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Identities involving special functions from hypergeometric solution of algebraic equations

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Abstract: From the algebraic solution of $x^m - x + t = 0$ for $m = 2, 3, 4$ and the corresponding solution in terms of hypergeometric functions, we obtain a set of reduction formulas for hypergeometric functions. By differentiation and integration of these results, and applying other known reduction formulas of hypergeometric functions, we derive new reduction formulas of special functions as well as the calculation of some definite integrals in terms of elementary functions.

Key words: Reduction formulas of special functions, hypergeometric functions, integrals of special functions, algebraic equations, trinomial equation

1. Introduction and Preliminaries

In existing literature, we found a large body of papers dealing with the trinomial equation (see [25] and the references therein). In fact, there are different versions of this kind of equation [2], [13, Chap.3 Sect.8], [19]. In this paper, we are interested in the following form of the trinomial equation:

$$x^m - x + t = 0. \tag{1.1}$$

Equation (1.1) was first solved by Lambert in 1758 as a series development for x in powers of t [16]. Euler's version of Lambert series [9] is connected to the *tree function* and the *Lambert W function* [6]. More recently, Glasser calculated the roots of (1.1) as a finite sum of generalized hypergeometric functions [11]. In many cases, one of the roots can be expressed as a single hypergeometric function. However, in 1770, Lagrange [15] applied his inversion formula [1, Appendix E] to derive a root $x_m(t)$ of the equation (1.1) as an expansion in powers of t. In modern notation, Lagrange's solution is written as:

$$x_m(t) = t\,{}_mF_{m-1}\left(\begin{array}{c} \frac{1}{m}, \ldots, \frac{m}{m} \\ \frac{2}{m-1}, \ldots, \frac{m}{m-1} \end{array} \middle| m \left(\frac{m\,t}{m-1} \right)^{m-1} \right). \tag{1.2}$$

where the generalized hypergeometric series is defined as:

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Definition 1.1 (Generalized hypergeometric series)

$$
{}_pF_q\left(\begin{array}{c} a_1,\ldots,a_p \\ b_1,\ldots,b_q \end{array}\middle| z\right)=\sum_{k=0}^{\infty}\frac{(a_1)_k\cdots(a_p)_k}{(b_1)_k\cdots(b_q)_k}\frac{z^k}{k!}. \tag{1.3}
$$

*If none of the parameters $b_1,\ldots b_q$ are non-positive integers and $p\leq q$, the series (1.3) converges for all finite values of z and defines an entire function**.

Note that, for $m=2,3,4$, (1.2) is reduced to (see also [22]):

$$
\begin{aligned}
x_2(t) &= t\,{}_2F_1\left(\begin{array}{c} \frac{1}{2},1 \\ 2 \end{array}\middle| 4t\right), & (1.4)\\
x_3(t) &= t\,{}_2F_1\left(\begin{array}{c} \frac{1}{3},\frac{2}{3} \\ \frac{3}{2} \end{array}\middle| 3\left(\frac{3t}{2}\right)^2\right), & (1.5)\\
x_4(t) &= t\,{}_3F_2\left(\begin{array}{c} \frac{1}{4},\frac{1}{2},\frac{3}{4} \\ \frac{2}{3},\frac{4}{3} \end{array}\middle| 4\left(\frac{4t}{3}\right)^3\right). & (1.6)
\end{aligned}
$$

Note as well that the roots of the trinomial equation (1.1) for $m=2,3,4$ are expressible in terms of elementary functions. Therefore, we can compare these well-known elementary solutions to the ones given in (1.4)-(1.6). The main goal of this paper is just to derive some new reduction formulas and definite integrals from this comparison. For this purpose, we present below some preliminary results. First, we will use the following differentiation formulas, that can be easily proved by induction:

$$
\begin{aligned}
\frac{d^n}{dt^n}\left(\frac{1}{t}\right) &= \frac{(-1)^n\,n!}{t^{n+1}}, & (1.7)\\
\frac{d^n}{dt^n}\left(\sqrt{1-t}\right) &= \left(-\frac{1}{2}\right)_n(1-t)^{1/2-n}. & (1.8)
\end{aligned}
$$

Taking $n=1$ in (1.8), and knowing that [20, Eqn. 18:5:7]

$$
(x)_{n+1}=x\,(x+1)_n, \tag{1.9}
$$

we also have

$$
\frac{d^n}{dt^n}\left(\frac{1}{\sqrt{1-t}}\right)=\left(\frac{1}{2}\right)_n(1-t)^{-1/2-n}. \tag{1.10}
$$

In addition, we will use Leibniz's differentiation formula [21, Eqn. 1.4.2] (for the historical origin of this formula, see [4, p. 143]),

$$
\frac{d^n}{dt^n}\left[f(t)\,g(t)\right]=\sum_{k=0}^{n}\binom{n}{k}f^{(k)}(t)\,g^{(n-k)}(t), \tag{1.11}
$$

Gauss summation formula [21, Eqn. 15.4.20] (for the original work of Gauss, see [10]),

$$
{}_2F_1\left(\begin{array}{c} a,b \\ c \end{array}\middle| 1\right)=\frac{\Gamma(c)\,\Gamma(c-a-b)}{\Gamma(c-a)\,\Gamma(c-b)}, \tag{1.12}
$$

$$
\mathrm{Re}\,(c-a-b)>0,
$$

*For the different cases of convergence of the generalized hypergeometric series see [21, Sect. 16.2].

and Whipple's sum [21, Eqn. 16.4.7],

$$
\begin{aligned}
& {}_3F_2\left(\begin{array}{c} a, 1-a, c \\ d, 2c-d+1 \end{array} \middle| 1\right) \\
= \quad & \frac{\pi\, 2^{1-2c}\,\Gamma(d)\,\Gamma(2c-d+1)}{\Gamma\left(c+\frac{a-d+1}{2}\right)\Gamma\left(c+1-\frac{a+d}{2}\right)\Gamma\left(\frac{a+d}{2}\right)\Gamma\left(\frac{d-a+1}{2}\right)}, \\
& \mathrm{Re}\,(c) > 0 \text{ or } a \in \mathbb{Z}.
\end{aligned} \tag{1.13}
$$

Finally, for the calculation of the definite integrals, we will use the following result [1, Ch. 2. Ex. 11]:

$$
\begin{aligned}
& \int_0^\infty e^{-st} t^{\alpha-1}\, {}_pF_q\left(\begin{array}{c} a_1, \ldots, a_p \\ b_1, \ldots, b_q \end{array} \middle| xt\right) dt = \frac{\Gamma(\alpha)}{s^\alpha}\, {}_{p+1}F_q\left(\begin{array}{c} a_1, \ldots, a_p, \alpha \\ b_1, \ldots, b_q \end{array} \middle| \frac{x}{s}\right), \\
& p \le q,\ \mathrm{Re}\, s > 0,\ \mathrm{Re}\,\alpha > 0.
\end{aligned} \tag{1.14}
$$

This paper is organized as follows. Section 2 equates the solution of (1.1) for $m = 2$ to the expression of $x_2(t)$ given in (1.4). From this result, and using some differentiation formulas of the ${}_2F_1$ function, we obtain a set of reduction formulas of some hypergeometric functions in terms of elementary functions. Whenever possible, we obtain alternative elementary representations of these hypergeometric functions by using known formulas in existing literature. As an application of these results, we calculate other reduction formulas of some special functions (i.e. incomplete beta function, Legendre function, and hypergeometric function) in terms of elementary functions, as well as the calculation of two definite integrals involving the lower incomplete gamma function. Section 3 equates the solution of (1.1) for $m = 3$ to be expression of $x_3(t)$ given in (1.5). From this comparison, we derive a new reduction formula of a ${}_2F_1$ hypergeometric function in terms of elementary functions, as well as an equivalent elementary representation in terms of a double finite sum by using a result found in the literature. As an application of the latter, we calculate a definite integral involving the parabolic cylinder function. In Section 4, we derive a reduction formula of a ${}_3F_2$ hypergeometric function in terms of elementary functions, equating the solution of (1.1) for $m = 4$ to the expression of $x_4(t)$ given in (1.6). From the latter reduction formula, we obtain an identity involving the product of two Legendre functions. Finally, Section 5 collects our conclusions. In the Appendices, we present the solution of the cubic and the quartic equations, which will be used throughout Sections 3 and 4 respectively; as well as the derivation of two finite sums that will be useful to us in Section 2.

2. Case $m = 2$

In this case, the algebraic solution of (1.1) is

$$
x_2(t) = \frac{1 \pm \sqrt{1-4t}}{2}, \tag{2.1}
$$

hence, selecting the proper root of (2.1), we can equate it to (1.4), to obtain

$$
{}_2F_1\left(\begin{array}{c} \frac{1}{2}, 1 \\ 2 \end{array} \middle| t\right) = \frac{2}{t}\left(1 - \sqrt{1-t}\right), \tag{2.2}
$$

which agrees with the result reported in the literature [24, Eqn. 7.3.2(84)]. Notice that (2.2) can be derived from the binomial theorem [20, Eqn. 6.14.1]. Indeed,

$$\sqrt{1-t} = \sum_{k=0}^{\infty} \left(-\frac{1}{2}\right)_k \frac{t^k}{k!} = 1 + \sum_{k=1}^{\infty} \left(-\frac{1}{2}\right)_k \frac{t^k}{k!},$$

hence, applying (1.9), we have

$$\frac{\sqrt{1-t}-1}{t} = \sum_{k=1}^{\infty} \left(-\frac{1}{2}\right)_k \frac{t^{k-1}}{k!} = -\frac{1}{2} \sum_{k=1}^{\infty} \left(\frac{1}{2}\right)_{k-1} \frac{(1)_{k-1}\, t^{k-1}}{(2)_{k-1}\,(k-1)!},$$

and the result follows. From (2.2), we obtain next a set of results using the formulas stated in the Introduction.

