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Second-order optimality conditions for semilinear elliptic control problems with constraints on the gradient of the state*

by

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Abstract: The aim of this paper is to state second order necessary and sufficient optimality conditions for distributed control problems governed by the Neumann problem associated to a semilinear elliptic partial differential equation. Bound constraints on the control are considered, as well as equality and inequality constraints of integral type on the gradient of the state.

Keywords: optimal control, second order conditions, semilinear elliptic PDE, state gradient constraints.

1 Introduction

In this paper we mainly discuss about second order necessary and sufficient optimality conditions for local solutions of a distributed control problem governed by the Neumann problem associated to a semilinear elliptic partial differential equation. Bound constraints on the control are considered, as well as equality and inequality constraints of integral type on the gradient of the state. The main tools to deal with this objetive are the necessary and sufficient optimality conditions for some abstract optimization problems in Banach spaces stated in

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Section 4. These can be viewed as the natural extention of the corresponding ones in finite dimensions, although the lack of compactness introduce some well-known extra difficulties. The rest of the paper is organized as follows: in Section 2, we study the existence, uniqueness and regularity of solution for the state equation; in Section 3, the C^2 character of the functionals involved in our control problem is established; finally, in Section 5, we verify that our control problem satisfies the assumptions required to the abstract optimization problem.

The control problem is stated as follows. Let Ω be a bounded open set in $\mathbb{R}^N$ with a C^1 boundary Γ. Let A be the operator given by

$$Ay = -\sum_{i,j=1}^{N} \frac{\partial}{\partial x_j}\left(a_{ij}(x)\frac{\partial y}{\partial x_i}\right),$$

with $a_{ij} \in C(\bar{\Omega})$ satisfying that

$$\mu_1 \|\xi\|^2_{\mathbb{R}^N} \leq \sum_{i,j=1}^{N} a_{ij}(x)\xi_i\xi_j \leq \mu_2 \|\xi\|^2_{\mathbb{R}^N} \quad \forall \xi \in \mathbb{R}^N, \quad \forall x \in \Omega,$$

for some positive constants μ_1 and μ_2.

Let $f : \mathbb{R}^2 \to \mathbb{R}$, $g_0 : \mathbb{R}^2 \to \mathbb{R}$ and $g_j : \mathbb{R}^N \to \mathbb{R}$ be continuous functions for $1 \leq j \leq n_e + n_i$, with $n_i, n_e \geq 1$. Let $u_a, u_b \in L^\infty(\Omega)$ with $u_a(x) \leq u_b(x)$ for almost every $x \in \Omega$. Our optimal control problem can be formulated as follows

$$\textbf{(P)} \begin{cases} \text{Minimize } J(u) \\ u_a(x) \leq u(x) \leq u_b(x) \quad \text{a.e. } x \in \Omega, \\ G_j(u) = 0, \ 1 \leq j \leq n_e, \\ G_j(u) \leq 0, \ n_e + 1 \leq j \leq n_e + n_i \end{cases}$$

where

$$J(u) = \int_\Omega g_0(y_u(x), u(x))dx,$$

with

$$\begin{cases} Ay_u = f(y_u, u) & \text{in } \Omega \\ \partial_{\nu_A} y_u = 0 & \text{on } \Gamma, \end{cases} \tag{1}$$

and

$$G_j(u) = \int_\Omega g_j(\nabla y_u(x))\, dx.$$

Remark 1 *The continuity assumption on the coefficients a_{ij} and the C^1 regularity of the boundary of the domain will allow us to consider quite general integral constraints G_j (see condition (7) below), thanks to the regularity result given in Proposition 1. Notice that we do not impose $a_{ij} = a_{ji}$. Nevertheless, if the coefficients a_{ij} are only bounded and the boundary Γ is Lipschitz, some results (similar to those obtained here) can be derived, assuming more restricted growth conditions on g_j.*

2 State equation

Let us begin by recalling the following result on the existence, uniqueness and regularity of the solution for the Neumann problem associated to a linear elliptic partial differential equation, see [6] for the proof:

Proposition 1 *Let p belong to $(1,+\infty)$, $\hat{f} \in (W^{1,p'}(\Omega))'$ with $p' = \frac{p}{p-1}$ and $g \in W^{-\frac{1}{p},p}(\Gamma)$. Then there exists a unique variational solution $y \in W^{1,p}(\Omega)$ to the Neumann's problem*

$$\left\{ \begin{array}{rcll} Ay + y & = & \hat{f} & \text{in } \Omega \\ \partial_{\nu_A} y & = & g & \text{on } \Gamma. \end{array} \right. \tag{2}$$

Moreover, the following estimate is satisfied.

$$\|y\|_{W^{1,p}(\Omega)} \le C \left(\|\hat{f}\|_{(W^{1,p'}(\Omega))'} + \|g\|_{W^{-\frac{1}{p},p}(\Gamma)} \right).$$

where C is a constant only depending on p, the dimension N, the coefficients a_{ij} and the domain Ω.

Remark 2 *As usual, by a variational solution of problem (2) we understand that y satisfies the variational equality*

$$\sum_{i,j=1}^{N} \int_\Omega a_{ij}(x) \frac{\partial y}{\partial x_i}(x) \frac{\partial \varphi}{\partial x_j}(x)\, dx + \int_\Omega y(x)\varphi(x)\, dx$$
$$= \langle f, \varphi \rangle_{(W^{1,p'}(\Omega))' \times W^{1,p'}(\Omega)} + \langle g, \gamma\varphi \rangle_{W^{-\frac{1}{p},p}(\Gamma) \times W^{\frac{1}{p},p'}(\Gamma)}$$

for all $\varphi \in W^{1,p'}(\Omega)$, where $\langle \cdot, \cdot \rangle_{X' \times X}$ denotes the duality product between the space X and its dual X', $\gamma : W^{1,p'}(\Omega) \to W^{\frac{1}{p},p'}(\Gamma)$ is the trace operator and $W^{-\frac{1}{p},p}(\Gamma) = (W^{\frac{1}{p},p'}(\Gamma))'$.

