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Covariant generalized conserved charges of General Relativity

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Abstract: Motivated by the current research of generalized symmetries and the construction of conserved charges in pure Einstein gravity linearized over Minkowski spacetime in Cartesian coordinates, we investigate, from a purely classical point of view, the construction of these charges in a coordinate- and frame-independent language in order to generalize them further. We show that all the charges constructed in that context are associated to the conformal Killing-Yano 2-forms of Minkowski spacetime. Furthermore, we prove that those associated to closed conformal Killing-Yano 2-forms are identical to the charges constructed by Kastor and Traschen for their dual Killing-Yano $(d-2)$-forms. We discuss the number of independent and non-trivial gravitational charges that can be constructed in this way.

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1 Introduction

The definition of the conserved charges of the solutions of gravitational theories is a fascinating topic of research that touches the foundations of our current understanding of the gravitational field.[1] It is our current understanding, at least at a classical level, that the conserved charges of a spacetime that asymptotes to another one which is treated as the vacuum, (this is the definition of an isolated system in the gravitational setting) are associated to the isometries of that vacuum spacetime just as the constants of motion of a point particle moving in that vacuum spacetime are. In a sense, very far away from the strong gravity region where the system whose charges we want to compute lies, that system can be viewed as such a particle moving in the vacuum spacetime and we know how to define the charges of that particle.

Mathematically, this associates the conserved charges of gravitational systems to the Killing vectors of the vacuum spacetime. However, it has long been known that particles and fields evolving in a spacetime may have other conserved charges as well. The simplest

[1] See refs. [1–9] for a somewhat *ad hoc* selection of milestones in the history of this field of research.

example is provided by massless particles, which have conserved charges associated to the conformal Killing vectors of the spacetime. There are charges associated to Killing-Yano forms (or tensors) [10, 11] as well and, in general, to conformal Killing-Yano p-forms [12].[2]

According to the preceding discussion, it should be possible to define conserved charges to gravitating isolated systems associated to the conformal Killing-Yano p-forms of the asymptotic vacuum spacetime and, indeed, in ref. [9] Kastor and Traschen found a definition of off-shell conserved charges associated to the Killing-Yano p-forms of the asymptotic vacuum spacetime, which did not include the more general conformal Killing-Yano p-forms, though.

More recently and with different goals, the definition of the conserved charges of quantum gravitational systems has been discussed in refs. [20–23]. In these references and, without any reference to classical symmetries of the vacuum spacetime, conserved 2- and $(d-2)$-form charges were constructed in terms of the Riemann tensor linearized over Minkowski spacetime (playing the role of vacuum). Since this construction is not connected to previous works in the classical setting and may include new charges not considered so far in it, it is interesting to understand them better from that point of view.

Thus, in this paper, we want to review the definitions of conserved gravitational charges made in refs. [20–23] and relate them to already existing definitions in the classical realm. We will refer mostly to the construction in d dimensions made in ref. [23]. We would like to stress that our approach is purely classical and we will not be concerned with the implications of our results in the context of the algebraic approach to Quantum Gravity.

We will start by reviewing in section 2 the definitions made in ref. [23] to reformulate them in a coordinate- and frame-independent way. As we are going to see, it is natural and unavoidable to generalize those definitions to p-form charges and to consider also the p-form charged defined by Kastor and Traschen. Fortunately, we will be able to prove a Lemma relating many of those charges, simplifying their classification and interpretation. In section 3 we prove the conservation of those charges under different assumptions concerning the properties of the p-form parameters used. Essentially we will be able to extend the definition of the Kastor-Traschen charges to the case in which the parameters are more general conformal Killing-Yano p-forms. In section 4 we apply the results of the previous section to asymptotically-flat spacetimes using the conformal Killing-Yano p-forms of Minkowski spacetime. In particular, we will show that all the charges constructed in ref. [23] are associated to these mathematical objects. We present our conclusions and directions for future work in section 5.

2 Definition of the charges and their relations

In order to illustrate certain ideas about duality in the context of generalized charges in QFT, the authors of ref. [23] (based on previous work and ideas in ref. [20] and generalizing the 4-dimensional case studied in refs. [21, 22]) proposed a construction of all possible 2-

[2]For a review with many references focused on the construction of conserved quantities of particles and fields evolving in spacetimes admitting conformal Killing-Yano p-forms see ref. [13] and the more recent ref. [14]. Further and later results in this area can be found in refs. [15–19].

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and $(d-2)$-form charges of pure d-dimensional Einstein gravity linearized over Minkowski spacetime. The construction uses the Cartesian coordinates of the Minkowski background, which obscures their geometrical and physical meaning and their properties. In order to study them it is necessary to rewrite them in a fully coordinate- and frame-independent form first.

It is not difficult to see that all those charges are contractions of the linearized Riemann tensor (treated as a 2-form) and its dual (treated as a $(d-2)$-form) with different 2-forms whose properties, combined with those of the linearized Riemann tensor,[3] guarantee the closedness of the charges.

In order to rewrite these charges in a fully coordinate- and frame-independent way, it is simpler to work first in the context of the non-linear theory and linearize later on. Thus, it is natural to consider the following 2- and $(d-2)$-forms

$$\mathbf{Q}[\sigma] \equiv R^{ab}\sigma_{ab}\,, \tag{2.1a}$$

$$\tilde{\mathbf{Q}}[\sigma] \equiv \star R^{ab}\sigma_{ab} = \star\mathbf{Q}[\sigma]\,, \tag{2.1b}$$

where R^{ab} is the Lorentz curvature 2-form defined in eqs. (A.5), $\star R_{ab}$ is its Hodge dual $(d-2)$-form defined in (A.23), and σ_{ab} are the components of a 2-form σ

$$\sigma = \tfrac{1}{2}\sigma_{\mu\nu}dx^{\mu}\wedge dx^{\nu} = \tfrac{1}{2}\sigma_{ab}e^{a}\wedge e^{b}\,. \tag{2.2}$$

Before we consider the conservation of these charges, some comments are in order:

1. As we are going to see, the conservation of these charges only depends on the Bianchi identities and/or on the vacuum Einstein equations. Thus, we can replace the Riemann tensor by any other tensor sharing similar properties. That is the case of the Riemann tensor linearized over Minkowski spacetime, which we will consider in section 4.

2. Some of these charges may be total derivatives. At the classical level, when, for instance $\mathbf{Q}[\sigma] = d\mathbf{X}[\sigma]$ for some 1-form $\mathbf{X}[\sigma]$, integrating $\mathbf{Q}[\sigma]$ over a close 2-dimensional surface will always give zero. However, one can integrate $\mathbf{X}[\sigma]$ over closed curves to get non-vanishing values of charges that may be associated to strings, for instance. At the classical level, these charges are well defined as long as $\mathbf{X}[\sigma]$ is invariant up to total derivatives under the local symmetries of the theory. If we are dealing with linearized gravity, this includes the spin-2 gauge transformations.

   From the point of view of refs. [20–23] and in the context of linearized gravity, though, the relevant objects are the local operators $\mathbf{Q}[\sigma]$ and $\mathbf{X}[\sigma]$, $\mathbf{Q}[\sigma]$ is strictly gauge-invariant under the spin-2 gauge transformations because it depends on the gauge-invariant linearized Riemann tensor. $\mathbf{X}[\sigma]$, however, may or may not be a function of that tensor. When it is not, it will not be strictly gauge invariant and it should not be taken into account.

[3] One only needs to use the same Bianchi identities that the full non-linear Riemann tensor satisfies.

Since we are only interested in the classical charges, we will not consider these aspects and we will consider the charges obtained by integrating a 1-form $\mathbf{X}[\sigma]$ invariant up to total derivatives as a well-defined charge.

3. As it turns out, in the context of pure Einstein gravity, these charges can be considered as the on-shell[4] expressions of the more general charges that include terms proportional to the Ricci scalar and the Ricci 1-form, considered by Kastor and Traschen in ref. [9]:

$$\mathbf{Q}_{\mathrm{KT}}[\sigma] \equiv \imath_b \imath_a \left[R^{ab} \wedge \sigma\right] = \left(R^{ab}\sigma_{ab} - 2\imath_a R^{ab} \wedge \imath_b \sigma + \imath_b \imath_a R^{ab} \sigma\right) \doteq \mathbf{Q}[\sigma]\,, \tag{2.3a}$$

$$\tilde{\mathbf{Q}}_{\mathrm{KT}}[\sigma] \equiv \star \mathbf{Q}_{\mathrm{KT}}[\sigma] \doteq \tilde{\mathbf{Q}}[\sigma]\,, \tag{2.3b}$$

These charges were not considered in ref. [23] because, on-shell, they are equivalent to those in eqs. (2.1a) and (2.1b) but they can be relevant because it can be shown that $\tilde{\mathbf{Q}}_{\mathrm{KT}}[\sigma]$ is conserved off-shell when σ is a Killing-Yano 2-form (KY2F) [9] while its on-shell equivalent $\tilde{\mathbf{Q}}[\sigma]$ is only conserved on-shell [23]. In the context of our general exploration of the possible conserved charges it is natural to consider their duals $\mathbf{Q}_{\mathrm{KT}}[\sigma]$ as well.