2.1. First differentiation formula

Theorem 2.1 *For $n = 0, 1, 2, \ldots$ and $t \in \mathbb{C}$, the following reduction formula holds true:*

$$
{}_2F_1\left(\left.\begin{array}{c} \frac{1}{2}+n, 1+n \\ 2+n \end{array}\right| t\right) \tag{2.3}
$$
$$
= \begin{cases} \dfrac{2(-1)^n (n+1)!}{\left(\frac{1}{2}\right)_n t^{n+1}} \left[1 - \sqrt{1-t} \displaystyle\sum_{k=0}^{n} \frac{\left(-\frac{1}{2}\right)_k}{k!} \left(\frac{t}{t-1}\right)^k\right], & t \neq 0, 1, \\ 1, & t = 0, \\ 2, & t = 1, n = 0, \\ \infty, & t = 1, n \geq 1. \end{cases}
$$

Proof For $t \neq 0, 1$, apply the following differentiation formula for the Gauss hypergeometric function [21, Eqn. 15.5.2]:

$$\frac{d^n}{dt^n}\left[{}_2F_1\left(\left.\begin{array}{c} a, b \\ c \end{array}\right| t\right)\right] = \frac{(a)_n (b)_n}{(c)_n} \, {}_2F_1\left(\left.\begin{array}{c} a+n, b+n \\ c+n \end{array}\right| t\right). \tag{2.4}$$
$$n = 0, 1, 2, \ldots$$

Therefore, taking $a = \frac{1}{2}$, $b = 1$ and $c = 2$ in (2.4) and using (2.2), we have

$$\frac{d^n}{dt^n}\left[{}_2F_1\left(\left.\begin{array}{c} \frac{1}{2}, 1 \\ 2 \end{array}\right| t\right)\right] = 2\left[\frac{d^n}{dt^n}\left(\frac{1}{t}\right) - \frac{d^n}{dt^n}\left(\frac{\sqrt{1-t}}{t}\right)\right].$$

Applying (1.7)-(1.8) and (1.11), after some algebra, we arrive at (2.3) for $t \neq 0, 1$. For $t = 0$, apply the definition of the hypergeometric series (1.3). For $t = 1$, apply the Gauss summation formula (1.12). This completes the proof. □

2.1.1. Other elementary representations

It is worth noting that we can derive different elementary representations of ${}_2F_1\left(\frac{1}{2}+n, 1+n; 2+n; t\right)$ by using known formulas given in existing literature.

Theorem 2.2 *For $n = 0, 1, 2, \ldots$ and $t \in \mathbb{C}$, the following reduction formula holds true:*

$$
\begin{aligned}
& {}_2F_1\left(\begin{array}{c} \frac{1}{2}+n, 1+n \\ 2+n \end{array} \middle| t \right) \qquad (2.5) \\
= & \begin{cases} \dfrac{2(-1)^n (n+1)!}{\left(\frac{1}{2}\right)_n t^{n+1}} \left[1 - (1-t)^{1/2-n} \displaystyle\sum_{k=0}^{n} \frac{\left(\frac{1}{2}-n\right)_k}{k!} t^k \right], & t \neq 0, 1, \\ 1, & t = 0, \\ 2, & t = 1, n = 0, \\ \infty, & t = 1, n \geq 1. \end{cases}
\end{aligned}
$$

Proof First we prove (2.5) for $t \neq 0, 1$. Apply Euler's transformation formula [21, Eqn. 15.8.1]:

$$
{}_2F_1\left(\begin{array}{c} \alpha, \beta \\ \gamma \end{array} \middle| z \right) = (1-z)^{\gamma-\alpha-\beta} \, {}_2F_1\left(\begin{array}{c} \gamma-\alpha, \gamma-\beta \\ \gamma \end{array} \middle| z \right), \qquad (2.6)
$$

to obtain

$$
{}_2F_1\left(\begin{array}{c} \frac{1}{2}+n, 1+n \\ 2+n \end{array} \middle| t \right) = (1-t)^{1/2-n} \, {}_2F_1\left(\begin{array}{c} \frac{3}{2}, 1 \\ 2+n \end{array} \middle| t \right). \qquad (2.7)
$$

We found in [24, Eqn. 7.3.1(123)] for $m = 1, 2, \ldots$, and $m - b \neq 1, 2, \ldots$, the formula:

$$
{}_2F_1\left(\begin{array}{c} 1, b \\ m \end{array} \middle| z \right) = \frac{(m-1)!\,(-z)^{1-m}}{(1-b)_{m-1}} \left[(1-z)^{m-b-1} - \sum_{k=0}^{m-2} \frac{(b-m+1)_k}{k!} z^k \right]. \qquad (2.8)
$$

Therefore, apply (2.8) to (2.7) with $m = n+2$ and $b = \frac{3}{2}$, taking into account (1.9) for $x = -\frac{1}{2}$, to arrive at (2.5) for $t \neq 0, 1$. Straightforward from (2.5) for $t \neq 0, 1$, we have a divergent result for $t = 1$, except for $n = 0$. For $t = 0$, we have an indeterminate expression on the RHS of (2.5), but the LHS of (2.5) is just 1. Nevertheless, if we calculate the limit $t \to 0$ of the RHS of (2.5) taking into account (2.12), we obtain also 1. Indeed,

$$
\begin{aligned}
& \frac{2(-1)^n (n+1)!}{\left(\frac{1}{2}\right)_n} \lim_{t\to 0} \frac{(1-t)^{1/2-n}}{t^{n+1}} \left[\frac{1}{(1-t)^{1/2-n}} - \sum_{k=0}^{n} \frac{\left(\frac{1}{2}-n\right)_k}{k!} t^k \right] \\
= \; & \frac{2(-1)^n (n+1)!}{\left(\frac{1}{2}\right)_n} \lim_{t\to 0} \frac{1}{t^{n+1}} \sum_{k=n+1}^{\infty} \frac{\left(\frac{1}{2}-n\right)_k}{k!} t^k \\
= \; & \frac{2(-1)^n \left(\frac{1}{2}-n\right)_{n+1}}{\left(\frac{1}{2}\right)_n} = 1,
\end{aligned}
$$

where we have applied the property $\Gamma(z)\Gamma(1-z) = \frac{\pi}{\sin \pi z}$ [17, Eqn. 1.2.2]. $\square$

Theorem 2.3 *For $n=0,1,2,\ldots$ and $t\in\mathbb{C}$, the following reduction formula holds true:*

$$
{}_2F_1\left(\left.\begin{array}{c} n+\frac{1}{2}, n+1 \\ n+2 \end{array}\right| t\right) \tag{2.9}
$$
$$
= \begin{cases} \dfrac{2^{2n+1} n\,(n+1)}{t^n\left(1+\sqrt{1-t}\right)} \displaystyle\sum_{k=0}^{n-1} \frac{\binom{n-1}{k}}{(n+1+k)\,(n+k)} \left(\frac{1-\sqrt{1-t}}{2\sqrt{1-t}}\right)^{n+k}, & t\neq 0,1, \\ 1, & t=0, \\ 2, & t=1, n=0, \\ \infty, & t=1, n\geq 1. \end{cases}
$$

Proof We need to prove (2.9) for $t\neq 0,1$. For this purpose, take $a=2n+1$, $k=n$, and $z=1-t$ in (C.5) to arrive at the desired result. □

2.1.2. Applications: Quadratic transformation, incomplete beta function, and definite integral

Next, we derive some results from the elementary representation of ${}_2F_1\left(\frac{1}{2}+n, 1+n; 2+n; t\right)$ given in (2.3).

Theorem 2.4 *For $n=0,1,2,\ldots$ and $t\in\mathbb{C}$, the following reduction formula holds true:*

$$
{}_2F_1\left(\left.\begin{array}{c} 1+2n, \frac{3}{2}+n \\ 3+2n \end{array}\right| t\right) \tag{2.10}
$$
$$
= \begin{cases} \dfrac{(-1)^n\,(n+1)!}{\left(\frac{1}{2}\right)_n} \left(\dfrac{2}{t}\right)^{2(n+1)} \left[2-t-2\sqrt{1-t}\displaystyle\sum_{k=0}^{n} \frac{\left(-\frac{1}{2}\right)_k}{k!} \left(\frac{t}{2\sqrt{t-1}}\right)^{2k}\right], & t\neq 0,1 \\ 1, & t=0, \\ 4, & t=1, n=0, \\ \infty, & t=1, n\geq 1. \end{cases}
$$

Proof Apply the quadratic transformation [1, Eqn. 3.1.7]:

$$
{}_2F_1\left(\left.\begin{array}{c} \alpha, \beta \\ 2\beta \end{array}\right| x\right) = \left(1-\frac{x}{2}\right)^{-\alpha} {}_2F_1\left(\left.\begin{array}{c} \frac{\alpha}{2}, \frac{\alpha+1}{2} \\ \beta+\frac{1}{2} \end{array}\right| \left(\frac{x}{2-x}\right)^2\right),
$$

taking $\alpha=2n+1$, $\beta=n+\frac{3}{2}$, and $x=\frac{2\sqrt{z}}{1+\sqrt{z}}$ to arrive at:

$$
{}_2F_1\left(\left.\begin{array}{c} 1+2n, \frac{3}{2}+n \\ 3+2n \end{array}\right| \frac{2\sqrt{z}}{1+\sqrt{z}}\right) = \left(1+\sqrt{z}\right)^{2n+1} {}_2F_1\left(\left.\begin{array}{c} \frac{1}{2}+n, 1+n \\ 2+n \end{array}\right| z\right). \tag{2.11}
$$

