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Tests of symmetry based on the bootstrap estimation of the asymptotic variance of a skewness coefficient

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A B S T R A C T

Testing for symmetry about an unknown median is a problem that has attracted the attention of statisticians for decades. Many of the existing tests of symmetry are based on the asymptotic distribution of a skewness coefficient, which typically is asymptotically normal and centered around zero for symmetric distributions. Unfortunately, the asymptotic variance depends on the underlying distribution and differs from one symmetric distribution to another one. A possible way out is to estimate this asymptotic variance from the sample by means of the bootstrap. In this paper, we explore this approach for six different skewness coefficients existing in the literature. Extensive experiments are performed for comparing the performance of the six associated tests of symmetry to that of two state-of-the-art symmetry tests in terms of preservation of the significance level under several symmetric distributions, power under asymmetric distributions and robustness in the presence of outliers. Even though the results show no clear best test, we conclude by providing some guidelines for choosing a test of symmetry based on the needs of the user.

1. Introduction

The term *symmetry* refers to the property of an object that is invariant under some geometrical transformation such as a reflection, a rotation or a translation, whereas the term *asymmetry* simply refers to the absence of this property. Since ancient times, the notion of symmetry has been linked to venerated ideals such as order, beauty and harmony, while, on the contrary, the notion of asymmetry has been known to instill dynamism and excitement. It is not surprising then that artists have used the presence or absence of symmetry as a recurring resource in their artworks. For instance, back in the 15th century, Leonardo da Vinci represented in his work *the Vitruvian Man* the ideal human body based on beauty standards such as symmetry and proportionality. On the contrary, the 19th century painting *The Starry Night* by Vincent van Gogh shows a clearly asymmetric design that, playing with colors, sizes and shapes, still manages to create a feeling of balance.

The notion of symmetry also appears in almost all fields of mathematics. For instance, in mathematical analysis, an even function is a function with a graph that is symmetric with respect to the y-axis and an odd function is a function with a graph that is rotationally symmetric with respect to the origin; in linear algebra, a symmetric matrix is a square matrix that is equal to its transpose; and, in set theory, a symmetric relation is a binary relation for which if the relation stands from x to y, then it also stands from y to x. In the field of statistics, which is the one of interest to this paper, symmetry of a random variable X simply means that the distribution of the random variables $X - x_0$ and $x_0 - X$ is the same for some real value x_0. A common example in which this property in fact

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turns out to be very handy arises when dealing with standard normal distribution tables, where the right-tail probabilities are easily computed from the left-tail probabilities due to the symmetry of the standard normal distribution about zero.

There is little doubt that symmetry is regarded as a desirable property for a random variable, either directly or indirectly through some more restrictive assumption (e.g., normality). For instance, we refer to two classical hypothesis tests such as Wilcoxon's signed-rank test [41] and Student's t-test [37], which under their most common preliminary assumptions require the underlying distribution to be symmetric in the first case and normal (and, thus, symmetric) in the second case. However, real-life data is not always perfect and the assumption of the data coming from a symmetric distribution may not always be reasonable. Unfortunately, if indeed the data does not come from a symmetric distribution, easy tasks such as data visualization by means of a boxplot [22] and outlier detection [21] become more involved, thus stimulating the necessity of designing specific techniques for non-symmetrical data. It is for this very reason that testing for symmetry has become an interesting study subject that has attracted the attention of statisticians for decades.

Two separate literatures on tests of symmetry exist depending on whether the point of symmetry is known or unknown. Some prominent symmetry tests of the former type are due to Butler [7] based on a statistic in the spirit of the Kolmogorov-Smirnov goodness-of-fit test, to McWilliams [28] based on a runs statistic, and to Thas, Rayner and Best [38] based on the Wilcoxon signed-rank statistic. Even though some authors such as Noughabi and Jarrahiferiz [32] or Xiong, Zhuang and Qiu [42] have recently proposed alternative approaches to the problem of testing for symmetry about an unknown point of symmetry, the literature on the latter type of symmetry tests is typically built around the notion of skewness coefficient, which is a measure of the degree of asymmetry of a distribution. The first author to call attention to the need for measuring the skewness of a distribution was Pearson [34] back in the 19th century, but a formalization of the notion of skewness coefficient is also the result of some further works such as those by van Zwet [40] and Oja [33]. Some prominent tests of symmetry about an unknown symmetry point based on the use of skewness coefficients are due to Gupta [17] based on the asymptotic distribution of the moment skewness coefficient, to Cabilio and Masaro [8] based on the asymptotic distribution of the nonparametric skewness coefficient, to Mira [30] based on a measure of skewness proposed by Bonferroni, and to Miao, Gel and Gastwirth [29] based on a robustification of Cabilio and Masaro's statistic.

A typical problem when dealing with symmetry tests based on the asymptotic distribution of a skewness coefficient is that this asymptotic distribution is not distribution-free in the sense that it is not the same for all symmetric distributions. In particular, skewness coefficients are typically asymptotically normal and centered around zero for symmetric distributions, yet the asymptotic variance usually depends on the underlying distribution. Some authors have proposed to consider a reference distribution (e.g., the normal distribution in [1] and the logistic distribution in [3]), however the most dominant approach simply aims at estimating said asymptotic variance. In this direction, we propose to resort to the bootstrap for estimating the asymptotic variance for the asymptotic distribution of several skewness coefficients. This will result in the introduction of several tests of symmetry that have not been explored before in the literature, even though, admittedly, the use of bootstrap techniques has already been considered in the context of tests of symmetry for estimating the distribution of the test statistic [26,44].

The remainder of the paper is structured as follows. Section 2 presents some preliminaries related to the notions of symmetry and skewness coefficient that will be necessary throughout the paper. Section 3 formalizes the presented bootstrap tests of symmetry based on the asymptotic distribution of a skewness coefficient. Six different skewness coefficients are considered and the power of the six associated tests of symmetry in different conditions is compared with two state-of-the-art methods in Section 4. A discussion on the influence of the number of bootstrap replications and the chosen type of bootstrap is presented in Section 5. We end with some conclusions and future work in Section 6.

2. Preliminaries on the notion of symmetry

2.1. A formalization of the notion of symmetry

A random variable X is called symmetric about x_0 if the distribution of $X - x_0$ is the same as that of $x_0 - X$, i.e.,

$$P(X \leq x_0 - x) = P(X \geq x_0 + x)\,, \text{ for any } x \in \mathbb{R}\,.$$

If X is a random variable with cumulative distribution function F for which there exists no point $x \in \mathbb{R}$ such that $P(X = x) > 0$, then symmetry about x_0 is equivalently defined by

$$F(x_0 - x) = 1 - F(x_0 + x)\,, \text{ for any } x \in \mathbb{R}\,.$$

Additionally, if X is a continuous random variable with a density function f, then symmetry about x_0 is also equivalently defined by

$$f(x_0 - x) = f(x_0 + x)\,, \text{ for any } x \in \mathbb{R}\,.$$

If one says that a random variable X is symmetric, it is meant that there exists $x_0 \in \mathbb{R}$ about which X is symmetric. The median (if it is unique) and the mean (if it exists) of a symmetric random variable coincide and are necessarily the point at which the symmetry occurs.

Back in 1981, Oja [33] introduced some desirable properties for measuring the asymmetry of a random variable, giving raise to a formalization of the notion of *skewness coefficient*. In the context of this work, we will say that a skewness coefficient is any function

$\gamma : \mathcal{F} \to \mathbb{R}$ assigning to any random variable ($\mathcal{F}$ denotes a set of random variables) a measurement of its skewness. Formally, it is typically required that a skewness coefficient satisfies the following properties:

(i) $\gamma(X) = 0$, if X is symmetric;
(ii) $\gamma(aX + b) = \gamma(X)$, for any $a > 0$ and $b \in \mathbb{R}$;
(iii) $\gamma(-X) = -\gamma(X)$.