4. The above charges eqs. (2.3a) and (2.3b) are particular cases of the p- and $(d-p)$-form charges defined by [9]

$$\mathbf{Q}_{\mathrm{KT}}[\sigma^{(p)}] \equiv \imath_b \imath_a \left[R^{ab} \wedge \sigma^{(p)}\right]\,, \tag{2.4a}$$

$$\tilde{\mathbf{Q}}_{\mathrm{KT}}[\sigma^{(p)}] \equiv \star \mathbf{Q}_{\mathrm{KT}}[\sigma^{(p)}]\,. \tag{2.4b}$$

where $\sigma^{(p)}$ is a p-form $(p > 0)$

$$\sigma^{(p)} = \frac{1}{p!}\sigma^{(p)}_{\mu_1\cdots\mu_p} dx^{\mu_1} \wedge \cdots \wedge dx^{\mu_p}\,. \tag{2.5}$$

The charges $\tilde{\mathbf{Q}}_{\mathrm{KT}}[\sigma^{(p)}]$ were shown to be conserved off-shell for $\sigma^{(p)}$s which are Killing-Yano p-forms (KYpFs) in ref. [9].

Notice that for $p = 1$

$$\mathbf{Q}_{\mathrm{KT}}[\sigma^{(1)}] = -2\sigma^{(1)\,a} G_{ab} e^b\,. \tag{2.6}$$

In its turn, this leads us to consider the p-form generalization of the charges eqs. (2.1a) and (2.1b)

$$\mathbf{Q}[\sigma^{(p)}] \equiv R^{ab} \wedge \imath_b \imath_a \sigma^{(p)}\,, \tag{2.7a}$$

$$\tilde{\mathbf{Q}}[\sigma^{(p)}] \equiv \star \mathbf{Q}[\sigma^{(p)}]\,, \tag{2.7b}$$

[4]We will denote identities which only hold on-shell with $\doteq$.

which are equivalent to the former on-shell in pure Einstein gravity:

$$\mathbf{Q}_{\mathrm{KT}}[\sigma^{(p)}] \doteq \mathbf{Q}[\sigma^{(p)}]\,, \tag{2.8a}$$

$$\tilde{\mathbf{Q}}_{\mathrm{KT}}[\sigma^{(p)}] \doteq \tilde{\mathbf{Q}}[\sigma^{(p)}]\,. \tag{2.8b}$$

Observe the definitions eqs. (2.7a) and (2.7b) require $p \geq 2$ to be non-trivial while the Kastor-Traschen charges can be defined for all p, but vanish on-shell for $p < 2$. We will not write a superscript for $p = 2$.

Thus, it may seem that we will have to study the charges $\mathbf{Q}[\sigma^{(p)}]$ and $\mathbf{Q}_{\mathrm{KT}}[\sigma^{(p)}]$ and their duals for all p, considering later the special case $p = 2$. However, the following lemma will allow us to place limits to the apparent proliferation of charges and focus only on the simplest of them, namely the $\mathbf{Q}[\sigma^{(p)}]$s and their duals.

Lemma. *For p- and $(d-p)$-form parameters related by*

$$\tilde{\sigma}^{(d-p)} = \star\sigma^{(p)}\,, \tag{2.9}$$

the charges (2.4b) and the charges eqs. (2.7a) are related by

$$\tilde{\mathbf{Q}}_{\mathrm{KT}}[\sigma^{(p)}] = \mathbf{Q}[\tilde{\sigma}^{(d-p)}]\,. \tag{2.10}$$

Proof. Using the property eq. (A.29) and the components of the curvature 2-form, we get

$$\begin{aligned}
\tilde{\mathbf{Q}}_{\mathrm{KT}}[\sigma^{(p)}] &= \star\imath_b\imath_a\left[R^{ab}\wedge\sigma^{(p)}\right] \\
&= e_a\wedge e_b\wedge\star\left[R^{ab}\wedge\sigma^{(p)}\right] \\
&= \tfrac{1}{2}R_{cd}{}^{ab}e_a\wedge e_b\wedge\star\left[e^c\wedge e^d\wedge\sigma^{(p)}\right] \\
&= R_{cd}\wedge\star\left[e^c\wedge e^d\wedge\sigma^{(p)}\right]\,,
\end{aligned} \tag{2.11}$$

where we have used the Bianchi identity eq. (A.7) in the last step. In order to use eq. (A.29) again, we Hodge-dualize twice $\sigma^{(p)}$ taking into account eq. (A.28)

$$\begin{aligned}
\tilde{\mathbf{Q}}_{\mathrm{KT}}[\sigma^{(p)}] &= (-1)^{p(d-p)}\det(\eta)R_{cd}\wedge\star\left[e^c\wedge e^d\wedge\star^2\sigma^{(p)}\right] \\
&= (-1)^{p(d-p)}\det(\eta)R_{cd}\wedge\star^2\left[\imath_d\imath_c\star\sigma^{(p)}\right] \\
&= R_{cd}\wedge\left[\imath_d\imath_c\star\sigma^{(p)}\right] \\
&= \mathbf{Q}[\tilde{\sigma}^{(d-p)}]\,,
\end{aligned} \tag{2.12}$$

quod erat demonstradum. □

This is an important result which, among other things, together with eqs. (2.8) implies the on-shell relation

$$\tilde{\mathbf{Q}}[\sigma^{(p)}] \doteq \mathbf{Q}[\tilde{\sigma}^{(d-p)}]\,. \tag{2.13}$$

For $p = 2$ in $d = 4$ it relates all the $\mathbf{Q}[\sigma]$ charges to their duals on-shell.

We conclude that it is enough to consider the conservation of the $\mathbf{Q}[\sigma^{(p)}]$s for all values of $p \geq 2$.

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3 The conservation of $Q[\sigma^{(p)}]$

We are going to study the conservation of the p-form charges $\mathbf{Q}[\sigma^{(p)}]$ defined in eq. (2.7a) assuming that $\sigma^{(p)}$ is a conformal Killing-Yano p-form (CKYpF).

By definition, a CKYpF satisfies an equation of the form (the CKYpF equation)[5]

$$\mathcal{D}_a \sigma^{(p)}_{b_1\cdots b_p} = \frac{1}{p+1}\aleph^{(p+1)}_{ab_1\cdots b_p} + (-1)^{d(p+1)}\frac{p}{(d-p+1)}\eta_{a[b_1}\xi^{(p-1)}_{b_2\cdots b_p]}\,, \tag{3.1}$$

for some $(p+1)$- and $(p-1)$-forms $\aleph, \xi$, which we can identify as

$$\aleph^{(p+1)} = d\sigma^{(p)}\,, \tag{3.2a}$$

$$\xi^{(p-1)} = \star d \star \sigma^{(p)}\,. \tag{3.2b}$$

If $\aleph^{(p+1)} = 0$, the CKYpF is a closed CKYpF (CCKYpF) and, if $\xi^{(p-1)} = 0$, it is a Killing-Yano p-form (KYpF). If both $\aleph^{(p+1)}$ and $\xi^{(p-1)}$ vanish, $\sigma^{(p)}$ is a covariantly constant CKYpF (CCCKYpF or, better, C3KYpF). This case is a particular sub-case of the CCKYpF and KYpF ones.

From the CKYpF equation we get

$$\mathcal{D}\imath_b\imath_a\sigma^{(p)} = \frac{p-1}{p+1}\imath_b\imath_a\aleph^{(p+1)}\,. \tag{3.3}$$

Then, using the Bianchi identity eq. (A.10) and the above equation we find

$$d\mathbf{Q}[\sigma^{(p)}] = \frac{p-1}{p+1}R^{ab}\wedge\imath_b\imath_a\aleph^{(p+1)}\,, \tag{3.4}$$

which vanishes trivially off-shell when $\sigma^{(p)}$ is a CCKYpF. When it is not closed, we can proceed as follows: we take the components of the dual of the above expression written in terms of the components of the dual of $\aleph^{(p+1)}$

$$\begin{aligned}\left(\star d\mathbf{Q}[\sigma^{(p)}]\right)_{c_1\cdots c_{d-p-1}} \sim\; & 2R^{f_1 f_2}{}_{f_1 f_2}\left(\star\aleph^{(p+1)}\right)_{c_1\cdots c_{d-p-1}} \\ & + 4(d-p-1)R^{f_1 f_2}{}_{[c_1|f_1}\left(\star\aleph^{(p+1)}\right)_{f_2|c_2\cdots c_{d-p-1}]} \\ & + (d-p-1)(d-p-2)R^{f_1 f_2}{}_{[c_1 c_2}\left(\star\aleph^{(p+1)}\right)_{c_3\cdots c_{d-p-1}]f_1 f_2}\,.\end{aligned} \tag{3.5}$$

The first two terms vanish on-shell for pure Einstein gravity but the third only vanishes in $d = p+1$ or $d = p+2$ dimensions.

