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ARTICLE

On parameters for inner products under imprecision and their applications to certain geometric and statistical problems

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ABSTRACT
This work aims to study the imprecision on the inner product of a real vector space. It starts defining the notion of imprecise inner product. This notion can also model the uncertainty about the variability of a multivariate random variable. The primary goal is to introduce parameters in order to provide information about the shape and the size of this kind of inner products.

Imprecision is inherent to any quantity which depends on the inner product (or the covariance matrix). Useful techniques are developed by means of the introduced parameters to obtain fast and easy approximations for that kind of quantities.

Throughout several examples, the convenience of the techniques is proven. These techniques can be applied to a wider class of statistical problems dealing with imprecision.

KEYWORDS
inner product; imprecision; precision parameters; generalized information theory; approximations

1. Introduction

A recent trend in the statistical literature concerns imprecision. In many real situations, imprecision is related to an inner product of a vector space. Tsai and Lenz (1989) consider a camera of a robot that needs to be calibrated. Moreover, the robot is programmed to make some movements, and each movement depends on the calibration. A crucial factor in this problem is the speed (according to the authors, the time for the execution of the underlying algorithms is of the order of miliseconds). Another crucial factor is the error, that is nicely controlled, see (Tsai and Lenz 1989, Section III). In this direction, we go further considering that the movements of the robot are determined by solving certain complex problem that depends on the calibration. This is not an intricate assumption, as each movement of the robot can be determined by solving a decision problem that can be posed as a variational one, for instance. Note that the calibration part can be formulated as certain imprecision on the inner product of the underlying three-dimensional Euclidean space. Then, from a theoretical viewpoint, we must address the problem of solving a complex problem that depends on an imprecise inner product and under the critical factors of speed and error control.

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From a more theoretical viewpoint, the imprecision of the inner product is inherited to any geometric quantity. When the computation of such a quantity for an exact inner product becomes complex, then the time factor becomes quite problematic. We want to obtain easy and fast approximations to bound all the possible values of the quantity.

An added problem can appear if it is necessary to fuse imprecise information (see a general introduction in Hall and Llinas (1997) and a related theoretical study in Yager (1997), for instance). In general, the calculus become more complex and consequently, the original problem grows in complexity.

A more general framework including the previous robot problem is the uncertainty in spatial systems. In (Leung 2013, Sec. 1.1-1.2), the authors discuss about the imprecision that shall to be included in the mathematical models. They also argue that the imprecision appears even in the original approximated data. Clearly, any decision procedure must take into account the different imprecisions that could occur. In this kind of spatial systems, the imprecision can arise from the inner product or can be formulated in these terms.

Summarizing, we consider an inner product under imprecision and from it we need to compute certain geometric quantity. The crucial problem arises with the time factor. In this direction, the main goal of this work is to provide several techniques to achieve nice and fast approximations for the geometric quantity. Throughout this work, we borrow the terminology of estimation to clarify our mathematical elements, even though no statistical reasoning is made (in this sense, we will say that we provide different estimations for the geometric quantity).

In this setting, the first step is to analyze the imprecision of an inner product. The "generalized information theory" is the field that studies the uncertainty and the related information under uncertainty. This area of study can be traced back to Klir (1991), where the author raises the related problems as a proper research program. From this starting point, several authors have focused on obtaining distinguished parameters for their imprecise elements. For example, the main goal of Bronevich and Klir (2010) is to obtain several measures of specificity or uncertainty for imprecise probabilities. For credal sets, we may highlight the work of Abellán and Klir (2005). In the context of this work, first we introduce the notion of an imprecise inner product (see Definition 2.1). Then, as a first approach, we introduce several parameters to analyze such an imprecise inner product. In the spirit of the aforementioned generalized information theory, these parameters inform us about the structure of the imprecise inner product. In fact, those parameters are related to the size and shape of an imprecise inner product.

Now, we want to bound the values of a geometric quantity from an imprecise inner product. We desire to obtain fast approximations about the values that such quantity can take, since as it was already commented, the time factor plays a central role. Then, our second aim is to provide several techniques to obtain these approximations. The idea is to simplify the original problem so that the approximations can take place in a simpler way. As a natural consequence, precision is the price paid for the speed. This formulation is widely extended; in (Leung 2013, p. 4) it is argued that *"(...) complexity and precision are mutually conflicting. A higher level of precision can only be achieved when a system is simple. The level of precision, however, decreases as our spatial systems become more complex"* [*sic*]. Surprisingly, the introduced parameters for an imprecise inner product are key to obtain such approximations. The general idea is to evaluate the quantity from an exact inner product belonging to the imprecise inner product (in some cases, it is better to enlarge the original imprecise inner product to simplify the evaluation; see Example 3.11). Then, the parameters of the imprecise inner product lead to certain error from which we calculate an interval for the values of the geometric quantity.

In another setting, imprecision can also appear in the covariance matrix of a multivariate random variable. Mathematically, a covariance matrix can be described as an inner product; consequently, an imprecise inner product can model a covariance matrix under imprecision. Several authors have studied decision problems when the covariance is imprecise (see, for instance, Cho (2011) or Chopra and Ziemba (2013)). We can apply the developed techniques to obtain fast approximations of statistical calculus when the covariance matrix is imprecise. Here, we find a deep connection between Statistics and Geometry.

Finally, we may relate this work with recent topics in the literature. For instance, Bronevich and Lepskiy (2015) studies, in a synthetic way, indices of imprecision. Montes, Miranda, and Destercke (2020a) and Montes, Miranda, and Destercke (2020b) consider imprecise probability models. Imprecision in the information provided by the expert can lead to serious problems; Pai and Prabhu Gaonkar (2020) provide some techniques to solve the problem of combining different information with a high degree of conflict. Doria (2021) studies several new properties of coherent upper conditional previsions. Dukes and Casey (2021) analyzes several completely general diversity metrics. And Li et al. (2021) studies attribute selection for heterogeneous data based on information entropy.

This paper is organized as follows. In Section 2, the main notions are introduced. We begin with the notion of an imprecise inner product, that generalizes the concept of an inner product. We examine the main properties of this new notion. Then, we introduce an order for inner products (a necessary tool throughout this work). This order arises as a comparison between measures from different inner products. Section 3 is aimed to introduce and analyse different parameters for an imprecise inner product. Each of them provides different information about an imprecise inner product. We analyse the main features of these parameters. Then, we show how these parameters serve in estimating a quantity from an imprecise inner product. Several quantities of interest are analysed in detail. In Section 3.1 we show how to obtain certain control on the error in the estimations. Section 4 is devoted to consider different simulations which illustrate the convenience of the developed techniques. Finally, a conclusion section ends this paper.

2. Imprecise inner products

Let us begin recalling some basic concepts. Let $\mathbb{V}$ be an n-dimensional real vector space. An inner product g on $\mathbb{V}$ is a symmetric bilinear map $g : \mathbb{V} \times \mathbb{V} \to \mathbb{R}$. If $g(x, x) \geq 0$ holds for any vector x, then g is said to be semi-definite positive. When a basis $\{e_1, \ldots, e_n\}$ of $\mathbb{V}$ is fixed, the inner product can be represented by a matrix $\mathcal{G}$, the Gram matrix of g. The (i, j)-entry of the Gram matrix is the inner product of the vectors e_i and e_j of the basis. Then, the inner product of two vectors x and y can be obtained by matrix multiplication; namely, $g(x, y) = x^\top \mathcal{G}\, y$, where the vectors x and

y are represented by column matrices and $^\top$ denotes transposition.

