Pontryagin's Principle for the Control of Parabolic Equations with Gradient State Constraints *

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Abstract

In this paper we are concerned with an optimal control problem governed by a semilinear parabolic equation, with boundary controls. Pointwise control constraints as well as integral constraints on the gradient of the state are considered. The aim of the paper is to prove a Pontryagin principle. To achieve this goal we use Ekeland's variational principle combined with a method of diffuse perturbations. A detailed analysis of the state and adjoint state equations is carried out. We obtain some regularity results, under minimal assumptions, which are necessary to treat the constraints on the gradient of the state.

Key Words. Parabolic equations, regularity results, optimal control, state constraints, Pontryagin principle.

1 Introduction.

Let Ω be an open, bounded and connected set in R^N. We denote by Γ the boundary of Ω. Let T be a positive real number. Set $Q = \Omega\times]0,T[$ and $\Sigma = \Gamma\times]0,T[$. Let us introduce the elliptic operator

$$Ay = -\sum_{i,j=1}^{N} \partial_{x_j}\left(a_{ij}(x,t)\partial_{x_i} y\right).$$

Let f, g, w be functions, $f : Q \times \mathrm{R} \times \mathrm{R} \longrightarrow \mathrm{R}$, $g : \Sigma \times \mathrm{R} \times \mathrm{R} \longrightarrow \mathrm{R}$ and $w : \Omega \longrightarrow \mathrm{R}$. We are going to study control problems for the parabolic equation

$$\begin{cases} \dfrac{\partial y}{\partial t} + Ay + f(x,t,y) = 0 & \text{in } Q, \\ \dfrac{\partial y}{\partial n_A} + g(s,t,y,v) = 0 & \text{on } \Sigma, \\ y(\cdot,0) = w & \text{in } \Omega. \end{cases} \tag{1}$$

We shall usually refer to this equation as the state equation. Let V_{ad} be a subset of $L^\infty(\Sigma)$. For every $v \in V_{ad}$ we denote by y_v the solution of equation (1). Consider functions $F : Q \times \mathrm{R} \longrightarrow \mathrm{R}$, $G : \Sigma \times \mathrm{R} \times \mathrm{R} \longrightarrow \mathrm{R}$ and

*The research of the first two authors was partially supported by Dirección General de Enseñanza Superior e Investigación Científica (Spain). This paper was partially written while the third author was visiting the Department of Applied Mathematics and Computer Science of Cantabria University.

$L : \Omega \times \mathrm{R} \longrightarrow \mathrm{R}$. Problem (P) consists in minimizing the cost functional

$$J(y_v, v) = \int_0^T \int_\Omega F(x, t, y_v)\ dx\ dt + \int_0^T \int_\Gamma G(s, t, y_v, v)\ ds\ dt + \int_\Omega L(x, y_v(x, T))\ dx$$

with the control constraint

$$v \in V_{ad},$$

and the state constraint

$$\nabla_x y \in C \subset (L^\tau(0, T; L^p(\Omega)))^N, \tag{2}$$

where τ and p belong to $(1, \infty)$, and C is a closed, convex subset of $(L^\tau(0, T; L^p(\Omega)))^N$ with a nonempty interior. For instance, if $C = \bar{B}_\delta(0)$, the closed ball of radius δ in $(L^\tau(0, T; L^p(\Omega)))^N$, then the constraint is

$$\int_0^T \left(\int_\Omega |\nabla_x y|^p dx \right)^{\tau/p} dt \le \delta^\tau.$$

Our main results is a Pontryagin principle for such problems (Theorem 3). In recent years there has been growing interest in Pontryagin principles for control problems governed by partial differential equations, with pointwise or integral constraints on the state variable. Among many others we could mention Casas [1], Fattorini [5, 7], Bei Hu and Yong [9], Li and Yong [11], Raymond and Zidani [13], Casas, Raymond and Zidani [3] and references therein. Only few results are avalaible for problems dealing with gradient state constraints. See for example Casas and Fernandez [2], Fattorini [6, 7], or White [18].