It should be noted that condition (i) does not require that $\gamma(X) = 0$ if and only if X is symmetric and, actually, one could find asymmetric distributions for which the value of any of the most prominent skewness coefficients equals zero.

Oja further discussed an additional condition concerning skewness-based orderings already discussed by van Zwet [40], but this condition is here left out since it is not fulfilled by all skewness coefficients that will be presented right after. Also, some more recent axiomatic definitions of skewness [5] aim at introducing an additional condition in which the coefficient is required to be bounded and normalized within the interval $[-1, 1]$.

2.2. Skewness coefficients

The most prominent skewness coefficient – hereinafter referred to as the *moment skewness coefficient* – is typically attributed to Pearson [34] (and also to Charlier [10] and Edgeworth [11]) and defined as follows:

$$\gamma_{\mathrm{M}}(X) = \frac{E\big((X-\mu)^3\big)}{\sigma^3},$$

where μ represents the population mean and σ represents the population standard deviation. A sample version of this coefficient is obtained by substituting the population parameters by their sample counterparts.

Another popular skewness coefficient that is oftentimes attributed to Pearson (and also to Yule [43]) – hereinafter referred to as the *nonparametric skewness coefficient* or as Fisher's moment skewness coefficient – is defined as follows:

$$\gamma_{\mathrm{NP}}(X) = \frac{\mu - \mathrm{Me}(X)}{\sigma},$$

where μ represents the population mean, σ represents the population standard deviation and $\mathrm{Me}(X)$ represents the population median. A sample version of the coefficient is also straightforwardly obtained by substituting the population parameters by the sample parameters. Note that, for historical reasons, some authors consider the nonparametric skewness coefficient multiplied by a constant term 3 since it turns out that for many distributions the difference between the mean and the mode is approximately three times the difference between the mean and the median (see page 121 of [43]).

An alternative to the nonparametric skewness coefficient (see, e.g., [16] and [2]) – hereinafter referred to as the *Groeneveld-Meeden skewness coefficient* – substitutes the standard deviation in the denominator by $E(|X - \mathrm{Me}(X)|)$, as follows:

$$\gamma_{\mathrm{GM}}(X) = \frac{\mu - \mathrm{Me}(X)}{E(|X - \mathrm{Me}(X)|)}.$$

A sample version of this coefficient was used by Miao, Gel and Gastwirth [29] for introducing a robust symmetry test:

$$\widehat{\gamma_{\mathrm{GM}}(\mathbf{x})} = \frac{\overline{\mathbf{x}} - \mathrm{Me}(\mathbf{x})}{\frac{1}{n}\sum_{i=1}^{n} |\mathbf{x}_i - \mathrm{Me}(\mathbf{x})|}.$$

Note that Miao, Gel and Gastwirth [29] actually considered the multiplicative constant $\sqrt{\pi/2}$ for the term in the denominator in order to have an unbiased estimator of the standard deviation at the normal distribution. However, said multiplicative constant does not actually play a big role since it just results in a rescaling of the skewness coefficient, thus it is here ignored.

A different skewness coefficient that avoids the need for finite moments is due to Bowley [4]. The *Bowley skewness coefficient* is defined as:

$$\gamma_{\mathrm{B}}(X) = \frac{Q_3(X) + Q_1(X) - 2\mathrm{Me}(X)}{Q_3(X) - Q_1(X)},$$

where $\mathrm{Me}(X)$ represents the population median and $Q_1(X)$ and $Q_3(X)$ represent the first and third population quartiles.

Note that the Bowley skewness coefficient belongs to a larger class of skewness coefficients introduced by Hinkley [18], parameterized by a given $p \in]0, 0.5[$, as follows:

$$\gamma_p(X) = \frac{\big(C_{1-p}(X) - C_{0.5}(X)\big) - \big(C_{0.5}(X) - C_p(X)\big)}{C_{1-p}(X) - C_p(X)},$$

where $C_q(X)$ represents the quantile of order q of X for any $q \in]0, 1[$. The case $p = 0.25$ corresponds to Bowley skewness coefficient. The case $p = 0.125$ has also attracted some attention from statisticians [6] and will be considered hereinafter under the name of *octile skewness coefficient* and denoted by γ_{OCT}. It is immediate to obtain a sample version of all coefficients in this family by substituting population quantiles by sample quantiles.

A more recent skewness coefficient – referred to as the *medcouple skewness coefficient* – was introduced by Brys, Hubert and Struyf [6] in the context of robust statistics. For a sample $\mathbf{x}$, we define the medcouple skewness coefficient as:

$$\gamma_{\mathrm{MC}}(\mathbf{x}) = \underset{x_i \le \mathrm{Me}(\mathbf{x}) \le x_j}{\mathrm{Me}} h(x_i, x_j), \tag{1}$$

where Me represents the median of the list of values and $h(x_i, x_j)$ is defined as:

$$h(x_i,x_j) = \begin{cases} \dfrac{(x_j - \mathrm{Me}(\mathbf{x})) - (\mathrm{Me}(\mathbf{x}) - x_i)}{x_j - x_i}, & \text{if } x_i \neq x_j, \\ \left.\begin{array}{ll} 1, & \text{if } i > j, \\ 0, & \text{if } i = j, \\ -1, & \text{if } i < j, \end{array}\right\} & \text{if } x_i = x_j = \mathrm{Me}(\mathbf{x}). \end{cases}$$

The computation of the medcouple requires to calculate all values $h(x_i, x_j)$ for all pairs of elements of the sample $\mathbf{x}$. For the experiments in this paper, the medcouple has been computed by using the function *mc()* of the R package *robustbase* [27]. Note that the definition of the medcouple in Eq. (1) actually is an estimate of the corresponding population skewness coefficient defined as:

$$\gamma_{\mathrm{MC}}(X) = H_F^{-1}(0.5),$$

where

$$H_F(u) = 4 \int_{\mathrm{Me}(X)}^{\infty} F\left(\frac{x(u-1) + 2\mathrm{Me}(X)}{u+1}\right) dF(x),$$

where $\mathrm{Me}(X)$ is the median of X.

With the exception of the medcouple, it has been proven that the sample versions of all the skewness coefficients above are asymptotically normal under some regularity conditions (see [17] for the moment skewness coefficient, [8] for the nonparametric skewness coefficient, [29] for the Groeneveld-Meeden skewness coefficient and [31] for skewness coefficients of the family introduced by Hinkley including the Bowley skewness coefficient and the octile skewness coefficient). Experimental results [5] and the fact that, as discussed in [6], the medcouple belongs to the class of incomplete generalized L-statistics of Hössjer [19] also hint at the asymptotic normality of the medcouple. Therefore, for any of the coefficients above, we expect that

$$\sqrt{n}\,\frac{\hat{\gamma} - \gamma_F}{\sqrt{V(\gamma, F)}} \rightsquigarrow N(0,1),$$

where $\hat{\gamma}$ is the sample version of the skewness coefficient γ, γ_F is the population value for the skewness coefficient γ at the distribution F, and $V(\gamma, F)$ denotes the asymptotic variance of γ at the distribution F. Note that $\gamma_F = 0$ for all symmetric distributions, however the asymptotic variance $V(\gamma, F)$ is dependent on the underlying distribution and varies even from one symmetric distribution to another one.