Thus, we have shown that

1. $d\mathbf{Q}[\sigma^{(p)}] = 0$ for CCKYpFs.
2. $d\mathbf{Q}[\sigma^{(p)}] \doteq 0$ for KYpFs only in $d = (p+1)$-dimensional spacetimes whose Ricci scalar vanishes.

[5]The following definitions and properties of CKYpFs can be found in the review ref. [13] and referenced therein.

3. $d\mathbf{Q}[\sigma^{(p)}] \doteq 0$ for KYpFs only in $d = (p+2)$-dimensional Ricci-flat spacetimes.

At this point, the following results become relevant to the discussion [13]:

1. If $\sigma^{(p)}$ is a CKYpF, then $\tilde{\sigma}^{(d-p)} = \star\sigma^{(p)}$ is a CKY$(d-p)$F.
2. If $\sigma^{(p)}$ is a CCKYpF, then $\tilde{\sigma}^{(d-p)} = \star\sigma^{(p)}$ is a KY$(d-p)$F and vice-versa. This implies that, if $\sigma^{(p)}$ is a C3KYpF, then $\tilde{\sigma}^{(d-p)} = \star\sigma^{(p)}$ is a C3KYpF.
3. The wedge product of a CCKYpF and a CCKYqF is a CCKY$(p+q)$. Observe that a CKY1F is the (metric) dual of a conformal Killing vector (CKV).
4. The maximal number of CCKYpFs is

$$\frac{(d+1)!}{p! \cdot (d-p+1)!}\,. \tag{3.6}$$

Then, according to our main result eq. (2.10), there is a one-to-one relation between the off-shell conserved Kastor-Traschen-type p-forms $\tilde{\mathbf{Q}}_{\text{KT}}[\tilde{\sigma}^{(d-p)}]$ constructed with KY$(d-p)$Fs $\tilde{\sigma}^{(d-p)}$ and the off-shell conserved p-forms $\mathbf{Q}[\sigma^{(p)}]$ constructed with CCKYpFs $\sigma^{(p)}$. They are, actually, identical.

Furthermore, in $d = p+1, p+2$, one can construct one on-shell conserved Kastor-Traschen-type p-form $\tilde{\mathbf{Q}}_{\text{KT}}[\tilde{\sigma}^{(d-p)}]$ with a CCKY$(d-p)$Fs $\tilde{\sigma}^{(d-p)}$ for each on-shell conserved p-form $\mathbf{Q}[\sigma^{(p)}]$ constructed with a KYpFs $\sigma^{(p)}$. Again, they are identical.

4 Abbott-Deser currents and charges

Following refs. [2, 9] let us consider metrics g that asymptote to a given metric $\bar{g}$ so that near infinity we can express them as perturbations h over the background metric $\bar{g}$, i.e.

$$g_{\mu\nu} = \bar{g}_{\mu\nu} + \chi h_{\mu\nu}\,, \tag{4.1}$$

where $\chi^2 = 16\pi G_N^{(d)}$.

Since we have chosen to work with Vielbein we have to study the linearization of the Vielbein, spin connection and Lorentz curvature tensor in this formalism first.[6]

4.1 Linearized gravity in the Vielbein formalism

The Vielbein $e^a = e^a{}_\mu dx^\mu$ satisfies the relations

$$\eta_{ab} e^a{}_\mu e^b{}_\nu = g_{\mu\nu}\,, \quad \text{and} \quad e^a{}_\mu e^b{}_\nu g^{\mu\nu} = \eta^{ab}\,, \tag{4.2}$$

and the background Vielbein field $\bar{e}^a = \bar{e}^a{}_\mu dx^\mu$ is assumed to satisfy analogous relations with respect to the background metric $\bar{g}_{\mu\nu}$ and to the same tangent space metric η_{ab}.

Then, we define the perturbation of the Vielbein $f^a = f^a{}_\mu dx^\mu$ by

$$e^a = \bar{e}^a + \frac{\chi}{2} f^a\,. \tag{4.3}$$

[6] The Abbott-Deser and Kastor-Traschen charges have been written in the Vielbein formalism in refs. [25, 26].

By definition,

$$g_{\mu\nu} = \bar{g}_{\mu\nu} + \chi \eta_{ab} \bar{e}^{\,a}{}_{(\mu} f^{b}{}_{\nu)} + \mathcal{O}(\chi^2)\,, \tag{4.4}$$

which requires, for consistency

$$\eta_{ab} \bar{e}^{\,a}{}_{(\mu} f^{b}{}_{\nu)} = f_{(\mu\nu)} = h_{\mu\nu} + \mathcal{O}(\chi)\,. \tag{4.5}$$

We use the background Vielbein field to convert tangent space indices into world indices and vice-versa, the background metric to raise and lower world indices and the flat Minkowski metric to raise and lower tangent space indices.

Observe that $f_{\mu\nu}$ has an antisymmetric part

$$\eta_{ab} \bar{e}^{\,a}{}_{[\mu} f^{b}{}_{\nu]} = f_{[\mu\nu]} \equiv b_{\mu\nu}\,, \tag{4.6}$$

which does not vanish in general.

Expanding the spin connection as

$$\omega^{a}{}_{b} = \bar{\omega}^{a}{}_{b} + \chi \omega_{L}{}^{a}{}_{b} + \mathcal{O}(\chi^2)\,, \tag{4.7}$$

the first Cartan structure equation with zero torsion leads to

$$\omega_{L\,ab} = \omega_{L\,cab} \bar{e}^{\,c} = \tfrac{1}{4}\left\{\imath_{\bar{c}}\imath_{\bar{a}}\bar{\mathcal{D}} f_b - \imath_{\bar{a}}\imath_{\bar{b}}\bar{\mathcal{D}} f_c + \imath_{\bar{b}}\imath_{\bar{c}}\bar{\mathcal{D}} f_a\right\}\bar{e}^{\,c}\,, \tag{4.8}$$

where the inner products $\imath_{\bar{a}}$ are taken with the background vector fields $\bar{e}_a = \bar{e}_a{}^{\mu}\partial_\mu$. In components, we have

$$\bar{\mathcal{D}} f_b = \bar{\mathcal{D}}_a f_{bc} \bar{e}^{\,a} \wedge \bar{e}^{\,c}\,, \quad \Rightarrow \quad \imath_{\bar{a}}\imath_{\bar{c}}\bar{\mathcal{D}} f_b = 2\bar{\mathcal{D}}_{[c|} f_{b|a]}\,, \tag{4.9}$$

and we find that the linearized connection is given by

$$\omega_{L\,ab} = \tfrac{1}{2}\left\{\bar{\mathcal{D}}_{[a} f_{b]} + \bar{\mathcal{D}}_{[a|} f_{c|b]} \bar{e}^{\,c} + \bar{\mathcal{D}} f_{[ab]}\right\}\,, \tag{4.10}$$

or

$$\omega_{L\,ab} = \tfrac{1}{2}\left\{\bar{\mathcal{D}}_a f_{(bc)} - \bar{\mathcal{D}}_b f_{(ac)} + \bar{\mathcal{D}}_c f_{[ab]}\right\}\bar{e}^{\,c}\,, \tag{4.11}$$

just to show that it does not depend on the symmetric part of f_{ab} only [27].