In order to obtain the desired result for $t\neq 0,1$, substitute (2.3) in (2.11) and perform the change of variables $t=\frac{2\sqrt{z}}{1+\sqrt{z}}$. According to this last result, the cases given in (2.10) for $t=1$ are straightforward. However, for $t=0$, we have an indeterminate expression on the RHS of (2.10), but the LHS of (2.10) is 1. Nevertheless, if we calculate the limit $t\to 0$ of the RHS of (2.10), taking into account the formula [20, Eqn. 18:3:4],

$$
\frac{1}{(1-t)^{\nu}} = \sum_{k=0}^{\infty} (\nu)_k \frac{t^k}{k!}, \tag{2.12}
$$

we obtain also 1,

$$
\begin{aligned}
& \frac{(-1)^n (n+1)!}{\left(\frac{1}{2}\right)_n} \lim_{t\to 0} \left(\frac{2}{t}\right)^{2(n+1)} \\
& \left\{2 - t - 2\sqrt{1-t}\left[\sum_{k=0}^{\infty} \frac{\left(-\frac{1}{2}\right)_k}{k!} \left(\frac{t^2}{4(t-1)}\right)^k - \sum_{k=n+1}^{\infty} \frac{\left(-\frac{1}{2}\right)_k}{k!} \left(\frac{t^2}{4(t-1)}\right)^k\right]\right\} \\
= \; & \frac{(-1)^n (n+1)!}{\left(\frac{1}{2}\right)_n} \lim_{t\to 0} \left(\frac{2}{t}\right)^{2(n+1)} \\
& \left\{2 - t - 2\sqrt{1-t}\left[\frac{2-t}{2\sqrt{1-t}} - \sum_{k=n+1}^{\infty} \frac{\left(-\frac{1}{2}\right)_k}{k!} \left(\frac{t^2}{4(t-1)}\right)^k\right]\right\} \\
= \; & -\frac{2\left(-\frac{1}{2}\right)_{n+1}}{\left(\frac{1}{2}\right)_n} = 1,
\end{aligned}
$$

where we have applied (1.9) for $x = -\frac{1}{2}$. □

Theorem 2.5 *For $n = 0, 1, 2, \ldots$ and $t \in \mathbb{C}$, the following reduction formula holds true:*

$$
\begin{aligned}
& \mathrm{B}\left(1+n, \frac{1}{2} - n, t\right) \qquad (2.13) \\
= \; & \begin{cases} \dfrac{2(-1)^n n!}{\left(\frac{1}{2}\right)_n} \left[1 - \sqrt{1-t}\displaystyle\sum_{k=0}^{n} \frac{\left(-\frac{1}{2}\right)_k}{k!} \left(\frac{t}{t-1}\right)^k\right], & t \neq 0, 1, \\ \frac{2(-1)^n n!}{\left(\frac{1}{2}\right)_n}, & t = 1, \\ 0, & t = 0. \end{cases}
\end{aligned}
$$

where $\mathrm{B}(\nu, \mu, z)$ denotes the incomplete beta function [20, Chap. 58].

Proof For $t \neq 0, 1$, in [24, Eqn. 7.3.1(28)], we found:

$$
{}_2F_1\left(\left.\begin{array}{c} a, b \\ b+1 \end{array}\right| t\right) = b\, t^{-b}\, \mathrm{B}(b, 1-a, t), \qquad (2.14)
$$

Therefore, take $a = \frac{1}{2} + n$ and $b = 1 + n$ in (2.14) and apply (2.3) to obtain (2.13). For $t = 1$, apply the properties of the incomplete beta function [20, Eqns. 58:3:1&58:1:1]

$$
\mathrm{B}(\nu, \mu, 1) = \mathrm{B}(\nu, \mu) = \frac{\Gamma(\nu)\,\Gamma(\mu)}{\Gamma(\nu+\mu)},
$$

and the formula of the gamma function [20, Eqn. 43:4:4]

$$
\Gamma\left(\frac{1}{2} - n\right) = \frac{(-1)^n}{\left(\frac{1}{2}\right)_n}\sqrt{\pi},
$$

to obtain the desired result. For $t=0$, apply the definition of the incomplete beta function [20, Eqn.58:3:1], and calculate the limit $t\to 0$ for $n\geq 0$, to obtain:

$$\lim_{t\to 0}\mathrm{B}\left(1+n,\mu,t\right)=\lim_{t\to 0}\int_0^t x^n\left(1-x\right)^{\mu-1}dx=0.$$

□

Theorem 2.6 *For $n=0,1,2,\ldots$ and $\mathrm{Re}\left(s+x\right)>0$, we have*

$$\begin{aligned}&\int_0^\infty \frac{e^{-st}}{t^{3/2}}\gamma\left(n+1,xt\right)dt \\ =\;& -2\sqrt{\pi}\,n!\left[\sqrt{s}-\sqrt{s+x}\sum_{k=0}^{n}\frac{\left(-\frac{1}{2}\right)_k}{k!}\left(\frac{x}{x+s}\right)^k\right],\end{aligned}\tag{2.15}$$

where $\gamma\left(\nu,z\right)$ denotes the lower incomplete gamma function [20, Chap. 45].

Proof Indeed, take $a_1=1+n$, $\alpha=\frac{1}{2}+n$ and $b_1=2+n$ in (1.14), consider the result (2.3), as well as [20, Eqn. 43:4:3]

$$\Gamma\left(n+\frac{1}{2}\right)=\left(\frac{1}{2}\right)_n\sqrt{\pi},$$

to obtain

$$\begin{aligned}&\int_0^\infty e^{-st}t^{n-1/2}\,{}_1F_1\left(\left.\begin{array}{c}1+n\\2+n\end{array}\right|xt\right)dt \\ =\;& \frac{2\sqrt{\pi}\left(-1\right)^n\left(n+1\right)!}{x^{n+1}}\left[\sqrt{s}-\sqrt{s-x}\sum_{k=0}^{n}\frac{\left(-\frac{1}{2}\right)_k}{k!}\left(\frac{x}{x-s}\right)^k\right].\end{aligned}\tag{2.16}$$

However, according to [24, Eqn. 7.11.1(13)], we have

$${}_1F_1\left(\left.\begin{array}{c}n\\1+n\end{array}\right|z\right)=\frac{\left(-1\right)^n n!}{z^n}\left[1-e^{-z}\sum_{k=0}^{n-1}\frac{\left(-1\right)^k z^k}{k!}\right],\tag{2.17}$$

and [20, Eqns. 45:4:2&26:12:2], we have as well

$$\Gamma\left(n,z\right)=\left(n-1\right)!\,e^{-z}e_{n-1}\left(z\right)=\left(n-1\right)!\,e^{-z}\sum_{k=0}^{n-1}\frac{z^k}{k!},\tag{2.18}$$

where $\Gamma\left(\nu,z\right)$ denotes the *upper incomplete gamma function* and $e_n\left(z\right)$ is the *exponential polynomial.* Therefore, from (2.17) and (2.18), and taking into account that the *lower incomplete gamma function* satisfies [20, Eqn. 45:0:1]

$$\gamma\left(\nu,z\right)=\Gamma\left(\nu\right)-\Gamma\left(\nu,z\right),$$

we conclude that

$${}_1F_1\left(\left.\begin{array}{c}n\\1+n\end{array}\right|z\right)=n\left(-z\right)^{-n}\gamma\left(n,-z\right),\tag{2.19}$$

hence, inserting (2.19) in (2.16), we arrive at (2.15), as we wanted to prove. □

Remark 2.7 *It is worth noting that we can obtain also (2.15) from [23, Eqn. 2.10.3(2)] and (2.3).*

2.2. Second differentiation formula

Definition 2.8 (Regularized hypergeometric function)

$$ {}_p\tilde{F}_q\left(\begin{array}{c} a_1,\ldots,a_p \\ b_1,\ldots,b_q \end{array}\middle| z\right) = \sum_{k=0}^{\infty} \frac{(a_1)_k \cdots (a_p)_k}{\Gamma(b_1+k)\cdots\Gamma(b_q+k)} \frac{z^k}{k!}. \tag{2.20} $$

When $p \leq q+1$ and z is fixed and not a branch point, (2.20) is an entire function of each of the parameters $a_1,\ldots,a_p,b_1,\ldots,b_q$ (see [21, Eqn. 15.2.2]).

Theorem 2.9 *For $n=0,1,2,\ldots$ and $t\in\mathbb{C}\backslash\{1\}$,*

$$ {}_2\tilde{F}_1\left(\begin{array}{c} \frac{1}{2},1 \\ 1-n \end{array}\middle| t\right) = \frac{\left(\frac{1}{2}\right)_n}{\sqrt{1-t}}\left(\frac{t}{1-t}\right)^n. \tag{2.21} $$

Proof In [21, Eqn. 15.5.4], we found the differentiation formula:

$$ \frac{d^n}{dt^n}\left[t^{c-1}\,{}_2F_1\left(\begin{array}{c} a,b \\ c \end{array}\middle| t\right)\right] = (c-n)_n\, t^{c-n-1}\,{}_2F_1\left(\begin{array}{c} a,b \\ c-n \end{array}\middle| t\right), \tag{2.22} $$
$$ n=0,1,2,\ldots $$

thus taking $a=\frac{1}{2}$, $b=1$ and $c=2$ in (2.22) and considering (2.2), we have

$$ 2\frac{d^{n+1}}{dt^{n+1}}\left[1-\sqrt{1-t}\right] = \frac{1}{\Gamma(1-n)\,t^n}\,{}_2F_1\left(\begin{array}{c} \frac{1}{2},1 \\ 1-n \end{array}\middle| t\right). \tag{2.23} $$

Apply (1.8)-(1.9), and the definition of the *regularized hypergeometric function* given in (2.20) in order to rewrite (2.23) as (2.21), as we wanted to prove. □