2.3. Skewness-based goodness-of-fit tests

As discussed in [6], a natural skewness-based goodness-of-fit test within a location-scale distribution family can be sketched as follows. Consider the following null and alternative hypothesis:

$$H_0: \quad X \text{ belongs to the location-scale distribution family } \mathcal{F},$$

$$H_1: \quad X \text{ does not belong to the location-scale distribution family } \mathcal{F}.$$

Since skewness coefficients are location and scale invariant, it holds that $\gamma_F = \gamma_{F'}$ and $V(\gamma, F) = V(\gamma, F')$, for any $F, F' \in \mathcal{F}$. Additionally, it follows for any asymptotically normal skewness coefficient that $\sqrt{n}\frac{\hat{\gamma}-\gamma_F}{\sqrt{V(\gamma,F)}} \rightsquigarrow N(0,1)$ under the null hypothesis (i.e., for any $F \in \mathcal{F}$), independently of the values of the location and scale parameters. Therefore, it suffices to compute γ_F and $V(\gamma, F)$ in order to define a rejection region based on the sampled value of $\hat{\gamma}$. Formally, the rejection region will be defined as follows:

$$RR = \left\{ \mathbf{x} \in \mathbb{R}^n \;\middle|\; |\gamma(\mathbf{x}) - \gamma_F| > \frac{\sqrt{V(\gamma, F)}}{\sqrt{n}} z_{1-\alpha/2} \right\},$$

where $z_{1-\alpha/2}$ is the quantile of order $1 - \alpha/2$ of a normal distribution.

As an example presented in [6], for a normality test based on the medcouple skewness coefficient, we need to consider $\gamma_\phi = 0$ and $V(\gamma_{\mathrm{MC}}, \phi) = 1.25$. Therefore, we can define a normality test at significance level α by means of the following rejection region:

$$RR = \left\{ \mathbf{x} \in \mathbb{R}^n \;\middle|\; |\gamma_{\mathrm{MC}}(\mathbf{x})| > \frac{\sqrt{1.25}}{\sqrt{n}} z_{1-\alpha/2} \right\}.$$

Unfortunately, skewness-based goodness-of-fit tests tend to exhibit a low power. For this very reason, it is common to further consider a test statistic that combines a skewness coefficient and a kurtosis coefficient. Two prominent examples of goodness-of-fit tests based on the combination of a skewness coefficient and a kurtosis coefficient are that of Jarque-Bera [24,25] and its robust alternative [14], which are specifically designed for testing for normality.

3. Symmetry tests based on skewness coefficients

A more elaborated statistical test could be built by establishing some hypothesis on the value of the chosen skewness coefficient γ:

$$H_0: \quad \gamma(X) = \gamma_0,$$

$$H_1: \quad \gamma(X) \neq \gamma_0.$$

In particular, setting $\gamma_0 = 0$ allows to present a symmetry test based on the skewness coefficient γ. It is important at this point that the reader bears in mind that the null hypothesis actually is more general than that of symmetry since there may exist non-symmetrical distributions for which the skewness coefficient takes the value zero.

A cumbersome problem for the definition of this type of tests is that, as has already been mentioned, the test statistic $\sqrt{n}\frac{\hat{\gamma}-\gamma_F}{\sqrt{V(\gamma,F)}}$ is not distribution-free since the asymptotic variance $V(\gamma, F)$ still depends on the underlying distribution of X.

In order to solve this problem, some authors have proposed to consider a reference distribution for which to compute $V(\gamma, F)$. For instance, as brought to attention by Cabilio and Masaro [8], the normal distribution [1] and the logistic distribution [3] have already been used for this purpose. A more elaborate solution consists in estimating $V(\gamma, F)$ based on the obtained sample. For instance, Gupta [17] proposed such an estimation for the test based on the moment skewness coefficient and Cabilio and Masaro [8] proposed such an estimation for the test based on the nonparametric skewness coefficient. However, the estimation of the asymptotic variance is typically designed specifically for each skewness coefficient and relies on estimations of the value of the density function at the symmetry point.

In this paper, we propose to use the bootstrap [12,13] to estimate the asymptotic variance of the test statistic. Although the bootstrap has already been considered within the context of tests of symmetry for the estimation of the distribution of some skewness coefficients such as the moment skewness coefficient (see, e.g., [26,44]), to the best of our knowledge it has not yet been considered in the context of the estimation of the asymptotic variance for most skewness coefficients presented in this paper.

Formally, the procedure for the bootstrap estimation of the asymptotic variance of the test statistic works as follows:

Input: A sample $\mathbf{x}$ of size n.
Step 1. Obtain B bootstrap samples $\mathbf{x}^{(1)}, \ldots, \mathbf{x}^{(B)}$ of size n by sampling with replacement from $\mathbf{x}$.
Step 2. For each bootstrap sample $\mathbf{x}^{(i)}$, compute $t_i = \sqrt{n}\big(\gamma(\mathbf{x}^{(i)}) - \gamma(\mathbf{x})\big)$.
Step 3. Estimate $V(\gamma, F)$ by $V^*(\gamma, \mathbf{x}) = \mathrm{Var}\big(t_1, \ldots, t_B\big)$.
Output: $V^*(\gamma, \mathbf{x})$.

Ultimately, the associated rejection region is defined as follows:

$$RR = \left\{ \mathbf{x} \in \mathbb{R}^n \;\middle|\; |\gamma(\mathbf{x}) - \gamma_0| > \frac{\sqrt{V^*(\gamma, \mathbf{x})}}{\sqrt{n}} z_{1-\alpha/2} \right\},$$

where, again, $z_{1-\alpha/2}$ is the quantile of order $1 - \alpha/2$ of a normal distribution.

Note that the bootstrap is here only used for the estimation of a parameter of the distribution of the test statistic, therefore yielding a method that relies strongly on the goodness of the asymptotic normality but not so strongly on the representativeness of the sample.

4. Experimental setup

In this section, we provide an experimental analysis of the power of the proposed symmetry tests under different circumstances. We work at significance level $\alpha = 0.05$. The power of the different tests is estimated by Monte Carlo simulation (10^4 replications) and the number of bootstrap replications is set to $B = 10^3$. Different sample sizes are explored ($n \in \{25, 50, 100, 200\}$). All tests are performed on the same samples and the asymptotic variances $V(\gamma, F)$ are estimated for all considered skewness coefficients γ based on the same bootstrap samples in order to minimize the influence of the sampling process. As a baseline, we compare the here-presented symmetry tests with the tests by Cabilio and Masaro [8] and Miao, Gel and Gastwirth [29], considering their m-out-of-n bootstrap counterparts implemented in the R package *lawstat* [23] (with the default options $B = 10^3$ and $q = 8/9$). Non-bootstrap implementations of the three tests considering a reference distribution for the computation of the asymptotic variance are not considered here due to their inefficiency at maintaining the significance level and/or excessively low power at many symmetric distributions. In all tables concerning the power of the different tests, the third to eighth columns correspond to the bootstrap tests associated with the skewness coefficient at the header considering the notation introduced in Section 2, whereas the ninth and tenth columns correspond to the Cabilio and Masaro test (CM) and the Miao, Gel and Gastwirth test (MGG).

Table 1
Power of the different tests at symmetric distributions. Values within the 95% confidence interval $[0.0458, 0.0545]$ are highlighted in boldface.