The proof of the Theorem 3 is based on Ekeland's variational principle. To obtain an approximate Pontryagin principle corresponding to optimality conditions following from Ekeland's variational principle, we use the method of diffuse perturbations, as in the papers of Raymond and Zidani [13] or Casas, Raymond and Zidani [3]. In this approach we have to prove some Taylor expansion (Theorem 2) for the solution of the state equation, with a remainder term converging to zero in the norm of $L^\tau(0, T; W^{1,p}(\Omega))$ (the norm corresponding to the state constraint). To establish this result, we use the compact embedding from $L^\tau(0, T; W^{1+\varepsilon,p}(\Omega)) \cap W^{1,\tau}(0, T, (W^{1,p'}(\Omega))')$ into $L^\tau(0, T; W^{1,p}(\Omega))$ (see the proof of Theorem 2). Thus we have to establish regular ity results in $L^\tau(0, T; W^{1+\varepsilon,p}(\Omega))$ for the linearized state equation. These results are stated in Section 2.

The rest of the paper is as follows. The adjoint equation is studied in Section 2. Assumptions on the control problem and the Taylor expansion are stated in Section 3. Section 4 is devoted to the proof of the main result and Section 5 deals with some extensions and examples.

2 Linearized and adjoint equations.

In the paper, whenever it does not lead to confusion, we shall use the following shortening: $L^\tau(W^{s,p})$, $L^2(H^1)$, $W^{1,\tau}((W^{1,p})')$, $L^{\tilde{k}}(L^k(\Omega))$, $L^{\tilde{\sigma}}(L^\sigma(\Gamma))$, and $C(C^{0,\varepsilon}(\bar{\Omega}))$ respectively for $L^\tau(0, T; W^{s,p}(\Omega))$, $L^2(0, T; H^1(\Omega))$, $W^{1,\tau}(0, T; (W^{1,p}(\Omega))')$, $L^{\tilde{k}}(0, T; L^k(\Omega))$, $L^{\tilde{\sigma}}(0, T; L^\sigma(\Gamma))$ and $C([0, T]; C^{0,\varepsilon}(\bar{\Omega}))$.

We suppose that $\tau \in (1, \infty)$ and $p \in (1, \infty)$ are given fixed throughout the paper.

We now state some hypotheses.

H1 - The boundary Γ is of class $C^{1,\hat{\varepsilon}}$ for some $0 < \hat{\varepsilon} < 1$.

H2 - The coefficients a_{ij} belong to $C([0, T]; C^{0,\hat{\varepsilon}}(\bar{\Omega}))$ and satisfy

$$m\|\xi\|^2 \le \sum_{i,j=1}^N a_{ij}(x, t)\xi_i\xi_j \le M\|\xi\|^2 \quad \text{for all } \xi \in \mathrm{R}^N \text{ and all } (x, t) \in Q$$

for some $m, M > 0$.

To deal with state constraints of the form (2), we need regularity results in $L^\tau(W^{1+\varepsilon,p})$ for the solution of the equation

$$\left\{ \begin{array}{rcll} \dfrac{\partial y}{\partial t} + Ay & = & \hat{f} & \text{in } Q, \\ \dfrac{\partial y}{\partial n_A} & = & \hat{g} & \text{on } \Sigma, \\ y(\cdot, 0) & = & 0 & \text{in } \Omega. \end{array} \right.$$