Now, let us denote by $G(\mathbb{V})$ the collection of all semi-definite positive inner products that $\mathbb{V}$ can be endowed with. At the same time, denote by $G^+(\mathbb{V})$ the collection of all Euclidean inner products that $\mathbb{V}$ can be endowed with (that is, definite-positive inner products).

Let us start with the notion of an imprecise inner product.

Definition 2.1. An *imprecise inner product* O is a (non-empty) set of $G(\mathbb{V})$.

Observe that the case of an exact inner product is included in the previous notion. Namely, $O = \{g\}$, $g \in G(\mathbb{V})$.

The interpretation of an imprecise inner product is clear: a set of inner products where we expect that the real one to be included. This notion arises naturally in the estimation of an inner product Bazley and Fox (1962); Cai and Yuan (2010, 2011); Rice and Silverman (1991). The inevitable error in this estimation supports our definition.

Remark 1. Let X be a random vector of an inner product space $(\mathbb{V}, g)$. Fixed an orthonormal basis $\{e_1, \ldots, e_n\}$ of $\mathbb{V}$, recall that the covariance matrix of X is the matrix whose (i,j) entry is $\mathbb{E}\left[g(X - \mathbb{E}[X], e_i)\, g(X - \mathbb{E}[X], e_j)\right]$ (since $g(X - \mathbb{E}[X], e_i)$ corresponds with the i-th component of the random vector $X - \mathbb{E}[X]$). Clearly, in this form, the covariance matrix is not coordinate-free, that is, the matrix depends on the chosen basis of $\mathbb{V}$. There exists an equivalent coordinate-free way to define the covariance matrix: the covariance of the random vector X is the symmetric bilinear map $Cov : \mathbb{V} \times \mathbb{V} \to \mathbb{R}$ defined by $Cov(x, y) = \mathbb{E}\left[g(X - \mathbb{E}[X], x)\, g(X - \mathbb{E}[X], y)\right]$. Note that the matrix representation of Cov in a basis is identified with the covariance matrix in such that basis. Observe also that the covariance is semi-definite positive. Consequently, the covariance can be viewed as an inner product of $\mathbb{V}$.

Example 2.2. (a) Consider the case of calibrating a camera in the plane. What we need is to estimate the inner product of the 2-dimensional Euclidean space.

(a.1) Assume that the angles can be measured with precision, but the lengths cannot. In this case, we can find an orthogonal basis of the corresponding vector space. In this basis, (the Gram matrices of) an imprecise inner product adapted to this problem is:

$$O = \left\{\operatorname{diag}(\lambda_1, \lambda_2) : \lambda_1, \lambda_2 \in L \subset \mathbb{R}^+, \right\} .$$

A distinguished case occurs when the elements of the diagonal vary on the same range; that is, we consider the same uncertainty in all the directions determined by the vectors of the basis. We agree to say that an imprecise inner product O is *round* if there exists $g \in G(\mathbb{V})$ such that

$$O = \{\lambda\, g : \lambda \in L \subset [0, \infty)\} .$$

(a.2) On the other hand, if we assume that the angles cannot be measured with precision but there exist two directions along which any length can be measured in a precise way. In this case, we can take a normal basis formed by two unitary vectors of those directions. In this basis, an appropiate imprecise inner product adopts the form:

$$O = \left\{ \begin{pmatrix} 1 & \cos\theta \\ \cos\theta & 1 \end{pmatrix} : \theta \in T \subset (0, \pi) \right\} .$$

(b) Let (X, Y) be a bivariate random variable whose exact joint distribution is unknown.

(b.1) Assume that X and Y are independent random variables. Under this condition, a round imprecise covariance can be appropriate to model the uncertainty about the covariance matrix. Observe certain analogy between 'precision measuring angles' (geometry) and 'statistical independence' (probability).

(b.2) Copulas are a useful tool to study the joint distribution of two or more random variables (see Nelsen (2007) and references therein). Roughly speaking, given two random variables X and Y, a copula allows us to build a bivariate random variable whose marginals are random variables identically distributed as X and Y. In fact, the converse is true, in the sense that the joint distribution can be decomposed into the marginal distributions and a copula (the Sklar's theorem, Nelsen (2007)). In this context, assume that the distributions of X and Y are known but their copula is not. Now, the uncertainty arising in the covariance matrix can be modelled by an imprecise inner product as in (a.2) in certain coordinate system (specifically, the uncertainty of the covariance matrix of $(X/\sqrt{Var(X)}, Y/\sqrt{Var(Y)})$ can be given in the form of (a.2)). Again, observe the analogy between the geometrical and probability objects. For further discussion concerning this example, we may remit to Montes et al. (2015).

Example 2.3. The mixture models have been extensively studied (see McLachlan and Peel (2004)). Let us consider m probability density functions on $\mathbb{R}^n$, which are denoted by $\rho_1, \ldots, \rho_m$. Denote by Cov_i the covariance matrix associated to the i-th probability density function. For simplicity in this example, assume that all these densities share the same expectation. Now, consider the mixed probability density function $\sum_{i=1}^m \pi_i \, \rho_i$ where $\sum_{i=1}^m \pi_i = 1$ and $\pi_i \geq 0$ for any i. Its covariance matrix is $\sum_{i=1}^m \pi_i \, Cov_i$. We have that the collection of all the possible mixtures provides an imprecise covariance: $O = \{\sum_{i=1}^m \pi_i \, Cov_i : 0 \leq \pi_i \leq 1, \sum_{i=1}^m \pi_i = 1\}$.

To understand the structure of an imprecise inner product, we need to introduce a partial order in $G(\mathbb{V})$, (see Salamanca (2018)):

Definition 2.4. Let $g_1, g_2 \in G(\mathbb{V})$. We say that g_1 *is greater than* g_2, denoted by $g_1 \succeq g_2$, if it holds

$$g_1(x, x) \geq g_2(x, x) \quad \forall x \in \mathbb{V}\,.$$

At the same time, we say that g_2 *is smaller than* g_1.

Remark 2. From two inner products, g_1, g_2, we can define the inner product $g_1 - g_2$ as follows: $(g_1 - g_2)(x, y) = g_1(x, y) - g_2(x, y)$ for any vectors x, y. Thus, we have that: $g_1 \succeq g_2$ if and only if $g_1 - g_2 \in G(\mathbb{V})$; namely, if $g_1 - g_2$ is a semi-definite positive inner product. It is clear that this condition can be easy to compute by means of the matrix representation.

The previous order identifies a distinguished class of imprecise inner products.

Definition 2.5. An imprecise inner product O is said to be *proper* if there exist $g_i, g_s \in G^+(\mathbb{V})$ such that

$$g_s \succeq g \succeq g_i\,, \quad \forall g \in O\,. \tag{1}$$

Remark 3. **(a)** Let O be a proper imprecise inner product. For any $x \in \mathbb{V}$, $x \neq \vec{0}$, there exist positive real numbers a, b (in general, depending on x) such that:

$$a \geq g(x,x) \geq b\,, \quad \forall g \in O\,.$$

(b) Let O be an imprecise covariance. If O is proper, then the distribution along each direction has a non-null, finite variance.

For an imprecise inner product, the existence of a minimum and a maximum can be very useful.

Definition 2.6. An imprecise inner product O is said to be *rhomboidal* if there exist $g_i, g_s \in O$ such that

$$g_s \succeq g \succeq g_i\,, \quad \forall g \in O\,.$$

Or equivalently, if a minimum and a maximum exist.

Note that if a minimum (or maximum) exists, then it is unique. The importance of a rhomboidal imprecise inner product will be shown throughout this work.

