} = \sup_{Lex1}\{[0.7, 0.8], [0.4, 0.7], [0.15, 0.35]\} = [0.7, 0.8]$.

We have to repeat the same procedure for x_2 and x_3 and we obtain that

$$C_3^X|_{B^Z}(x_2) = [0.5, 0.6] \quad \textit{and} \quad C_3^X|_{B^Z}(x_3) = [0.4, 0.7].$$

Then, we have that the interval-valued fuzzy set $C_3^X|_{B^Z}$ is given by

$$C_3^X|_{B^Z} = \{\langle x_1, [0.7, 0.8]\rangle, \langle x_2, [0.5, 0.6]\rangle, \langle x_3, [0.4, 0.7]\rangle\}.$$

Finally, the decision is $D''_{Lex1} = G \cap_{Lex1} C_1 \cap_{Lex1} f(C_2^Y) \cap_{Lex1} C_3^X|_{B^Z}$, that is, the decision if the interval-valued fuzzy set D''_{Lex1} defined as:

$$D''_{Lex1} = \{\langle x_1, [0.2, 0.7]\rangle, \langle x_2, [0.5, 0.6]\rangle, \langle x_3, [0.3, 0.9]\rangle\}.$$

Thus, x_2 is again the optimal decision.

It is time to combine a decision-making problem with Proposition 3.38 and Theorem 3.41.

Corollary 3.48 *[39, 40] Let $\preceq_o$ be a total order on $L([0,1])$ with least element, let $G_1,\ldots,G_p$ be the interval-valued fuzzy goals, $C_1,\ldots,C_m$ the interval-valued fuzzy constraints, and $D = G_1 \cap \ldots \cap G_p \cap C_1 \cap \ldots \cap C_m$ be the resulting decision.*

- *If the interval-valued fuzzy goals and constraints are o-convex interval-valued fuzzy set, then the resulting decision D is an o-convex interval-valued fuzzy set and the set of maximizing decisions of the interval-valued fuzzy set D is a convex crisp set.*
- *If the interval-valued fuzzy goals and constraints are strictly o-convex interval-valued fuzzy set, then the resulting decision D is a strictly o-convex interval-valued fuzzy set and the cardinal of the set of maximizing decisions of D is no more than two.*

Let us summarize the decision-making problem of Example 3.47 in the following example.

Example 3.49 *In the previous examples we consider one interval-valued fuzzy goal*

$$G = \{\langle x_1, [0.2, 0.7]\rangle, \langle x_2, [0.6, 0.7]\rangle, \langle x_3, [0.4, 0.8]\rangle\}$$

and three interval-valued fuzzy constraints

$$C_1 = \{\langle x_1, [0.5, 0.6]\rangle, \langle x_2, [0.5, 0.9]\rangle, \langle x_3, [0.3, 0.9]\rangle\}$$

$$f(C_2^Y) = \{\langle x_1, [0.7, 0.9]\rangle, \langle x_2, [0.7, 1]\rangle, \langle x_3, [0.6, 0.9]\rangle\}$$

$$C_3^X|_{B^Z} = \{\langle x_1, [0.7, 0.8]\rangle, \langle x_2, [0.5, 0.6]\rangle, \langle x_3, [0.4, 0.7]\rangle\}$$

If we suppose $x_1 < x_2 < x_3$, it is clear that G, C_1, $f(C_2^Y)$ and $C_3^X|_{B^Z}$ are strictly convex interval-valued fuzzy set with respect to the lexicographical order type 1, so the decision D''_{Lex1} is also a convex interval-valued fuzzy set w.r.t. the same order. It is easy to check it, since

$$D''_{Lex1} = \{\langle x_1, [0.2, 0.7]\rangle, \langle x_2, [0.5, 0.6]\rangle, \langle x_3, [0.3, 0.9]\rangle\}$$

We can apply the previous result to assert that x_2 is a global maximizer.

We can also find an illustration of the goal and constraints for x_1, x_2 and x_3 in Figures 3.5, 3.6 and 3.7, respectively. The importance of the choice of the order is clearly shown in these figures.