Recall the following regularity results. Assume that the boundary Γ is of class C^2. Set $\bar{a}_{ij} = a_{ij}(\bar{x},\bar{t})$ and $\bar{A}y = -\sum_{i,j=1}^N \partial_{x_j}(\bar{a}_{ij}\partial_{x_i}y)$, where $(\bar{x},\bar{t})$ is any point in $\bar{Q}$. Then the mapping that associates $\hat{f}$ with the solution y of

$$\begin{cases} \dfrac{\partial y}{\partial t} + \bar{A}y &= \hat{f} \quad \text{in } Q, \\ \dfrac{\partial y}{\partial n_{\bar{A}}} &= 0 \quad \text{on } \Sigma, \\ y(\cdot,0) &= 0 \quad \text{in } \Omega, \end{cases}$$

is continuous from $L^{\tilde{k}_1}(L^{k_1}(\Omega))$ into $L^\tau(W^{1+\varepsilon_k,p})$ when one of the following conditions is satisfied

$$0 < \frac{\varepsilon_k}{2} < \frac{N}{2p} + \frac{1}{\tau} + \frac{1}{2} - \frac{N}{2k_1} - \frac{1}{\tilde{k}_1}, \quad \text{if } k_1 \le p \text{ and } \tilde{k}_1 \le \tau, \tag{3}$$

$$0 < \frac{\varepsilon_k}{2} < \frac{N}{2p} + \frac{1}{2} - \frac{N}{2k_1}, \quad \text{if } k_1 \le p \text{ and } \tilde{k}_1 > \tau, \tag{4}$$

$$0 < \frac{\varepsilon_k}{2} < \frac{1}{\tau} + \frac{1}{2} - \frac{1}{\tilde{k}_1}, \quad \text{if } k_1 > p \text{ and } \tilde{k}_1 \le \tau, \tag{5}$$

$$0 < \varepsilon_k < 1, \quad \text{if } k_1 > p \text{ and } \tilde{k}_1 > \tau. \tag{6}$$

For non homogeneous boundary data, the mapping that associates $\hat{g}$ with the solution y of

$$\begin{cases} \dfrac{\partial y}{\partial t} + \bar{A}y &= 0 \quad \text{in } Q, \\ \dfrac{\partial y}{\partial n_{\bar{A}}} &= \hat{g} \quad \text{on } \Sigma, \\ y(\cdot,0) &= 0 \quad \text{in } \Omega, \end{cases}$$

is continuous from $L^{\tilde{\sigma}_1}(L^{\sigma_1}(\Gamma))$ into $L^\tau(W^{1+\varepsilon_\sigma,p})$ when one of the following conditions is satisfied:

$$0 < \frac{\varepsilon_\sigma}{2} < \frac{N}{2p} + \frac{1}{\tau} - \frac{N-1}{2\sigma_1} - \frac{1}{\tilde{\sigma}_1}, \quad \text{if } \sigma_1 \le p \text{ and } \tilde{\sigma}_1 \le \tau, \tag{7}$$

$$0 < \frac{\varepsilon_\sigma}{2} < \frac{N}{2p} - \frac{N-1}{2\sigma_1}, \quad \text{if } \sigma_1 \le p \text{ and } \tilde{\sigma}_1 > \tau, \tag{8}$$

$$0 < \frac{\varepsilon_\sigma}{2} < \frac{1}{2p} + \frac{1}{\tau} - \frac{1}{\tilde{\sigma}_1}, \quad \text{if } \sigma_1 > p \text{ and } \tilde{\sigma}_1 \le \tau, \tag{9}$$

$$0 < \varepsilon_\sigma < \frac{1}{p}, \quad \text{if } \sigma_1 > p \text{ and } \tilde{\sigma}_1 > \tau. \tag{10}$$

The previous regularity results may be proved by using the same techniques as in [12, Prop. 3.2].

In all what follows $\varepsilon > 0$ is given fixed, strictly less than $\min(\hat{\varepsilon}, 2/\tau, 2/p)$, and less or equal than $\min(\varepsilon_\sigma, \varepsilon_k)$, where $\varepsilon_\sigma,\ \varepsilon_k$ are chosen as in (3)–(10). We make the following hypotheses on $\tilde{k}_1$, k_1, $\tilde{\sigma}_1$, σ_1.