The linearized curvature tensor follows from the Palatini identity

$$R_{L\,ab} = \bar{\mathcal{D}}\omega_{L\,ab} = \tfrac{1}{2}\bar{\mathcal{D}}\left\{\bar{\mathcal{D}}_{[a} f_{b]} + \bar{\mathcal{D}}_{[a|} f_{c|b]} \bar{e}^{\,c} + \bar{\mathcal{D}} f_{[ab]}\right\}\,, \tag{4.12}$$

and its components are

$$R_{L\,cd\,ab} \overset{[cd]\,[ab]}{=} \tfrac{1}{2}\bar{\mathcal{D}}_c\left\{\bar{\mathcal{D}}_a f_{bd} + \bar{\mathcal{D}}_a f_{db} + \bar{\mathcal{D}}_d f_{ab}\right\}\,, \tag{4.13}$$

where the notation $\overset{[cd]\,[ab]}{=}$ indicates that the pairs of indices cd and ab are antisymmetrized in right-hand side. Then,

$$R_{L\,ab} \wedge \bar{e}^{\,a} = \tfrac{1}{2}\bar{R}_b{}^{c} \wedge f_c\,, \tag{4.14}$$

after use of the Bianchi identity eq. (A.6) for the Riemann tensor of the background spacetime. Furthermore,

$$\bar{\mathcal{D}}R_L{}^{ab} = -2\bar{R}^{[a|}{}_c \wedge \omega_L{}^{c|b]}\,. \tag{4.15}$$

Hence, for flat background spacetime ($\bar{R}^{ab}=0$), we recover the Bianchi identities

$$\bar{\mathcal{D}}R_L{}^{ab} = 0\,, \tag{4.16a}$$

$$R_{L\,ab} \wedge \bar{e}^{\,a} = 0\,, \tag{4.16b}$$

which we have used to construct the conserved charges in the previous section. and, therefore, we will restrict ourselves to that case from now onwards.

We also have (using eqs. (A.24) and (A.25))

$$\bar{\mathcal{D}}\bar{\star}R_L{}^{ab} = (-1)^{d-1}\bar{\mathcal{D}}_d R_{L\,ab}{}^{cd}\imath_{\bar{c}}\bar{\omega}\,, \tag{4.17a}$$

$$\bar{\star}R_{L\,ab} \wedge e^a = R_{\mathrm{ic}\,L}{}^e{}_b \imath_{\bar{e}}\bar{\omega}\,, \tag{4.17b}$$

both of which vanish on-shell for pure (linearized) gravity.

Under diffeomorphisms generated by vector fields $\xi = \bar{\xi} + \frac{\chi}{2}\epsilon$, and local Lorentz transformations generated by parameters $\sigma^{ab} = \bar{\sigma}^{\,ab} + \frac{\chi}{2}s^{ab}$ to lowest order in χ we find that

$$\delta\bar{e}^{\,a} = -\pounds_{\bar{\xi}}\bar{e}^{\,a} + \bar{\sigma}^{\,a}{}_b\bar{e}^{\,b}\,, \tag{4.18a}$$

$$\delta f^a = -\pounds_{\bar{\xi}}f^a + \bar{\sigma}^{\,a}{}_b f^b - \pounds_\epsilon \bar{e}^{\,a} + s^a{}_b\bar{e}^{\,b}\,. \tag{4.18b}$$

Thus, both $\bar{e}^{\,a}$ and f^a transform as 1-forms defined over the background spacetime under diffeomorphisms generated by the background vector fields $\bar{\xi}$ and also as Lorentz vectors with respect to the Lorentz transformations of the tangent space of the background spacetime, generated by $\bar{\sigma}^{\,ab}$. On top of this, there are gauge symmetries generated by the vector fields ϵ and local Lorentz parameters s^{ab} which act on the f^a as

$$\begin{aligned}\delta f^a &= -\pounds_\epsilon\bar{e}^{\,a} + s^a{}_b\bar{e}^{\,b}\\ &= -\bar{\mathcal{D}}\epsilon^a + \left(s^a{}_b - \imath_\epsilon\bar{\omega}^{\,a}{}_b\right)\bar{e}^{\,b}\,.\end{aligned} \tag{4.19}$$

Observe that

$$\delta f_{(ab)} = -\bar{\mathcal{D}}_{(b}\epsilon_{a)}\,, \tag{4.20a}$$

$$\delta f_{ab} = -\bar{\mathcal{D}}_{[b}\epsilon_{a]} + \left(s_{ab} - \imath_\epsilon\bar{\omega}_{ab}\right)\,, \tag{4.20b}$$

which means that the symmetric part transforms as a spin-2 field while the antisymmetric part transforms as a Kalb-Ramond 2-form with an additional Stückelberg transformation with a 2-form parameter s_{ab} which can be used to remove the antisymmetric part of f_{ab}, as expected.[7] We will just fix the s_{ab} symmetry setting

$$s_{ab} = \imath_\epsilon\bar{\omega}_{ab}\,, \tag{4.21}$$

[7] In absence of local Lorentz symmetry this is not possible and theories constructed in terms of the Vielbein describe a spin-2 and a spin-1 field. See section 4.6.1 of ref. [27].

in order to simplify the gauge transformations of f^a:

$$\delta_\epsilon f^a = -\bar{\mathcal{D}}\epsilon^a \,. \tag{4.22}$$

The linearized connection transforms as a 1-form under the diffeomorphisms of the background spacetime generated by the vector fields $\bar{\xi}$ and as a Lorentz tensor under the local Lorentz transformations of the background spacetime generated by the parameters $\bar{\sigma}^{ab}$. Under the spin-2 gauge transformations eqs. (4.22)

$$\delta_\epsilon \omega_{L\,dab} = -\tfrac{1}{2}\left\{\bar{\mathcal{D}}_{[a}\bar{\mathcal{D}}_{b]}\epsilon_d + \bar{\mathcal{D}}_{[a|}\bar{\mathcal{D}}_d\epsilon_{|b]} + \bar{\mathcal{D}}_d\bar{\mathcal{D}}_{[a}\epsilon_{b]}\right\} \,. \tag{4.23}$$

In Minkowski spacetime this expression can be simplified

$$\delta_\epsilon \omega_{L\,ab} = -\bar{\mathcal{D}}\bar{\mathcal{D}}_{[a}\epsilon_{b]} \,, \tag{4.24}$$

and the transformation of the linearized Riemann tensor is

$$\delta_\epsilon R_{L\,ab} = \bar{\mathcal{D}}\delta_\epsilon\omega_{L\,ab} = -\bar{\mathcal{D}}\bar{\mathcal{D}}\bar{\mathcal{D}}_{[a}\epsilon_{b]} = 0 \,. \tag{4.25}$$

4.2 Asymptotic AD charges

As we have stated before, we can simply replace in the definitions of the charges the Riemann tensor by the Riemann tensor linearized over Minkowski spacetime using $\sigma^{(p)}$s which are CKYpFs of the background Minkowski spacetime (now denoted by $\bar{\sigma}^{(p)}$s) and obtain charges which are conserved under the conditions we discussed for the charges constructed with full Riemann tensors. Also, we have seen that it is enough to consider the $\mathbf{Q}[\sigma^{(p)}]$ charges for $p \geq 2$.

Thus, we have to consider the p-forms

$$\mathbf{Q}_L[\bar{\sigma}^{(p)}] \equiv R_L^{ab} \wedge \imath_{\bar{b}}\imath_{\bar{a}}\bar{\sigma}^{(p)} \,, \tag{4.26}$$

where the inner products $\imath_{\bar{a}}$ are taken with the background vector fields $\bar{e}_a = \bar{e}_a{}^\mu\partial_\mu$ and we assume that the background satisfies the linearized Einstein equations so that, in particular,

$$\imath_{\bar{a}}R_L^{ab} = \imath_{\bar{b}}\imath_{\bar{a}}R_L^{ab} \doteq 0 \,. \tag{4.27}$$

Observe that these charges are equal to the dual linearized KT-type ones

$$\tilde{\mathbf{Q}}_{\mathrm{KT}\,L}[\tilde{\bar{\sigma}}^{(d-p)}] \equiv \bar{\star}\imath_{\bar{b}}\imath_{\bar{a}}\left[R_L^{ab} \wedge \tilde{\bar{\sigma}}^{(d-p)}\right] \,. \tag{4.28}$$

We want to obtain the explicit form of these asymptotic AD-type charges and, in particular, we want to know when they are total derivatives. This means that we will have to study the duals as well.

If $\bar{\sigma}^{(p)}$ is a CKYpF of the background metric

$$\begin{aligned}\mathbf{Q}_L[\bar{\sigma}^{(p)}] &= \bar{\mathcal{D}}\omega_L^{ab} \wedge \imath_{\bar{b}}\imath_{\bar{a}}\bar{\sigma}^{(p)} \\ &= d\left\{\omega_L^{ab} \wedge \imath_{\bar{b}}\imath_{\bar{a}}\bar{\sigma}^{(p)}\right\} + \frac{p-1}{p+1}\omega_L^{ab} \wedge \imath_{\bar{b}}\imath_{\bar{a}}\aleph^{(p+1)}\end{aligned} \tag{4.29}$$

where we have used eq. (3.3). The last term will always vanish for CCKYpFs and, due to the relations that we have established, these are the charges associated to KY$(d-p)$Fs in ref. [9].