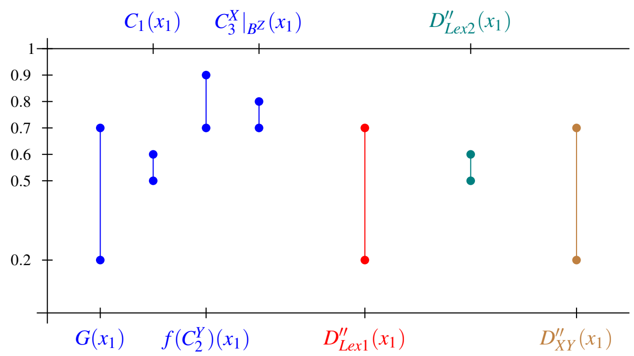

Figure 3.5: Visualization of examples (x_1)

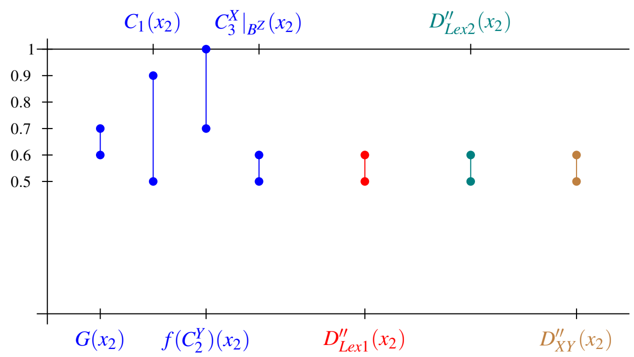

Figure 3.6: Visualization of examples (x_2)

We demonstrate what occurs when we choose lexicographical order type 2 in the following example because switching the order could also be interesting.

Example 3.50 *Using the same interval-valued fuzzy sets for the goal and constraints from the previous examples but using the lexicographical order type 2 in-*

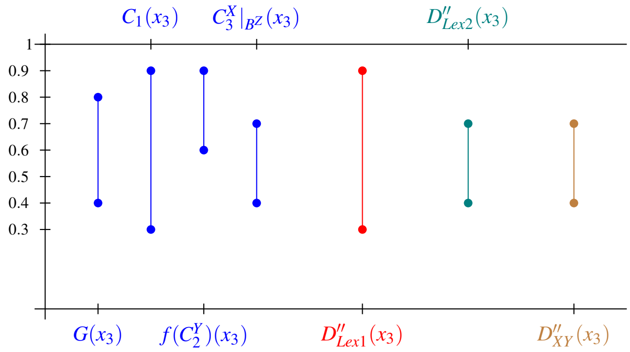

Figure 3.7: Visualization of examples (x_3)

stead of type 1, the decision-making problem is

$$G = \{\langle x_1, [0.2, 0.7]\rangle, \langle x_2, [0.6, 0.7]\rangle, \langle x_3, [0.4, 0.8]\rangle\}$$

$$C_1 = \{\langle x_1, [0.5, 0.6]\rangle, \langle x_2, [0.5, 0.9]\rangle, \langle x_3, [0.3, 0.9]\rangle\}$$

$$f(C_2^Y) = \{\langle x_1, [0.7, 0.9]\rangle, \langle x_2, [0.7, 1]\rangle, \langle x_3, [0.6, 0.9]\rangle\}$$

and

$$C_3^X|_{B^Z} = \{\langle x_1, [0.7, 0.8]\rangle, \langle x_2, [0.4, 0.7]\rangle, \langle x_3, [0.3, 0.9]\rangle\}$$

It should be noticed that there are changes in the constraint $C_3^X|_{B^Z}$ *because we used lexicographical order type 2 and it affects the supremum and the minimum.*

We can also see that D''_{Lex2} *is a* Lex2*-convex interval-valued fuzzy set, as*

$$D''_{Lex2} = \{\langle x_1, [0.5, 0.6]\rangle, \langle x_2, [0.4, 0.7]\rangle, \langle x_3, [0.4, 0.8]\rangle\}$$

Thus, D''_{Lex2} *is not only convex but strictly* Lex2*-convex, so we can assure that* x_3 *is the optimal decision. This happens even in the case* $C_3^X|_{B^Z}$ *is not a* Lex2*-convex interval-valued fuzzy set.*

3.2.6 Ranking method based on interval-valued fuzzy sets

In this section, we follow similar steps to the case of hesitant fuzzy sets presented in Subsection 2.3.4. We will introduce a ranking method for interval-valued fuzzy sets.