Distribution	n	γ_{M}	γ_{NP}	γ_{GM}	γ_{B}	γ_{OCT}	γ_{MC}	CM	MGG
Normal	25	0.0652	0.0174	0.0167	0.0036	0.0169	0.0086	0.0298	0.0313
	50	0.0809	0.0269	0.0265	0.0152	0.0298	0.0158	0.0293	0.0315
	100	0.0739	0.0366	0.0355	0.0278	0.0360	0.0247	0.0426	**0.0467**
	200	0.0654	0.0420	0.0414	0.0363	0.0425	0.0330	0.0419	**0.0458**
Cauchy	25	0.4092	0.1329	0.2136	0.0105	**0.0531**	0.0262	0.2673	0.0304
	50	0.4185	0.1527	0.2216	0.0281	0.0633	0.0323	0.3638	0.0421
	100	0.4189	0.1511	0.2090	0.0389	0.0640	0.0356	0.4314	0.0563
	200	0.4216	0.1407	0.1980	0.0360	0.0584	0.0359	0.4231	0.0453
Student's t_2	25	0.2966	0.0623	0.0779	0.0042	0.0269	0.0133	0.0837	0.0322
	50	0.3316	0.0813	0.0879	0.0183	0.0388	0.0173	0.1050	0.0438
	100	0.3258	0.0869	0.0818	0.0266	0.0433	0.0235	0.1189	0.0656
	200	0.3136	0.0923	0.0785	0.0365	0.0451	0.0311	0.1091	0.0681
Student's t_3	25	0.2272	0.0381	0.0438	0.0038	0.0184	0.0114	**0.0485**	0.0304
	50	0.2472	**0.0535**	**0.0528**	0.0150	0.0338	0.0157	**0.0492**	0.0347
	100	0.2253	0.0597	0.0563	0.0274	0.0397	0.0221	0.0679	0.0567
	200	0.1969	0.0568	**0.0512**	0.0326	0.0441	0.0312	0.0564	**0.0502**
Student's t_5	25	0.1433	0.0271	0.0290	0.0037	0.0193	0.0103	0.0356	0.0309
	50	0.1624	0.0380	0.0378	0.0175	0.0316	0.0163	0.0360	0.0330
	100	0.1530	0.0439	0.0423	0.0276	0.0392	0.0235	**0.0477**	**0.0483**
	200	0.1177	**0.0459**	0.0442	0.0324	0.0413	0.0279	0.0438	0.0442
Student's t_8	25	0.1110	0.0217	0.0227	0.0031	0.0191	0.0091	0.0345	0.0303
	50	0.1281	0.0356	0.0346	0.0159	0.0314	0.0157	0.0349	0.0332
	100	0.1158	0.0391	0.0382	0.0284	0.0361	0.0261	0.0446	**0.0470**
	200	0.0906	0.0408	0.0403	0.0334	0.0411	0.0304	0.0415	0.0430
Logistic	25	0.1108	0.0198	0.0192	0.0023	0.0163	0.0086	0.0299	0.0261
	50	0.1268	0.0363	0.0358	0.0155	0.0299	0.0171	0.0351	0.0329
	100	0.1144	0.0384	0.0377	0.0279	0.0373	0.0228	0.0415	0.0453
	200	0.0902	0.0425	0.0422	0.0338	0.0407	0.0315	0.0437	0.0447
Laplace	25	0.1653	0.0343	0.0372	0.0045	0.0253	0.0127	0.0401	0.0299
	50	0.1586	0.0435	0.0428	0.0196	0.0366	0.0171	0.0394	0.0360
	100	0.1474	**0.0492**	**0.0475**	0.0328	0.0438	0.0241	**0.0516**	**0.0488**
	200	0.1147	**0.0485**	**0.0479**	0.0356	0.0456	0.0301	**0.0486**	**0.0480**
Uniform	25	0.0308	0.0248	0.0235	0.0038	0.0269	0.0134	**0.0480**	0.0553
	50	0.0445	0.0422	0.0403	0.0228	0.0453	0.0287	**0.0499**	0.0554
	100	**0.0492**	**0.0484**	**0.0462**	0.0301	**0.0491**	0.0387	0.0590	0.0714
	200	**0.0494**	**0.0521**	**0.0502**	0.0391	**0.0535**	**0.0463**	**0.0525**	0.0604
Beta(2, 2)	25	0.0296	0.0145	0.0155	0.0024	0.0171	0.0082	0.0332	0.0361
	50	0.0443	0.0328	0.0319	0.0183	0.0385	0.0213	0.0378	0.0409
	100	0.0408	0.0418	0.0404	0.0283	0.0395	0.0297	**0.0459**	**0.0544**
	200	**0.0523**	**0.0460**	0.0454	0.0357	**0.0462**	0.0382	**0.0481**	**0.0526**
Beta(3, 3)	25	0.0342	0.0140	0.0144	0.0027	0.0179	0.0093	0.0312	0.0365
	50	**0.0498**	0.0297	0.0287	0.0145	0.0306	0.0175	0.0337	0.0390
	100	**0.0505**	0.0385	0.0373	0.0271	0.0375	0.0258	0.0451	**0.0537**
	200	**0.0517**	0.0431	0.0429	0.0328	0.0404	0.0347	0.0443	**0.0459**
Beta(4, 4)	25	0.0403	0.0142	0.0146	0.0026	0.0157	0.0071	0.0313	0.0343
	50	**0.0535**	0.0327	0.0319	0.0190	0.0333	0.0205	0.0349	0.0379
	100	**0.0502**	0.0363	0.0356	0.0278	0.0376	0.0256	0.0406	**0.0522**
	200	0.0555	0.0415	0.0409	0.0357	0.0405	0.0331	0.0426	0.0456

4.1. Size of the tests

In this subsection, we analyze the preservation of the significance level ($\alpha = 0.05$) at different symmetric distributions and present the results in Table 1. For historical reasons, we use the symmetric distributions considered by Cabilio and Masaro in [8] (note that contaminated normal distributions are presented in a later subsection concerning experiments on the robustness of the tests in the presence of outliers).

Since the experimentation is performed at significance level $\alpha = 0.05$, the power of the different symmetry tests must be around this value under the null hypothesis of symmetry. Analyzing the third column in Table 1 for the Cauchy, Student's t, Logistic and Laplace distributions, we notice that the powers of the test based on the moment skewness coefficient γ_{M} are considerably higher than $\alpha = 0.05$, specially in the case of the Cauchy distribution where the powers reach the value 0.4 for all sample sizes. This means

Table 2
Combinations of parameters for the eight considered members of the generalized lambda distribution family.

Distribution	λ_1	λ_2	λ_3	λ_4
GLD7	0.000000	1.000000	1.400000	0.250000
GLD8	0.000000	1.000000	0.000070	0.100000
GLD9	3.586508	0.043060	0.025213	0.094029
GLD10	0.000000	−1.000000	−0.007500	−0.030000
GLD11	−0.116734	−0.351663	−0.130000	−0.160000
GLD12	0.000000	−1.000000	−0.100000	−0.180000
GLD13	0.000000	−1.000000	−0.001000	−0.130000
GLD14	0.000000	−1.000000	−0.000100	−0.170000

that this test will reject the null hypothesis of symmetry in these families of probability distributions more often than it should do. This situation is due to the fact that the mean is not properly defined for the Cauchy distribution and, thus, the moment skewness coefficient γ_{M} is also not properly defined. Additionally, one should note that the moment skewness coefficient γ_{M} is also not properly defined for the Student's t distribution with 2 and 3 degrees of freedom, therefore justifying the obtained powers at these distributions. Similarly, all other tests based on statistics that are obtained as a function of the mean will typically come to an incorrect conclusion at the Cauchy distribution. For instance, it is shown in Table 1 that the power of the symmetry tests based on the nonparametric skewness coefficient γ_{NP} and the Groeneveld-Meeden skewness coefficient γ_{GM} are around 0.15 and 0.2, respectively, at the Cauchy distribution.

Regarding all other tests and distributions, we observe that, in general, all powers are close to the significance level α. In fact, the bigger the considered sample size is, the closer the power is to the significance value. This situation is to be expected as the asymptotic distributions for the statistics will tend to be more accurate as the sample size increases. Unfortunately, most skewness coefficients exhibit a slow convergence at most distributions as can be concluded from the fact that most values in Table 1 do not belong to the classical confidence interval $[0.0458, 0.0545]$ with confidence level 0.95 for the Bernoulli distribution for samples of size 10^4 with mean 0.05, even for large values of n. It should be pointed out that the uniform distribution is the distribution at which the powers adjust the best to the significance level α for most tests and sample sizes. Among the two tests used for control, the Cabilio and Masaro test (CM) does not succeed in maintaining the significance level for some distributions, whereas the Miao, Gel and Gastwirth test (MGG) does succeed in maintaining the significance level at almost all distributions.