Since these charges are exact p-forms, when we integrate them over compact p-dimensional surfaces, we will always get zero. As it is well known, in these cases one has to define the conserved quantities as integrals over closed $(p-1)$-dimensional surfaces of the background spacetime of the $(p-1)$-form whose total derivative we have obtained:

$$\mathcal{Q}[\bar{\sigma}^{(p)}] \sim \int_{\Sigma^{(p-1)}} \omega_L^{ab} \wedge \imath_{\bar{b}} \imath_{\bar{a}} \bar{\sigma}^{(p)} \,. \tag{4.30}$$

By construction, these charges are invariant under diffeomorphisms and local Lorentz transformations of the background spacetime but also under spin-2 gauge transformations eq. (4.22) because the integrand is invariant up to a total derivative:

$$\begin{aligned}
\delta_\epsilon \mathcal{Q}[\bar{\sigma}^{(p)}] &\sim -\int_{\Sigma^{(p-1)}} \bar{\mathcal{D}}\bar{\mathcal{D}}^a \epsilon^b \wedge \imath_{\bar{b}} \imath_{\bar{a}} \bar{\sigma}^{(p)} \\
&= -\int_{\Sigma^{(p-1)}} \left\{ d\left[\bar{\mathcal{D}}^a \epsilon^b \wedge \imath_{\bar{b}} \imath_{\bar{a}} \bar{\sigma}^{(p)}\right] - \frac{p-1}{p+1} \bar{\mathcal{D}}^a \epsilon^b \imath_{\bar{b}} \imath_{\bar{a}} \aleph^{(p+1)} \right\} \\
&= -\int_{\Sigma^{(p-1)}} d\left[\bar{\mathcal{D}}^a \epsilon^b \wedge \imath_{\bar{b}} \imath_{\bar{a}} \bar{\sigma}^{(p)}\right] \\
&= 0\,.
\end{aligned} \tag{4.31}$$

where we have used eq. (3.3) again, we have assumed that $\sigma^{(p)}$ is a CCKYpF and we have used Stokes' theorem.

When the integration surface $\Sigma^{(p-1)}$ is not closed, the result of the integral depends on the boundary conditions satisfied by the gauge parameters ϵ^b. This may not be acceptable from the point of view of the algebraic approach to generalized symmetries in QFT adopted in refs. [20–23]. As we have stated in the introduction, our approach is purely classical and we will not discussed the implications of our results in that respect.

Thus, from the purely classical point of view, the off-shell conserved 2-forms associated to CCKY2Fs are exact and actually lead to 1-form charges. The off-shell conserved $(d-2)$-forms are associated to non-CCKY$(d-2)$Fs, including KY$(d-2)$Fs and, in general, they are not exact.

In order to get 2-form charges one may have to consider CCKY3Fs, and one could also consider CCKY$(d-1)$Fs to get $(d-2)$-form charges. We will study all these possibilities in detail in the next sections, comparing our results to those in ref. [23]. We start with the simplest case, $d=4$.

4.3 The $d = 4$ case

According to the previous discussion, it is not enough to consider the CKY2Fs of the Minkowski spacetime background to construct 2-form charges: some of them will give rise to 1-form charges, because our construction gives total derivatives and, (perhaps) to compensate the problem, some CFKY3Fs and CKVs will give 2-forms because our construction, again, gives total derivatives.

We start by finding all the CKY2Fs of $d = 4$ Minkowski spacetime, to show that the solutions cover all the 2-forms used in ref. [23].

4.3.1 The CKY2Fs of $d = 4$ Minkowski spacetime

The most general CKY2F σ satisfies the equation

$$\mathcal{D}\sigma_{ab} = \tfrac{1}{3}\left(\aleph_{abc} + 2\eta_{c[a}\xi_{b]}\right)e^c\,, \tag{4.32}$$

where $\xi = \xi_a e^a$ is a 1-form and $\aleph = \frac{1}{3!}\aleph_{abc}e^a \wedge e^b \wedge e^c$ is a 3-form which characterize the closedness and co-closedness of the CKY2F σ. If we write these equations in the form

$$d\sigma = \aleph\,, \tag{4.33a}$$

$$d\star\sigma = \star\xi\,, \tag{4.33b}$$

it is evident that the most general solution to eq. (4.32) is characterized by an exact 3-form $\aleph$, a co-exact 1-form ξ and a covariantly constant (hence closed and co-closed and, therefore, harmonic) 2-form $a = \frac{1}{2}a_{ab}e^a \wedge e^b$.[8] On the other hand, the interchange between σ and $\tilde{\sigma}$ corresponds to the interchange between $\aleph$ and $\star\xi$ and between a and $\star a$.

Since linear combinations of CKY2Fs with constant coefficients give CKY2Fs, we may consider separately those which are covariantly constant $\sigma = c$, those which are closed, $\aleph = 0$, and those which are KY2Fs, $\xi = 0$. However, not all the CKY2Fs can be constructed as linear combinations of CKY2Fs of these three classes, basically because the integrability conditions allow for solutions which demand for both non-vanishing $\aleph$ and non-vanishing ξ. This class of solutions must be closed under Hodge duality just as the class of C3KY2Fs does.

We are going to see how all this is realized in the case of the 4-dimensional Minkowski spacetime considered in ref. [23]. Thus, we use Cartesian coordinates x^μ and Vielbein $e^a = \delta^a{}_\mu dx^\mu$. In this basis the spin connection vanishes and $\mathcal{D} = d$.

We are going to consider, in this order, the covariantly constant, the closed CKY2Fs, the KY2Fs and the case of σs which have $\xi \neq 0$ and $\aleph \neq 0$.

1. The covariantly constant bivectors $\sigma^{ab} = a^{ab}$ are purely constant bivectors with 6 independent components. σ is an exact 2-form and

   $$\sigma = d\left(\tfrac{1}{2}a_{\mu\nu}x^\mu dx^\nu\right)\,. \tag{4.34}$$

   In other words: the constant σ^{ab}s are the *Killing bivectors* or *momentum maps* [24] of the vectors that generate Lorentz transformations $k_{ab} = k_{ab}{}^\mu\partial_\mu$ with $k_{ab}{}^\mu = \eta_{ab}{}^\mu{}_\nu x^\nu$:

   $$\partial^a\left(-a^{cd}k_{cd}{}^b\right) = \partial^a\left(-a^{cd}\eta_{cd}{}^b{}_\nu x^\nu\right) = -a^{cd}\eta_{cd}{}^{ba} = a^{ab}\,. \tag{4.35}$$

   This class of 2-forms is evidently closed under Hodge duality.

   For each of the 6 independent as we get an off-shell conserved charge $\mathbf{Q}_L[a]$ which is exact and which leads to non-trivial 1-form charges only. Our main result eq. (2.10)

[8] This is reminiscent of the Hodge decomposition of differential forms.

applied to the charges constructed with the linearized Riemann tensors tells us that $\mathbf{Q}_L[a] = \tilde{\mathbf{Q}}_{\mathrm{KT}\,L}[\tilde{a}]$ and that this charge is also a total derivative.

Furthermore, on-shell $\mathbf{Q}_L[a] = \tilde{\mathbf{Q}}_{\mathrm{KT}\,L}[\tilde{a}] \doteq \tilde{\mathbf{Q}}[\tilde{a}]$ will have the same value on-shell and does not give an independent charge. (This is just eq. (2.13)).

Finally, $\tilde{\mathbf{Q}}[\tilde{a}] = \mathbf{Q}_{\mathrm{KT}\,L}[a]$ and this last charge is not independent, either.

2. The CCKY2Fs ($\aleph = 0$) satisfy the equation

$$d\sigma^{ab} = \tfrac{2}{3}\delta^{[a}{}_{\mu}\xi^{b]}dx^{\mu}\,. \tag{4.36}$$

The integrability condition is

$$\delta^{[a}{}_{\mu}\delta^{b]}{}_{\nu}d\xi^{\nu} \wedge dx^{\mu} = 0\,, \tag{4.37}$$

which is solved by vectors with constant components ξ^a. These are the 4 Killing vectors that generate translations and their dual 1-forms are exact: $\xi_a e^a = d(\xi_\mu x^\mu)$.

Then, redefining $\xi^a \to 3\xi^a$ the solutions are of the form

$$\sigma^{ab} = 2\delta^{[a}{}_{\mu}\xi^{b]}x^{\mu} \equiv b^{ab}\,, \tag{4.38}$$

where the components ξ^b are constant, (up to constant bivectors which we have already taken into account). Actually, we can view the 2-form $b = \frac{1}{2}b_{ab}e^a \wedge e^b$ as the exterior product of (dual of) the CKVs that generates dilatations $\eta_{\mu\nu}x^\mu dx^\nu$ and the (dual 1-form) of the KVs that generate translations

$$b = 2\left(\tfrac{1}{2}\eta_{\mu\nu}x^{\mu}dx^{\nu}\right) \wedge (\xi_\rho dx^\rho) = d\left(x^2\right) \wedge d\left(\xi_\rho x^\rho\right)\,, \tag{4.39}$$

and are obviously exact.