H3 - The pair $(\tilde{k}_1, k_1)$ satisfies one of the conditions (3)–(6) and

$$\frac{N}{2k_1} + \frac{1}{\tilde{k}_1} < 1. \tag{11}$$

H4 - The pair $(\tilde{\sigma}_1, \sigma_1)$ satisfies one of the conditions (7)–(10) and

$$\frac{N-1}{2\sigma_1} + \frac{1}{\tilde{\sigma}_1} < \frac{1}{2}. \tag{12}$$

Remark 1 *Conditions* (11) *and* (12) *are needed to prove Propositions 5 and 7.*

A regularity result in $L^\tau(W^{1+\varepsilon,p})$ for the linearized state equation is proved in Proposition 5. We first establish some preliminary estimates.

Proposition 1 *Assume that the boundary* Γ *is of class* C^2. *Set* $\bar{a}_{ij} = a_{ij}(\bar{x}, \bar{t})$ *and* $\bar{A}y = -\sum_{i,j=1}^{N} \partial_{x_j}(\bar{a}_{ij}\partial_{x_i} y)$, *where* $(\bar{x}, \bar{t})$ *is any point in* $\bar{Q}$. *Let* $\hat{f}$ *be in* $L^{\tilde{k}_1}(L^{k_1}(\Omega))$ *and* $\hat{g}$ *be in* $L^{\tilde{\sigma}_1}(L^{\sigma_1}(\Gamma))$. *Then the weak solution* y *to the equation*

$$\begin{cases} \dfrac{\partial y}{\partial t} + \bar{A}y = \hat{f} & \text{in } Q, \\ \dfrac{\partial y}{\partial n_{\bar{A}}} = \hat{g} & \text{on } \Sigma, \\ y(\cdot, 0) = 0 & \text{in } \Omega, \end{cases} \tag{13}$$

belongs to $L^\tau(W^{1+\varepsilon,p}) \cap L^2(H^1)$, *and satisfies*

$$\|y\|_{L^\tau(W^{1+\varepsilon,p})\cap L^2(H^1)} \le C(\|\hat{f}\|_{L^{\tilde{k}_1}(L^{k_1}(\Omega))} + \|\hat{g}\|_{L^{\tilde{\sigma}_1}(L^{\sigma_1}(\Gamma))}), \tag{14}$$

where C *depends on* Ω, T, ε, $\tilde{k}_1$, k_1, $\tilde{\sigma}_1$, *and* σ_1 *but is independent of the point* $(\bar{x}, \bar{t})$.

Proof. The proof may be performed by using estimates on analytic semigroup as in [12, Proposition 3.2]. Observe that the conditions stated in H3-H4, linking $\tilde{k}_1$, k_1, $\tilde{\sigma}_1$, and σ_1, with p, τ, ε_σ and ε_k are needed to prove the above estimate. □

Proposition 2 *Suppose that the boundary* Γ *is of class* C^2, *and define the coefficients* $\bar{a}_{ij}$ *as in Proposition 1. Let* $\vec{f}$ *be in* $(L^\tau(W^{\varepsilon,q}) \cap L^2(Q))^N$, *with* $min(p, \frac{2N}{N-2+2\varepsilon}) \le q \le p$. *Then the weak solution* y *to the variational equation*

$$-\int_Q y\frac{\partial \phi}{\partial t}\, dx\, dt + \int_Q \sum_{i,j=1}^{N} \bar{a}_{ij}\partial_{xi} y \partial_{x_j}\phi\, dx\, dt = \int_Q \vec{f}\cdot\nabla\phi\, dx\, dt \text{ for all } \phi \in C^1(\bar{Q}) \text{ such that } \phi(T) = 0,$$

belongs to $L^\tau(W^{1+\varepsilon,q}) \cap L^2(H^1)$ *and satisfies*

$$\|y\|_{L^\tau(W^{1+\varepsilon,q})\cap L^2(H^1)} \le C\|\vec{f}\|_{(L^\tau(W^{\varepsilon,q})\cap L^2(Q))^N},$$

where C *is independent of* $(\bar{x}, \bar{t}) \in \bar{Q}$ *and of* $q \in [min(p, \frac{2N}{N-2+2\varepsilon}), p]$.