Once we have shown how to model the decision-making problems in the previous section, the method is almost done. We must obtain the decision D and order their components. We keep the largest element as the first element of the ranking and delete it from the others. We iterate this process until there are no elements on the list. In case two or more elements have the same value we must decide on one randomly.

For the sake of clarity, we introduce here a pair of practical cases.

Example 3.51 *Let us consider a situation similar to the one described in Example 2.65. Thus, we have the four national airlines: UNI Air, Transasia, Mandarin and Daily Air, and the four criteria: the booking and ticket service (C_1), the check-in and boarding process (C_2), the cabin service (C_3) and the responsiveness of the company (C_4). In this case, we can consider that the experts only propose a lower bound and an upper bound for the different membership values. Thus, we will deal with interval-valued fuzzy sets. The data is shown in Table 3.6.*

	C_1	C_2	C_3	C_4
UNI Air	[0.6,0.9]	[0.6,0.8]	[0.3,0.9]	[0.4,0.9]
Transasia	[0.7,0.9]	[0.5,0.9]	[0.4,0.8]	[0.5,0.7]
Mandarin	[0.5,0.8]	[0.6,0.9]	[0.3,0.7]	[0.5,0.7]
Daily Air	[0.6,0.9]	[0.7,0.9]	[0.2,0.7]	[0.4,0.5]

Table 3.6: Interval-valued fuzzy decision matrix.

Now we should transform this into a decision-making problem and then compute the intersection of the criteria. If we consider lexicographical order type 1, then we obtain the results given in Table 3.7.

	UNI Air	Transasia	Mandarin	Daily Air
$C_1 \cap_{Lex1} C_2 \cap_{Lex1} C_3 \cap_{Lex1} C_4$	[0.3,0.9]	[0.4,0.8]	[0.3,0.7]	[0.2,0.7]

Table 3.7: $Lex1$-intersection of the criteria.

So therefore, our ranking with lexicographical order type 1 is

$$Transasia > UNIAir > Mandarin > DailyAir$$

as

$$[0.2,0.7] \preceq_{Lex1} [0.3,0.7] \preceq_{Lex1} [0.3,0.9] \preceq_{Lex1} [0.4,0.8].$$

On the other hand, if we consider the lexicographical order type 2, the intersection is clearly different, as we can see in Table 3.8.

	UNI Air	Transasia	Mandarin	Daily Air
$C_1 \cap_{Lex2} C_2 \cap_{Lex2} C_3 \cap_{Lex2} C_4$	[0.6,0.8]	[0.5,0.7]	[0.3,0.7]	[0.4,0.5]

Table 3.8: *Lex*2-intersection of the criteria.

And the ranking using the lexicographical order type 2 is

$$UNIAir > Transasia > Mandarin > DailyAir.$$

Example 3.52 *In we consider now a similar scenario to the Example 2.66, but again with a piece of information given by means of intervals. Thus, we have seven universities: Stanford, Harvard, Oxford, Cambridge, California-Berkeley, Princeton and Yale. We also have five fields: Arts and Humanities (AH), Life Sciences and Medicine (LM), Engineering and Technology (ET), Natural Science and Mathematics (SCI) and Social Sciences (SOC).*

The data is the was presented in Table 3.9.