The total number of CCKY2Fs (including the C3KY2Fs) is $6+4$, in agreement with the general result eq. (3.6). However, all these give off-shell conserved 2-form charges which turn out to be exact. Therefore, one can only define with them 10 non-trivial 1-form charges.

3. Next, let us consider the KY2Fs ($\xi = 0$), which satisfy the equation

$$d\sigma_{ab} = \tfrac{1}{3}\aleph_{ab\mu}dx^{\mu}\,. \tag{4.40}$$

The integrability condition of this equation reads

$$\partial_{[\mu}\aleph_{\nu]ab}dx^{\mu} \wedge dx^{\nu} = 0\,, \tag{4.41}$$

and its only non-trivial solution is a tensor with constant components, and the components of σ are

$$\sigma_{ab} = \tfrac{1}{3}\aleph_{ab\mu}x^{\mu} \equiv c_{ab}\,. \tag{4.42}$$

There are 4 constant 3-forms in 4 dimensions, and they are dual to constant 1-forms b

$$\aleph_{abc} \sim \varepsilon_{abcd}\xi^{d}\,. \tag{4.43}$$

This is the duality between the KY2Fs (c) and the CCKY2Fs (b).

With the 4 KY2Fs c we can construct 4 on-shell conserved 2-form charges $\mathbf{Q}[c]$ which are not total derivatives in 4 dimensions.

4. Finally, let us consider the general equation

$$d\sigma_{ab} = \tfrac{1}{3}\left(\aleph_{ab\mu}dx^{\mu} + 2\eta_{\mu[a}\xi_{b]}dx^{\mu}\right)\,, \tag{4.44}$$

whose integrability condition is

$$\left(\partial_{[\mu}\aleph_{\nu]}{}^{ab} + 2\eta^{[a}{}_{[\nu}\partial_{\mu]}\xi^{b]}\right)dx^{\mu}\wedge dx^{\nu} = 0\,. \tag{4.45}$$

This equation admits solutions which are not combinations of those belonging to the previous cases (general CCKY2Fs) [23]:

$$\aleph_{\nu}{}^{ab} = \eta_{\nu\rho}x^{[\rho}a^{ab]}\,, \qquad \xi^{b} = \tfrac{1}{3}a^{b}{}_{\mu}x^{\mu}\,, \tag{4.46}$$

with a constant, antisymmetric a^{ab}, and the solution of the original equation is just

$$\sigma^{ab} = \tfrac{1}{2}a^{ab}x^{2} + 2a^{[a}{}_{\mu}\delta^{b]}{}_{\nu}x^{\mu}x^{\nu} \equiv d^{ab}\,, \tag{4.47}$$

in agreement with ref. [23]. There is an independent solution for each independent choice of d^{ab}, that is 6 in 4 dimensions, and for each of them there is an on-shell conserved charge $\mathbf{Q}_L[d]$ in 4 dimensions.

Thus, in $d = 4$ dimensions, using only CKY2Fs we can only construct 10 independent on-shell conserved 2-form charges and no off-shell 2-forms whatsoever. However we still have to consider the CCKY3Fs.

4.3.2 The CCKY3Fs of $d = 4$ Minkowski spacetime

CKY3Fs satisfy the equation

$$\partial_{a}\sigma^{(3)}_{bcd} = \tfrac{3}{2}\eta_{a[b}\xi^{(2)}_{cd]}\,, \tag{4.48}$$

which admits two classes of solutions:

1. Constant 3-forms ($\xi^{(2)} = 0$), of which there are 4 independent in 4 dimensions, dual to constant vectors

$$\sigma^{(3)}_{abc} = \varepsilon_{abcd}\xi^{d} \equiv f_{abc}\,. \tag{4.49}$$

These are, actually, the CCKY3Fs dual to the 4 translational KVs (KY1Fs) ξ. One can construct with them 4 off-shell conserved 3-form charges $\mathbf{Q}_L[f] \doteq \tilde{\mathbf{Q}}_L[b]$ which would be the 3-forms considered in [9]. They are exact and give rise to 4 off-shell conserved 2-form charges.

2. 3-forms associated to the dual of the Killing vectors k that generate Lorentz transformations with constant parameters a_{ab}:

$$\sigma^{(3)}_{abc} = -3\tilde{a}_{[ab}\eta_{c]\mu}x^{\mu} \equiv l_{abc}\,. \tag{4.50}$$

Indeed, if k is such a Killing vector

$$(\star k)^a = \tilde{\sigma}^{\,(1)\,a} = a^a{}_{\mu}x^{\mu}\,. \tag{4.51}$$

On the other hand, these CCKY3Fs can be seen as the exterior product of the constant CCKY2Fs a and the CKV that generated dilatations.

There are 6 of these and, again they give off-shell conserved 3-form charges of the type considered in [9] $\mathbf{Q}_L[l] = \tilde{\mathbf{Q}}_{\mathrm{KT}\,L}[\xi]$ which are exact and give rise to 6 off-shell conserved 2-form charges.

Thus, we find $4+6$ off-shell additional conserved 2-form charges.

It is worth discussing these charges in some more detail, because, upon integration at infinity, they are the standard gravitational conserved charges of asymptotically-flat spacetimes. As we have shown, the CCKY3Fs are dual to the 10 KY1Fs of the spacetime, that is, to its Killing vectors, which we can generically denote by $\bar{k}$. Then, using the duality eq. (2.10) and eq. (2.6), we get

$$\mathbf{Q}[\tilde{\bar{k}}] = -2\bar{k}^{\,a}G_{L\,ab}\star\bar{e}^{b}\,. \tag{4.52}$$

This expression vanishes on-shell and, if we are only interested in strictly gauge-invariant charges constructed with the linearized Riemann tensor as in refs. [20–23], they should not be considered. However, as we have shown,[9] this expression is the total derivative of a 2-form and the integral of this 3-form does not necessarily vanish on-shell and is also gauge-invariant (4.31).

$$\begin{aligned}
\mathcal{Q}[\tilde{\bar{k}}] &\sim \int_{\Sigma^{(2)}} \varepsilon_{abcd}\,\omega_{L\,e}{}^{ab}\bar{k}^{\,d}\bar{e}^{\,e}\wedge\bar{e}^{\,c}\\
&= -\tfrac{1}{6}\int_{\Sigma^{(2)}}\left\{2\omega_{L\,c}{}^{ca}\bar{k}^{\,b} + \omega_{L\,c}{}^{ab}\bar{k}^{\,c}\right\}\imath_{\bar{a}}\imath_{\bar{b}}\bar{\omega}\,,\\
&= -\tfrac{1}{24}\int_{\Sigma^{(2)}}\left\{2\left[\bar{\mathcal{D}}_c f^{ac} + \bar{\mathcal{D}}_c f^{ca} - \bar{\mathcal{D}}^a f^c{}_c - \bar{\mathcal{D}}^a f_c{}^c + \bar{\mathcal{D}}_c f^{ca} - \bar{\mathcal{D}}_c f^{ac}\right]\bar{k}^{\,b}\right.\\
&\qquad\left. + \left[\bar{\mathcal{D}}^a f^b{}_c + \bar{\mathcal{D}}^a f_c{}^b - \bar{\mathcal{D}}^b f^a{}_c - \bar{\mathcal{D}}^b f_c{}^a + \bar{\mathcal{D}}_c f^{ab} - \bar{\mathcal{D}}_c f^{ba}\right]\bar{k}^{\,c}\right\}\imath_{\bar{a}}\imath_{\bar{b}}\bar{\omega}\\
&= -\tfrac{1}{24}\int_{\Sigma^{(2)}}\left\{2\left[2\bar{\mathcal{D}}_c f^{ca} - \bar{\mathcal{D}}^a f^c{}_c - \bar{\mathcal{D}}^a f_c{}^c\right]\bar{k}^{\,b}\right.\\
&\qquad\left. + \left[2\bar{\mathcal{D}}^a f^b{}_c + 2\bar{\mathcal{D}}^a f_c{}^b + 2\bar{\mathcal{D}}_c f^{ab}\right]\bar{k}^{\,c}\right\}\imath_{\bar{a}}\imath_{\bar{b}}\bar{\omega}\\
&= -\tfrac{1}{12}\int_{\Sigma^{(2)}}\left\{\left[2\bar{\mathcal{D}}_c f^{ca} - \bar{\mathcal{D}}^a f^c{}_c - \bar{\mathcal{D}}^a f_c{}^c\right]\bar{k}^{\,b}\right.\\
&\qquad\left. + \left[\bar{\mathcal{D}}^a f^b{}_c + \bar{\mathcal{D}}^a f_c{}^b + \bar{\mathcal{D}}_c f^{ab}\right]\bar{k}^{\,c}\right\}\imath_{\bar{a}}\imath_{\bar{b}}\bar{\omega}\,.
\end{aligned} \tag{4.53}$$

[9] As it has also been shown, for instance, in [2].