Proof. The estimate in $L^2(H^1)$, when $\vec{f}$ belongs to $(L^2(Q))^N$ is classical. Let us prove the estimate in $L^\tau(W^{1+\varepsilon,q})$. From maximal regularity results for equations with regular coefficients, we deduce that the mapping $\vec{f} \mapsto y_{\vec{f}}$ (where $y_{\vec{f}}$ denotes the solution to the equation) is continuous from $L^\tau(W^{1,q})$ into $L^\tau(W^{2,q})$, and from $L^\tau(L^q(\Omega))$ into $L^\tau(W^{1,q})$ (see [17]). Moreover the constant in the corresponding estimates may be chosen independent of $q \in [min(p, \frac{2N}{N-2+2\varepsilon}), p]$. Since $(L^\tau(W^{1,q}), L^\tau(W^{2,q}))_{\varepsilon,q} \equiv L^\tau(W^{1+\varepsilon,q})$ (see Triebel [16], or Daners and Medina [4]), the result follows by means of real interpolation. □

Proposition 3 *Suppose that the boundary* Γ *is of class* C^2, *and define the coefficients* $\bar{a}_{ij}$ *as in Proposition 1. Let* f *be in* $L^2(Q)$, *and let* y *be the weak solution in* $L^2(H^1)$ *to the variational equation*

$$-\int_Q y\frac{\partial \phi}{\partial t}\, dx\, dt + \int_Q \sum_{i,j=1}^{N} \bar{a}_{ij}\partial_{x_i} y\, \partial_{x_j}\phi\, dx\, dt = \int_Q f\phi\, dx\, dt \text{ for all } \phi \in C^1(\bar{Q}) \text{ such that } \phi(T) = 0.$$

If $p \le 2$, *then*

$$\|y\|_{L^\tau(W^{1+\varepsilon,p})\cap L^2(H^1)} \le C\|f\|_{L^2(Q)}.$$

If $\tau \le 2$ *and* $p > 2$, *then*

$$\|y\|_{L^\tau(W^{1+\varepsilon,q})\cap L^2(H^1)} \le C\|f\|_{L^2(Q)},$$

with $q = \frac{2N}{N-2+2\varepsilon}$. *If* $\tau > 2$ *and* $p > 2$, *then*

$$\|y\|_{L^\tau(W^{1+\varepsilon,q})\cap L^2(H^1)} \le C\|f\|_{L^2(Q)},$$

for any $q \ge 2$ *satisfying* $\frac{N}{4} + \frac{1}{2} < \frac{N}{2q} + \frac{1}{\tau} + \frac{1}{2} - \frac{\varepsilon}{2}$. *Moreover, in the above estimates, the constants* C *are independent of* $(\bar{x}, \bar{t}) \in \bar{Q}$.

Proof. If $p \le 2$, using estimates on analytic semigroups, we can prove that y belongs to $L^\tau(W^{1+\varepsilon,2})$ for every $\tau \ge 2$ such that $1/2 < 1/\tau + 1/2 - \varepsilon/2$. Since $\varepsilon < 2/\tau$, y belongs to $L^\tau(W^{1+\varepsilon,2})$ for every $\tau \ge 2$. If $\tau \le 2$ and $p > 2$, then y belongs to $L^2(W^{2,2})$. In this case, the estimate follows from Sobolev embeddings. The last case can also be treated by using estimates on analytic semigroups. □

Proposition 4 *Suppose that the boundary Γ is of class C^3, and define the coefficients $\bar{a}_{ij}$ as in Proposition 1. Let f be in $L^\tau(W^{\varepsilon,q}) \cap L^2(Q)$, with $min(p, \frac{2N}{N-2+2\varepsilon}) \le q \le p$. Then the weak solution y to the variational equation*

$$-\int_Q y\frac{\partial\phi}{\partial t}\,dx\,dt + \int_Q \sum_{i,j=1}^{N} \bar{a}_{ij}\partial_{x_i}y\,\partial_{x_j}\phi\,dx\,dt = \int_Q f\phi\,dx\,dt \text{ for all } \phi \in C^1(\bar{Q}) \text{ such that } \phi(T) = 0,$$