4.2. *Power under asymmetry*

In this subsection, we compare the power of the different tests for asymmetric distributions.

4.2.1. *The generalized lambda family*

For historical reasons, we firstly study eight asymmetric distributions already considered by Cabilio and Masaro in [8]. In particular, those eight distributions are members of the generalized lambda family discussed in [35]. This family is modeled by four parameters $(\lambda_1, \lambda_2, \lambda_3, \lambda_4)$ and its members are defined in terms of their inverse cumulative distribution function:

$$F^{-1}(u) = \lambda_1 + \frac{u^{\lambda_3} - (1-u)^{\lambda_4}}{\lambda_2},$$

for $0 < u < 1$. The eight considered distributions are listed as types seven to fourteen in [36] and correspond to the combinations of parameters shown in Table 2. Note that the first six types of distributions listed in [36] are symmetric and, therefore, not considered here.

Table 3 presents the power of the different tests at the eight mentioned members of the generalized lambda distribution. As the considered distributions are asymmetric, high powers are expected.

As a first comparative result, it should be pointed out that the power of the symmetry test based on the Bowley skewness coefficient γ_{B} at all distributions is much lower than the powers of all other symmetry tests at the same distribution, which reveals a suboptimal performance of the former test for detecting asymmetry. Since the distributions have a different type of behavior in terms of its asymmetry, for a more detailed comparison of the other symmetry tests we will discuss these distributions in three separate groups. The first group will be formed by GLD8, GLD13 and GLD14, whereas the second group will be formed by GLD7, GLD9, GLD10 and GLD12. Finally, GLD11 will be studied apart.

For the first group of distributions (GLD8, GLD13 and GLD14), most powers are higher than 0.75 when $n = 50$, $n = 100$ and $n = 200$, however the tests based on the octile coefficient γ_{OCT} and the medcouple coefficient γ_{MC} (robust skewness coefficients) lead to powers that are a little lower than 0.75. For the smallest sample size $n = 25$, we observe a considerable decrease in power for all tests except for the one based on the moment skewness coefficient γ_{M}, whose powers are around 1 for all sample sizes. Note that the two tests used for control exhibit a lower power than the tests based on the skewness coefficients γ_{M}, γ_{NP} and γ_{GM}.

For the second group of distributions (GLD7, GLD9, GLD10 and GLD12), we can also highlight the test based on the moment skewness coefficient γ_{M} since it obtains the highest powers. On the contrary, some of the other tests do not reject the null hypothesis of symmetry as many times as they should. For example, at GLD7, the powers range from 0.0534 to 0.1056 when $n = 25$, and from

Table 3
Power of the different tests at asymmetric distributions. The three most powerful tests for each distribution and sample size are highlighted in boldface.

Distribution	n	γ_M	γ_{NP}	γ_{GM}	γ_B	γ_{OCT}	γ_{MC}	CM	MGG
GLD7	25	**0.2585**	0.1056	0.1010	0.0106	0.0935	0.0534	**0.1346**	**0.1355**
	50	**0.5975**	**0.2485**	0.2340	0.0602	0.2315	0.1427	0.2387	**0.2514**
	100	**0.9077**	0.4273	0.4129	0.1250	0.4238	0.2902	**0.4623**	**0.4867**
	200	**0.9976**	0.6839	0.6763	0.2292	0.7041	0.5111	**0.7320**	**0.7411**
GLD8	25	**0.8351**	**0.4148**	0.4113	0.0351	0.2903	0.1662	**0.4124**	0.3548
	50	**0.9927**	**0.7548**	0.7397	0.1612	0.6250	0.3856	**0.7491**	0.7248
	100	**1.0000**	0.9506	0.9457	0.3363	0.8996	0.6643	**0.9719**	**0.9715**
	200	**1.0000**	0.9981	0.9980	0.6201	0.9937	0.9135	**0.9995**	**0.9998**
GLD9	25	**0.3894**	0.1187	**0.1199**	0.0067	0.0795	0.0381	**0.1329**	0.1176
	50	**0.6908**	**0.3056**	**0.2976**	0.0528	0.2201	0.1160	0.2715	0.2650
	100	**0.9272**	0.5597	0.5506	0.1113	0.4379	0.2345	**0.5691**	**0.5719**
	200	**0.9955**	0.8420	0.8366	0.2244	0.7259	0.4466	**0.8664**	**0.8656**
GLD10	25	**0.5967**	0.2185	**0.2260**	0.0133	0.1323	0.0648	**0.2235**	0.1841
	50	**0.8789**	**0.5007**	**0.4903**	0.0762	0.3293	0.1726	0.4593	0.4336
	100	**0.9876**	0.7995	0.7901	0.1799	0.6325	0.3691	**0.8141**	**0.8080**
	200	**0.9998**	0.9732	0.9711	0.3489	0.8995	0.6410	**0.9819**	**0.9815**
GLD11	25	**0.2097**	0.0410	**0.0449**	0.0038	0.0255	0.0150	**0.0477**	0.0355
	50	**0.2550**	**0.0727**	**0.0718**	0.0196	0.0462	0.0239	0.0657	0.0566
	100	**0.2676**	**0.1064**	0.1026	0.0362	0.0703	0.0410	**0.1107**	0.1024
	200	**0.2929**	**0.1680**	0.1622	0.0468	0.1021	0.0585	**0.1672**	0.1630
GLD12	25	**0.3781**	0.1136	**0.1250**	0.0078	0.0601	0.0302	**0.1188**	0.0849
	50	**0.5497**	**0.2603**	**0.2562**	0.0417	0.1437	0.0741	0.2317	0.1999
	100	**0.7081**	**0.4696**	0.4544	0.0805	0.2777	0.1420	**0.4676**	0.4486
	200	**0.8597**	**0.7633**	0.7520	0.1528	0.5164	0.2891	**0.7676**	0.7600
GLD13	25	**0.9443**	**0.6180**	**0.6246**	0.0537	0.4334	0.2432	0.6108	0.4539
	50	**0.9995**	**0.9186**	0.9134	0.2447	0.7809	0.5206	**0.9167**	0.8817
	100	**1.0000**	0.9960	0.9953	0.4899	0.9772	0.8132	**0.9983**	**0.9977**
	200	**1.0000**	**1.0000**	**1.0000**	0.7947	0.9996	0.9727	**1.0000**	**1.0000**
GLD14	25	**0.9500**	**0.6687**	**0.6781**	0.0702	0.4711	0.2726	0.6581	0.4750
	50	**0.9995**	**0.9393**	0.9358	0.2651	0.8185	0.5577	**0.9394**	0.8991
	100	**1.0000**	0.9977	**0.9978**	0.5308	0.9811	0.8385	**0.9988**	0.9974
	200	**1.0000**	0.9999	**1.0000**	0.8286	0.9998	0.9813	**1.0000**	0.9999

0.5111 to 0.7041 when $n = 200$. Again, we note how sample size largely affects the obtained powers. Note that the two tests used for control show an intermediate behavior, exhibiting a lower power than the test based on the moment skewness coefficient γ_M but a higher power than all other presented tests.

For the GLD11 distribution, the reader may be surprised by the information provided in Table 3, as the powers obtained for this distribution are lower than 0.2929 for all sample sizes and tests. However, these results are due to the fact that this distribution is just slightly asymmetric.