The main technique related to the previous order is the following result by Salamanca (2018). It provides a comparison between measures in the corresponding affine space $\mathbb{R}^n$ (see, for instance, Nomizu, Katsumi, and Sasaki (1994)). Here, we consider the length of a curve, the volume of a set and the variance of a random variable. Let us recall these notions formally. For this, fix a Euclidean space $\mathbb{R}^n$, letting g be its inner product. Given a (smooth) curve $\gamma : I \subseteq \mathbb{R} \to \mathbb{R}^n$, its length is $\int_I \sqrt{g\left(\gamma'(t), \gamma'(t)\right)}\, dt$. Fixing a Cartesian coordinate system $\{x_1, \ldots, x_n\}$, the volume of a set U is $\int_U \sqrt{\det(g)}\, dx_1 \cdots dx_n$. Finally, the variance of a random variable X is $Var\,[X] := \inf_{p \in \mathbb{R}^n} \mathbb{E}[d(X,p)^2] = \mathbb{E}\left[g\left(X - \mathbb{E}[X], X - \mathbb{E}[X]\right)\right]$. Note that all these three measures depend on the inner product g.

Proposition 2.7. *Let $\mathbb{R}^n$ be an affine space and g_1 and g_2 be two semi-definite positive inner products from which $\mathbb{R}^n$ can be endowed with. If $g_1 \succeq g_2$ holds, then*

(i) the g_1-length of a curve is not smaller than its g_2-length,
(ii) the g_1-volume of a set is not smaller than its g_2-volume, and
(iii) the g_1-variance of a random variable is not smaller than its g_2-variance.

As aforementioned, given an imprecise inner product, we wish to obtain several parameters which provide us information about it. Then, we will be able to make comparisons between imprecise inner products, or make decisions about them (see Gajdos et al. (2008) and Bronevich and Klir (2010)). Hence, our primary objective is to describe such parameters and to develop the characteristics provided by these parameters.

In our framework, it is natural to consider a quantity Q which depends on the inner product of the vector space, that is, a measurand Q. Assume that $Q(g_2) \leq Q(g_1)$ whenever $g_1 \succeq g_2$; let us focus on any of the cases of Proposition 2.7. The uncertainty of the inner product yields to uncertainty of Q. If this uncertainty is modelled by an imprecise inner product O, then we have several values of Q -perhaps, one for each inner product of Q. The set $\{Q(g) : g \in O\}$ can be provided to estimate Q. However, it may carry huge computational costs. In fact, we may compute

$$\omega(Q) := \inf_{g\in O} Q(g) \quad \text{and} \quad \alpha(Q) := \sup_{g\in O} Q(g)\,.$$

Then, $[\omega(Q), \alpha(Q)]$ is an *interval estimation for* Q. From a practical viewpoint, $\alpha(Q)$ and $\omega(Q)$ can be difficult or even laborious to compute (see the numerical simulation in the last section). However, for rhomboidal imprecise inner products, there exists no difficulty since it is enough to compute $[Q(g_i), Q(g_s)]$, where g_i and g_s are the minimal and maximal elements (see Proposition 2.7).

Instead, we prefer to compute $Q(g)$ for a single, suitable, $g \in G(\mathbb{V})$ and to give positive numbers e_i, e^s to assure that $[\alpha(Q), \omega(Q)] \subseteq [e_i\, Q(g),\, e^s\, Q(g)]$ holds. Hence, $[e_i\, Q(g),\, e^s\, Q(g)]$ can be used to provide the interval estimation. In this case, we interpret $(e^s - e_i)\,/(e^s + e_i)$ as the relative error of $Q(g)$ (by relative error of an interval we mean the quotient between the semi-amplitude and the middle value). Surprisingly, our precision parameters are related to this error (see Equation (6)).

In any case, we need to expand our initial imprecise inner product. For instance, let O have only two inner products g_1, g_2 and neither $g_1 \succeq g_2$ nor $g_2 \succeq g_1$. Then, $g_3 = 0.5\, g_1 + 0.5\, g_2$ can be used to provide point estimations; notice that g_3 does not belong to O in general. Another example which supports the same idea is the following: fix $g \in G^+(\mathbb{V})$ and let $O = \{\lambda\, g : \lambda \in [a,b] \cup [c,d]\}$ $(a, c > 0)$ be a round imprecise inner product. In this case, a candidate to make any estimation can be: $g' = (0.5\, a + 0.5\, d)\, g$, which may not belong to O. Another interesting example is Example 2.9, which simplifies computations as in Example 3.11.

For these reasons, we expand O in the following sense:

Definition 2.8. Let O be an imprecise inner product. The imprecise inner product $G^0(O)$ is

$$G^0(O) = \left\{ g \in G(\mathbb{V}) : \inf_{g'\in O} g'(x,x) \leq g(x,x) \leq \sup_{g'\in O} g'(x,x), \forall x \in \mathbb{V} \right\}\,. \tag{2}$$

Note that $G^0(O)$ is a convex set -which is usually larger than the convex hull of O. More concretely, for any $g_1, g_2 \in G^0(O)$ and $\rho \in [0,1]$, it holds: $\rho\, g_1 + (1-\rho)\, g_2 \in G^0(O)$. From Equation (2), we interpret $G^0(O)$ as a convex set similar to O which does not increase critically its imprecision; in the sense that, for a vector x, the range for its length computed by O (that is, $[\inf_{g\in O} \sqrt{g(x,x)}, \sup_{g\in O} \sqrt{g(x,x)}]$) coincides with the range for its length computed by $G^0(O)$.

Remark 4. Let O be an imprecise inner product and $g \in G(\mathbb{V})$. If there exist $g_1, g_2 \in O$ such that

$$g_1 \succeq g \quad \text{and} \quad g \succeq g_2$$

then $g \in G^0(O)$.

Example 2.9. Fixed a basis $\{e_1, e_2\}$ of a two-dimensional vector space, consider the imprecise inner product:

$$O = \left\{ \begin{pmatrix} a & c \\ c & b \end{pmatrix} : (a-2)^2 + (b-2)^2 + c^2 = 1, |c| > 0.1 \right\}\,.$$

By Remark 4, we have that

$$g = \begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix}$$

belongs to $G^0(O)$. On the one hand, if we consider

$$g_1 = \begin{pmatrix} 2+\sqrt{12}/5 & 1/5 \\ 1/5 & 2+\sqrt{12}/5 \end{pmatrix} \in O$$

we have that $g_1 \succeq g$. To prove it, it is enough to check that the matrix

$$(g_1 - g) = \begin{pmatrix} \sqrt{12}/5 & 1/5 \\ 1/5 & \sqrt{12}/5 \end{pmatrix}$$

is definite positive (see Remark 2). On the other hand, if we consider

$$g_2 = \begin{pmatrix} 1.4 & \sqrt{0.28} \\ \sqrt{0.28} & 1.4 \end{pmatrix} \in O$$

we have that $g \succeq g_2$. This is an immediate consequence of the matrix

$$(g - g_2) = \begin{pmatrix} 0.6 & -\sqrt{0.28} \\ -\sqrt{0.28} & 0.6 \end{pmatrix}$$

being definite positive.

This example is considered again in several parts of this work. In particular, Example 3.11 reveals the importance of $G^0(O)$.

3. Precision parameters

The origin of parameters to measure imprecision goes back to the origin of imprecision theory. Several papers (see Abellan and Masegosa (2008); Abellán and Klir (2005); Bronevich and Klir (2010) and references therein) exemplify the necessity and utility of this kind of parameters. In these references, different kinds of axioms are imposed. However, we limit ourselves to a minimum set of hypotheses, since our aim is not to provide an exhaustive synthetic approach but a more pragmatical one.