belongs to $L^\tau(W^{1+\varepsilon,\tilde{q}}) \cap L^2(H^1)$ with $\tilde{q} = \frac{Nq}{N-q}$ if $q < N$, $q = p$ if $q \ge N$, and satisfies

$$\|y\|_{L^\tau(W^{1+\varepsilon,\tilde{q}})\cap L^2(H^1)} \le C\|f\|_{L^\tau(W^{\varepsilon,q})\cap L^2(Q)},$$

where C is independent of $(\bar{x},\bar{t}) \in \bar{Q}$ and of $q \in [min(p, \frac{2N}{N-2+2\varepsilon}), p]$.

Proof. Using real interpolation, as in the proof of Proposition 2, we can first prove that

$$\|y\|_{L^\tau(W^{2+\varepsilon,q})\cap L^2(H^1)} \le C\|f\|_{L^\tau(W^{\varepsilon,q})\cap L^2(Q)}.$$

We conclude with Sobolev embeddings. □

Lemma 1 *Let $\varepsilon < \tilde{\varepsilon} < \hat{\varepsilon}$. For all $q \in [min(p, \frac{2N}{N-2+2\varepsilon}), p]$, all $a \in C([0,T]; C^{0,\tilde{\varepsilon}}(\bar{\Omega}))$, all $y \in L^\tau(W^{\varepsilon,q})$, ay belongs to $L^\tau(W^{\varepsilon,q})$, and*

$$\|ay\|_{L^\tau(W^{\varepsilon,q})} \le C\|a\|_{C([0,T];C^{0,\tilde{\varepsilon}}(\bar{\Omega}))}\|y\|_{L^\tau(W^{\varepsilon,q})},$$

where C does not depend on $q \in [min(p, \frac{2N}{N-2+2\varepsilon}), p]$.

Proof. Using the definition of the norm in $L^\tau(W^{\varepsilon,q})$, with straightforward calculations we obtain

$$\begin{aligned}
\|ay\|^\tau_{L^\tau(W^{\varepsilon,q})} &= \int_0^T \left(\int_{\Omega\times\Omega} \frac{|a(x,t)y(x,t) - a(x',t)y(x',t)|^q}{|x-x'|^{n+\varepsilon q}}\,dx\,dx'\right)^{\tau/q} dt \\
&\le C\int_0^T \left(\int_{\Omega\times\Omega} \frac{|a(x,t) - a(x',t)|^q}{|x-x'|^{\tilde{\varepsilon} q}} \frac{|y(x,t)|^q}{|x-x'|^{n+(\varepsilon-\tilde{\varepsilon})q}}\,dx\,dx'\right)^{\tau/q} dt \\
&\quad + C\int_0^T \left(\int_{\Omega\times\Omega} |a(x',t)|^q \frac{|y(x,t) - y(x',t)|^q}{|x-x'|^{n+\varepsilon q}}\,dx\,dx'\right)^{\tau/q} dt
\end{aligned}$$

$$\le C\|a\|^\tau_{C(C^{0,\tilde{\varepsilon}}(\bar{\Omega}))}\max_{\xi\in\bar{\Omega}}\left(\int_\Omega \frac{dx'}{|\xi - x'|^{n+(\varepsilon-\tilde{\varepsilon})q}}\right)^{\tau/q} \int_0^T \left(\int_\Omega |y(x,t)|^q dx\right)^{\tau/q} dt + C\|a\|^\tau_{C(\bar{Q})}\|y\|^\tau_{L^\tau(W^{\varepsilon,q})}.$$