In order to deal with the state equation (1) and to obtain a C^2 relation control-state, we assume that the function f belongs to $C^2(\mathbb{R}^2)$ and satisfies

$$\frac{\partial f}{\partial y}(y,u) \le -\mu_1 < 0, \quad \forall (y,u) \in \mathbb{R}^2. \tag{3}$$

Under this hypothesis, we can prove the following theorem

Theorem 1 *For every $u \in L^\infty(\Omega)$ there exists a unique variational solution $y_u \in W^{1,p}(\Omega)$ for all $p \in (1,+\infty)$ of the problem (1). Moreover, the mapping $G : L^\infty(\Omega) \to W^{1,p}(\Omega)$ is of class C^2 for all $p \in [1,+\infty)$. If u, $h \in L^\infty(\Omega)$ $y_u = G(u)$ and $z_h = G'(u)h$, then z_h is the solution of*

$$\left\{ \begin{array}{rcll} Az & = & \frac{\partial f}{\partial y}(y_u,u)z + \frac{\partial f}{\partial u}(y_u,u)h & \text{in } \Omega \\ \partial_{\nu_A} z & = & 0 & \text{on } \Gamma. \end{array} \right. \tag{4}$$

Finally, if we take $h_1, h_2 \in L^\infty(\Omega)$, $z_i = G'(u)h_i$ *and* $z_{12} = G''(u)[h_1, h_2]$, *we have*

$$\left\{\begin{array}{rcll} Az_{12} &=& \dfrac{\partial f}{\partial y}(y_u,u)z_{12} + \dfrac{\partial^2 f}{\partial y^2}(y_u,u)z_1z_2 + \dfrac{\partial^2 f}{\partial u^2}(y_u,u)h_1h_2 & \\ && +\dfrac{\partial^2 f}{\partial y \partial u}(y_u,u)(z_1h_2 + z_2h_1) & \text{in } \Omega \\ \partial_{\nu_A} z_{12} &=& 0 & \text{on } \Gamma. \end{array}\right. \tag{5}$$

Proof. For a bounded function f, the existence of a unique solution y_u in $H^1(\Omega)$ is classical. Moreover, by using the monotonicity of f with respect to y and an standard technique (see Stampacchia [8]), it can be proved that $y_u \in L^\infty(\Omega)$. In the general case, the result follows from the previous case via a truncation method. Since $y_u, u \in L^\infty(\Omega)$, then $f(y_u, u) \in L^\infty(\Omega) \subset (W^{1,p'}(\Omega))'$ for all $1 < p < \infty$, the regularity result for linear equations (Proposition 1), assures that $y_u \in W^{1,p}(\Omega)$ for all $1 < p < \infty$. Hence, the mapping G is well defined. To check that G is of class C^2, we take

$$V(A) = \left\{y \in W^{1,p}(\Omega) :\ Ay \in L^\infty(\Omega),\ \partial_{\nu_A} y = 0\right\}$$

endowed with the norm

$$\|y\|_{V(A)} = \|y\|_{W^{1,p}(\Omega)} + \|Ay\|_{L^\infty(\Omega)}$$

(recall that

$$\partial_{\nu_A} y(x) = \sum_{i,j=1}^{N} a_{ij}(x) \frac{\partial y}{\partial x_i}(x)\nu_j(x),$$

where $\nu(x) = (\nu_1(x), \ldots, \nu_N(x))$ denotes the unit outward normal vector to Γ at x.)

Let us now define the function $F : V(A) \times L^\infty(\Omega) \to L^\infty(\Omega)$, $F(y, u) = Ay - f(y, u)$. It is an exercise to show that F is of class C^2. Moreover $\dfrac{\partial F}{\partial y}(y, u) = A - \dfrac{\partial f}{\partial y}(y, u)$ is an isomorphism from $V(A)$ to $L^\infty(\Omega)$. Taking into account that $F(y, u) = 0$ if and only if $y = G(u)$, we can apply the implicit function theorem (see for instance [2]) to deduce that G is of class C^2 and satisfies

$$F(G(u), u) = 0. \tag{6}$$

From this identity (4) and (5) follows easily. □

3 Functionals involved in the control problem.

As we have pointed out from the beginning, the aim of this work is to deduce second order optimality conditions for problem **(P)**. In order to deal with this task, we will assume that $g_0 \in C^2(\mathbb{R}^2)$, $g_j \in C^2(\mathbb{R}^N)$ for each $j = 1, \ldots, n_e + n_i$, and

$$\sum_{i=1}^{N} \left(\left| \frac{\partial g_j}{\partial \eta_i}(\eta) \right| + \sum_{k=1}^{N} \left| \frac{\partial^2 g_j}{\partial \eta_i \partial \eta_k}(\eta) \right| \right) \leq \mu_2 (1 + \|\eta\|^r) \qquad \forall \eta \in \mathbb{R}^N \tag{7}$$

for some exponent $r \in [1, +\infty)$ and $\mu_2 > 0$.

We now study the differentiability of J and G_j.

Theorem 2 *The functional $J : L^\infty(\Omega) \to \mathbb{R}$ is of class C^2. Moreover, for every $u, h \in L^\infty(\Omega)$*

$$J'(u)h = \int_\Omega \left(\frac{\partial g_0}{\partial u}(y_u, u) + \varphi_{0u} \frac{\partial f}{\partial u}(y_u, u) \right) h \, dx \tag{8}$$

and for every $u, h_1, h_2 \in L^\infty(\Omega)$

$$J''(u)h_1h_2 =$$

$$\int_\Omega \left[\frac{\partial^2 g_0}{\partial y^2}(y_u, u) z_1 z_2 + \frac{\partial^2 g_0}{\partial y \partial u}(y_u, u)(z_1 h_2 + z_2 h_1) + \frac{\partial^2 g_0}{\partial u^2}(y_u, u) h_1 h_2 \right.$$

$$\left. + \varphi_{0u} \left(\frac{\partial^2 f}{\partial y^2}(y_u, u) z_1 z_2 + \frac{\partial^2 f}{\partial y \partial u}(y_u, u)(z_1 h_2 + z_2 h_1) + \frac{\partial^2 f}{\partial u^2}(y_u, u) h_1 h_2 \right) \right] dx \tag{9}$$

where $y_u = G(u)$, $\varphi_{0u} \in W^{1,p}(\Omega)$ for all $p \in (1, +\infty)$ is the unique solution of the problem

$$\left\{ \begin{array}{rcll} A^*\varphi & = & \dfrac{\partial f}{\partial y}(y_u, u)\varphi + \dfrac{\partial g_0}{\partial y}(y_u, u) & \text{in } \Omega \\ \partial_{\nu_{A^*}} \varphi & = & 0 & \text{on } \Gamma, \end{array} \right. \tag{10}$$

where A^* is the adjoint operator of A

$$A^*\varphi = - \sum_{i,j=1}^{N} \frac{\partial}{\partial x_j} \left(a_{ji}(x) \frac{\partial \varphi}{\partial x_i} \right),$$

and $z_i = G'(u)h_i$, $i = 1, 2$.