Remark 2.10 *According to (1.12), note that for $t=1$, both sides of (2.21) are divergent.*

Remark 2.11 *We can also derive (2.21) taking $a=1$, $k=n+1$, and $z=1-t$ in (C.5), as well as applying the formula [20, 2:12:3].*

$$ (2n)! = 4^n\, n!\,(1/2)_n. \tag{2.24} $$

2.3. Third differentiation formula

Theorem 2.12 *For $n=0,1,2,\ldots$ and $t\in\mathbb{C}$, the following reduction formula holds true:*

$$ {}_2F_1\left(\begin{array}{c} \frac{1}{2},1 \\ 2+n \end{array}\middle| t\right) \tag{2.25} $$
$$ = \begin{cases} \dfrac{2\,(n+1)!}{\left(\frac{3}{2}\right)_n\sqrt{1-t}}\left(\dfrac{t-1}{t}\right)^{n+1}\left[1-\dfrac{1}{\sqrt{1-t}}\displaystyle\sum_{k=0}^{n}\frac{\left(\frac{1}{2}\right)_k}{k!}\left(\frac{t}{t-1}\right)^k\right], & t\neq 0,1, \\ \dfrac{2\,(n+1)}{2n+1}, & t=1, \\ 1, & t=0. \end{cases} $$

Proof For the case $t \neq 0,1$, set $a=\frac{1}{2}$, $b=1$ and $c=2$ in the differentiation formula [21, Eqn. 15.5.6],

$$\begin{aligned} & \frac{d^n}{dt^n}\left[(1-t)^{a+b-c}\, {}_2F_1\left(\begin{array}{c} a,b \\ c \end{array}\middle| t\right)\right] \\ = & \frac{(c-a)_n\,(c-b)_n}{(c)_n}\,(1-t)^{a+b-c-n}\, {}_2F_1\left(\begin{array}{c} a,b \\ c+n \end{array}\middle| t\right), \\ & n=0,1,2,\ldots \end{aligned} \tag{2.26}$$

and use the result (2.2) to arrive at

$$2\frac{d^n}{dt^n}\left[\frac{1}{t}\frac{1}{\sqrt{1-t}}-\frac{1}{t}\right] = \frac{\left(\frac{3}{2}\right)_n}{(n+1)\,(1-t)^{n+1/2}}\, {}_2F_1\left(\begin{array}{c} \frac{1}{2},1 \\ 2+n \end{array}\middle| t\right).$$

Apply now Leibniz's differentiation formula (1.11) and the differentiation formulas (1.7) and (1.10). After some algebra, we obtain (2.25), as we wanted to prove. For $t=1$, apply Gauss summation formula (1.12), to obtain

$${}_2F_1\left(\begin{array}{c} \frac{1}{2},1 \\ 2+n \end{array}\middle| 1\right) = \frac{2\,(n+1)}{2n+1}, \tag{2.27}$$

where (2.27) only converges for $\mathrm{Re}\left(\frac{1}{2}+n\right)>0$, i.e. for $n=0,1,\ldots$, as we wanted to prove. Finally, according to (1.3), we obtain the value of 1 for $t=0$. □

2.3.1. Other elementary representations

It is worth noting that we can provide other elementary representations for ${}_2F_1\left(1/2,1;2+n;t\right)$ by using known formulas given in existing literature.

Theorem 2.13 *For $n=0,1,2,\ldots$ and $t\in\mathbb{C}$, the following reduction formula holds true:*

$$\begin{aligned} & {}_2F_1\left(\begin{array}{c} \frac{1}{2},1 \\ 2+n \end{array}\middle| t\right) \\ = & \begin{cases} \dfrac{2\,(n+1)!}{\left(\frac{3}{2}\right)_n} \\ \left[\dfrac{2}{\sqrt{1-t}}\left(\dfrac{t-1}{t}\right)^{n+1} + \dfrac{1}{1-\sqrt{1-t}}\displaystyle\sum_{k=0}^{n}\frac{(n+1)_k}{k!\,2^{k+n}}\left(1-\frac{1}{\sqrt{1-t}}\right)^{k-n}\right], & t\neq 0,1, \\ \dfrac{2\,(n+1)}{2n+1}, & t=1, \\ 1, & t=0. \end{cases} \end{aligned} \tag{2.28}$$

Proof We need to prove (2.28) for $t\neq 0,1$. In [26], we found

$${}_2F_1\left(\begin{array}{c} \frac{\alpha}{2},\frac{\alpha+1}{2} \\ \alpha+n+1 \end{array}\middle| t\right) = \left(\frac{1+\sqrt{1-t}}{2}\right)^{-\alpha} {}_2F_1\left(\begin{array}{c} -n,\alpha \\ \alpha+n+1 \end{array}\middle| \frac{1-\sqrt{1-t}}{1+\sqrt{1-t}}\right),$$

hence for $\alpha=1$, we obtain

$${}_2F_1\left(\begin{array}{c} \frac{1}{2},1 \\ 2+n \end{array}\middle| t\right) = \frac{2}{1+\sqrt{1-t}}\, {}_2F_1\left(\begin{array}{c} -n,1 \\ 2+n \end{array}\middle| \frac{1-\sqrt{1-t}}{1+\sqrt{1-t}}\right).$$

Now apply [24, Eqn. 7.3.1(179)]

$$ {}_2F_1\left(\begin{array}{c} -n,1 \\ m \end{array}\middle| z\right) = \frac{-n!}{(m)_n\, z}\left(\frac{z-1}{z}\right)^{m-2}\left[(1-z)^{n+1} - \sum_{k=0}^{m-2}\frac{(n+1)_k}{k!}\left(\frac{z}{z-1}\right)^k\right], $$

taking $m = n+2$ and $z = \frac{1-\sqrt{1-t}}{1+\sqrt{1-t}}$. Knowing that $(n+2)_n = \frac{2^{2n}}{n+1}\left(\frac{3}{2}\right)_n$, after some algebra, we arrive at (2.28) for $t \neq 0,1$, as we wanted to prove. □

Theorem 2.14 *For $n = 0,1,2,\ldots$ and $t \in \mathbb{C}$, the following reduction formula holds true:*

$$ {}_2F_1\left(\begin{array}{c} \frac{1}{2},1 \\ 2+n \end{array}\middle| t\right) \tag{2.29} $$

$$ = \begin{cases} \dfrac{2\,(n+1)!}{\left(\frac{3}{2}\right)_n (-t)^{n+1}}\left[(1-t)^{n+1/2} - \displaystyle\sum_{k=0}^{n}\frac{\left(-n-\frac{1}{2}\right)_k}{k!}t^k\right], & t \neq 0,1, \\ \dfrac{2\,(n+1)}{2n+1}, & t = 1, \\ 1, & t = 0. \end{cases} $$

Proof We need to prove (2.29) for $t \neq 0,1$. For this purpose, apply (2.8) taking $m = n+2$ and $b = \frac{1}{2}$ and use (1.9) for $x = \frac{1}{2}$. □

Theorem 2.15 *For $n = 0,1,2,\ldots$ and $t \in \mathbb{C}$, the following reduction formula holds true:*

$$ {}_2F_1\left(\begin{array}{c} \frac{1}{2},1 \\ n+2 \end{array}\middle| t\right) \tag{2.30} $$

$$ = \begin{cases} \dfrac{2^{2n+1}\,(n+1)!}{1+\sqrt{1-t}}\left(\dfrac{1-t}{t}\right)^n \displaystyle\sum_{k=0}^{n}\frac{1}{k!\,(n-k)!\,(n+1+k)}\left(\frac{1-\sqrt{1-t}}{2\sqrt{1-t}}\right)^{n+k}, & t \neq 0,1, \\ \dfrac{2\,(n+1)}{2n+1}, & t = 1, \\ 1, & t = 0. \end{cases} $$

Proof We need to prove (2.30) for $t \neq 0,1$. For this purpose, take $a = 1$, $k = n$, and $z = 1-t$ in (C.2) to arrive at the desired result. □

2.3.2. Applications: Legendre function and definite integral

Next, we derive some results from the elementary representation of ${}_2F_1(1/2,1;2+n;t)$ given in (2.25).

Theorem 2.16 *For $n = 1,2,\ldots$ and $t \in \mathbb{C}\setminus\{1\}$, we have*

$$ P_{-n}^{-n}\left(\frac{1}{\sqrt{1-t}}\right) \tag{2.31} $$

$$ = \begin{cases} \dfrac{1}{2^n\left(\frac{1}{2}\right)_n}\left(\dfrac{t-1}{t}\right)^{n/2}\left[1 - \dfrac{1}{\sqrt{1-t}}\displaystyle\sum_{k=0}^{n-1}\frac{\left(\frac{1}{2}\right)_k}{k!}\left(\frac{t}{t-1}\right)^k\right], & t \neq 0, \\ 0, & t = 0. \end{cases} $$

where $P_\nu^\mu(z)$ denotes the Legendre function [7, Chap. III].