This section is aimed to introduce and study several precision parameters for imprecise inner products. Moreover, we show how these parameters are related to the relative error in several estimations, as commented in the previous section.

A precision parameter must fulfil some natural axioms. To state them properly, we need the notion of more imprecise inner product,

Definition 3.1. Let O and O' be two imprecise inner products. We say that O' is *more imprecise than* O if $O \subseteq O'$ holds.

The previous concept has a clear meaning: if O is a set of possible candidates to be the exact inner product, then O' contains such candidates and other ones.

Now, our axioms for a precision parameter are:

(A1) It must be non-negative.
(A2) It must vanish for any exact inner product.
(A3) If O' is more imprecise than O, then the precision parameter evaluated on O' is not smaller than its evaluation on O.

Given an imprecise inner product O, the map $\psi_O : \mathbb{V}\backslash \left\{\vec{0}\right\} \to [1, \infty)$ is defined by

$$\psi_O(x) = \sup_{g_1, g_2 \in O} \frac{g_1(x,x)}{g_2(x,x)} \tag{3}$$

when it provides a finite value. This map gives us information about the imprecision in each direction. Indeed, it fulfils that $\psi_O(x) = \psi_O(\lambda\, x)$ for any $\lambda \neq 0$. Moreover, if O is proper, then ψ_O is a bounded map. In fact, we have:

$$\psi_O(x) \leq \frac{g_s(x,x)}{g_i(x,x)}\,, \quad \forall g \in O\,, \tag{4}$$

where $g_s, g_i \in G^+(\mathbb{V})$ satisfy Equation (1).

Remark 5. On $\mathbb{V}$, denote by $\sim$ the equivalence relation: $x \sim y$ if and only if there exists $t \neq 0$ such that $x = t\, y$. The space $\mathbb{V}/\sim$ is a projective vector space $P(\mathbb{V})$. The function ψ_O induces canonically a function $\widehat{\psi_O} : P(\mathbb{V}) \to \mathbb{R}$, $\widehat{\psi_O}([x]) = \psi_O(x)$. Note that the functions ψ_O and $\widehat{\psi_O}$ are equivalent; in particular, they share the maximum values. In this context, usually a parametrization is quite helpful. Recall that a parametrization of a topological space $\mathcal{M}$ is a set of variables, the coordinates, $\{x_1, \ldots, x_n\}$, defined on a subset U of $\mathbb{R}^n$ and a map $\phi : U \to \mathcal{M}$ such that ϕ is bijective. For instance, the usual Cartesian coordinates of $\mathbb{R}^n$ or the usual geographic coordinates of the sphere.

Now, we will introduce our first precision parameter which fulfils our requirements.

Definition 3.2. Let O be an imprecise inner product of a vector space $\mathbb{V}$. Its *absolute imprecision parameter* (AIP) is

$$AIP(O) = \max_{x \in \mathbb{V}} \psi_O(x) - 1\,. \tag{5}$$

We can show that AIP satisfies Axioms $(A1)$-$(A2)$-$(A3)$. Besides, we can obtain a sufficient condition to assume a finite value.

Lemma 3.3. (i) *The absolute imprecision parameter is non-negative and it is zero if and only if the information is exact.* (ii) *For a proper imprecise inner product, it takes a finite value.* (iii) *If O' is more imprecise than O, then $AIP(O') \geq AIP(O)$.*

Proof. (i) Since O is non-empty, it must contain an inner product g. Then, $\psi_O(x) \geq g(x,x)/g(x,x) = 1$. Therefore, $AIP(O) \geq 0$. If $O = \{g\}$, then

$$AIP(O) = \max_{x \in \mathbb{V}} g(x,x)/g(x,x) - 1 = 0\,.$$

For the converse, let us assume that O is not exact. Hence, O must contain two different inner products, g_1 and g_2. Hence, there exists $x^* \in \mathbb{V}$ such that $g_1(x^*, x^*) \neq g_2(x^*, x^*)$. Therefore, either $g_1(x^*, x^*)/g_2(x^*, x^*)$ or $g_2(x^*, x^*)/g_1(x^*, x^*)$ is bigger than 1, let us say $g_1(x^*, x^*)/g_2(x^*, x^*)$. Hence, $AIP(O) \geq g_1(x^*, x^*)/g_2(x^*, x^*) > 1$. (ii) Note that $\psi_O(x) \leq g_s(x,x)/g_i(x,x)$ for any $x \in \mathbb{V}$, $x \neq \vec{0}$, where $g_i, g_s \in G^+(\mathbb{V})$ satisfy Equation (1). Recalling that the space $P(\mathbb{V})$ is compact (see Remark 5), the function $\widehat{\psi_O}$ must take its maximum at least at a class $[x']$. Since $g_s(x', x')/g_i(x', x')$ bounds the function from above, and $g_s, g_i \in G^+(\mathbb{V})$, we have that $g_s(x', x')/g_i(x', x')$ takes a finite value. Finally, this fact implies that the maximum of ψ_O must be finite. (iii) This follows from Equation (3). □

Before giving several examples, let us interpret the meaning of the AIP. In the case of a proper imprecise inner product, the proof of (ii) in Lemma 3.3 allows to state: there exists $x^* \in \mathbb{V}$ such that $AIP(O) = -1 + (\sup_{g\in O} g(x^*, x^*))/(\inf_{g \in O} g(x^*, x^*))$. In our approach, the interval estimation for the norm of the vector x^* (see Page 7) is

$$\left[\inf_{g' \in O} g(x^*, x^*), \sup_{g\in O} g(x^*, x^*)\right] .$$

The relative error ϵ_r (the quotient between the semi-amplitude of the interval and its middle point) is:

$$\begin{aligned}
\epsilon_r &= \frac{\left(\sup_{g\in O} g(x^*, x^*) - \inf_{g'\in O} g(x^*, x^*)\right)/2}{\left(\sup_{g\in O} g(x^*, x^*) + \inf_{g'\in O} g(x^*, x^*)\right)/2} \\
&= 1 - \frac{2}{1 + \sup_{g\in O} g(x^*, x^*)/\inf_{g'\in O} g(x^*, x^*)} = 1 - \frac{2}{2 + AIP(O)} .
\end{aligned} \tag{6}$$

The previous equation leads to interpret the AIP as a measure of the maximum relative error when measuring the norm of a vector.

Example 3.4. Let $O = \{\lambda\, g : \lambda \in L \subset \mathbb{R}\}$ be an imprecise inner product. The map ψ_O is the constant map at $\sup L/\inf L$. Consequently, its AIP is $-1 + \sup L/\inf L$.