Proof. Let us consider the function $F_0 : C(\bar{\Omega}) \times L^\infty(\Omega) \to \mathbb{R}$ defined by

$$F_0(y, u) = \int_\Omega g_0(y(x), u(x)) \, dx.$$

Due to the assumptions on g_0 it is straightforward to prove that F_0 is of class C^2. Now, applying the chain rule to $J(u) = F_0(G(u), u)$ and using Theorem 1 and the fact that $W^{1,p}(\Omega) \subset C(\bar{\Omega})$ for every $p > n$ we get that J is of class C^2 and

$$J'(u)h = \int_\Omega \left(\frac{\partial g_0}{\partial y}(y_u, u) z_h + \frac{\partial g_0}{\partial u}(y_u, u) h \right) dx.$$

Taking φ_{0u} as the solution of (10), we deduce (8) from previous identity and (4). Let us remark that the assumptions on f and g_0 and the Proposition 1 imply the regularity of φ_{0u}. The second derivative can be deduced in a similar way, using Theorem 1 once more. □

Theorem 3 *For each j, the functional $G_j : L^\infty(\Omega) \to \mathbb{R}$ is of class C^2. Moreover, for every $u, h \in L^\infty(\Omega)$*

$$G_j'(u)h = \int_\Omega \varphi_{ju} \frac{\partial f}{\partial u}(y_u, u) h \, dx \tag{11}$$

and for every $u, h_1, h_2 \in L^\infty(\Omega)$

$$\begin{aligned} G_j''(u)h_1h_2 = \int_\Omega \Bigg[& \nabla z_2 \frac{\partial^2 g_j}{\partial \eta^2}(\nabla y_u) \nabla z_1 \\ & + \varphi_{ju} \left(\frac{\partial^2 f}{\partial y^2}(y_u, u) z_1 z_2 + \frac{\partial^2 f}{\partial y \partial u}(y_u, u)(z_1 h_2 + z_2 h_1) + \frac{\partial^2 f}{\partial u^2}(y_u, u) h_1 h_2 \right) \Bigg] dx \end{aligned} \tag{12}$$

where $y_u = G(u)$, $\varphi_{ju} \in W^{1,p}(\Omega)$ for all $p \in (1, +\infty)$ is the unique solution of the problem

$$\left\{ \begin{array}{rcll} A^* \varphi_{ju} & = & \dfrac{\partial f}{\partial y}(y_u, u) \varphi_{ju} - div\left(\dfrac{\partial g_j}{\partial \eta}(\nabla y_u) \right) & \text{in } \Omega \\ \partial_{\nu_{A^*}} \varphi_{ju} & = & 0 & \text{on } \Gamma, \end{array} \right. \tag{13}$$

and $z_i = G'(u)h_i$, $i = 1, 2$.

Proof. Given $p > r + 2$ (see the condition (7)), it is enough to consider the function of class C^2 $F_j : W^{1,p}(\Omega) \to \mathbb{R}$ defined by

$$F_j(y) = \int_\Omega g_j(\nabla y(x)) \, dx.$$

Taking into account Theorem 1, we know that $y_u \in W^{1,p}(\Omega)$ for each $p \in (1, +\infty)$. Moreover, thanks to the assumption (7),

$$\frac{\partial g_j}{\partial \eta_i}(\nabla y_u) \in L^p(\Omega) \quad \forall p \in (1, +\infty);$$

hence, Proposition 1 can be used in order to deduce that φ_{ju} is well defined and belongs to $W^{1,p}(\Omega)$ for every $p \in (1, +\infty)$. □

Remark 3 *The solution of equation* (13) *must be interpreted in the following variational sense*

$$\sum_{i,j=1}^{N}\int_\Omega a_{ji}(x)\frac{\partial\varphi_{ku}}{\partial x_i}(x)\frac{\partial\psi}{\partial x_j}(x)\,dx = \int_\Omega \frac{\partial f}{\partial y}(y_u,u)\varphi_{ku}\psi\,dx$$

$$+\sum_{j=1}^{N}\int_\Omega \frac{\partial g_k}{\partial\eta_j}(\nabla y_u)\frac{\partial\psi}{\partial x_j}(x)\,dx$$

for all $\psi\in W^{1,p'}(\Omega)$.

4 First and second order optimality conditions for optimization problems.

In this section we present some results on the optimality conditions for abstract optimization problems that have been mainly obtained by Casas and Tröltzsch [3].

Let us consider the following optimization problem

$$\textbf{(Q)}\left\{\begin{array}{l} \text{Minimize } J(u)\\ u_a(x)\le u(x)\le u_b(x)\quad \text{a.e. } x\in\Omega,\\ G_j(u)=0,\ 1\le j\le n_e,\\ G_j(u)\le 0,\ n_e+1\le j\le n_e+n_i\end{array}\right.$$

where $u_a,u_b\in L^\infty(\Omega)$ and $J,G_j : L^\infty(\Omega)\longrightarrow\mathbb{R}$ are given functions, $1\le j\le n_e+n_i$.

We will assume that $\bar u$ is a local solution of **(Q)**, i.e. there exists a real number $\rho>0$ such that for every feasible point of **(Q)**, with $\|u-\bar u\|_{L^\infty(\Omega)}<\rho$, we have that $J(\bar u)\le J(u)$.

For every $\varepsilon>0$, we denote

$$\Omega_\varepsilon=\{x\in\Omega : u_a(x)+\varepsilon\le\bar u(x)\le u_b(x)-\varepsilon\}.$$

We make the following regularity assumption

$$\left\{\begin{array}{l}\exists\varepsilon_{\bar u}>0 \text{ and } \{h_j\}_{j\in I_0}\subset L^\infty(\Omega),\ \text{with } \operatorname{supp} h_j\subset\Omega_{\varepsilon_{\bar u}},\ \text{such that}\\ G_i'(\bar u)h_j=\delta_{ij},\quad i,j\in I_0,\end{array}\right. \tag{14}$$

where

$$I_0=\{j\le n_e+n_i\,|\,G_j(\bar u)=0\}.$$

I_0 is the set of indices corresponding to active constraints. We also denote the set of non active constraints by

$$I_-=\{j\le n_e+n_i\,|\,G_j(\bar u)<0\}.$$

Under this assumption we can derive the first order necessary conditions for optimality satisfied by $\bar{u}$. For the proof the reader is referred to Bonnans and Casas [1] or Clarke [4]).