If we use the gauge symmetry to eliminate the antisymmetric part of f_{ab} and identifying $f_{ab} = 2h_{ab}$ this expression simplifies further

$$\mathcal{Q}[\tilde{\tilde{k}}] \sim -\tfrac{1}{3} \int_{\Sigma^{(2)}} \left\{ \bar{k}^{\,b} \bar{\mathcal{D}}_c h^{ca} - \bar{k}^{\,b} \bar{\mathcal{D}}^a h_c{}^c + \bar{k}^{\,c} \bar{\mathcal{D}}^a h^b{}_c \right\} \imath_{\bar{a}} \imath_{\bar{b}} \bar{\omega} \,. \tag{4.54}$$

In Cartesian coordinates x^μ and in the Vielbein basis $\bar{e}^{\,a} = \delta^a{}_\mu dx^\mu$ ($\bar{\mathcal{D}}_a = \partial_a$) and for the timelike Killing vector $\bar{k} = \partial_0$

$$\mathcal{Q}[\tilde{\tilde{k}}] \sim \tfrac{1}{6} \int_{\Sigma^{(2)}} \left\{ \partial^i h^j{}_j - \partial^j h^i{}_j \right\} \varepsilon_{ikl} dx^k \wedge dx^l \,, \tag{4.55}$$

which is, up to adequate normalization, the ADM mass [1].

The charges associated to the rest of the KVs of Minkowski spacetime give the other 9 conserved quantities that characterize asymptotically-flat spacetimes.

The overall situation is summarized in table 1.

4.4 The $d = 5$ case

Instead of considering the arbitrary $d > 4$ case, we will just consider the $d = 5$ case which already exhibits the main features of the general case and is somewhat easier to handle. Using the dualities and on-shell relations we have uncovered, it is enough to focus on the $\mathbf{Q}_L[\sigma^{(p)}]$ charges with $p = 2, 3, 4$ if we are just interested in 2- and 3-form conserved, independent and nontrivial charges.

It is easy to see that the CKY2Fs do not give any: for the constant (10) and closed (5) CKY2Fs the $\mathbf{Q}_L[\sigma^{(2)}]$s are exact and only give 1-form charges. The (10) KY2Fs and the (10) CKY2Fs which are not closed do not give any conserved charges in 5 dimensions.

The (10) constant and (10) closed CKY3Fs give the off-shell conserved nontrivial 2-form charges studied in ref. [9] while the (10) KY3Fs and the (10) CKY3Fs which are not closed do give on-shell conserved charges in 5 dimensions.

Finally, the (5) constant and (10) closed CKY4Fs give additional off-shell conserved nontrivial 3-form charges while the KY4Fs and the CKY4Fs which are not closed do not give any conserved charges in 5 dimensions. The existence of the off-shell conserved 3-form charges does not follow the pattern of on-shell conserved $(d-2)$-forms and off-shell conserved 2-forms. However, since the CCKY4Fs are dual to the KVs, these are the conventional gravitational charges of asymptotically-flat 5-dimensional spacetimes, as we showed in the 4-dimensional case.

5 Conclusions

In this paper we have managed to relate and extend the definitions of the conserved charges of gravitating systems made in refs. [20–23] and ref. [9]. In particular, we have shown how the definitions of conserved charges of particles and fields evolving in "vacuum" spacetimes admitting conformal Killing-Yano p-forms can be extended to definitions of gravitational charges of spacetimes that asymptote to them. In the construction of the Abbott-Deser-type charges, though, we have considered only asymptotically-flat spacetimes. However, the

	BBM	Here	KT	exact?	on/off-shell?	#
i	A	$\tilde{\mathbf{Q}}_L[a]$	$\mathbf{Q}_{\mathrm{KT}\,L}[\tilde{a}] \doteq \mathbf{Q}_L[\tilde{a}]$	yes	on-shell	6
ii	B	$\tilde{\mathbf{Q}}_L[b]$	$\mathbf{Q}_{\mathrm{KT}\,L}[c] \doteq \mathbf{Q}_L[c]$		on-shell	4
iii	C	$\tilde{\mathbf{Q}}_L[c]$	$\mathbf{Q}_{\mathrm{KT}\,L}[b] \doteq \mathbf{Q}_L[b]$	yes	on-shell	4
iv	D	$\tilde{\mathbf{Q}}_L[d]$	$\mathbf{Q}_{\mathrm{KT}\,L}[\tilde{d}] \doteq \mathbf{Q}_L[\tilde{d}]$		on-shell	6
v	$\star A$	$\mathbf{Q}_L[a]$	$\tilde{\mathbf{Q}}_{\mathrm{KT}\,L}[\tilde{a}] \doteq \tilde{\mathbf{Q}}_L[\tilde{a}]$	yes	off-shell	6
vi	$\star B$	$\mathbf{Q}_L[b]$	$\tilde{\mathbf{Q}}_{\mathrm{KT}\,L}[c] \doteq \tilde{\mathbf{Q}}_L[c]$	yes	off-shell	4
vii	$\star C$	$\mathbf{Q}_L[c]$	$\tilde{\mathbf{Q}}_{\mathrm{KT}\,L}[b] \doteq \tilde{\mathbf{Q}}_L[b]$		on-shell	4
viii	$\star D$	$\mathbf{Q}_L[d]$	$\tilde{\mathbf{Q}}_{\mathrm{KT}\,L}[\tilde{d}] \doteq \tilde{\mathbf{Q}}_L[\tilde{d}]$		on-shell	6
ix		$\mathbf{Q}_L[f]$	$\tilde{\mathbf{Q}}_{\mathrm{KT}\,L}[\tilde{f}] \doteq \tilde{\mathbf{Q}}_L[\tilde{f}]$	yes	off-shell	4
x		$\mathbf{Q}_L[l]$	$\tilde{\mathbf{Q}}_{KT\,L}[\tilde{l}] \doteq \tilde{\mathbf{Q}}_L[\tilde{l}]$	yes	off-shell	6

Table 1. In this table we represent all the charges that can be constructed with the CKY2Fs of 4-dimensional Minkowski spacetime using the linearized Riemann tensor. In the second column we write the charge as it is referred to in ref. [23] (BBM). The form which is actually integrated is the dual. In the third column we write the same charge in our notation and in the fourth we write the Kastor-Traschen-type charge [9] which is strictly equivalent to the other two upon use of eq. (2.10) and, next to it, the charge which is equivalent to it on-shell. In the next columns we indicate whether the charge is exact, conserved on- or off-shell and the number of charges of that kind. The charges in the rows i and v, iv and viii, ii and vii and iii and vi are related by duality and have the same values. Only half of them are independent and a half of that half are exact. Thus, only the pairs ii-v and iv-vii are independent, not exact, 2-form charges and they all turn out to be conserved on-shell. The rows ix and x describe exact, off-shell conserved charges associated to 3-forms which give rise to 2-form charges.

main result in Abbott and Deser's seminal paper ref. [2] was, precisely, the extension of these ideas to asymptotically-ADS spacetimes. It is natural to search for a similar extension of the charges studied here and work in this direction is currently underway.

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A Some definitions and identities

In this paper we use the conventions of ref. [27] throughout and differential-form language. We collect here the main definitions and identities used throughout the text.

A.1 The curvature tensor and Bianchi identities

We define the Vielbein and spin connection 1-forms

$$e^a = e^a{}_\mu dx^\mu\,, \qquad\qquad \omega^{ab} = \omega_\mu{}^{ab} dx^\mu = -\omega^{ba}\,, \tag{A.1}$$

satisfying

$$\mathcal{D}e^a \equiv de^a - \omega^a{}_b \wedge e^b = 0\,, \tag{A.2}$$

where $\mathcal{D}$ is the exterior Lorentz-covariant derivative.