Proof For $t \neq 0, 1$, we found, in [24, Eqn. 7.3.1(101)],

$$ {}_2F_1\left(\begin{array}{c} a, a+\frac{1}{2} \\ c \end{array} \middle| t \right) = 2^{c-1}\Gamma(c)(-t)^{(1-c)/2}(1-t)^{(c-1)/2-a}\, P_{2a-c}^{1-c}\left(\frac{1}{\sqrt{1-t}}\right). \tag{2.32} $$

Therefore, taking $a = \frac{1}{2}$ and $c = 2+n$ in (2.32), and considering (2.25) and (1.9), we eventually arrive at (2.31), as we wanted to prove. For $t = 0$, apply the hypergeometric representation of the Legendre function [21, Eqn. 14.3.15]

$$ P_\nu^{-\mu}(x) = 2^{-\mu}\left(x^2-1\right)^{\mu/2} {}_2F_1\left(\begin{array}{c} \mu-\nu, \mu+\nu+1 \\ \mu+1 \end{array} \middle| \frac{1-x}{2} \right), $$

to conclude that $P_{-n}^{-n}(1) = 0$, for $n = 1, 2, \ldots$ □

Theorem 2.17 *For $n = 1, 2, \ldots$ and $x, p \in \mathbb{C}$, we have*

$$ \int_0^\infty \frac{e^{-pt}}{t^{1/2+n}}\gamma(n, xt)\, dt \tag{2.33} $$

$$ = \begin{cases} \dfrac{-\sqrt{\pi}\,(n-1)!\,(-p)^{n-1}}{\left(\frac{1}{2}\right)_n}\left[\sqrt{p} - \sqrt{p+x}\displaystyle\sum_{k=0}^{n-1}\frac{\left(\frac{1}{2}\right)_k}{k!}\left(-\frac{x}{p}\right)^k\right], & p \neq 0, \\ \dfrac{2\sqrt{\pi}\,x^{n-1/2}}{2n-1}, & p = 0. \end{cases} $$

Proof For $p \neq 0$, take $a_1 = 1$, $\alpha = \frac{1}{2}$ and $b_1 = 1+n$ in (1.14), consider the reduction formula of the Kummer function [24, Eqn. 7.11.1(14)], i.e.

$$ {}_1F_1\left(\begin{array}{c} 1 \\ 1+n \end{array} \middle| xt \right) = \frac{n\,e^{xt}}{(xt)^n}\gamma(n, xt), $$

and apply the result given in (2.25) and the property (1.9), to arrive after some algebra at (2.33), as we wanted to prove. For $p = 0$, rewrite the result obtained above as

$$ \int_0^\infty \frac{e^{-pt}}{t^{1/2+n}}\gamma(n, xt)\, dt $$

$$ = \frac{\sqrt{\pi}\,(n-1)!\,(-1)^n}{\left(\frac{1}{2}\right)_n}\left\{p^{n-1/2} - \sqrt{p+x}\left[\sum_{k=0}^{n-2}\frac{\left(\frac{1}{2}\right)_k}{k!}(-x)^k p^{n-1-k} + \frac{\left(\frac{1}{2}\right)_{n-1}}{(n-1)!}(-x)^k\right]\right\}, $$

and take $p = 0$, to obtain the desired result. □

Remark 2.18 *It is worth noting that we can obtain (2.33) from [23, Eqn. 2.10.3(2)] and (2.25).*

2.4. Fourth differentiation formula

Theorem 2.19 *For $n = 1, 2, \ldots$ and $t \in \mathbb{C}$, we have*

$$ {}_2\tilde{F}_1\left(\begin{array}{c} \frac{1}{2}-n, 1-n \\ 2-n \end{array} \middle| t \right) = \begin{cases} 2\left(\dfrac{1}{2}\right)_n t^{n-1}, & n \geq 1, \\ 1, & n = 1. \end{cases}, \tag{2.34} $$

Proof Set $a = \frac{1}{2}$, $b = 1$ and $c = 2$ in the differentiation formula [21, Eqn. 15.5.9],

$$\begin{aligned} & \frac{d^n}{dt^n}\left[t^{c-1}(1-t)^{a+b-c}\,{}_2F_1\left(\begin{array}{c} a,b \\ c \end{array}\middle| t\right)\right] \\ = \;& (c-n)_n\, t^{c-n-1}(1-t)^{a+b-c-n}\,{}_2F_1\left(\begin{array}{c} a-n,b-n \\ c-n \end{array}\middle| t\right), \\ & n = 0,1,2,\ldots \end{aligned} \tag{2.35}$$

and apply the result given in (2.2), to obtain

$$2\frac{d^n}{dt^n}\left(\frac{1}{\sqrt{1-t}}-1\right) = \frac{t^{1-n}(1-t)^{-1/2-n}}{\Gamma(2-n)}\,{}_2F_1\left(\begin{array}{c} \frac{1}{2}-n,1-n \\ 2-n \end{array}\middle| t\right).$$

According to (1.10) for $n \geq 1$ and the definition of the regularized generalized hypergeometric function given in (2.20), we finally get (2.34). For $t = 0$ and $n = 1$, we obtain an indeterminate expression. However, according to (1.3), we have that

$${}_2F_1\left(\begin{array}{c} a,0 \\ b \end{array}\middle| t\right) = 1,$$

thus we obtain the desired result for $n = 1$. ☐

Remark 2.20 *We can also derive (2.34) taking $a = 1-2n$, $k = n$, and $z = 1-t$ in (C.2), and applying (2.24).*

2.4.1. Application: Legendre function

Theorem 2.21 *The following identity holds true for $n = 1,2,\ldots$ and $t \in \mathbb{C}$,*

$$P_n^{n-1}(t) = -(-2)^n\left(\frac{1}{2}\right)_n t\left(1-t^2\right)^{(n-1)/2}. \tag{2.36}$$

Proof Set $a = \frac{1}{2} - n$ and $c = 2 - n$ in (2.32), and take into account (2.34), to obtain

$$P_{-n-1}^{n-1}\left(\frac{1}{\sqrt{1-t}}\right) = -\frac{(-2)^n\left(\frac{1}{2}\right)_n}{\sqrt{1-t}}\left(\frac{t}{t-1}\right)^{(n-1)/2},$$

which, according to the property [7, Eqn. 3.3.1(1)]:

$$P_{-\nu-1}^{\mu}(z) = P_{\nu}^{\mu}(z), \tag{2.37}$$

is equivalent to (2.36). ☐

3. Case $m = 3$

In this case, (1.1) becomes

$$x^3 - x + t = 0. \tag{3.1}$$

In order to solve (3.1), we apply the solution of the cubic equation given in Appendix A, considering in (A.1) the negative sign '$-$', $m=\frac{1}{3}$ and $n=\frac{t}{2}$, i.e.

$$x_3(t)=\frac{1}{\sqrt{3}}\begin{cases}\cosh\left(\frac{1}{3}\cosh^{-1}\sqrt{z}\right)-i\sqrt{3}\sinh\left(\frac{1}{3}\cosh^{-1}\sqrt{z}\right), & z\geq 1,\\ \cos\left(\frac{1}{3}\cos^{-1}\sqrt{z}\right)-\sqrt{3}\sin\left(\frac{1}{3}\cos^{-1}\sqrt{z}\right), & z\leq 1.\end{cases}\tag{3.2}$$

where $z=3\left(\frac{3t}{2}\right)^2$. Therefore, from (1.5) and (3.2) we have

$$\begin{aligned}&{}_2F_1\left(\begin{array}{c}\frac{1}{3},\frac{2}{3}\\ \frac{3}{2}\end{array}\middle| z\right)\qquad(3.3)\\ =\;&\frac{3}{2\sqrt{z}}\begin{cases}\cosh\left(\frac{1}{3}\cosh^{-1}\sqrt{z}\right)-i\sqrt{3}\sinh\left(\frac{1}{3}\cosh^{-1}\sqrt{z}\right), & z\geq 1,\\ \cos\left(\frac{1}{3}\cos^{-1}\sqrt{z}\right)-\sqrt{3}\sin\left(\frac{1}{3}\cos^{-1}\sqrt{z}\right), & z\leq 1.\end{cases}\end{aligned}$$

Note that we can simplify (3.3) considering that

$$\begin{aligned}\frac{3}{\sqrt{z}}\sin\left(\frac{1}{3}\sin^{-1}\sqrt{z}\right)&=\frac{3}{\sqrt{z}}\sin\left(\frac{\pi/2-\cos^{-1}\sqrt{z}}{3}\right)\\ &=\frac{3}{2\sqrt{z}}\left\{\cos\left(\frac{\cos^{-1}\sqrt{z}}{3}\right)-\sqrt{3}\sin\left(\frac{\cos^{-1}\sqrt{z}}{3}\right)\right\}.\end{aligned}$$

Since

$$\cos^{-1}x=\begin{cases}i\cosh^{-1}x, & x\geq 1,\\ -i\cosh^{-1}x, & x\leq 1,\end{cases}$$

and $\cos(ix)=\cosh x$, and $\sin(ix)=i\sinh x$, we conclude that $\forall z\in\mathbb{C}$,

$${}_2F_1\left(\begin{array}{c}\frac{1}{3},\frac{2}{3}\\ \frac{3}{2}\end{array}\middle| z\right)=\frac{3}{\sqrt{z}}\sin\left(\frac{1}{3}\sin^{-1}\sqrt{z}\right).\tag{3.4}$$

The result given (3.4) can be obtained from [7, Eqn. 2.8(12)]:

$${}_2F_1\left(\begin{array}{c}\frac{1+a}{2},\frac{1-a}{3}\\ \frac{3}{2}\end{array}\middle| \sin^2 z\right)=\frac{\sin a\,z}{a\sin z},$$

taking $a=\frac{1}{3}$. Nonetheless, by differentiation, we obtain from (3.4) the following interesting identity.