THEOREM 4 *Let us assume that* (14) *holds and* J *and* $\{G_j\}_{j=1}^{n_e+n_i}$ *are of class* C^1 *in a neighbourhood of* $\bar{u}$. *Then there exist real numbers* $\{\bar{\lambda}_j\}_{j=1}^{n_e+n_i}$ *such that*

$$\bar{\lambda}_j \geq 0, \quad n_e + 1 \leq j \leq n_e + n_i, \ \bar{\lambda}_j = 0 \ \text{if } j \in I_-; \tag{15}$$

$$\langle J'(\bar{u}) + \sum_{j=1}^{n_e+n_i} \bar{\lambda}_j G_j'(\bar{u}), u - \bar{u}\rangle \geq 0 \quad \text{for all } u_a \leq u \leq u_b. \tag{16}$$

Since we want to give some second order optimality conditions useful for the study of the control problem **(P)**, we need to take into account the two-norm discrepancy; for this question see for instance Ioffe [5] and Maurer [7]. Then we have to impose some additional assumptions on the functions J and G_j.

(A1) There exist functions $\phi, \psi_j \in L^2(\Omega)$, $1 \leq j \leq n_e + n_i$, such that for every $h \in L^\infty(\Omega)$

$$J'(\bar{u})h = \int_\Omega \phi(x)h(x)dx \ \text{ and } \ G_j'(\bar{u})h = \int_\Omega \psi_j(x)h(x)dx, \ 1 \leq j \leq n_e + n_i. \tag{17}$$

(A2) If $\{h_k\}_{k=1}^\infty \subset L^\infty(\Omega)$ is bounded, $h \in L^\infty(\Omega)$ and $h_k(x) \to h(x)$ a.e. in Ω, then

$$[J''(\bar{u}) + \sum_{j=1}^{n_e+n_i} \bar{\lambda}_j G_j''(\bar{u})]h_k^2 \to [J''(\bar{u}) + \sum_{j=1}^{n_e+n_i} \bar{\lambda}_j G_j''(\bar{u})]h^2. \tag{18}$$

If we define

$$\mathcal{L}(u,\lambda) = J(u) + \sum_{j=1}^{n_e+n_i} \lambda_j G_j(u) \text{ and } d(x) = \phi(x) + \sum_{j=1}^{n_e+n_i} \bar{\lambda}_j \psi_j(x), \tag{19}$$

then

$$\frac{\partial \mathcal{L}}{\partial u}(\bar{u},\bar{\lambda})h = [J'(\bar{u}) + \sum_{j=1}^{n_e+n_i} \bar{\lambda}_j G_j'(\bar{u})]h = \int_\Omega d(x)h(x)dx \ \ \forall h \in L^\infty(\Omega). \tag{20}$$

From (16) we deduce that

$$d(x) = \begin{cases} 0 & \text{for a.e. } x \in \Omega \text{ such that } u_a(x) < \bar{u}(x) < u_b(x), \\ \geq 0 & \text{for a.e. } x \in \Omega \text{ such that } \bar{u}(x) = u_a(x), \\ \leq 0 & \text{for a.e. } x \in \Omega \text{ such that } \bar{u}(x) = u_b(x). \end{cases} \tag{21}$$

Associated with d we set

$$\Omega^0 = \{x \in \Omega : |d(x)| > 0\}. \tag{22}$$

Given $\{\bar{\lambda}_j\}_{j=1}^{n_e+n_i}$ by Theorem 4 we define

$$C_{\bar{u}}^0 = \{h \in L^\infty(\Omega) \ \text{ satisfying } \ (24) \ \text{ and } \ h(x) = 0 \ \text{ a.e. } \ x \in \Omega^0\}, \tag{23}$$

with

$$\begin{cases} G_j'(\bar{u})h = 0 \text{ if } (j \le n_e) \text{ or } (j > n_e,\ G_j(\bar{u}) = 0 \text{ and } \bar{\lambda}_j > 0); \\ \\ G_j'(\bar{u})h \le 0 \ \text{ if } j > n_e,\ G_j(\bar{u}) = 0 \text{ and } \bar{\lambda}_j = 0; \\ \\ h(x) = \begin{cases} \ge 0 & \text{if } \bar{u}(x) = u_a(x); \\ \le 0 & \text{if } \bar{u}(x) = u_b(x). \end{cases} \end{cases} \tag{24}$$

In the following theorem we state the necessary second order optimality conditions.

THEOREM 5 *Let us assume that* (14), **(A1)** *and* **(A2)** *hold,* $\{\bar{\lambda}_j\}_{j=1}^{n_e+n_i}$ *are the Lagrange multipliers satisfying* (15) *and* (16) *and* J *and* $\{G_j\}_{j=1}^{n_e+n_i}$ *are of class* C^2 *in a neighbourhood of* $\bar{u}$. *Then the following inequality is satisfied*

$$\frac{\partial^2 \mathcal{L}}{\partial u^2}(\bar{u}, \bar{\lambda})h^2 \ge 0 \quad \forall h \in C_{\bar{u}}^0. \tag{25}$$

Now $\bar{u}$ is a given feasible element for the problem **(Q)** satisfying first order necessary conditions. Motivated again for the considerations on the two-norm discrepancy we have to make some assumptions involving the $L^\infty(\Omega)$ and $L^2(\Omega)$ norms,

(A3) There exists a positive number $\rho > 0$ such that J and $\{G_j\}_{j=1}^{n_e+n_i}$ are of class C^2 in the $L^\infty(\Omega)$-ball $B_\rho(\bar{u})$ and for every $\delta > 0$ there exists $\varepsilon \in (0, \rho)$ such that for each $u \in B_\rho(\bar{u})$, $\|v - \bar{u}\|_{L^\infty(\Omega)} < \varepsilon$, $h, h_1, h_2 \in L^\infty(\Omega)$ and $1 \le j \le n_e + n_i$ we have

$$\begin{cases} \left| \left[\frac{\partial^2 \mathcal{L}}{\partial u^2}(v, \bar{\lambda}) - \frac{\partial^2 \mathcal{L}}{\partial u^2}(\bar{u}, \bar{\lambda}) \right] h^2 \right| \le \delta \|h\|_{L^2(\Omega)}^2, \\ \\ |J'(u)h| \le M_{0,1}\|h\|_{L^2(\Omega)}, \quad |G_j'(u)h| \le M_{j,1}\|h\|_{L^2(\Omega)}, \\ \\ |J''(u)h_1h_2| \le M_{0,2}\|h_1\|_{L^2(\Omega)}\|h_2\|_{L^2(\Omega)}, \\ \\ |G_j''(u)h_1h_2| \le M_{j,2}\|h_1\|_{L^2(\Omega)}\|h_2\|_{L^2(\Omega)}, \end{cases} \tag{26}$$

Analogously to (22) and (23) we define for every $\tau > 0$

$$\Omega^\tau = \{x \in \Omega : |d(x)| > \tau\} \tag{27}$$

and

$$C^\tau_{\bar u} = \{h \in L^\infty(\Omega) \ \text{ satisfying } \ (24) \ \text{ and } \ h(x) = 0 \ \text{ a.e. } \ x \in \Omega^\tau\}. \tag{28}$$

The next theorem provides the second order sufficient optimality conditions of **(Q)**.