Example 3.5. Let us compute several bounds for the AIP of Example 2.9. For this purpose, let O' be following imprecise inner product:

$$O' = \left\{ \begin{pmatrix} a & c \\ c & b \end{pmatrix} : (a-2)^2 + (b-2)^2 + c^2 = 1 \right\} .$$

We have that O' is more imprecise than O. Now, define

$$g_1 = \begin{pmatrix} 3.5 & 0 \\ 0 & 3.5 \end{pmatrix}, \quad g_2 = \begin{pmatrix} 0.5 & 0 \\ 0 & 0.5 \end{pmatrix} .$$

It is easy to prove that: $g_1 \succeq g \succeq g_2$ for any $g \in O'$. This fact implies that $O'' := O' \cup \{g_1, g_2\}$ is a rhomboidal imprecise inner product, whose maximal and minimal elements are g_1 and g_2, respectively. We deduce: $AIP(O) \leq AIP(O'') = 6$. As this is a rough approximation, no computational effort was needed. Nevertheless, we desire to get the exact value of $AIP(O')$. This value serves us as an approximation to $AIP(O)$ for the remaining. Recall that the function ψ_O can be defined on $P(\mathbb{V})$ without losing any information. We can parametrize the classes of $P(\mathbb{V})$ as follows:

$$\{\cos\theta\, e_1 + \sin\theta\, e_2 : \theta \in [0,\pi)\} \,.$$

Now, parametrize O' by the parameters $\{\alpha, \beta\}$ in the following way:

$$O' = \left\{ \begin{pmatrix} 2+\cos\alpha\, \cos\beta & \sin\alpha \\ \sin\alpha & 2+\cos\alpha\, \sin\beta \end{pmatrix} : \alpha, \beta : \alpha \in [0,\pi], \beta \in [0, 2\,\pi) \right\} \,.$$

At this point, we compute:

$$AIP(O') = -1 + \max_{\theta\in[0,\pi)} \sup\nolimits_{\alpha,\alpha'\in[0,\pi],\beta,\beta'\in[0,2\pi)}\ \phi \tag{7}$$

where the function ϕ is

$$\frac{\cos^2\theta\,(2+\cos\alpha\, \cos\beta) + \sin^2\theta\,(2+\cos\alpha\, \sin\beta) + 2\,\sin\theta\, \cos\theta\, \sin\alpha}{\cos^2\theta\,(2+\cos\alpha'\, \cos\beta') + \sin^2\theta\,(2+\cos\alpha'\, \sin\beta') + 2\,\sin\theta'\, \cos\theta'\, \sin\alpha'}$$

To solve the optimization problem in Equation (7), we used R, which provides us a solution: $AIP(O') = 2.5$. Observe not only the difference with the previous rough estimation but also the computational costs.

To show how the AIP is related to different measures and relative errors, we need the following result.

Proposition 3.6. *Let O be an imprecise inner product and let $g \in O$. For any $g' \in G^0(O)$, it holds:*

(i) $(1 + AIP)\, g' \succeq g$, *and*
(ii) $g \succeq \frac{1}{1+AIP}\, g'$.

Proof. (i) By contradiction. Assume that there exists $x \in \mathbb{V}$ such that $(1 + AIP)\, g'(x,x) < g(x,x)$. Hence, we can write:

$$\sup\nolimits_{g_1,g_2\in O} \frac{g_1(x,x)}{g_2(x,x)} < \frac{g(x,x)}{g'(x,x)} \,.$$

Since $g' \in G^0(O)$, we can find a sequence $(g_i)_i$ of elements of O such that

$$g'(x,x) \geq \lim_{i\to\infty} g_i(x,x) \,.$$

Combining the previous two equations and recalling that $g \in O$, we have

$$\sup\nolimits_{g_1,g_2\in O} \frac{g_1(x,x)}{g_2(x,x)} < \lim_{i\to\infty} \frac{g(x,x)}{g_(x,x)} \leq \sup\nolimits_{g_1,g_2\in O} \frac{g_1(x,x)}{g_2(x,x)} \,.$$

A contradiction.

To prove (ii), we can use a similar reasoning taking now into account that we can find a sequence fulfilling

$$g'(x,x) \leq \lim_{i\to\infty} g_i(x,x) \,.$$

$\square$

Remark 6. The orders in Proposition 3.6 are sharp, as the following example shows. Fix $g \in G^+(\mathbb{V})$, and let $O = \{g_\lambda = \lambda\, g : \lambda \in [a,b]\}$, $a > 0$, be a round imprecise inner product. We have seen that $AIP = b/a - 1$. Taking g_a, from Proposition 3.6, we deduce that $\frac{b}{a}\, g_a \succeq g_\lambda$ for any $\lambda \in [a,b]$; equivalently, $b\, g \succeq g_\lambda$ for any $\lambda \in [a,b]$, that is, $b \geq \lambda$ for any $\lambda \in [a,b]$. At the same time, taking g_b, the same proposition leads to: $g_\lambda \succeq \frac{a}{b}\, g_b$; equivalently, $g_\lambda \succeq a\, g_\lambda$ for any $\lambda \in [a,b]$, that is, $a \leq \lambda$ for any $\lambda \in [a,b]$.

Let us see that Proposition 3.6 can be applied to different statistical problems.

Remark 7. (See and compare with Couso et al. (2007), Dubois et al. (2008), Quaeghebeur (2008) and Walley (1991).) Let X and Y be two random vectors of a Euclidean vector space $(\mathbb{V}, g)$. Assume that $\mathbb{E}^X[X] = \vec{0}$, where $\mathbb{E}^X$ denotes the expectation operator associated to X. Denote the covariance of X by Cov_X. Following Remark 1, we have: $Cov_X(y,y) = \mathbb{E}^X\left[g\left(X - \mathbb{E}[X], y\right)^2\right] = \mathbb{E}^X\left[g(X,y)^2\right]$. Consequently, $Cov_X(Y,Y) = \mathbb{E}^X\left[g(X,Y)^2\right]$. This fact means that the variance of the random variable $g(X,Y)$ is:

$$Var[g(X,Y)] = \mathbb{E}\left[g(X,Y)^2\right] - \left(g\left(\mathbb{E}[X], \mathbb{E}[Y]\right)\right)^2 = \mathbb{E}^Y\left[Cov_X(Y,Y)\right]. \tag{8}$$

Assume now that we have uncertainty about the distribution of X. Moreover, the uncertainty of the covariance of X is modelled by an imprecise covariance O. Proposition 3.6 implies the following fact -that holds when $\mathbb{E}[X] = 0$:

$$\frac{1}{1 + AIP(O)}\, \mathbb{E}^Y\left[Cov_X^*(Y,Y)\right] \leq Var[g(X,Y)]) \leq (1{+}AIP(O))\, \mathbb{E}^Y\left[Cov_X^*(Y,Y)\right], \tag{9}$$

where $Cov_X^* \in G^0(O)$. Let us see that Equation (9) is useful. For this end, let us take an example.

Example 3.7. Consider two random vectors X and Y of a two-dimensional vector space, with $\mathbb{E}[X] = 0$. The probability distributions are unknown. However, some piece of information bounds the covariance matrix Cov_X of X, which must satisfy the following equation:

$$Cov_X \in O = \left\{ \begin{pmatrix} a & c \\ c & b \end{pmatrix} : (a-2)^2 + (b-2)^2 + c^2 = 1 \right\}.$$

The distribution of Y is also unknown. We know that Y is atomic and uniformly distributed at its three atoms, which are:

$$\begin{aligned} \{z_1 = \sin(\omega)\, e_1 + \cos(\omega)\, e_2,\ & z_2 = \sin(\omega + 2\,\pi/3)\, e_1 + \cos(\omega + 2\,\pi/3)\, e_2 \\ & z_3 = \sin(\omega + 4\,\pi/3)\, e_1 + \cos(\omega + 4\,\pi/3)\, e_2 : \omega \in [\omega_0, \omega_1] \subset [0, 2\,\pi)\}\,. \end{aligned}$$

We desire to know the maximum and the minimum values of the variance of $g(X,Y)$. In fact, we achieve bounds for these numbers. From Equation (8), the problem for the maximum value of this variance can be read as follows:

$$
\begin{array}{ll}
\max\limits_{\substack{\omega\in[\omega_1,\omega_2]\\ a,b,c\in\mathbb{R}}} & \frac{1}{3}\,a\,(\sin^2(\omega)+\sin^2(\omega+\frac{2\pi}{3})+\sin^2(\omega+\frac{4\pi}{3}))\\
& +\frac{1}{3}\,b\,(\cos^2(\omega)+\cos^2(\omega+\frac{2\pi}{3})+\cos^2(\omega+\frac{4\pi}{3}))\\
& +\frac{1}{3}\,c\,(\sin(2\,\omega)+\sin(2\,\omega+\frac{2\pi}{3})+\sin(2\,\omega+\frac{4\pi}{3}))\\
\text{s.t.} & (a-2)^2+(b-2)^2+c^2=1\,.
\end{array}
$$