Theorem 3.1 *For $n=1,2,\ldots$ and $z\in\mathbb{C}\backslash\{0,1\}$, we have:*

$$\begin{aligned}&{}_2\tilde{F}_1\left(\begin{array}{c}\frac{1}{3},\frac{2}{3}\\ \frac{3}{2}-n\end{array}\middle| z\right)&(3.5)\\ =\;&\frac{6\,z^{n-1/2}}{\sqrt{\pi}}\frac{d^n}{dz^n}\left[\sin\left(\frac{1}{3}\sin^{-1}\sqrt{z}\right)\right]&(3.6)\\ =\;&\frac{6\,z^{n-1/2}}{\sqrt{\pi}}\sum_{k=1}^{n}\sin\left(\frac{\sin^{-1}\sqrt{z}}{3}+\frac{\pi k}{2}\right)B_{n,k}\left(h_1(z),\ldots,h_{n-k+1}(z)\right),&(3.7)\end{aligned}$$

where $B_{n,k}(x_1,\ldots,x_{n-k+1})$ *denotes the Bell polynomial [5, p. 133]. Also, we have defined*

$$h_s(z)=\frac{(-i)^{s-1}(s-1)!}{6[z(1-z)]^{s/2}}\,P_{s-1}\left(\frac{1-2z}{2\sqrt{z(z-1)}}\right),$$

where $P_n(x)$ *is a Legendre polynomial.*

Proof Set $a=\frac{1}{3}$, $b=\frac{2}{3}$, and $c=\frac{3}{2}$ in (2.22) to obtain

$$\frac{1}{3}\frac{d^n}{dz^n}\left[\sqrt{z}\,{}_2F_1\left(\left.\begin{array}{c}\frac{1}{3},\frac{2}{3}\\ \frac{3}{2}\end{array}\right|z\right)\right]=\frac{\sqrt{\pi}\,z^{1/2-n}}{6}\,{}_2\tilde{F}_1\left(\left.\begin{array}{c}\frac{1}{3},\frac{2}{3}\\ \frac{3}{2}-n\end{array}\right|z\right),\tag{3.8}$$

and substitute (3.4) in (3.8), to get

$${}_2\tilde{F}_1\left(\left.\begin{array}{c}\frac{1}{3},\frac{2}{3}\\ \frac{3}{2}-n\end{array}\right|z\right)=\frac{6\,z^{n-1/2}}{\sqrt{\pi}}\frac{d^n}{dz^n}\left[\sin\left(\frac{1}{3}\sin^{-1}\sqrt{z}\right)\right].\tag{3.9}$$

In order to calculate the n-th derivative given in (3.9), we apply Faà di Bruno's formula [5, p. 137]:

$$\frac{d^n}{dz^n}f[g(z)]=\sum_{k=1}^{n}f^{(k)}[g(z)]\;B_{n,k}\left(g'(z),g''(z),\ldots,g^{(n-k+1)}(z)\right),\tag{3.10}$$

Set $f(z)=\sin z$ and $g(z)=\frac{1}{3}\sin^{-1}\sqrt{z}$ in (3.10) and take into account the differentiation formula [3, Eqn. 1.1.7(7)]:

$$\begin{aligned}&\frac{d^n}{dz^n}\sin^{-1}\left(a\sqrt{z}\right)=\frac{(-i)^{n-1}}{2}(n-1)!a^n\left(z-a^2z^2\right)^{-n/2}P_{n-1}\left(\frac{1-2a^2z}{2a\sqrt{a^2z^2-z}}\right),\\ &n\geq 1,\end{aligned}$$

to arrive at (3.5), as we wanted to prove. □

Remark 3.2 *On the one hand, according to the Gauss summation formula (1.12), the regularized hypergeometric function is given in (3.5) is divergent for* $z=1$ *and* $n=1,2,\ldots$ *On the other hand, for* $z=0$*, (3.6) and (3.7) yield indeterminate expressions. However, according to (1.3),* ${}_2F_1(1/2,2/3;3/2-n;0)=1$ *for* $n=1,2,\ldots$

Next, we provide the elementary representations of (3.5) for $n=1,2$:

$${}_2F_1\left(\left.\begin{array}{c}\frac{1}{3},\frac{2}{3}\\ \frac{1}{2}\end{array}\right|z\right)=\frac{\cos\left(\frac{1}{3}\sin^{-1}\sqrt{z}\right)}{\sqrt{1-z}},$$

and

$$\begin{aligned}&{}_2F_1\left(\left.\begin{array}{c}\frac{1}{3},\frac{2}{3}\\ -\frac{1}{2}\end{array}\right|z\right)\\ =&\;\frac{(3-6z)\cos\left(\frac{1}{3}\sin^{-1}\sqrt{z}\right)+\sqrt{z(1-z)}\sin\left(\frac{1}{3}\sin^{-1}\sqrt{z}\right)}{3(1-z)^{3/2}}.\end{aligned}$$

3.1. Other elementary representations

Theorem 3.3 *For $n = 1, 2, \ldots$ and $t \in \mathbb{C} \setminus \{0, 1\}$, the following elementary representation holds true:*

$$
\begin{aligned}
& {}_2F_1\left(\left. \begin{array}{c} \frac{1}{3}, \frac{2}{3} \\ \frac{3}{2} - n \end{array} \right| z \right) \qquad (3.11) \\
= & \frac{\left(\frac{1}{2}\right)_{n-1}}{\left(\frac{5}{3} - n\right)_{n-1}} \sum_{p=0}^{n-1} \sum_{q=0}^{n-1} \frac{2^{p+q} (1-n)_p (1-n)_q \left(\frac{7}{3} - 2n\right)_{p+q}}{(2-2n)_p (2-2n)_q \, p! q!} \\
& z^{p/2} (1-z)^{(q+1)/2-n} \cos\left(\left(\frac{7}{3} - 2n + p + q \right) \sin^{-1} \sqrt{z} - \frac{\pi}{2} p \right).
\end{aligned}
$$

Proof Take $a = \frac{7}{3} - 2n$, $k = n-1$, $\ell = n-1$, and $z = \sin^2 x$ in [26, Theorem 6.1],

$$
\begin{aligned}
& \frac{\left(\frac{a+1}{2}\right)_\ell}{\left(\frac{1}{2}\right)_\ell} {}_2F_1\left(\left. \begin{array}{c} \frac{a}{2}, -\frac{a}{2} - k - \ell \\ \frac{1}{2} - k \end{array} \right| \sin^2 x \right) \\
= & \sum_{p=0}^{k} \sum_{q=0}^{\ell} \frac{2^{p+q} (-k)_p (-\ell)_q (a)_{p+q}}{(-2k)_p (-2\ell)_q \, p! q!} \sin^p x \cos^q x \cos\left(ax + (p+q) x - \frac{\pi}{2} p \right),
\end{aligned}
$$

to arrive at

$$
\begin{aligned}
& \frac{\left(\frac{5}{3} - n\right)_{n-1}}{\left(\frac{1}{2}\right)_{n-1}} {}_2F_1\left(\left. \begin{array}{c} \frac{7}{6} - n, \frac{5}{6} - n \\ \frac{3}{2} - n \end{array} \right| z \right) \\
= & \sum_{p=0}^{n-1} \sum_{q=0}^{n-1} \frac{2^{p+q} (1-n)_p (1-n)_q \left(\frac{7}{3} - 2n\right)_{p+q}}{(2-2n)_p (2-2n)_q \, p! q!} \\
& z^{p/2} (1-z)^{q/2} \cos\left(\left(\frac{7}{3} - 2n + p + q \right) \sin^{-1} \sqrt{z} - \frac{\pi}{2} p \right).
\end{aligned}
$$

Finally, apply Euler transformation formula (2.6), to obtain (3.11), as we wanted to prove. □

3.2. Application: Definite integral

Theorem 3.4 *For $\mathrm{Re}\,(2p - x) > 0$, the following infinite integral holds true:*

$$
\begin{aligned}
& \int_0^\infty \frac{e^{-pt}}{t^{5/6}} D_{1/3}\left(-\sqrt{2xt} \right) dt \qquad (3.12) \\
= & \frac{2\,\Gamma\left(\frac{1}{3}\right)}{(2p+x)^{1/6}} \left[\cos\left(\frac{1}{3} \cos^{-1} \sqrt{\frac{2x}{2p+x}} \right) - \sin\left(\frac{1}{3} \sin^{-1} \sqrt{\frac{2x}{2p+x}} \right) \right],
\end{aligned}
$$

where $D_\nu(z)$ denotes the parabolic cylinder function [8, Chap. VIII].

Proof Set $a_1 = \frac{1}{3}$, $b_1 = \frac{3}{2}$, and $\alpha = \frac{2}{3}$ in (1.14)[†], taking into account (3.4), to obtain

$$
\int_0^\infty \frac{e^{-st}}{t^{1/3}} {}_1F_1\left(\left. \begin{array}{c} \frac{1}{3} \\ \frac{3}{2} \end{array} \right| xt \right) dt = \frac{3\,\Gamma\left(\frac{2}{3}\right)}{s^{1/6}\sqrt{x}} \sin\left(\frac{1}{3} \sin^{-1} \sqrt{\frac{x}{s}} \right). \qquad (3.13)
$$

[†]It is worth noting that the other choice, i.e. $a_1 = \frac{2}{3}$ and $\alpha = \frac{1}{3}$, leads to nonconvergent integrals.