Theorem 6 *Let $\bar u$ be a feasible point for problem* **(Q)** *satisfying first order necessary optimality conditions, and let us suppose that assumptions* (14), **(A1)** *and* **(A3)** *hold. Let us also assume that*

$$\frac{\partial^2 \mathcal{L}}{\partial u^2}(\bar u, \bar\lambda)h^2 \geq \delta \|h\|^2_{L^2(\Omega)} \quad \forall h \in C^\tau_{\bar u} \tag{29}$$

for some $\delta > 0$ and $\tau > 0$ given. Then there exist $\varepsilon > 0$ and $\alpha > 0$ such that $J(\bar u) + \alpha\|u - \bar u\|^2_{L^2(\Omega)} \leq J(u)$ for every feasible point u for **(Q)**, *with $\|u - \bar u\|_{L^\infty(\Omega)} < \varepsilon$.*

5 First and second order optimality conditions for problem (P).

In this section we assume that $\bar u$ is a local solution for problem **(P)**. We denote by $\bar y = G(\bar u)$ the state associated to the optimal control and by $\bar\varphi_j = \varphi_{j\bar u}$ the function satisfying (13) for $u = \bar u$. The same notation introduced in the Section 4 will be used.

5.1 First order necessary conditions for (P)

First order necessary conditions satisfied by $\bar u$ can be deduced very easily from the abstract Theorem 4 with the help of Theorems 2 and 3.

Theorem 7 *Assume* (14) *is satisfied. Then there exist real numbers $\bar\lambda_j, j = 1, \ldots, n_i + n_e$ and functions $\bar y, \bar\varphi \in W^{1,p}(\Omega)$ for all $p < \infty$ such that*

$$\bar\lambda_j \geq 0 \quad n_e + 1 \leq j \leq n_e + n_i, \qquad \bar\lambda_j \int_\Omega g_j(\nabla \bar y(x))\, dx = 0, \tag{30}$$

$$\left\{ \begin{array}{rcll} A\bar y & = & f(\bar y(x), \bar u(x)) & \text{in } \Omega \\ \partial_{\nu_A}\bar y & = & 0 & \text{on } \Gamma, \end{array} \right. \tag{31}$$

$$\left\{\begin{array}{rcll} A^*\bar{\varphi} & = & \dfrac{\partial f}{\partial y}(\bar{y},\bar{u})\bar{\varphi} + \dfrac{\partial g_0}{\partial y}(\bar{y},\bar{u}) - \displaystyle\sum_{j=1}^{n_e+n_i} div\left(\dfrac{\partial g_j}{\partial \eta}(\nabla\bar{y})\right) & in\ \Omega \\ \partial_{\nu_{A^*}}\bar{\varphi} & = & 0 & on\ \Gamma, \end{array}\right. \tag{32}$$

and

$$\int_\Omega \left(\frac{\partial g_0}{\partial u}(\bar{y},\bar{u}) + \bar{\varphi}\frac{\partial f}{\partial u}(\bar{y},\bar{u})\right)(u-\bar{u})dx \geq 0 \quad \textit{for all } u_a \leq u \leq u_b. \tag{33}$$

Moreover, if $\bar{\varphi}_0 = \varphi_{0\bar{u}}$ *and* $\bar{\varphi}_j = \varphi_{j\bar{u}}$ *for* $1 \leq j \leq n_e + n_i$ *are the solutions of* (10) *and* (13) *respectively for* $u = \bar{u}$ *, then*

$$\bar{\varphi} = \bar{\varphi}_0 + \sum_{j=1}^{n_e+n_i} \bar{\lambda}_j \bar{\varphi}_j. \tag{34}$$

Remark 4

1. *Equation* (32) *must be interpreted in the same sense to that of Remark 3.*
2. *In our case, assumption* **(A1)** *is satisfied with* $\phi = \dfrac{\partial g_0}{\partial u}(\bar{y},\bar{u}) + \bar{\varphi}_0\dfrac{\partial f}{\partial u}(\bar{y},\bar{u})$ *and* $\psi_j = \bar{\varphi}_j\dfrac{\partial f}{\partial u}(\bar{y},\bar{u})$.
3. *The regularity assumption* (14) *is equivalent to: There exists* $\bar{\varepsilon} > 0$ *such that the set of functions* $\{\psi_j :\ j \in I_0\}$ *is linearly independent in* $L^1(\Omega_{\bar{\varepsilon}})$.

   This condition looks very similar to the corresponding one in finite dimensions, with the identification $G_j'(\bar{u}) = \psi_j$.

5.2 Second order necessary conditions for problem (P)

Taking into account Theorems 2 and 3 together with the conditions imposed over f, g_0, g_j is not difficult to show that the assumptions for Theorem 5 are satisfied by the problem **(P)**. Moreover in this case we can identify

$$d(x) = \frac{\partial g_0}{\partial u}(\bar{y}(x),\bar{u}(x)) + \bar{\varphi}(x)\frac{\partial f}{\partial u}(\bar{y}(x),\bar{u}(x)).$$

So we arrive to the following theorem.

THEOREM 8 *Let the hypotheses of Theorem 7 be satisfied. Then*

$$\begin{aligned}\frac{\partial^2 \mathcal{L}}{\partial u^2}(\bar{u},\bar{\lambda})h^2 = \int_\Omega \left(\frac{\partial^2 g_0}{\partial y^2}(\bar{y},\bar{u}) + \bar{\varphi}\frac{\partial^2 f}{\partial y^2}(\bar{y},\bar{u})\right) z_h^2\,dx +\\ 2\int_\Omega \left(\frac{\partial^2 g_0}{\partial y\partial u}(\bar{y},\bar{u}) + \bar{\varphi}\frac{\partial^2 f}{\partial y\partial u}(\bar{y},\bar{u})\right) h z_h\,dx +\\ \int_\Omega \left(\frac{\partial^2 g_0}{\partial u^2}(\bar{y},\bar{u}) + \bar{\varphi}\frac{\partial^2 f}{\partial u^2}(\bar{y},\bar{u})\right) h^2\,dx +\\ \sum_{j=1}^{n_i+n_e}\bar{\lambda}_j\int_\Omega \nabla z_h \frac{\partial^2 g_j}{\partial \eta^2}(\nabla\bar{y})\nabla z_h\,dx \geq 0\end{aligned} \tag{35}$$

for all $h \in L^\infty(\Omega)$ *satisfying* $h(x) = 0$ *for almost all* $x \in \Omega^0$ *and*

$$\begin{cases} \displaystyle\int_\Omega \bar{\varphi}_j \frac{\partial f}{\partial u}(\bar{y},\bar{u})h\,dx = 0 \text{ if } (j \leq n_e) \text{ or } (j > n_e, \int_\Omega g_j(\nabla\bar{y}) = 0 \text{ and } \bar{\lambda}_j > 0)\\ \displaystyle\int_\Omega \bar{\varphi}_j \frac{\partial f}{\partial u}h\,dx \leq 0 \text{ if } n_e+1 \leq j \leq n_i+n_e \text{ and } \int_\Omega g_j(\nabla\bar{y}) = 0 \text{ and } \bar{\lambda}_j = 0\\ h(x) \geq 0 \text{ if } \bar{u}(x) = u_a(x)\\ h(x) \leq 0 \text{ if } \bar{u}(x) = u_b(x).\end{cases} \tag{36}$$