The Lorentz curvature 2-form

$$R_{ab} \equiv \tfrac{1}{2} R_{\mu\nu\, ab} dx^\mu \wedge dx^\nu = \tfrac{1}{2} R_{cd\, ab} e^c \wedge e^d\,, \tag{A.3}$$

can be defined vie the Ricci identity

$$\mathcal{D}\mathcal{D}\xi^a = -R^a{}_b \xi^b\,, \tag{A.4}$$

for an arbitrary Lorentz vector ξ^a, and it is given by

$$R^{ab} = d\omega^{ab} - \omega^a{}_c \wedge \omega^{cb}\,. \tag{A.5}$$

Acting on the Vielbein, we get the Bianchi identity

$$R^a{}_b \wedge e^b = -\tfrac{1}{2} R_{[cd\, b]}{}^a e^c \wedge e^d \wedge e^b = 0\,, \quad \Rightarrow \quad R_{[cd\, b]}{}^a = 0\,, \tag{A.6}$$

which also implies

$$R_{ab\, cd} = R_{cd\, ab}\,. \tag{A.7}$$

Acting on $\mathcal{D}\xi^b$ for an arbitrary Lorentz vector ξ^b, we get

$$\mathcal{D}\mathcal{D}\mathcal{D}\xi^a = -R^a{}_b \wedge \mathcal{D}\xi^b\,, \tag{A.8}$$

but acting on both sides of eq. (A.4) we get

$$\mathcal{D}\mathcal{D}\mathcal{D}\xi^a = -\mathcal{D}R^a{}_b \mathcal{D}\xi^b - R^a{}_b \wedge \mathcal{D}\xi^b\,, \tag{A.9}$$

which implies the Bianchi identity

$$\mathcal{D}R^{ab} = 0\,, \quad \Leftrightarrow \mathcal{D}_{[e} R_{cd]}{}^{ab} = 0\,. \tag{A.10}$$

The Ricci tensor is defined by

$$R_{\mu\nu} \equiv R_{\mu\rho\nu}{}^{\rho}\,, \tag{A.11}$$

and analogously using tangent space indices:

$$R_{ab} \equiv R_{acb}{}^{c}\,, \tag{A.12}$$

However, in order to avoid confusion with the curvature tensor 2-form, when using tangent space indices we will write $R_{\text{ic}\,ab}$.

Contracting the indices d and b of eq. (A.10) the Bianchi identity takes the form

$$2\mathcal{D}_{[e}R_{\text{ic}\,c]}{}^{a} + \mathcal{D}_{b}R_{ec}{}^{ab} = 0\,, \tag{A.13}$$

and contracting now the indices e and a we arrive at the famous contracted Bianchi identity

$$\mathcal{D}_{a}G^{ab} = 0\,, \quad \text{where} \quad G^{ab} \equiv R_{\text{ic}}{}^{ab} - \tfrac{1}{2}g^{ab}R\,, \tag{A.14}$$

is the Einstein tensor.

The Ricci identity for an antisymmetric Lorentz tensor is

$$\mathcal{D}\mathcal{D}\sigma^{ab} = 2\sigma^{[a|}{}_{c}R^{c|b]} = \delta_{\sigma}R^{ab}\,, \tag{A.15}$$

where δ_{σ} is an infinitesimal local Lorentz transformation generated by the parameter σ^{ab}. The same transformation acts on the spin connection as

$$\delta_{\sigma}\omega^{ab} = \mathcal{D}\sigma^{ab}\,. \tag{A.16}$$

We also introduce the Levi-Civita affine connection $\Gamma_{\mu\nu}{}^{\rho}$, whose components are given by the Christoffel symbols, and the total (Lorentz and general) covariant derivative, denoted by ∇, which satisfies the first Vierbein postulate

$$\nabla e^{a} = \mathcal{D}e^{a} - \Gamma_{\mu\nu}{}^{a}dx^{\mu}\wedge dx^{\nu} = 0\,. \tag{A.17}$$

As a consequence, the Riemann curvature tensor

$$R_{\mu\nu\rho}{}^{\sigma}(\Gamma) \equiv 2\partial_{[\mu}\Gamma_{\nu]\rho}{}^{\sigma} + 2\Gamma_{[\mu|\lambda}{}^{\sigma}\Gamma_{|\nu]\rho}{}^{\lambda}\,, \tag{A.18}$$

is related to the Lorentz curvature tensor we have defined before by

$$R_{\mu\nu\,ab}(\omega) = R_{\mu\nu\,\rho\sigma}(\Gamma)e_{a}{}^{\rho}e_{b}{}^{\sigma}\,, \tag{A.19}$$

so we can treat both objects as one and the same.

The Einstein equation in vacuum can be written in terms of the curvature tensor as follows:

$$3g^{cde}{}_{abf}R_{cd}{}^{ab} = -2G_{f}{}^{e} = 0\,. \tag{A.20}$$

In $d = 3$ dimensions

$$g^{cde}{}_{abf} = \varepsilon^{cde}\varepsilon_{abf}\,, \tag{A.21}$$

and the above relation between the Einstein and Riemann tensors can be inverted

$$\begin{aligned} -2G_f{}^e g_{egh}{}^{fij} &= 3\varepsilon^{cde}\varepsilon_{egh}\varepsilon_{abf}\varepsilon^{fij}R_{cd}{}^{ab} \\ &= 12g^{cd}{}_{gh}g_{ab}{}^{ij}R_{cd}{}^{ab} \\ &= 12R_{gh}{}^{ij}\,, \end{aligned} \tag{A.22}$$

which implies that all the 3-dimensional solutions to the vacuum Einstein equations are locally flat.

A.2 On-shell identities involving the dual of the curvature tensor

Here we refer to the Hodge dual of the Lorentz curvature 2-form

$$\star R^{ab} \equiv \frac{\varepsilon_{\mu_1\cdots\mu_{d-2}}{}^{\rho\sigma}R_{\rho\sigma}{}^{ab}}{2\cdot(d-2)!\sqrt{|g|}}dx^{\mu_1}\wedge\cdots\wedge dx^{\mu_{d-2}} = \frac{\varepsilon_{c_1\cdots c_{d-2}}{}^{cd}R_{cd}{}^{ab}}{2\cdot(d-2)!}e^{c_1}\wedge\cdots\wedge e^{c_{d-2}}\,. \tag{A.23}$$

as the dual curvature tensor $(d-2)$-form.

Acting with the exterior Lorentz derivative on it, we get

$$\mathcal{D}\star R_{ab} = (-1)^{d-1}\mathcal{D}_d R_{ab}{}^{cd}\imath_c\omega\,, \tag{A.24}$$

where ω is the d-dimensional volume form defined in eq. (A.26) and $\imath_c$ stands for the inner product with the vector field $e_c = e_c{}^{\mu}\partial_\mu$ and where we have used the identity eq. (A.27a). The above expression vanishes on-shell for pure gravity due to the Bianchi identity eq. (A.13).

Also, using eq. (A.27a)

$$\star R_{ab}\wedge e^a = R_{\text{ic}}{}^e{}_b\imath_e\omega\,, \tag{A.25}$$

which also vanishes on-shell.

A.3 Identities involving the d-dimensional volume form ω

Another set of identities. First, the definition of the volume form:

$$\omega \equiv e^0\wedge e^1\wedge\cdots\wedge e^{d-1} = \frac{(-1)^{d-1}\varepsilon_{a_1\cdots a_d}}{d!}e^{a_1}\wedge\cdots\wedge e^{a_d}\,. \tag{A.26}$$

The $(-1)^{d-1}$ factor is associated to the mostly minus signature that we are using. Then, we can prove the following identities:

$$e^{c_1}\wedge\cdots\wedge e^{c_{d-1}} = (-1)^{d-1}\varepsilon^{c_1\cdots c_{d-1}b}\imath_b\omega\,, \tag{A.27a}$$

$$e^{c_1}\wedge\cdots\wedge e^{c_{d-2}} = -\tfrac{1}{2}\varepsilon^{c_1\cdots c_{d-2}b_1b_2}\imath_{b_1}\imath_{b_2}\omega\,, \tag{A.27b}$$

$$e^{c_1}\wedge\cdots\wedge e^{c_{d-3}} = -\frac{(-1)^{d-1}}{3!}\varepsilon^{c_1\cdots c_{d-3}b_1b_2b_3}\imath_{b_1}\imath_{b_2}\imath_{b_3}\omega\,, \tag{A.27c}$$

$$e^{c_1}\wedge\cdots\wedge e^{c_{d-n}} = \frac{(-1)^{[n/2]}(-1)^{n(d-1)}}{n!}\varepsilon^{c_1\cdots c_{d-n}b_1\cdots b_n}\imath_{b_1}\cdots\imath_{b_n}\omega\,. \tag{A.27d}$$

A.4 Other identities

With our conventions, for any p-form $F^{(p)}$,

$$\star^2 F^{(p)} = (-1)^{p(d-p)} \det(\eta) F^{(p)} \,, \tag{A.28}$$

where $\det(\eta)$ is the determinant if the tangent-space metric η_{ab} (which equals $(-1)^{d-1}$ for a d-dimensional Lorentzian metric with mostly minus signature) and, if $p \geq n$,

$$\star \imath_{a_1} \cdots \imath_{a_n} F^{(p)} = e^{a_n} \wedge \cdots \wedge e^{a_1} \wedge \star F^{(p)} \,. \tag{A.29}$$

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