The proof is complete. □

Proposition 5 *Let a be in $L^{\tilde{k}_1}(L^{k_1}(\Omega))$, b be in $L^{\tilde{\sigma}_1}(L^{\sigma_1}(\Gamma))$, $\hat{f}$ be in $L^{\tilde{k}_1}(L^{k_1}(\Omega))$ and $\hat{g}$ be in $L^{\tilde{\sigma}_1}(L^{\sigma_1}(\Gamma))$. Then the solution y in $L^2(H^1) \cap C([0,T]; L^2)$ to the equation*

$$\left\{\begin{array}{rcll}
\dfrac{\partial y}{\partial t} + Ay + ay &=& \hat{f} & \text{in } Q, \\
\dfrac{\partial y}{\partial n_A} + by &=& \hat{g} & \text{on } \Sigma, \\
y(\cdot,0) &=& 0 & \text{in } \Omega,
\end{array}\right. \tag{15}$$

satisfies the estimate

$$\|y\|_{L^\tau(W^{1+\varepsilon,p})} \le C(\|\hat{f}\|_{L^{\tilde{k}_1}(L^{k_1}(\Omega))} + \|\hat{g}\|_{L^{\tilde{\sigma}_1}(L^{\sigma_1}(\Gamma))}), \tag{16}$$

where C only depends on Ω, T, A and an upper bound for $\|a\|_{L^{\tilde{k}_1}(L^{k_1}(\Omega))} + \|b\|_{L^{\tilde{\sigma}_1}(L^{\sigma_1}(\Gamma))}$.

Proof. Due to (11) and (12), first notice that $y \in L^\infty(Q)$ (see Casas, Raymond and Zidani [3]), and that

$$\|y\|_{L^\infty(Q)} \le C(\|\hat{f}\|_{L^{\tilde{k}_1}(L^{k_1}(\Omega))} + \|\hat{g}\|_{L^{\tilde{\sigma}_1}(L^{\sigma_1}(\Gamma))}), \tag{17}$$

where C depends on an upper bound for $\|a\|_{L^{\tilde{k}_1}(L^{k_1}(\Omega))} + \|b\|_{L^{\tilde{\sigma}_1}(L^{\sigma_1}(\Gamma))}$. Therefore it is sufficient to consider the case where $a \equiv 0$ and $b \equiv 0$. We now suppose that we are in this case. To prove (16) when the coefficients a_{ij} satisfy H2, we use a technique of freezing coefficients as in Vespri [17, Theorem 3.1]. Up to Step 3, we suppose that the boundary Γ is regular.

Step 1. First we prove an estimate in $L^\tau(W^{\varepsilon,p})$. From Ladyženskaja et al. [10, Chapter 3, Theorem 5.1], we know that the weak solution to (15) belongs to $L^2(0,T;H^1(\Omega)) \cap C([0,T];L^2(\Omega))$, and satisfies

$$\|y\|_{L^2(0,T;H^1(\Omega))\cap C([0,T];L^2(\Omega))} \le C(\|\hat{f}\|_{L^{\tilde{k}_1}(L^{k_1}(\Omega))} + \|\hat{g}\|_{L^{\tilde{\sigma}_1}(L^{\sigma_1}(\Gamma))}). \tag{18}$$

Choose $\tilde{r}$ and r, such that $\frac{\tilde{\varepsilon}}{2} + \frac{1-\tilde{\varepsilon}}{r} = \frac{1}{p}$, and $\frac{\tilde{\varepsilon}}{2} + \frac{1-\tilde{\varepsilon}}{\tilde{r}} = \frac{1}{\tau}$, where $\tilde{\varepsilon}$ is an exponent strictly greater than ε. Since $\|y\|_{L^{\tilde{r}}(L^r(\Omega))} \le C\|y\|_{L^\infty(Q)}$ and $[L^r(\Omega), W^{1,2}(\Omega)]_{\tilde{\varepsilon}} \hookrightarrow W^{\varepsilon,p}(\Omega)$, from (17) and (18), and by interpolation it follows that

$$\|y\|_{L^\tau(W^{\varepsilon,p})} \le C(\|\hat{f}\|_{L^{\tilde{k}_1}(L^{k_1}(\Omega))} + \|\hat{g}\|_{L^{\tilde{\sigma}_1}(L^{\sigma_1}(\Gamma))}).$$