5.3 Second order sufficient conditions for problem (P).

Clearly, here we are going to apply Theorem 6. Let us see that the assumptions for this theorem are satisfied by our problem. The main difficulty appears to prove that **(A3)** holds. Let $\bar{u}$ be a feasible control satisfying first order necessary conditions (30)–(33). Given $v \in L^\infty(\Omega)$, we denote $\varphi_v = \varphi_{0v} + \sum_{j=1}^{n_e+n_i}\bar{\lambda}_j\varphi_{jv}$, where φ_{0v} and φ_{jv} are the solutions of (10) and (13) for $u = v$, respectively. Take $h \in L^\infty(\Omega)$ and $\delta > 0$.

Let us verify the first inequality in (26). In fact, we will state that

$$\left|\left[\frac{\partial^2\mathcal{L}}{\partial u^2}(v,\bar{\lambda}) - \frac{\partial^2\mathcal{L}}{\partial u^2}(\bar{u},\bar{\lambda})\right]h^2\right| \leq$$

$$\int_\Omega \left|\frac{\partial^2 g_0}{\partial u^2}(y_v,v) + \varphi_v\frac{\partial^2 f}{\partial u^2}(y_v,v) - \frac{\partial^2 g_0}{\partial u^2}(\bar{y},\bar{u}) - \bar{\varphi}\frac{\partial^2 f}{\partial u^2}(\bar{y},\bar{u})\right| h^2\,dx +$$

$$\int_\Omega \left| \left(\frac{\partial^2 g_0}{\partial y \partial u}(y_v, v) + \varphi_v \frac{\partial^2 f}{\partial y \partial u}(y_v, v) \right) z_h - \left(\frac{\partial^2 g_0}{\partial y \partial u}(\bar{y}, \bar{u}) + \bar{\varphi} \frac{\partial^2 f}{\partial y \partial u}(\bar{y}, \bar{u}) \right) \bar{z} \right| |h|$$

$$+ \int_\Omega \left| \left(\frac{\partial^2 g_0}{\partial y^2}(y_v, v) + \varphi_v \frac{\partial^2 f}{\partial y^2}(y_v, v) \right) z_h^2 - \left(\frac{\partial^2 g_0}{\partial y^2}(\bar{y}, \bar{u}) + \bar{\varphi} \frac{\partial^2 f}{\partial y^2}(\bar{y}, \bar{u}) \right) \bar{z}^2 \right| \, dx +$$

$$\sum_{j=1}^{n_e+ni} |\bar{\lambda}_j| \int_\Omega \left| \nabla z_h \frac{\partial^2 g_j}{\partial \eta^2}(\nabla y_v) \nabla z_h - \nabla z_h \frac{\partial^2 g_j}{\partial \eta^2}(\nabla \bar{y}) \nabla z_h \right| \, dx \leq \delta \|h\|^2_{L^2(\Omega)} \quad (37)$$

supposed that $\|v - \bar{u}\|_{L^\infty(\Omega)} < \varepsilon$ with ε small enough, where

$$\begin{cases} A\bar{z} = \dfrac{\partial f}{\partial y}(\bar{y}, \bar{u})\bar{z} + \dfrac{\partial f}{\partial u}(\bar{y}, \bar{u})h & \text{in } \Omega \\ \partial_{\nu_A} \bar{z} = 0 & \text{on } \Gamma. \end{cases} \quad (38)$$

$$\begin{cases} Az_h = \dfrac{\partial f}{\partial y}(y_v, v)z_h + \dfrac{\partial f}{\partial u}(y_v, v)h & \text{in } \Omega \\ \partial_{\nu_A} z_h = 0 & \text{on } \Gamma. \end{cases} \quad (39)$$

We can carry out the argumentation working with each term in a separate way. Let us emphasize that the main ingredients to prove (37) are the continuity of the functional G, the C^2- regularity of f and g_j $j = 0, 1, \ldots, n_e + n_i$ and the assumptions (3) and (7).

Given $\tilde{\delta} > 0$, for the first term of the left hand side of (37) it is easy to establish that

$$\left\| \frac{\partial^2 g_0}{\partial u^2}(y_v, v) + \varphi_v \frac{\partial^2 f}{\partial u^2}(y_v, v) - \frac{\partial^2 g_0}{\partial u^2}(\bar{y}, \bar{u}) - \bar{\varphi} \frac{\partial^2 f}{\partial u^2}(\bar{y}, \bar{u}) \right\|_{L^\infty(\Omega)} < \tilde{\delta}$$

provided that $\|v - \bar{u}\|_{L^\infty(\Omega)}$ is sufficiently small: this is a direct consequence of the continuous dependence of φ_v with respect to v in the $L^\infty(\Omega)$-norm, which can be obtained with the help of Proposition 1.