4.2.2. *The Weibull family*

Additionally, we explore the power of the tests for varying sample size at the Weibull distribution $W(\lambda, k)$, where λ is the scale parameter and k is the shape parameter. Since changes of scale do not affect skewness, we set without loss of generality $\lambda = 1$ and simply explore the power of the different tests for varying values of k from 0.5 to 5 considering steps of 0.5. Table 4 presents the obtained powers for sample sizes $n = 25$ and $n = 200$.

As discussed in [15], the skewness of the Weibull distribution decreases as k increases. More precisely, the skewness is clearly positive for values of k smaller than 3, is close to zero for values of k within the interval $[3, 4]$, and is clearly negative for values of k greater than 4. In particular, it is stated in [15] that the value at which the median and the mean coincide (and, thus, the value at which the nonparametric skewness coefficient γ_{NP} and the Groeneveld-Meeden skewness coefficient γ_{GM} equal zero) is found numerically to be $k = 3.3125$ and the value at which the moment skewness coefficient γ_M equals zero is found numerically to be $k = 3.6023$. These expected results can be confirmed by looking at Table 4, where the power of all tests is initially very high for $k = 0.5$, quickly decreases as k approaches the value 3, and slightly increases again when $k > 4$. It is observed that the symmetry test based on the moment skewness coefficient γ_M exhibits the highest power at this distribution for all values of k and n.

4.3. *Robustness in the presence of outliers*

In this subsection, we study the robustness of the different tests in the presence of outliers. For such purpose, we recreate the presence of outliers by considering a Tukey-Huber contamination model (see [39] and [20]). This model assumes that the distribution

Table 4
Power of the different tests at the Weibull distribution. The three most powerful tests for each value of the shape parameter k and sample size are highlighted in boldface.

n	k	γ_{M}	γ_{NP}	γ_{GM}	γ_{B}	γ_{OCT}	γ_{MC}	CM	MGG
	0.5	**0.9965**	0.9537	**0.9604**	0.3465	0.8892	0.6727	**0.9599**	0.5126
	1.0	**0.8950**	**0.5099**	**0.5107**	0.0412	0.3539	0.2015	0.5041	0.4045
	1.5	**0.5627**	0.1701	**0.1725**	0.0109	0.1055	0.0554	**0.1853**	0.1684
	2.0	**0.2524**	0.0591	0.0590	0.0060	0.0429	0.0210	**0.0770**	**0.0748**
25	2.5	**0.1100**	0.0283	0.0287	0.0036	0.0243	0.0121	**0.0460**	**0.0453**
	3.0	**0.0548**	0.0174	0.0178	0.0028	0.0186	0.0085	**0.0326**	**0.0365**
	3.5	**0.0472**	0.0137	0.0145	0.0028	0.0145	0.0084	**0.0276**	**0.0293**
	4.0	**0.0608**	0.0160	0.0177	0.0021	0.0198	0.0088	**0.0330**	**0.0317**
	4.5	**0.0733**	0.0208	0.0212	0.0028	0.0194	0.0088	**0.0360**	**0.0362**
	5.0	**0.0956**	0.0233	0.0248	0.0041	0.0246	0.0105	**0.0424**	**0.0415**
	0.5	**1.0000**	0.9991	**1.0000**	0.9989	**1.0000**	0.9998	**1.0000**	0.9993
	1.0	**1.0000**	0.9998	0.9998	0.7221	0.9993	0.9515	**1.0000**	**1.0000**
	1.5	**1.0000**	0.9299	0.9259	0.2658	0.8343	0.5907	**0.9544**	**0.9552**
	2.0	**0.9874**	0.5428	0.5354	0.0999	0.3798	0.2332	**0.5636**	**0.5652**
200	2.5	**0.7085**	0.1847	0.1809	0.0519	0.1286	0.0797	**0.1880**	**0.1927**
	3.0	**0.2228**	0.0657	0.0647	0.0357	0.0525	0.0391	**0.0664**	**0.0686**
	3.5	**0.0608**	0.0401	0.0394	0.0326	**0.0414**	0.0303	0.0413	**0.0426**
	4.0	**0.1096**	0.0596	0.0598	0.0407	0.0569	0.0429	**0.0616**	**0.0659**
	4.5	**0.2574**	0.0980	0.0969	0.0438	0.0881	0.0590	**0.1010**	**0.1047**
	5.0	**0.4095**	0.1446	0.1430	0.0494	0.1183	0.0726	**0.1504**	**0.1535**

Table 5
Power of the different tests at different contaminated distributions for sample size $n = 25$. Values within the 95% confidence interval $[0.0458, 0.0545]$ are highlighted in boldface.

Cont.	ε	γ_{M}	γ_{NP}	γ_{GM}	γ_{B}	γ_{OCT}	γ_{MC}	CM	MGG
	0.00	0.0639	0.0178	0.0180	0.0031	0.0174	0.0092	0.0315	0.0321
	0.01	0.0877	0.0185	0.0197	0.0028	0.0177	0.0080	0.0315	0.0285
(a)	0.05	0.1528	0.0245	0.0263	0.0028	0.0155	0.0077	0.0341	0.0299
	0.10	0.1997	0.0353	0.0369	0.0028	0.0198	0.0109	0.0407	0.0325
	0.20	0.2244	0.0374	0.0419	0.0029	0.0196	0.0102	0.0444	0.0316
	0.00	0.0656	0.0158	0.0158	0.0031	0.0183	0.0073	0.0285	0.0305
	0.01	0.0915	0.0179	0.0208	0.0037	0.0161	0.0086	0.0380	0.0300
(b)	0.05	0.1805	0.0232	0.0383	0.0030	0.0167	0.0092	0.0618	0.0270
	0.10	0.2582	0.0340	0.0563	0.0031	0.0169	0.0088	0.0876	0.0265
	0.20	0.3653	0.0556	0.1002	0.0029	0.0178	0.0112	0.1482	0.0258
	0.00	0.0665	0.0170	0.0173	0.0030	0.0154	0.0088	0.0313	0.0323
	0.01	0.1476	0.0257	0.0286	0.0032	0.0192	0.0095	0.0365	0.0307
(c)	0.05	0.4738	0.1058	0.1143	0.0047	0.0352	0.0184	0.1084	0.0741
	0.10	0.7436	0.2698	0.2903	0.0097	0.1358	**0.0515**	0.2755	0.2178
	0.20	0.7629	0.5590	0.5828	0.0619	0.4604	0.2160	0.5912	0.5333

function F of the contaminated distribution is given by $F = (1-\varepsilon)\,G + \varepsilon\, H$, where G is the distribution function of the uncontaminated distribution, H is the distribution function of the contamination model and $0 \le \varepsilon < 1$ is a small value representing the probability that a single observation is contaminated. For control, we set the uncontaminated distribution to be the standard normal distribution and vary the contamination model to be (a) a normal distribution with mean zero and standard deviation three (as in [8]); (b) a standard Cauchy distribution; (c) a normal distribution with mean five and standard deviation one. Note that the first two resulting contaminated distributions are symmetric (the second one being heavy-tailed), whereas the third resulting contaminated distribution is slightly asymmetric. Therefore, we expect the power to be similar to the significance level at the first two distributions, whereas the power at the third distribution shall be close to the significance level for robust tests but large for tests that are sensitive to outliers. We explore varying values of ε, and restrict to $n = 25$ (see Table 5) and $n = 200$ (see Table 6).

On the one hand, the first five rows of both tables make reference to the (symmetric) distribution (a), in which no big differences can be appreciated for each sample size when increasing ε. In particular, all tests keep the power under the significance level α, except for the test based on the moment skewness coefficient γ_{M} for which values up to 0.22 ($n = 25$) and 0.13 ($n = 200$) are obtained when $\varepsilon = 0.2$. A comparison of the powers of every test for both sample sizes reveals that all powers are closer to the significance level when n increases.