University	AH	LM	ET	SCI	SOC
Stanford	[0.868, 0.871]	[0.694, 0.912]	[0.919, 0.933]	[0.899, 0.925]	[0.801, 0.936]
Harvard	[0.861 , 0.897]	[0.913, 1]	[0.651, 0.857]	[0.902, 1]	[0.919, 1]
Oxford	[0.844, 0.991]	[0.609, 0.923]	[0.644, 0.876]	[0.723, 0.904]	[0.599, 0.942]
Cambridge	[0.839, 0.935]	[0.756, 0.918]	[0.748, 0.905]	[0.888, 0.97]	[0.594, 0.912]
California, Berkeley	[0.814 , 0.872]	[0.580, 0.856]	[0.868, 0.906]	[0.899, 0.963]	[0.796, 0.873]
Princeton	[0.812, 0.865]	[0.248, 0.741]	[0.711, 0.895]	[0.892, 0.937]	[0.764, 0.911]
Yale	[0.812, 0.89]	[0.624, 0.886]	[0.491, 0.752]	[0.652, 0.843]	[0.728, 0.900]

Table 3.9: Interval-valued Fuzzy Data.

Then we have to obtain the intersection using the Xu and Yager order, which is shown in Table 3.10.

So, therefore, our ranking with the Xu and Yager order is:

University	$AH \cap_{XY} LM \cap_{XY} ET \cap_{XY} SCI \cap_{XY} SOC$
Stanford	[0.694, 0.912]
Harvard	[0.651, 0.857]
Oxford	[0.644, 0.876]
Cambridge	[0.594, 0.912]
California, Berkeley	[0.580, 0.856]
Princeton	[0.248, 0.741]
Yale	[0.491, 0.752]

Table 3.10: XY-intersection.

$$Stanford > Oxford > Harvard > Cambridge > California, Berkeley \\ > Yale > Princeton.$$

It is clear, that this ranking could be different if we consider a different order to manage the information. Thus, for the lexicographical order type 1, the ranking remains equal for the last positions, but we obtain that:

$$Stanford > Harvard > Oxford > Cambridge$$

and for the lexicographical order type 2, these first positions are:

$$Cambridge > Stanford > Oxford > Harvard$$

These differences are the clear consequence of the different points of view considered by the chosen orders.

Despite the fact we obtain the same results as in the hesitant case, this is not always fulfilled as we can see in the following example.

Example 3.53 *Let us consider* $X = \{x_1, x_2, x_3\}$ *and the following hesitant fuzzy set:*

$$A = \{\langle x_1, \{0.1, 0.6, 0.7\}\rangle, \langle x_2, \{0.2, 0.7\}\rangle, \langle x_3, \{0.2, 0.6, 0.8\}\rangle\}$$

If we transform this hesitant fuzzy set into an interval-valued fuzzy set A' by using the minimum and maximum values for defining the extremes of the interval, we would obtain the following interval-valued fuzzy set:

$$A = \{\langle x_1, [0.1, 0.7]\rangle, \langle x_2, [0.2, 0.7]\rangle, \langle x_3, [0.2, 0.8]\rangle\}$$

It is easy to check that, if we use lexicographical order 2 we could obtain different maximums. In the case of the typical hesitant fuzzy set the maximum is attained in x_2, while in the interval-valued case is in x_3.

Also with this example, we showed that the loss of information when considering just the minimum and the maximum could affect the final maximizer.

Conclusions

In this thesis we have done a deep study of the convexity for two of the most used extensions of the fuzzy sets. In all the cases we have tried to keep the original ideas for crisp and fuzzy sets. However, as we have more degree of uncertainty, this also increases the difficulty of the related studies. Apart from theoretical studies, some possible applications have also been shown. More precisely, about hesitant fuzzy sets:

- First, we have presented a definition of convex hesitant fuzzy sets in a way that is consistent with the conventional understanding of convexity and is based on aggregation functions.
- We have found that the convexity of hesitant fuzzy sets is conserved under intersections when we employ as an aggregation function the minimum and the maximum.
- We have characterized the behaviour of any aggregation function with respect to the preservation of the convexity under intersections in all the cases that it is possible.
- Once we created the theory about the convexity of hesitant fuzzy sets using aggregation functions, we have considered a different approach for the cases the use of aggregation functions fails. This is done by considering all the original information given at the membership function. For this reason, we need to propose some operations on hesitant fuzzy sets, such as the intersection, union and level sets, to restore the traditional concept of these operations based on admissible orders, which are total orders that improve the lattice order.