For the second term of (37), the Hölder's inequality leads us to

$$\int_\Omega \left| \left(\frac{\partial^2 g_0}{\partial y \partial u}(y_v, v) + \varphi_v \frac{\partial^2 f}{\partial y \partial u}(y_v, v) \right) z_h - \left(\frac{\partial^2 g_0}{\partial y \partial u}(\bar{y}, \bar{u}) + \bar{\varphi} \frac{\partial^2 f}{\partial y \partial u}(\bar{y}, \bar{u}) \right) \bar{z} \right| |h|$$

$$\leq \|h\|_{L^2(\Omega)} \left(\left\| \frac{\partial^2 g_0}{\partial y \partial u}(y_v, v) - \frac{\partial^2 g_0}{\partial y \partial u}(\bar{y}, \bar{u}) \right\|_{L^\infty(\Omega)} \|z_h\|_{L^2(\Omega)} \right.$$

$$+ \left\| \frac{\partial^2 g_0}{\partial y \partial u}(\bar{y}, \bar{u}) \right\|_{L^\infty(\Omega)} \|z_h - \bar{z}\|_{L^2(\Omega)}$$

$$+ \left\| \varphi_v \frac{\partial^2 f}{\partial y \partial u}(y_v, v) - \bar{\varphi} \frac{\partial^2 f}{\partial y \partial u}(\bar{y}, \bar{u}) \right\|_{L^\infty(\Omega)} \|z_h\|_{L^2(\Omega)}$$

$$\left. + \left\| \bar{\varphi} \frac{\partial^2 f}{\partial y \partial u}(\bar{y}, \bar{u}) \right\|_{L^\infty(\Omega)} \|z_h - \bar{z}\|_{L^2(\Omega)} \right)$$

Argumentation can be now completed by taking into account the estimations

$$\|z_h\|_{L^2(\Omega)} + \|\bar{z}\|_{L^2(\Omega)} \leq C_1 \|h\|_{L^2(\Omega)} \quad \text{and} \tag{40}$$

$$\|z_h - \bar{z}\|_{L^2(\Omega)} \leq \tilde{\delta} \|h\|_{L^2(\Omega)}, \tag{41}$$

when $\|v - \bar{u}\|_{L^\infty(\Omega)}$ is small.

Following the same scheme we have

$$\int_\Omega \left| \left(\frac{\partial^2 g_0}{\partial y^2}(y_v, v) + \varphi_v \frac{\partial^2 f}{\partial y^2}(y_v, v) \right) z_h^2 - \left(\frac{\partial^2 g_0}{\partial y^2}(\bar{y}, \bar{u}) + \bar{\varphi} \frac{\partial^2 f}{\partial y^2}(\bar{y}, \bar{u}) \right) \bar{z}^2 \right| \, dx \leq$$

$$\leq \left\| \frac{\partial^2 g_0}{\partial y^2}(y_v, v) - \frac{\partial^2 g_0}{\partial y^2}(\bar{y}, \bar{u}) \right\|_{L^\infty(\Omega)} \|z_h\|^2_{L^2(\Omega)}$$

$$+ \left\| \frac{\partial^2 g_0}{\partial y^2}(\bar{y}, \bar{u}) \right\|_{L^\infty(\Omega)} \|z_h - \bar{z}\|_{L^2(\Omega)} \|z_h + \bar{z}\|_{L^2(\Omega)}$$

$$+ \left\| \varphi_v \frac{\partial^2 f}{\partial y^2}(y_v, v) - \bar{\varphi} \frac{\partial^2 f}{\partial y^2}(\bar{y}, \bar{u}) \right\|_{L^\infty(\Omega)} \|z_h\|^2_{L^2(\Omega)}$$

$$+ \left\| \bar{\varphi} \frac{\partial^2 f}{\partial y^2}(\bar{y}, \bar{u}) \right\|_{L^\infty(\Omega)} \|z_h - \bar{z}\|_{L^2(\Omega)} \|z_h + \bar{z}\|_{L^2(\Omega)},$$

which together with (40)-(41) allow us to deal with the third term of (37)

We study the last term, by decomposing it as follows and using Hölder's inequality once more

$$\int_\Omega \left| \nabla z_h \frac{\partial^2 g_j}{\partial \eta^2}(\nabla y_v) \nabla z_h - \nabla z_h \frac{\partial^2 g_j}{\partial \eta^2}(\nabla \bar{y}) \nabla z_h \right| \, dx \leq$$

$$\leq \int_\Omega \left| \nabla z_h \left(\frac{\partial^2 g_j}{\partial \eta^2}(\nabla y_v) - \frac{\partial^2 g_j}{\partial \eta^2}(\nabla \bar{y}) \right) \nabla z_h \right| dx$$

$$+ \int_\Omega \left| (\nabla z_h - \nabla \bar{z}) \frac{\partial^2 g_j}{\partial \eta^2}(\nabla \bar{y})(\nabla z_h + \nabla z_h) \right| dx \leq$$

$$\leq \|\nabla z_h\|^2_{L^p(\Omega)} \left\| \frac{\partial^2 g_j}{\partial \eta^2}(\nabla y_v) - \frac{\partial^2 g_j}{\partial \eta^2}(\nabla \bar{y}) \right\|_{L^q(\Omega)^{N^2}}$$

$$+ \|\nabla z_h - \nabla \bar{z}\|_{L^p(\Omega)} \|\nabla z_h + \nabla \bar{z}\|_{L^p(\Omega)} \left\| \frac{\partial^2 g_j}{\partial \eta^2}(\nabla \bar{y}) \right\|_{L^q(\Omega)^{N^2}}$$

with $p = 2N/(N-2)$ (if $N > 2$), $p = 3$ (if $N = 1$ or 2) and $q = pp'/(p - p')$.

The exponent p has been chosen such that $L^2(\Omega) \subset (W^{1,p'}(\Omega))'$. Hence, using Proposition 1, we have that

$$\|\nabla z_h\|_{L^p(\Omega)} + \|\nabla \bar{z}\|_{L^p(\Omega)} \leq C_2 \|h\|_{L^2(\Omega)}. \tag{42}$$

when $\|v - \bar{u}\|_{L^\infty(\Omega)}$ is bounded. Moreover, in this case, subtracting the equations (38) and (39) and using Proposition 1 once more, we can derive that

$$\|\nabla z_h - \nabla \bar{z}\|_{L^p(\Omega)} \leq \tilde{\delta} \|h\|_{L^2(\Omega)}.$$

Finally, we can deduce that

$$\left\| \frac{\partial^2 g_j}{\partial \eta^2}(\nabla y_v) - \frac{\partial^2 g_j}{\partial \eta^2}(\nabla \bar{y}) \right\|_{L^q(\Omega)^{N^2}} < \tilde{\delta}$$

for small enough $\|v - \bar{u}\|_{L^\infty(\Omega)}$ uniformly with respect to v. Let us show this in detail: by the continuity of the functional G and the assumption (7), fixed $\tilde{q} > q$, there exists a positive constant C_3 such that

$$\|\nabla y_v\|_{L^{r\tilde{q}}(\Omega)} + \|\nabla \bar{y}\|_{L^{r\tilde{q}}(\Omega)} + \left\| \frac{\partial^2 g_j}{\partial \eta^2}(\nabla y_v) \right\|_{L^{\tilde{q}}(\Omega)^{N^2}} + \left\| \frac{\partial^2 g_j}{\partial \eta^2}(\nabla \bar{y}) \right\|_{L^{\tilde{q}}(\Omega)^{N^2}} \leq C_3,$$

the exponent r being the one introduced in (7) for every feasible point v. Given $M > 0$, let us introduce the following sets $E_1^M = \{x \in \Omega : \|\nabla y_v(x)\| \geq M\}$ and $E_2^M = \{x \in \Omega : \|\nabla \bar{y}(x)\| \geq M\}$. Clearly E_1^M and E_2^M depend on v and $\bar{u}$, respectively, but we will not emphasized this. Here, it is important to point out the obvious inequality

$$m(E_1^M) \leq \frac{1}{M} \int_\Omega \|\nabla y_v(x)\| dx \leq \frac{C_4}{M}.$$

The same argument holds for E_2^M.