On the other hand, considering now the (symmetric) distribution (b), we notice that some tests do not succeed at maintaining the significance level α, specially the one based on the moment skewness coefficient γ_{M}, since as has been explained in Subsection 4.1 the mean is not defined for the Cauchy distribution. In addition, we note significant differences between both tables because powers are, generally, greater for large values of n (i.e., in Table 6) since the presence of Cauchy tails becomes more patent as the sample

Table 6
Power of the different tests at different contaminated distributions for sample size $n = 200$. Values within the 95% confidence interval [0.0458, 0.0545] are highlighted in boldface.

Cont.	ε	γ_{M}	γ_{NP}	γ_{GM}	γ_{B}	γ_{OCT}	γ_{MC}	CM	MGG
	0.00	0.0673	0.0412	0.0408	0.0363	0.0419	0.0324	0.0414	0.0415
	0.01	0.0755	**0.0461**	0.0456	0.0372	0.0417	0.0332	0.0453	**0.0460**
(a)	0.05	0.1481	0.0455	0.0430	0.0323	0.0392	0.0285	0.0446	**0.0459**
	0.10	0.1540	**0.0469**	0.0445	0.0346	0.0369	0.0303	**0.0463**	0.0449
	0.20	0.1305	**0.0475**	**0.0470**	0.0332	0.0373	0.0304	**0.0501**	**0.0476**
	0.00	0.0700	0.0369	0.0364	0.0319	0.0429	0.0304	0.0387	0.0407
	0.01	0.1391	**0.0491**	0.0456	0.0327	0.0438	0.0320	0.0645	0.0426
(b)	0.05	0.3558	0.0824	0.0735	0.0328	0.0421	0.0328	0.1391	0.0444
	0.10	0.4479	0.0994	0.1000	0.0319	0.0417	0.0284	0.2122	0.0439
	0.20	0.4503	0.1346	0.1523	0.0322	0.0405	0.0307	0.3093	0.0597
	0.00	0.0684	0.0427	0.0427	0.0364	0.0404	0.0345	0.0434	0.0443
	0.01	0.4099	0.0886	0.0826	0.0308	0.0420	0.0331	0.0829	0.0838
(c)	0.05	0.9954	0.6845	0.6692	0.0430	0.0955	0.0675	0.6744	0.6640
	0.10	1.0000	0.9832	0.9819	0.0772	0.3048	0.2448	0.9842	0.9819
	0.20	1.0000	1.0000	1.0000	0.3170	0.9967	0.9118	1.0000	1.0000

size increases. Additionally, it is necessary to mention that the tests based on robust coefficients such as γ_{B}, γ_{OCT} and γ_{MC} succeed in maintaining the significance level α, independently of the value of ε. It should be mentioned that, among the two tests used for control, the Cabilio and Masaro test (CM) does not succeed in maintaining the significance level whereas the Miao, Gel and Gastwirth test (MGG) does succeed in maintaining the significance level.

Finally, considering now the (asymmetric) distribution (c), we clearly observe that all powers considerably increase when increasing the probability of contamination ε. As can be seen in Table 6, except for the test based on the Bowley skewness coefficient γ_{B}, all powers are near 1 when considering $\varepsilon = 0.2$ and $n = 200$, which points out that the null hypothesis of symmetry is clearly rejected. If we consider lower values of ε instead, the power of all tests decreases and the rejection of the null hypothesis is actually not so strong for some of the tests. For instance, tests based on robust skewness coefficients (such as γ_{B}, γ_{OCT} and γ_{MC}) must be mentioned as they are not so sensitive to the outliers and do not reject the null hypothesis of symmetry for small values of ε (and in those cases all powers are closer to the significance level α). For example, when $\varepsilon = 0.05$ and $n = 200$, these tests attain powers ranging from 0.043 (for γ_{B}) to 0.0955 (for γ_{OCT}) while all other tests attain a power of at least 0.6, even 0.99 for the one based on the moment skewness coefficient γ_{M}. In case $n = 25$ (shown in Table 5), for almost all tests the powers are close to the significance level α when $\varepsilon \le 0.01$ and considerably greater than the significance level α when $\varepsilon \ge 0.1$. For the case $\varepsilon = 0.05$, all tests based on robust skewness coefficients have a power that is close to the significance level α, whereas all other tests lead to a higher power. It must be remarked that the test based on the medcouple skewness coefficient γ_{MC} succeeds in maintaining the significance level up to a percentage of contamination of $\varepsilon = 0.1$, whereas the test based on the Bowley skewness coefficient γ_{B} succeeds in maintaining the significance level up to a percentage of contamination of $\varepsilon = 0.2$.

In summary, depending on the test used, the presence of outliers will have a different influence on the decision of rejecting or maintaining the null hypothesis of symmetry. More precisely, tests based on non-robust skewness coefficients will detect the asymmetry for small values of ε and n, whereas tests based on robust skewness coefficients will tend to maintain the null hypothesis of symmetry even for large values of ε or n. It thus remains as a personal decision for the user whether to consider robust or non-robust skewness coefficients.

5. Final considerations on the bootstrap

5.1. Number of bootstrap replications

Almost all of the presented tests positioned favorably with respect to the Cabilio and Masaro test [8] when considering its m-out-of-n bootstrap counterpart implemented in the R package *lawstat* [23]. Interestingly, the Miao, Gel and Gastwirth test [29] (also considering its m-out-of-n bootstrap counterpart implemented in the R package *lawstat* [23]) exhibited an intermediate behavior among the here-presented tests as it succeeded at maintaining the significance level at all symmetric distributions, led to a moderate power at asymmetric distributions and exhibited certain robustness in the presence of outliers. The main difference between the Cabilio and Masaro and the Miao, Gel and Gastwirth tests and the tests presented in this paper is that the former do not rely on an asymptotic distribution but, instead, rely more strongly on the bootstrap estimation. In this direction, one may note that the here-presented tests only consider the bootstrap for the estimation of a parameter and still behave reasonably in case the number of bootstrap replications is greatly reduced (e.g., $B = 10$). On the contrary, the Cabilio and Masaro test and the Miao, Gel and Gastwirth test are no longer able to maintain the significance level in such case under the null hypothesis of symmetry, as can be seen in Table 7.

Table 7
Power of the different tests at symmetric distributions when $B = 10$. Values within the 95% confidence interval $[0.0458, 0.0545]$ are highlighted in boldface.

Distribution	n	γ_M	γ_{NP}	γ_{GM}	γ_B	γ_{OCT}	γ_{MC}	CM	MGG
Normal	25	0.1079	**0.0460**	0.0453	0.0265	**0.0482**	0.0372	0.1793	0.2003
	50	0.1183	0.0590	0.0573	0.0445	0.0583	0.0437	0.1774	0.1997
	100	0.1076	0.0676	0.0669	0.0583	0.0673	0.0547	0.1768	0.1912
	200	0.0978	0.0708	0.0700	0.0642	0.0661	0.0634	0.1827	0.1875
Cauchy	25	0.4303	0.1867	0.2408	0.0393	0.0928	**0.0525**	0.4593	0.2037
	50	0.4316	0.2037	0.2492	0.0573	0.0943	0.0593	0.5343	0.2200
	100	0.4380	0.2014	0.2351	0.0666	0.0933	0.0630	0.5410	0.1860
	200	0.4328	0.1996	0.2285	0.0696	0.0885	0.0635	0.5246	0.1674
Logistic	25	0.1592	**0.0481**	**0.0489**	0.0231	0.0456	0.0336	0.1739	0.1945
	50	0.1609	0.0622	0.0607	0.0406	0.0573	**0.0458**	0.1712	0.1947
	100	0.1442	0.0671	0.0654	0.0553	0.0660	0.0563	0.1830	0.1869
	200	0.1303	0.0693	0.0688	0.0643	0.0704	0.0590	0.1871	0.1886
Laplace	25	0.2092	0.0668	0.0670	0.0320	0.0547	0.0384	0.1901	0.1883
	50	0.1973	0.0705	0.0712	**0.0469**	0.0668	0.0439	0.1862	0.1865
	100	0.1809	0.0757	0.0763	0.0595	0.0757	**0.0538**	0.1948	0.1903
	200	0.1466	0.0810	0.0803	0.0657	0.0775	0.0595	0.1976	0.1896
Uniform	25	0.0650	0.0621	0.0592	0.0306	0.0623	**0.0484**	0.2111	0.2489
	50	0.0745	0.0841	0.0794	0.0546	0.0801	0.0645	0.2082	0.2536
	100	0.0788	0.0862	0.0833	0.0629	0.0827	0.0760	0.1998	0.2204
	200	0.0787	0.0872	0.0853	0.0716	0.0850	0.0812	0.1965	0.2019