Let us compact notation to solve the previous problem. Let us write it as follows:

$$
\sup Var(g(X,Y)) = \max_{\omega\in[\omega_1,\omega_2]} \sup_{Cov_X\in O} \frac{1}{3}\sum_{i=1}^{3} Cov_X(z_i,z_i)\,.
$$

Now, making use of Equation (9), we obtain,

$$
\sup Var(g(X,Y)) \leq \frac{1+AIP(O)}{3} \sup_{\omega\in[\omega_1,\omega_2]} \sum_{i=1}^{3} Cov_X(z_i,z_i)\,,
$$

where $Cov_X \in G^0(O)$. In particular, since $Cov_X^* = \mathrm{diag}(2,2) \in G^0(O)$ (see Example 2.9) and recalling that $AIP(O) = 2.5$, we conclude:

$$
\begin{aligned}
\sup Var(g(X,Y)) &\leq \frac{1+AIP(O)}{3} \sup_{\omega\in[\omega_1,\omega_2]} \sum_{i=1}^{3} Cov_X^*(z_i,z_i)\\
&= \frac{2\,(1+AIP(O))}{3} \sup_{\omega\in[\omega_1,\omega_2]} \sum_{i=1}^{3} g(z_i,z_i)\\
&= 2+2\,AIP(O)\\
&= 7\,.
\end{aligned}
$$

Following a similar reasoning, it can be checked that:

$$
\inf Var(g(X,Y)) \geq \frac{2}{3.5}\,.
$$

The remaining of this section is aimed at providing results to estimate other quantities of geometrical or statistical interest. Let us begin with the variance.

Theorem 3.8. *Let O be an imprecise inner product and $g \in G^0(O)$. An interval estimation of the variance of a random vector X is*

$$
\left[\frac{1}{1+AIP}\,Var_g(X),(1+AIP)\,Var_g(X)\right]\,. \tag{10}
$$

That is, for any $g' \in O$, the g'-variance of X must be contained in Equation (10).

Proof. Given a random vector X of $(\mathbb{V},g)$, its variance is $Var_g[X] := \mathbb{E}[g(X-\mu,X-\mu)]$, where μ denotes the expectation of X -we assume it exists. Combining Proposition

2.7 and Lemma 3.6, we have,

$$\frac{1}{1+AIP}\, Var_g[X] \leq Var_{g'}[X] \leq (1+AIP)\, Var_g[X] \quad \forall g' \in O\,,$$

which ends the proof. $\square$

Remark 8. In the case of a round imprecise inner product, the study of the variance becomes simple. In fact, assuming $O = \{\lambda\, g : \lambda \in [a,b] \subset \mathbb{R}^+, g \in G(\mathbb{V})\}$, we have

$$\left[\inf_{g'\in O} Var_{g'}(X), \sup_{g'\in O} Var_{g'}(X)\right] = [a\, Var_g(X), b\, Var_g(X)]\ .$$

Recalling that $AIP(O) = (b-a)/a$, applying Theorem 3.8 to g_b we obtain:

$$\left[\inf_{g'\in O} Var_{g'}(X), \sup_{g'\in O} Var_{g'}(X)\right] \subseteq \left[a\, Var_g(X), \frac{b^2}{a}\, Var_g(X)\right]\ .$$

Applying Theorem 3.8 to g_a we obtain:

$$\left[\inf_{g'\in O} Var_{g'}(X), \sup_{g'\in O} Var_{g'}(X)\right] \subseteq \left[\frac{a^2}{b}\, Var_g(X), b\, Var_g(X)\right]\ .$$

Combining the last two equations,

$$\left[\inf_{g'\in O} Var_{g'}(X), \sup_{g'\in O} Var_{g'}(X)\right] \subseteq [a\, Var_g(X), b\, Var_g(X)]\ .$$

That is, the techniques work sharply.

Example 3.9. Let O be the imprecise inner product of Examples 2.9 and 3.5. Let Z be a 2-dimensional vector which is normally distributed with covariance matrix:

$$\Sigma = \begin{pmatrix} \sigma_1^2 & \sigma_{12} \\ \sigma_{12} & \sigma_2^2 \end{pmatrix}\ .$$

It is showed that $g_1 := \mathrm{diag}(2,2)$ belongs to $G^0(O)$ (see Example 3.5). We have that the g_1-variance of Z is $2\,(\sigma_1^2+\sigma_2^2)$. In the same remark, it is also shown that $AIP(O) = 2.5$. Applying Theorem 3.8, we obtain that, for any $g \in O$, the g-variance of Z must be contained in

$$\left[\frac{2\,(\sigma_1^2+\sigma_2^2)}{3.5}, 7\,(\sigma_1^2+\sigma_2^2)\right]\ .$$

Notice that no computational cost was needed.

In a more geometrical setting, let us consider the lengths of the curves:

Theorem 3.10. *Let O be an imprecise inner product and $g \in G^0(O)$. An interval estimation of the length of a curve γ is*

$$\left[\frac{1}{\sqrt{1+AIP}}\, g-\mathrm{length}(\gamma),\ \sqrt{1+AIP}\, g-\mathrm{length}(\gamma)\right]\ . \tag{11}$$

That is, for any $g' \in O$, the g'-length of γ must be contained in Equation (11).

Proof. First, observe the following fact. Let $g \in G(\mathbb{V})$ and $\lambda \in (0, \infty)$. Define $\widehat{g} = \lambda\, g$. The $\widehat{g}$-length of a curve $\gamma : I \to \mathbb{V}$ is related to its g-length by means of a simple formula,

$$\begin{aligned} \widehat{g} - \text{length} &= \int_I \sqrt{\widehat{g}(\gamma'(t), \gamma'(t))}\, dt = \sqrt{\lambda} \int_I \sqrt{g(\gamma'(t), \gamma'(t))}\, dt \\ &= \sqrt{\lambda}\, g - \text{length}\,. \end{aligned}$$

Denote $\overline{g} = (1 + AIP)\, g$ and $\underline{g} = (1 + AIP)^{-1}\, g$. Making use of the previous lemma and Proposition 2.7, we have: the $\overline{g}$-length of a curve is not smaller than its g'-length for any $g' \in O$; and the $\underline{g}$-length of a curve is not bigger than its g'-length for any $g' \in O$. From the above observation, for any $g' \in O$ we have:

$$\frac{1}{\sqrt{1 + AIP}}\, g - \text{length} \leq g' - \text{length} \quad \text{and} \quad \sqrt{1 + AIP}\, g - \text{length} \geq g' - \text{length}\,.$$

□

Example 3.11. Consider again the imprecise inner product of Examples 2.9 and 3.5. In the corresponding basis, consider the curve: $\gamma(t) = (\sin(t), \cos(t))$, $t \in [0, 1]$. Its velocity is $\gamma'(t) = (\cos(t), -\sin(t))$. For any

$$g = \begin{pmatrix} a & c \\ c & b \end{pmatrix}$$

which belongs to O, the g-length of α is:

$$g - \text{length} = \int_0^1 \sqrt{a\, \cos^2(t) + b\, \sin^2(t) - 2\, c\, \sin(t)\, \cos(t)}\, dt\,.$$

Note that the previous equation is hard to compute even for a single $g \in O$. In fact, it is an elliptic integral (Abramowitz and Stegun (1965)). Nevertheless, for $g_1 = \text{diag}(2, 2)$, $g_1 \in G^0(O)$, the previous length is easy to obtain:

$$g_1 - \text{length} = \sqrt{2}\,.$$

As a consequence, for any $g \in O$, the g-length of γ must be contained in $[\sqrt{2/3.5}, \sqrt{7}]$. Undoubtedly, $G^0(O)$ has considerably simplified the computations.