Thanks to the regularity of g_j, the second order derivatives are uniform continuity in the ball of $\mathbb{R}^N$ with center at the origin and radius M. Hence, there exists $\epsilon_1 > 0$ such that always that $\|\eta - \tilde{\eta}\|_{\mathbb{R}^N} \leq \epsilon_1$ with $\|\eta\|_{\mathbb{R}^N}, \|\tilde{\eta}\|_{\mathbb{R}^N} \leq M$, we have

$$\left\| \frac{\partial^2 g_j}{\partial \eta^2}(\eta) - \frac{\partial^2 g_j}{\partial \eta^2}(\tilde{\eta}) \right\|_{\mathbb{R}^{N^2}} < \left(\frac{\tilde{\delta}}{4m(\Omega)} \right)^{1/q}.$$

Using again the continuity of the functional G, there exists $\epsilon_2 > 0$ such that when $\|v - \bar{u}\|_{L^\infty(\Omega)} \leq \epsilon_2$, then

$$\int_\Omega \|\nabla y_v(x) - \nabla \bar{y}(x)\| dx \leq \epsilon_1 \frac{C_4}{M}.$$

Let us introduce another set $E_3^M = \{x \in \Omega : \|\nabla y_v(x) - \nabla \bar{y}(x)\| > \epsilon_1\}$. Arguing as before, we derive

$$\epsilon_1 m(E_3^M) \leq \int_\Omega \|\nabla y_v(x) - \nabla \bar{y}(x)\| dx.$$

In particular, the last two relations imply $m(E_3^M) \leq \frac{C_4}{M}$. Combining the previous estimations and using Hölder's inequality with $s = \tilde{q}/q$, we get

$$\int_\Omega \left\| \frac{\partial^2 g_j}{\partial \eta^2}(\nabla y_v) - \frac{\partial^2 g_j}{\partial \eta^2}(\nabla \bar{y}) \right\|^q dx \leq \int_{E_1^M} \left\| \frac{\partial^2 g_j}{\partial \eta^2}(\nabla y_v) - \frac{\partial^2 g_j}{\partial \eta^2}(\nabla \bar{y}) \right\|^q dx +$$

$$\int_{E_2^M} \left\| \frac{\partial^2 g_j}{\partial \eta^2}(\nabla y_v) - \frac{\partial^2 g_j}{\partial \eta^2}(\nabla \bar{y}) \right\|^q dx + \int_{E_3^M} \left\| \frac{\partial^2 g_j}{\partial \eta^2}(\nabla y_v) - \frac{\partial^2 g_j}{\partial \eta^2}(\nabla \bar{y}) \right\|^q dx +$$

$$\int_{\Omega \setminus (E_1^M \cup E_2^M \cup E_M^3)} \left\| \frac{\partial^2 g_j}{\partial \eta^2}(\nabla y_v) - \frac{\partial^2 g_j}{\partial \eta^2}(\nabla \bar{y}) \right\|^q dx \leq$$

$$\frac{\tilde{\delta}}{4} + \left(\sum_{j=1}^3 m(E_j^M)^{1/s'} \right) \left(\int_\Omega \left\| \frac{\partial^2 g_j}{\partial \eta^2}(\nabla y_v) - \frac{\partial^2 g_j}{\partial \eta^2}(\nabla \bar{y}) \right\|^{\tilde{q}} dx \right)^{1/s}$$

$$\leq \frac{\tilde{\delta}}{4} + 3 \left(\frac{C_4}{M} \right)^{1/s'} 2^{q+1/s} C_3^q$$

This right hand term can be taken less than $\tilde{\delta}$, provided that M is sufficiently large.

From all the above considerations, we can assure that the first condition on the continuity of the second derivative of the Lagrangian in (26) is satisfied. The rest of the conditions follows easily from the properties of the functions f and g_j, $j = 0, 1, \ldots, n_e + n_i$.

THEOREM 9 *Let $\bar{u}$ be a feasible point for problem* **(P)** *and let us suppose that it satisfies the regularity assumption* (14) *and the first order necessary conditions stated in Theorem 7. Let us also assume that*

$$\int_\Omega \left(\frac{\partial^2 g_0}{\partial y^2}(\bar{y},\bar{u}) + \bar{\varphi}\frac{\partial^2 f}{\partial y^2}(\bar{y},\bar{u}) \right) z_h^2 dx + \tag{43}$$

$$2\int_\Omega \left(\frac{\partial^2 g_0}{\partial y \partial u}(\bar{y},\bar{u}) + \bar{\varphi}\frac{\partial^2 f}{\partial y \partial u}(\bar{y},\bar{u}) \right) h z_h dx$$

$$+ \int_\Omega \left(\frac{\partial^2 g_0}{\partial u^2}(\bar{y},\bar{u}) + \bar{\varphi}\frac{\partial^2 f}{\partial u^2}(\bar{y},\bar{u}) \right) h^2\, dx +$$

$$\sum_{j=1}^{n_i+n_e} \bar{\lambda}_j \int_\Omega \nabla z_h \frac{\partial^2 g_j}{\partial \eta^2}(\nabla \bar{y}) \nabla z_h\, dx \geq \delta \|h\|^2_{L^2(\Omega)}$$

for all $h \in L^\infty(\Omega)$ satisfying (36) *and $h(x) = 0$ for almost every $x \in \Omega^\tau$ and some $\delta > 0$ and $\tau > 0$ given. Then there exist $\varepsilon > 0$ and $\alpha > 0$ such that $J(\bar{u}) + \alpha\|u - \bar{u}\|^2_{L^2(\Omega)} \leq J(u)$ for every feasible control u with $\|u - \bar{u}\|_{L^\infty(\Omega)} < \varepsilon$.*

There is no difficulty in extending our results to more general situations, where the nonlinear term f of the state equation depends on (x, y_u, u), the cost functional J is given by an integrand g_0 depending on $(x, y_u, \nabla y_u, u)$ as well as the integral constraints G_j. Clearly, in this case, some appropiate growth conditions have to be imposed in order to apply the abstract framework (see Section 4).

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