5.2. *Type of bootstrap*

In all previous experiments the most **basic** bootstrap has been considered for the estimation of the asymptotic variance of the skewness coefficient. In this section, we explore two alternative types of bootstrap for some selected distributions and sample sizes. In particular, we follow the spirit of [9] and, instead of resampling from the original sample $\mathbf{x}$, we resample from the symmetrized sample $\mathbf{x}' = (\mathbf{x}, 2\overline{\mathbf{x}} - \mathbf{x})$, where $\overline{\mathbf{x}}$ denotes the arithmetic mean of $\mathbf{x}$. In this way, we assure ourselves that we are computing the distribution of the test statistic under the null hypothesis. More specifically, we consider a **symmetrized** version of the bootstrap presented in this paper for computing the asymptotic variance of the skewness coefficient while still considering its asymptotic distribution and, additionally, we simply consider the **classical** bootstrap for the estimation of the whole distribution of the test statistic.

The obtained results are presented in Table 8. On the one hand, we have considered two symmetric distributions (normal and uniform). It can be seen that the basic bootstrap leads to powers that are closer to the significance level than the other two types of bootstrap. On the other hand, we have considered two asymmetric distributions (GLD7 and GLD8). It can be seen that the basic bootstrap leads to powers that are higher than the other two types of bootstrap, specially for small sample sizes. Overall, the use of the basic bootstrap is slightly encouraged over the two other alternatives.

6. Conclusions and future work

In this paper, six tests of symmetry based on the asymptotic distribution of a skewness coefficient have been presented. Experimental results have shown that the test based on the moment skewness coefficient γ_M exhibits the highest power at asymmetric distributions, yet it does not succeed in maintaining the significance level at some symmetric distributions such as Student's t distributions with 1 (Cauchy), 2 and 3 degrees of freedom. The tests based on both the nonparametric skewness coefficient γ_{NP} and the Groeneveld-Meeden skewness coefficient γ_{GM} also exhibit a high power at asymmetric distributions, and only fail to maintain the significance level at the Cauchy distribution among all analyzed symmetric distributions (also noting a slightly greater power at the Student's t distribution with 2 degrees of freedom for the test based on the nonparametric skewness coefficient γ_{NP}). On the contrary, tests based on robust skewness coefficients such as the Bowley skewness coefficient γ_B, the octile skewness coefficient γ_{OCT} and the medcouple skewness coefficient γ_{MC} typically succeed at maintaining the significance level at symmetric distributions, but exhibit a lower power at asymmetric distributions. It should be borne in mind that the latter tests are also more robust, meaning that they will tend to maintain the significance level in the presence of a small number of outliers, even when those outliers are only present at one of the tails of the distribution (thus making the distribution slightly asymmetric).

Overall, it remains a personal decision for the user to consider a skewness coefficient that will result in a greater power at asymmetric distributions or, on the contrary, a skewness coefficient that will succeed at maintaining the significance level at heavy-tailed symmetric distributions and/or contaminated symmetric distributions. In case the former goal is favored, the user should resort to the test based on the moment skewness coefficient γ_M, whereas in case the latter goal is favored the user should resort to the test based on the octile skewness coefficient γ_{OCT}. In any case, the use of the test based on the Bowley skewness coefficient γ_B is discouraged since it is clearly less powerful than all the other five presented tests at all studied asymmetric distributions.

Table 8
Comparison of the different types of bootstrap at selected distributions and sample sizes. For the normal and uniform distributions, values within the 95% confidence interval $[0.0458, 0.0545]$ are highlighted in boldface. For the GLD7 and GLD8 distributions, the type of bootstrap leading to the most powerful test for each distribution, skewness coefficient and sample size are highlighted in boldface.

Distribution	n	Type	γ_M	γ_{NP}	γ_{GM}	γ_B	γ_{OCT}	γ_{MC}
Normal	25	Basic	0.0652	0.0174	0.0167	0.0036	0.0169	0.0086
		Symmetrized	0.0156	0.0035	0.0062	0.0067	0.0100	0.0044
		Classical	0.0158	0.0066	0.0102	0.0102	0.0145	0.0080
	200	Basic	0.0654	0.0420	0.0414	0.0363	0.0425	0.0330
		Symmetrized	0.0457	0.0233	0.0245	0.0257	0.0329	0.0200
		Classical	**0.0485**	0.0307	0.0319	0.0291	0.0363	0.0253
Uniform	25	Basic	0.0308	0.0248	0.0235	0.0038	0.0269	0.0134
		Symmetrized	0.0315	0.0050	0.0068	0.0051	0.0111	0.0056
		Classical	0.0342	0.0130	0.0186	0.0094	0.0186	0.0091
	200	Basic	**0.0494**	**0.0521**	**0.0502**	0.0391	**0.0535**	**0.0463**
		Symmetrized	**0.0512**	0.0259	0.0267	0.0286	0.0347	0.0224
		Classical	0.0452	0.0297	0.0305	0.0294	0.0369	0.0256
GLD7	25	Basic	**0.2585**	**0.1056**	**0.1010**	**0.0106**	**0.0935**	**0.0534**
		Symmetrized	0.1908	0.0276	0.0422	0.0063	0.0346	0.0135
		Classical	0.1906	0.0528	0.0723	0.0100	0.0575	0.0256
	200	Basic	**0.9976**	0.6839	0.6763	**0.2292**	0.7041	0.5111
		Symmetrized	0.9959	0.6904	0.6941	0.1724	0.6934	0.4856
		Classical	0.9965	**0.7051**	**0.7089**	0.1803	**0.7056**	**0.5119**
GLD8	25	Basic	**0.8351**	**0.4148**	**0.4113**	**0.0351**	**0.2903**	**0.1662**
		Symmetrized	0.3777	0.1275	0.2184	0.0048	0.1170	0.0402
		Classical	0.4920	0.1669	0.2771	0.0090	0.1757	0.0612
	200	Basic	**1.0000**	0.9981	0.9980	**0.6201**	0.9937	0.9135
		Symmetrized	0.9966	**1.0000**	**1.0000**	0.5839	**0.9983**	0.9433
		Classical	0.9983	0.9996	0.9996	0.6053	0.9981	**0.9529**

As future work, the authors highlight the study of tests of symmetry based on different skewness coefficients belonging to the class introduced by Hinkley [18] in which a parameter $p \in]0, 0.5[$ models the robustness of the skewness coefficient in the presence of outliers. As it has already been discussed, both the Bowley skewness coefficient ($p = 0.25$) and the octile skewness coefficient ($p = 0.125$) belong to this class. Since both skewness coefficients exhibit a very different behavior, it becomes natural to study, as a function of p, the performance of the symmetry tests associated with skewness coefficients belonging to the class introduced by Hinkley.

CRediT authorship contribution statement

Both authors contributed equally to this paper.

Declaration of competing interest

The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

Data availability

No data was used for the research described in the article.

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