To close this example note that, from an analytical viewpoint, we have approximated the solutions of the following optimization problem:

$$\begin{array}{rc} \text{Extremal values of} & \int_0^1 \sqrt{a\, \cos^2(t) + b\, \sin^2(t) - 2\, c\, \sin(t)\, \cos(t)}\, dt \\ \text{subject to} & (a-2)^2 + (b-2)^2 + c^2 = 1 \\ & |c| > 0.1\,. \end{array}$$

To end this section we focus on the volume of a set. For this quantity, the corresponding bound of the error can be significantly improved by means of a new precision parameter.

Consider an imprecise inner product O of an n-dimensional vector space $\mathbb{V}$. Let $W = \{e_1^*, \ldots, e_n^*\}$ be a basis of the dual vector space of $\mathbb{V}$, denoted by $\mathbb{V}^*$ (we assume that the basis of $\mathbb{V}$ is positively oriented; that is, we measure the volume of a set according to the order of the vectors of this basis, see O'neill (1983) for further details). For each $g \in O$, there exists a positive number δ_g such that the volume element of $(\mathbb{V}, g)$, dV_g, is $\delta_g\, e_1^* \wedge \ldots \wedge e_n^*$. We can define two positive numbers, $\mu^0|_W = \sup\{\mu > 0 : \exists g \in O, dV_g = \mu\, e_1^* \wedge \ldots \wedge e_n^*\}$, and $\mu_0|_W = \inf\{\mu > 0 : \exists g \in O, dV_g = \mu\, e_1^* \wedge \ldots \wedge e_n^*\}$. Clearly, $\mu_0|_W$ and $\mu^0|_W$ depend on the chosen basis of $\mathbb{V}^*$. However, its quotient does not. That is, $\mu^0|_W/\mu_0|_W$ does not depend on the basis of $\mathbb{V}^*$ considered. To prove that, let $V = \{a_1^*, \ldots, a_n^*\}$ be another basis of $\mathbb{V}^*$ (also positively oriented). Then, $e_1^* \wedge \ldots \wedge e_n^* = \alpha\, a_1^* \wedge \ldots \wedge a_n^*$ for some positive constant α. Hence, $\mu_0|_V = \alpha\, \mu_0|_W$ and $\mu^0|_V = \alpha\, \mu^0|_W$. Finally, $\mu^0|_W/\mu_0|_W = \mu^0|_V/\mu_0|_V$. We define the following parameter in an equivalent way than $\mu^0|_W/\mu_0|_W - 1$,

Definition 3.12. Let O be an imprecise inner product. Its *volume precision parameter (VPP)* is

$$VPP(O) = \sup_{g_1, g_2 \in O} \{\lambda : \lambda\, dV_{g_1} = dV_{g_2}\} - 1\,.$$

Let $\{e_1, \ldots, e_n\}$ be a basis of $\mathbb{V}$ and O be an imprecise inner product. The VPP measures the quotient between the biggest and the smallest volume of the parallelepiped spanned by such basis when this volume is measured by the elements of O. Note again that this is independent of the basis considered.

The VPP also provides information about the shape of an imprecise inner product. Notice that the AIP takes into account the maximum error in a direction. In broad terms, the VPP is a measure about the imprecision averaging on all directions.

The VPP fulfils Axioms (A1)-(A2)-(A3):

Lemma 3.13. (i) *The volume precision parameter is non-negative.* (ii) *If the information is exact, then its value is* 0. (iii) *If O' is more imprecise than O, then $VPP(O') \geq VPP(O)$.*

The proof is straightforward.

Observe that a non-exact imprecise inner product can have a null volume precision parameter:

Example 3.14. Let $\{e_1, e_2\}$ be a basis of a vector space. Consider the following 1-parametric family of inner products,

$$O_\lambda = \left\{ g_\lambda : g_\lambda(e_1, e_1) = \lambda,\, g_\lambda(e_1, e_2) = 0,\, g_\lambda(e_2, e_2) = \frac{1}{\lambda} \right\},$$

where $\lambda \in (1, 2)$. We have that the volume element of any $g \in O_\lambda$ is $e_1^* \wedge e_2^*$. Hence, $SPP(O_\lambda) = 0$.

The VPP also plays a fundamental role in the volume estimations.

Theorem 3.15. *Let O be an imprecise inner product of $\mathbb{V}$. An interval estimation of* *the volume of a measurable set U is*

$$\left[\frac{1}{1+VPP}\,Vol_g(U),(1+VPP)\,Vol_g(U)\right]\,, \tag{12}$$

where $g \in G(O)$. That is, for any $g' \in O$, the g'-volume of U must be contained in Equation (12).

Proof. It follows from the definitions that, for any $g, g' \in O$, it holds: $dV_g \leq (1 + VPP)\,dV_{g'}$, in the sense that there exists $\alpha \geq 1$ such that $\alpha\,dV_g = (1+VPP)\,dV_{g'}$. $\square$

To close this section, we give a nice relationship between the AIP and the VPP:

Proposition 3.16. *For any imprecise inner product of $\mathbb{V}$, it holds:*

$$(1 + AIP)^{n/2} \geq VPP + 1\,. \tag{13}$$

Proof. Fix a basis $\{e_1, \ldots, e_n\}$ of $\mathbb{V}$. Recall that for any inner product g, $dV_g = \sqrt{\det(g)}e_1^* \wedge \ldots \wedge e_n^*$ -where the determinant is computed from the Gram matrix of g in the fixed basis of $\mathbb{V}$. Now, let $g_1, g_2 \in O$. From Proposition 3.6, we have: $(1 + AIP)^{n/2}\,dV_{g_1} \geq dV_{g_2}$. Recalling Definition 3.12, the result is proven. $\square$

3.1. *Controlling the error*

In all the interval estimations made, no mention about minimizing the error was made. For instance, in Theorem 3.10, to find an interval with minimum length, we must search for an inner product of O which minimizes the length of the curve. However, this is another optimization problem which may be hard to solve (perhaps, worse than computing the lengths of different inner products of the imprecise inner product). Example 3.11 is very illustrative in this sense. Other approach consists in evaluating the measurand with several elements of the imprecise inner product and obtaining several intervals. In Remark 8 we did this approach. A very valuable extra strategy consists in obtaining different intervals from different inner products of the imprecise inner product. Then, the intersection of these intervals can be used to improve the final estimation. Better information must be obtained.

At this point, we would need some information about the potential of improving the estimations made by our techniques. The following note can be applied to improve the estimations in some particular cases.

Remark 9. Let O be an imprecise inner product. Consider a curve (or a random variable) whose support is contained in a subspace H. Instead considering $AIP(O)$, we can restrict this quantity onto H. More precisely, denote by π_H the projection onto H. We have that $\pi_H\,O$ is an imprecise inner product of H. Observe that $AIP(\pi_H\,O) \leq AIP(O)$. Then, it is possible to replace the AIP with $AIP(\pi_H\,O)$ to play its corresponding role of error for such measures.

In the general case, we do not have the structure shown in the previous remark. Hence, we do not know how good is our interval estimation.

In another setting, observe that the previous parameter is not related to the shape of O. We claim a parameter related to this. In the case of a round imprecise inner product, it must vanish.

The following notion fulfils all our claims here.