- We apply convexity to optimization or decision-making problems and develop a novel ranking method by analyzing the convexity found in the literature and proposing an appropriate definition that is compatible with the intersection.

In addition, we conclude here the results we obtain for interval-valued fuzzy sets:

- In this thesis we propose a definition for the intersection of interval-valued fuzzy sets. Because the inclusion relation changes with the chosen order, there are multiple definitions of intersection. It should come as no surprise that not all of the usually considered relations to order the intervals are suitable for defining the intersection, despite the fact that the lattice order generates a definition of the intersection that is consistent with the standard definition found in the literature. However, it appears that admissible orders are better suited for our purposes.
- In a similar way, we introduce the union and level sets for interval-valued fuzzy sets and they work well with admissible orders. We look at some interesting properties and proposed a proper definition of convexity, which is preserved through intersections.
- Moreover, we prove a decomposition theorem for interval-valued fuzzy sets in order to characterize them through their level sets.
- We also presented a natural cutworthy property-satisfying definition of convexity of interval-valued fuzzy set, based on an order relation between intervals.
- Finally, we present a strategy for applying convexity and interval-valued fuzzy sets to optimization or decision-making issues.

Conclusiones

En esta tesis hemos realizado un estudio en profundidad sobre la convexidad de dos de las extensiones más utilizadas de los conjuntos difusos. En todos los casos hemos tratado de mantener las ideas originales para conjuntos nítidos y difusos. Sin embargo, como tenemos un grado de incertidumbre mayor, esto hace que aumente la dificultad de los estudios realizados. Además de un análisis teórico, también se han mostrado algunas posibles aplicaciones. Más precisamente, sobre conjuntos difusos *hesitant*:

- Hemos presentado una definición de conjunto difuso *hesitant* convexo que es coherente con las ideas clásicas de convexidad y que está basada en las funciones de agregación.

- Hemos concluido que se preserva la convexidad para la interseción de conjuntos difusos *hesitant* cuando la función de agregación que se utiliza es el mínimo o el máximo.

- Hemos caracterizado el comportamiento de cualquier función de agregación respecto a la conservación de la convexidad bajo intersecciones en todos en los casos que esto es posible.

- Concluimos la teoría sobre la convexidad de los conjuntos difusos *hesitant* usando funciones de agregación, poniendo de manifiesto sus debilidades y dando una alternativa, en la que se considera toda la información original contenida en la función de pertenencia. Motivado por esto, hemos definido algunas operaciones sobre conjuntos difusos *hesitant*, como son la intersección, la unión y alfa-cortes, con el objetivo de recuperar el concepto tradicional de estas operaciones basadas en órdenes admisibles, que son órdenes totales que mejoran el orden reticular.

- Aplicamos la convexidad a problemas de optimización o toma de decisiones y desarrollamos un método de *ranking* novedoso analizando la convexidad encontrada en la literatura y proponiendo una definición apropiada que sea compatible con la intersección.

Además, para los conjuntos difusos intervalo-valorados, hemos obtenido lo siguiente:

- Proponemos una definición para la intersección de conjuntos difusos intervalo-valorados. Debido a que la relación de inclusión cambia con el orden elegido, existen múltiples definiciones de intersección. No debería sorprender que no todas las relaciones habitualmente consideradas para ordenar intervalos sean adecuados para definir la intersección, a pesar de que el orden reticular genera una definición de la intersección que es consistente con la definición estándar encontrada en la literatura. Sin embargo, parece que los órdenes admisibles son los más adecuados para nuestros propósitos en este campo.