Apply now the following formula with $a=\frac{1}{3}$ [24, Eqn. 7.11.1(10)]:

$$ {}_1F_1\left(\begin{array}{c} a \\ \frac{3}{2} \end{array}\middle| z\right) = \frac{2^{a-5/2}}{\sqrt{\pi}\,z}\Gamma\left(a-\frac{1}{2}\right)e^{z/2}\left[D_{1-2a}\left(-\sqrt{2z}\right)-D_{1-2a}\left(\sqrt{2z}\right)\right], $$

hence the RHS of (3.13) becomes:

$$ \begin{aligned} &\int_0^\infty \frac{e^{-st}}{t^{1/3}}\,{}_1F_1\left(\begin{array}{c} \frac{1}{3} \\ \frac{3}{2} \end{array}\middle| xt\right)dt \qquad (3.14) \\ =\;& \frac{2^{-13/6}}{\sqrt{\pi}\,x}\Gamma\left(\frac{-1}{6}\right)\left[\int_0^\infty \frac{e^{-(s-x/2)t}}{t^{5/6}}D_{1/3}\left(-\sqrt{2xt}\right)dt-\int_0^\infty \frac{e^{-(s-x/2)t}}{t^{5/6}}D_{1/3}\left(\sqrt{2xt}\right)dt\right]. \end{aligned} $$

Consider now the infinite integral [8, Eqn. 8.3(11)]:

$$ \int_0^\infty \frac{e^{-zt}}{t^{1-\beta/2}}D_{-\nu}\left(2\sqrt{kt}\right)dt = \frac{2^{1-\beta-\nu/2}\sqrt{\pi}\,\Gamma(\beta)}{\Gamma\left(\frac{\nu+\beta+1}{2}\right)(z+k)^{\beta/2}}\,{}_2F_1\left(\begin{array}{c} \frac{\nu}{2},\frac{\beta}{2} \\ \frac{\nu+\beta+1}{2} \end{array}\middle| \frac{z-k}{z+k}\right), $$

$$ \mathrm{Re}\,\beta>0,\mathrm{Re}\,z/k>0, $$

and the reduction formula [24, Eqn. 7.3.1(83)]:

$$ {}_2F_1\left(\begin{array}{c} a,-a \\ \frac{1}{2} \end{array}\middle| z\right) = \cos\left(2a\sin^{-1}\sqrt{z}\right), $$

to arrive at

$$ \int_0^\infty \frac{e^{-(s-x/2)t}}{t^{5/6}}D_{1/3}\left(\sqrt{2xt}\right)dt = \frac{2^{5/6}\Gamma\left(\frac{1}{3}\right)}{s^{1/6}}\cos\left(\frac{1}{3}\cos^{-1}\sqrt{\frac{x}{s}}\right). \qquad (3.15) $$

Therefore, taking into account (3.13)-(3.15), as well as [17, Eqns. 1.2.1&3]:

$$ \frac{\Gamma\left(\frac{2}{3}\right)}{\Gamma\left(-\frac{1}{6}\right)\Gamma\left(\frac{1}{3}\right)} = \frac{-1}{6\times 2^{1/3}\sqrt{\pi}}, $$

after some algebra, we conclude (3.12), as we wanted to prove. □

4. Case $m = 4$

In this case, (1.1) becomes

$$ x^4-x+t=0. \qquad (4.1) $$

To solve (4.1), we consider $p=0$, $q=-1$ and $r=t$ in the solution of the quartic equation given in Appendix B, i.e. (B.1). Thereby, (B.5) and (B.6) become

$$ \gamma = \frac{1}{2}\left(\alpha^2+\frac{1}{\alpha}\right), \qquad (4.2) $$

$$ \beta = \frac{t}{\gamma}. \qquad (4.3) $$

Therefore, setting $\xi = \alpha^2$, the resolvent cubic (B.4) is

$$\xi^3 - 4t\xi - 1 = 0,$$

which can be solved taking in (A.1) the '$-$' sign, $m = \frac{4t}{3}$ and $n = -\frac{1}{2}$. Thereby, according to (A.2) and (A.3), and defining $z = 4\left(\frac{4t}{3}\right)^3$, we arrive at

$$\xi(z) = \begin{cases} -2^{2/3} z^{1/6} \cosh\left(\frac{1}{3}\cosh^{-1}\left(\frac{-1}{\sqrt{z}}\right)\right), & z \leq 1, \\ -2^{2/3} z^{1/6} \cos\left(\frac{1}{3}\cos^{-1}\left(\frac{-1}{\sqrt{z}}\right)\right), & z \geq 1. \end{cases} \tag{4.4}$$

Note that both branches in (4.4) are equivalent, if we consider $z \in \mathbb{C}$, thus let us define the following function:

Definition 4.1

$$g(z) = -z^{1/6} \cosh\left(\frac{1}{3}\cosh^{-1}\left(\frac{-1}{\sqrt{z}}\right)\right). \tag{4.5}$$

By inspection, the solution of (1.1) for $n = 4$ corresponding to (1.6) is just the solution x_1 in (B.2), i.e.

$$x_1 = \frac{1}{2}\left(-\alpha + \sqrt{\alpha^2 - 4\beta}\right). \tag{4.6}$$

Therefore, from (1.6) on the one hand, and from (4.2)-(4.6) on the other hand, we finally obtain:

Theorem 4.2 *For $z \in \mathbb{C}$, we have*

$${}_3F_2\left(\left.\begin{array}{c} \frac{1}{4}, \frac{1}{2}, \frac{3}{4} \\ \frac{2}{3}, \frac{4}{3} \end{array}\right| z\right) = \frac{4}{3} z^{-1/3}\left[\sqrt{g(z) + \frac{3z^{1/3}\sqrt{g(z)}}{1 - 2\left[g(z)\right]^{3/2}}} - \sqrt{g(z)}\right]. \tag{4.7}$$

Remark 4.3 *We can calculate the LHS of (4.7) for $z = 1$, taking $a = \frac{1}{4}$, $c = \frac{1}{2}$ and $d = \frac{1}{3}$ in Whipple's sum (1.13), and using the gamma values given in [27], to obtain:*

$${}_3F_2\left(\left.\begin{array}{c} \frac{1}{4}, \frac{1}{2}, \frac{3}{4} \\ \frac{2}{3}, \frac{4}{3} \end{array}\right| 1\right) = \frac{\pi\,\Gamma\left(\frac{2}{3}\right)\Gamma\left(\frac{4}{3}\right)}{\Gamma\left(\frac{11}{24}\right)\Gamma\left(\frac{17}{24}\right)\Gamma\left(\frac{19}{24}\right)\Gamma\left(\frac{25}{24}\right)} = \frac{4}{3},$$

which is the result that we obtain on the RHS of (4.7).

4.1. Applications: Quadratic transformation and definite integral

Theorem 4.4 *For $|z| < 1$, we have*

$${}_3F_2\left(\left.\begin{array}{c} \frac{1}{2}, \frac{5}{6}, \frac{1}{6} \\ \frac{2}{3}, \frac{4}{3} \end{array}\right| z\right) = \frac{1}{\sqrt{1-z}} H\left(\frac{-4z}{(1-z)^2}\right), \tag{4.8}$$

where

$$H(t) = \frac{4}{3} t^{-1/3}\left[\sqrt{g(t) + \frac{3\,t^{1/3}\sqrt{g(t)}}{1 - 2\left[g(t)\right]^{3/2}}} - \sqrt{g(t)}\right].$$

Proof Take $\alpha = \frac{1}{4}$, $\lambda = \frac{1}{3}$, and $\mu = -\frac{1}{3}$ in the the quadratic transformation [14]:

$$\begin{aligned}
& {}_3F_2\left(\begin{array}{c} 2\alpha, 2\alpha+\lambda, 2\alpha+\mu \\ 1-\lambda, 1-\mu \end{array} \middle| x \right) \\
= \; & (1-x)^{-2\alpha} \, {}_3F_2\left(\begin{array}{c} \alpha, \alpha+\frac{1}{2}, 1-2\alpha-\lambda-\mu \\ 1-\lambda, 1-\mu \end{array} \middle| \frac{-4x}{(1-x)^2} \right),
\end{aligned}$$

to obtain,

$$ {}_3F_2\left(\begin{array}{c} \frac{1}{2}, \frac{5}{6}, \frac{1}{6} \\ \frac{2}{3}, \frac{4}{3} \end{array} \middle| x \right) = \frac{1}{\sqrt{1-x}} \, {}_3F_2\left(\begin{array}{c} \frac{1}{4}, \frac{3}{4}, \frac{1}{2} \\ \frac{2}{3}, \frac{4}{3} \end{array} \middle| \frac{-4x}{(1-x)^2} \right). \tag{4.9}$$

From (4.7) and (4.9), we arrive at (4.8), as we wanted to prove. □

Remark 4.5 *We can calculate the LHS of (4.8) for the branch point $z = 1$ taking $a = \frac{1}{6}$, $c = \frac{1}{2}$, and $d = \frac{2}{3}$ in (1.13), as well as the gamma values found in [27], resulting in*

$$ {}_3F_2\left(\begin{array}{c} \frac{1}{2}, \frac{5}{6}, \frac{1}{6} \\ \frac{2}{3}, \frac{4}{3} \end{array} \middle| 1 \right) = \frac{\pi \, \Gamma\left(\frac{2}{3}\right) \Gamma\left(\frac{4}{3}\right)}{\Gamma\left(\frac{5}{12}\right) \Gamma^2\left(\frac{3}{4}\right) \Gamma\left(\frac{13}{12}\right)} = 2^{3/2} \, 3^{-3/4}.$$

Theorem 4.6 *From the result (4.7), we obtain the following identity involving the product of two Legendre functions:*

$$\begin{aligned}
& P_{-1/6}^{1/3}\left(\sqrt{\frac{2}{1+\sqrt{1-z}}} \right) \, P_{-1/6}^{-1/3}\left(\sqrt{\frac{2}{1+\sqrt{1-z}}} \right) \\
= \; & \frac{\sqrt{6\left(1+\sqrt{1-z}\right)}}{\pi \; z^{1/3}} \left[\sqrt{g(z) + \frac{3 z^{1/3} \sqrt{g(z)}}{1-2\left[g(z)\right]^{3/2}}} - \sqrt{g(z)} \right].
\end{aligned} \tag{4.10}$$