Definition 3.17. Let O be an imprecise inner product. Its *shape precision parameter (SPP)* is

$$SPP(O) = 1 - \frac{\min_{x\in\mathbb{V}}\psi_O(x)}{\max_{x\in\mathbb{V}}\psi_O(x)}\,.$$

We note that the SPP is contained in $[0, 1]$.

Note also that any round imprecise inner product vanishes its SPP. Recall that for the round imprecise inner products, the interval estimations are sharp. A SPP near to 0 means that the imprecise inner product is close to be like a round imprecise inner product. From another point of view, the SPP is a measure of the anisotropy of the imprecise inner product.

A SPP close to 0 also means that the AIP is enough as a measure of error, and no extra study is needed. Opposite, a SPP close to 1 implies that it may be useful a deeper study of the AIP for a particular geometric quantity, if we desire to sharpen the error estimation. The simulations in the last section will illustrate this fact.

Example 3.18. Fix a basis of $\mathbb{V}$. In this basis, consider the following collection of imprecise inner products O_t $(t \geq 1)$:

$$O = \{\mathrm{diag}(1, s) : s \in [1, t]\}\ .$$

Its SPP is equal to $1 - 1/t$. As $t \approx 1$, it takes values close to 0, guaranteeing that the estimations depend *weakly* on directions. As t becomes bigger, the dependence of the error on the directions becomes stronger. Let us observe, in the first case, $t \approx 1$, the length of the vector e_2 is 1 approximately, as the exact length of the vector e_1; in the second case, the length of e_1 is still exact but the length of e_2 has a high range: $[1, t]$ and, consequently, more error.

Unfortunately, increasing imprecision does not imply that the SPP increases,

Example 3.19. Consider the following imprecise inner products:

$$O = \{\mathrm{diag}(1, s) : s \in [1, 2]\}\ , O' = \{\mathrm{diag}(r, s) : r, s \in [1, 2]\}\ .$$

Clearly, O' is more imprecise than O. However, $SSP(O') = 0$ and $SPP(O) = 1/2$.

4. Simulations

In this section, we study numerically Example 3.11 (with O'). Recall all the ingredients. From Example 2.9, fixed a basis $\{e_1, e_2\}$ of a two-dimensional vector space, we consider the imprecise inner product:

$$O' = \left\{ \begin{pmatrix} a & c \\ c & b \end{pmatrix} : (a-2)^2 + (b-2)^2 + c^2 = 1 \right\}\ .$$

In Example 3.11, we consider the curve: $\gamma(t) = (\sin(t), \cos(t))$, $t \in [0, 1]$. Its g-length is:

$$g - \text{length}(\gamma) = \int_0^1 \sqrt{a\, \cos^2(t) + b\, \sin^2(t) - 2\, c\, \sin(t)\, \cos(t)}\, dt\,. \qquad (14)$$

We estimated that the length of γ must be contained in $[\sqrt{2/3.5}, \sqrt{7}]$.

In this simulation, we compute Equation (14) for different inner products of O. But first, let us compute its SPP value. By a computer program it can be estimated in 0.9 at least.

We can parametrize O:

$$O = \left\{ \begin{pmatrix} 2 + \cos\alpha\, \cos\beta & \sin\beta \\ \sin\beta & \cos\alpha\, \sin\beta \end{pmatrix} : \alpha \in [0, \pi], \beta \in [0, 2\,\pi] \right\}.$$

We take a collection U of elements of O built by a grid on the parameters α and β.

For each inner product of U we compute (14). Our data satisfy Theorem 3.10, as we show:

$$[\min(U), \max(U)] = [0.87, 1.79] \subset [0.75, 2.65] = \text{Theoretical}\,. \qquad (15)$$

We observe that we accurately estimate the infimum of the values of Equation (14). Recalling the values of the SPP, we may have certain margin to improve the prediction of the maximum of Equation (14).

Now, we show simulations of the interval estimation of the length of γ for other imprecise inner products. We show their parameters and we can compare the predicted interval with the obtained by several elements of each imprecise inner product.

These imprecise inner products are obtained by mixing two inner products; namely, fixed $g_1, g_2 \in G^+(\mathbb{V})$, we define the *mixed imprecise inner product by* g_1, g_2, denoted by $Mi(g_1, g_2)$ by

$$Mi(g_1, g_2) := \bigcup \{\rho\, g_1 + (1 - \rho)\, g_2 : \rho \in [0, 1]\}\,.$$

Note that $Mi(g_1, g_2)$ is proper (since $g_1, g_2 \in G^+(\mathbb{V})$). Note also that, from a statistical point of view, this corresponds to mixture models (see Example 2.3).

For all the simulations, we fix $g_1 = \text{diag}(1, 1)$. The g_1-length of γ is 1. Consequently, Theorem 3.10 implies that: for any $g' \in Mi(g_1, g_2)$, it holds: $g' - \text{length}(\gamma) \in [(1 + AIP)^{-1/2}, (1 + AIP)^{1/2}]$. The results of the simulations are presented in Table 1.

Finally, note that the bigger the SPP, the more error is made in the interval estimation.

5. Conclusions

There exist real situations where imprecision of an inner product or a covariance appears. We define the notion of an imprecise inner product as a model to deal with this situation. More precisely, an imprecise inner product (or an imprecise covariance) is a set of inner products (or covariances). We have introduced several precision parameters which are related to the size and shape of an imprecise inner product. The first one,

g_2	AIP	Theoretic $[(1+AIP)^{-0.5}, (1+AIP)^{0.5}]$	Simulated interval	SPP
$\begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix}$	1	$[0.707, 1.414]$	$[1, 1.414]$	0
$\begin{pmatrix} 1.2 & 0.1 \\ 0.1 & 0.9 \end{pmatrix}$	0.23	$[0.901, 1.109]$	$[1, 1.090]$	0.186
$\begin{pmatrix} 0.8 & 0.1 \\ 0.1 & 0.9 \end{pmatrix}$	1.35	$[0.859, 1.164]$	$[0.947, 1]$	0.232
$\begin{pmatrix} 3 & 0.01 \\ 0.01 & 0.4 \end{pmatrix}$	2	$[0.576, 1.733]$	$[1, 1.525]$	0.662
$\begin{pmatrix} 2 & 1 \\ 1 & 0.7 \end{pmatrix}$	5.35	$[0.396, 2.521]$	$[1, 1.533]$	0.84
$\begin{pmatrix} 1.2 & 1 \\ 1 & 0.9 \end{pmatrix}$	24.6	$[0.197, 5.058]$	$[1, 1.348]$	0.961

Table 1. The results of the simulations.

the AIP, refers to the maximum relative error which occurs when measuring the length of an arbitrary vector. It can also be seen as a measure of how big an imprecise inner product is. The second and last one, the VPP refers to the error of measuring sets. It can be seen as a measure of the relative error averaging on directions. We have also introduced an extra parameter, the SPP, which has served us to measure the shape of an imprecise inner product. It is not a precision parameter in our sense (since it does not satisfy (A3)). However, important information can be derived from it; in fact, it informs about the anisotropy of the imprecise inner product.

In another setting, we have a quantity that we desire to estimate (as a geometrical quantity, like the length of a curve). Clearly, one approach consists in obtaining its extremal values for the elements of the imprecise inner product. This approach can yield high computational costs. Instead, we are able to provide an interval estimation by means of our precision parameters. In fact, the results in Section 2 are proof of it. The value of the SPP provides us hints about how good are our approaches.

The main advantage of our techniques is clear: the computational costs decrease significantly. These techniques can be applied to numerous cases. Not only the quantities we have focused on this article but other ones which admit a study like we have done. Therefore, we observe a great potential in applications in this work.

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