- De manera similar, presentamos los conjuntos de nivel y la unión para conjuntos difusos intervalo-valorados, las cuales tienen buenas propiedades si se trabaja con órdenes admisibles. Con todo lo anterior, proposimos una definición adecuada de convexidad, que se preserva por intersecciones.

- Además, demostramos un teorema de descomposición para conjuntos difusos intervalo-valorados, que nos permite caracterizarlos a través de sus conjuntos de nivel o alfa-cortes.

- También presentamos una definición de convexidad para conjuntos difusos intervalo-valorados que satisface la propiedad de conservación por alfa-cortes, basada en una relación de orden entre intervalos.

- Finalmente, presentamos un método para aplicar la convexidad y los conjuntos difusos intervalo-valorados a problemas de optimización o toma de decisiones.

Závery

V tejto práci sme podrobne skúmali konvexnosť pre dve z najpoužívanejších rozšírení fuzzy množín. Vo všetkých prípadoch sme sa snažili zachovať pôvodné koncepty zaužívané pre ostré a fuzzy množiny. Keďže v skúmaných štruktúrach je vyššia miera neistoty, zvyšuje to aj náročnosť súvisiaceho výskumu. Okrem teoretických výsledkov sme ukázali aj niektoré možné aplikácie. Konkrétne, pre hesitant fuzzy množiny:

- Uviedli sme definíciu konvexnej hesitant fuzzy množiny spôsobom, ktorý je v súlade s konvenčným chápaním konvexnosti a je založený na agregačných funkciách.
- Zistili sme, že konvexnosť hesitant fuzzy množín je zachovaná pri prieniku, keď ako agregačné funkcie použijeme minimum a maximum.
- Charakterizovali sme správanie akejkoľvek agregačnej funkcie vzhľadom na zachovanie konvexnosti prieniku vo všetkých prípadoch, kedy je to možné.
- Okrem vytvorenej teórie konvexnosti hesitant fuzzy množín pomocou agregačných funkcií sme použili odlišný prístup v prípadoch, kde použitie agregačných funkcií zlyháva. Tu využívame úplnú informáciu z funkcie príslušnosti. Za týmto účelom zavádzame niektoré operácie pre hesitant fuzzy množiny, ako napríklad zjednotenie, prienik a rezy, aby sme zachovali obvyklý koncept týchto operácií založený na prípustných usporiadaniach, čo sú úplné usporiadannia rozširujúce zväzové usporiadanie.
- Aplikujeme konvexnosť na optimalizačné alebo rozhodovacie problémy a vyvíjame novú metódu ohodnocovnia analyzovaním konvexnosti v literatúre

a navrhnutím vhodnej definície, ktorá je kompatibilná s prienikom.

V ďalšom uvádzame výsledky, ktoré sme získali pre intervalovohodnotové fuzzy množiny:

- Navrhujeme definíciu prieniku intervalovohodnotových fuzzy množín. Pretože relácia inklúzie závisí od zvolenej definície, existuje viacero možných prienikov. Nie je prekvapujúce, že nie všetky obvykle používané usporiadania intervalov sú vhodné na definovanie prieniku, napriek tomu, že usporiadanie vo zväze generuje prienik, ktorý je v súlade so štandardnou definíciou. Zdá sa však, že na tento účel sú vhodnejšie prípustné usporiadania.

- Podobným spôsobom zavedieme zjednotenie a hladiny pre intervalovohodnotovú fuzzy množinu, ktoré sú v súlade s prípustnými usporiadaniami. Navrhli sme správnu definíciu konvexnosti, ktorá je zachovaná pri prienku.

- Dokazujeme dekompozičnú vetu pre intervalovohodnotové fuzzy množiny, ktorá umožňuje charakterizovať ich prostredníctvom ich hladín.

- Navrhujeme definíciu konvexnosti pre intervalovohodnotové fuzzy množiny, ktorá je kompatibilná s vlastnosťami rezov, založenú na usporiadaní intervalov.

- Uvádzame spôsob použitia konvexných intervalovohodnotovných fuzzy množín v úlohách optimalizácie alebo rozhodovania.

List of Figures

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