Proof We found in the literature [24, Eqn. 7.4.1(10)]:

$$\begin{aligned}
& {}_3F_2\left(\begin{array}{c} a, 1-a, \frac{1}{2} \\ b, 2-b \end{array} \middle| z \right) \\
= \; & {}_2F_1\left(\begin{array}{c} a, 1-a \\ 2-b \end{array} \middle| \frac{1-\sqrt{1-z}}{2} \right) {}_2F_1\left(\begin{array}{c} a, 1-a \\ b \end{array} \middle| \frac{1-\sqrt{1-z}}{2} \right),
\end{aligned}$$

thus, taking $a = \frac{1}{4}$ and $b = \frac{3}{3}$, we have

$$\begin{aligned}
& {}_3F_2\left(\begin{array}{c} \frac{1}{4}, \frac{1}{2}, \frac{3}{4} \\ \frac{2}{3}, \frac{4}{3} \end{array} \middle| z \right) \\
= \; & {}_2F_1\left(\begin{array}{c} \frac{1}{4}, \frac{3}{4} \\ \frac{4}{3} \end{array} \middle| \frac{1-\sqrt{1-z}}{2} \right) {}_2F_1\left(\begin{array}{c} \frac{1}{4}, \frac{3}{4} \\ \frac{2}{3} \end{array} \middle| \frac{1-\sqrt{1-z}}{2} \right).
\end{aligned} \tag{4.11}$$

Also, setting $a=\frac{1}{4}$ and $c=\frac{2}{3},\frac{4}{3}$ in (2.32), we have

$$
\begin{aligned}
{}_2F_1\left(\left.\begin{array}{c}\frac{1}{4},\frac{3}{4}\\ \frac{2}{3}\end{array}\right| z\right) &= 2^{-1/3}\Gamma\left(\frac{2}{3}\right) z^{1/6}\left(1-z\right)^{-5/12} P_{-1/6}^{1/3}\left(\frac{1}{\sqrt{1-z}}\right), \qquad (4.12)\\
{}_2F_1\left(\left.\begin{array}{c}\frac{1}{4},\frac{3}{4}\\ \frac{4}{3}\end{array}\right| z\right) &= 2^{1/3}\Gamma\left(\frac{4}{3}\right) z^{-1/6}\left(1-z\right)^{-1/12} P_{-5/6}^{-1/3}\left(\frac{1}{\sqrt{1-z}}\right). \qquad (4.13)
\end{aligned}
$$

Therefore, inserting (4.12) and (4.13) in (4.11), taking into account the property (2.37), and knowing, according to [20, Eqn. 43:4:5], that $\Gamma\left(\frac{2}{3}\right)\Gamma\left(\frac{4}{3}\right)=\frac{2\pi}{3\sqrt{3}}$, we obtain (4.10), as we wanted to prove. □

5. Conclusions

We have considered the solution of $x^m-x+t=0$ for $m=2,3,4$, both in terms of hypergeometric functions as well as in terms of elementary functions. Thereby, we have obtained some reduction formulas of hypergeometric functions. In order to extend the latter results, we have applied the differentiation formulas (2.4), (2.22), (2.26) and (2.35), as well as the integration formula stated in (1.14). Consequently, we have derived new identities and infinite integrals involving special functions, i.e. the incomplete beta function, the lower incomplete gamma function, the parabolic cylinder function, and the Legendre function. Whenever possible, we have derived other elementary representations of the hypergeometric functions presented throughout the paper by applying formulas found in the literature, but not explicitly provided in it. All the results have been checked with MATHEMATICA and are available at https://bit.ly/2PyPz6Y.

A. The solution of the cubic equation

According to [18], in the solution of the depressed cubic equation:

$$
x^3 \pm 3mx + 2n = 0, \quad m>0, \qquad (A.1)
$$

we may distinguish the following cases:

Case I Sign '+' in (A.1). One real root and two complex roots:

$$
\begin{aligned}
x_1 &= -2\sqrt{m}\sinh\left(\frac{\sinh^{-1}\left(n\,m^{-3/2}\right)}{3}\right),\\
x_{2,3} &= \sqrt{m}\left[\sinh\left(\frac{\sinh^{-1}\left(n\,m^{-3/2}\right)}{3}\right)\pm i\sqrt{3}\cosh\left(\frac{\sinh^{-1}\left(n\,m^{-3/2}\right)}{3}\right)\right].
\end{aligned}
$$

Case II Sign '−' in (A.1) and $n^2-m^3>0$. One real root and two complex roots.

$$
\begin{aligned}
x_1 &= -2\sqrt{m}\cosh\left(\frac{\cosh^{-1}\left(n\,m^{-3/2}\right)}{3}\right), \qquad (A.2)\\
x_{2,3} &= \sqrt{m}\left[\cosh\left(\frac{\cosh^{-1}\left(n\,m^{-3/2}\right)}{3}\right)\pm i\sqrt{3}\sinh\left(\frac{\cosh^{-1}\left(n\,m^{-3/2}\right)}{3}\right)\right].
\end{aligned}
$$

Case III Sign '$-$' in (A.1) and $n^2 - m^3 < 0$. Three real roots.

$$\begin{aligned} x_1 &= -2\sqrt{m}\cos\left(\frac{\cos^{-1}\left(n\,m^{-3/2}\right)}{3}\right), \\ x_{2,3} &= \sqrt{m}\left[\cos\left(\frac{\cos^{-1}\left(n\,m^{-3/2}\right)}{3}\right) \pm \sqrt{3}\sin\left(\frac{\cos^{-1}\left(n\,m^{-3/2}\right)}{3}\right)\right]. \end{aligned} \tag{A.3}$$

B. The solution of the quartic equation

According to Descartes solution of the quartic equation [12], the four solutions of the depressed quartic equation:

$$x^4 + p\,x^2 + q\,x + r = 0, \tag{B.1}$$

are given by:

$$x_{1,2} = \frac{1}{2}\left(-\alpha \pm \sqrt{\alpha^2 - 4\beta}\right), \tag{B.2}$$

$$x_{3,4} = \frac{1}{2}\left(\alpha \pm \sqrt{\alpha^2 - 4\gamma}\right), \tag{B.3}$$

where α is a solution of the *resolvent bicubic equation*:

$$\alpha^6 + 2p\alpha^4 + (p - 4r)\,\alpha^2 - q^2 = 0, \tag{B.4}$$

and

$$\gamma = \frac{1}{2}\left(p + \alpha^2 + \frac{q}{\alpha}\right), \tag{B.5}$$

$$\beta = \frac{r}{\gamma}. \tag{B.6}$$

Note that the resolvent equation can be solved in α^2 with the solution described in Appendix A.

C. Finite sums

According to [26, Eqn. 3.1], we have

$$\begin{aligned} &{}_2F_1\left(\left.\begin{array}{c} \frac{a}{2}, \frac{a+1}{2} + \ell \\ a + k + \ell + 1 \end{array}\right| 1 - z\right) \\ &= z^{k/2}\left(\frac{1+\sqrt{z}}{2}\right)^{-a-k-\ell} F_3\left(\left.\begin{array}{c} k+1, \ell+1; -k, -\ell \\ a + k + \ell + 1 \end{array}\right| \frac{\sqrt{z}-1}{2\sqrt{z}}, \frac{1-\sqrt{z}}{2}\right), \end{aligned} \tag{C.1}$$

where the Apell F_3 bivariate function is defined as [21, Eqn. 16.13.3]:

$$F_3\left(\left.\begin{array}{c} a_1, a_2; b_1, b_2 \\ c \end{array}\right| x, y\right) = \sum_{p=0}^{\infty}\sum_{q=0}^{\infty} \frac{(a_1)_p\,(a_2)_q\,(b_1)_p\,(b_2)_q}{(c)_{p+q}\; p!q!}\, x^p\, y^q.$$

Proposition C.1 *For $k = 0, 1, 2, \ldots$ we have*

$$\begin{aligned} & {}_2F_1\left(\begin{array}{c} \frac{a}{2}, \frac{a+1}{2} \\ a+k+1 \end{array} \middle| 1-z \right) \\ = \; & z^{k/2} \left(\frac{2}{1+\sqrt{z}} \right)^{a+k} \sum_{p=0}^{k} \frac{(k+p)!}{p!\,(k-p)!\,(a+k+1)_p} \left(\frac{1-\sqrt{z}}{2\sqrt{z}} \right)^p . \end{aligned} \tag{C.2}$$

Proof Take $\ell = 0$ in (C.1) recalling that

$$(0)_n = \begin{cases} 1, & n = 0, \\ 0, & n \geq 1, \end{cases}$$

to obtain

$$\begin{aligned} & {}_2F_1\left(\begin{array}{c} \frac{a}{2}, \frac{a+1}{2} \\ a+k+1 \end{array} \middle| 1-z \right) \\ = \; & z^{k/2} \left(\frac{2}{1+\sqrt{z}} \right)^{a+k} \sum_{p=0}^{\infty} \frac{(k+1)_p\,(-k)_p}{p!\,(a+k+1)_p} \left(\frac{\sqrt{z}-1}{2\sqrt{z}} \right)^p , \end{aligned} \tag{C.3}$$

Now, apply the property [20, Eqn. 18:5:1]

$$(-x)_n = (-1)^n\,(x-n+1)_n\,, \tag{C.4}$$

to rewrite (C.3) as (C.2), as we wanted to prove. □

Proposition C.2 *For $k = 1, 2, \ldots$ we have*

$$\begin{aligned} & {}_2F_1\left(\begin{array}{c} \frac{a}{2}, \frac{a+1}{2} \\ a-k+1 \end{array} \middle| 1-z \right) \\ = \; & z^{-k/2} \left(\frac{2}{1+\sqrt{z}} \right)^{a-k} \sum_{p=0}^{k-1} \frac{(k+p-1)!}{p!\,(k-p-1)!\,(a-k+1)_p} \left(\frac{1-\sqrt{z}}{2\sqrt{z}} \right)^p . \end{aligned} \tag{C.5}$$

Proof Perform the substitution $k \to -k$ in (C.3) and apply again (C.4) to complete the proof. □

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