Step 2. For any $\rho > 0$, let $0 = t_1 < t_2 < \ldots < t_k < \ldots < t_K = T$ be a regular subdivision of $[0,T]$, such that $t_k - t_{k-1} = \ell(\rho)$ and

$$\max\{\|a_{ij}(t,\cdot) - a_{ij}(t',\cdot)\|_{C^{0,\bar{\varepsilon}}(\bar{\Omega})} \mid t \in [t_{k-1},t_k], t' \in [t_{k-1},t_k], 1 \le i,j \le N, 2 \le k \le K\} \le \rho.$$

Let $\{C_\rho^s\}_{s=1}^\mu$ be a collection of open sets of class C^∞, of diameter less or equal than $\rho > 0$ such that

$$\bar{\Omega} \subset \cup_{s=1}^\mu C_\rho^s,$$

and let $\{\varphi_s\}_{s=1}^\mu$ be a partition of unity subordinate to this covering. Let ψ_k be the continuous function on $[0,T]$, affine on each interval $[t_k, t_{k+1}]$, which is equal to 1 on t_k and 0 on t_j if $j \ne k$. For a given fixed point $x_s \in C_\rho^s$, set

$$\bar{a}_{ij}^{sk} = a_{ij}(x_s, t_k) \quad \text{and} \quad y_{sk}(x,t) = \psi_k(t)\varphi_s(x)y(x,t) \text{ for } 1 \le s \le \mu,\ 1 \le k \le K. \tag{19}$$

Let us fix $1 \le k \le K$ and $1 \le s \le \mu$. For every $\xi \in L^2(H^1)$, define the operator T_ξ^{ks} by

$$\begin{aligned} T_\xi^{ks}(\phi) \quad = \quad & \int_Q \psi_k\varphi_s\hat{f}\phi\,dx\,dt + \int_\Sigma \psi_k\varphi_s\hat{g}\phi\,ds\,dt \\ & + \int_Q \psi_k \sum_{i,j=1}^N a_{ij}y\partial_{x_i}\varphi_s\partial_{x_j}\phi\,dx\,dt - \int_Q \psi_k \sum_{i,j=1}^N a_{ij}\partial_{x_i}y\partial_{x_j}\varphi_s\phi\,dx\,dt \\ & + \int_Q \varphi_s y\frac{\partial\psi_k}{\partial t}\phi\,dx\,dt + \int_{t_{k-1}}^{t_{k+1}} \int_{C_\rho^s} \sum_{i,j=1}^N (\bar{a}_{ij}^{sk} - a_{ij})\partial_{x_i}\xi\partial_{x_j}\phi\,dx\,dt, \end{aligned}$$

with the convention $t_0 = t_1 = 0$ and $t_{K+1} = t_K = T$. For every $\xi \in L^2(H^1)$, let $z(\xi)$ be the unique solution in $L^2(H^1)$ to the variational equation

$$-\int_Q z\frac{\partial\phi}{\partial t}\,dx\,dt + \int_Q \sum_{i,j=1}^N \bar{a}_{ij}^{sk}\partial_{x_i}z\partial_{x_j}\phi\,dx\,dt = T_\xi^{ks}(\phi) \ \text{ for all } \phi \in C^1(\bar{Q}) \text{ such that } \phi(T) = 0. \tag{20}$$

Observe that $z(y_{sk}) \equiv y_{sk}$. Let us prove that, if ρ is small enough, then the mapping $\xi \mapsto z(\xi)$ admits a fixed point in $L^\tau(W^{1+\varepsilon,p_1}) \cap L^2(H^1)$, where $p_1 = \min(p, \frac{2N}{N-2+2\varepsilon})$. Due to Lemma 1, if $\xi \in L^\tau(W^{1+\varepsilon,p_1}) \cap L^2(H^1)$, then $\sum_{i=1}^N (\bar{a}_{ij}^{st} - a_{ij})\partial_{x_i}\xi$ belongs to $L^\tau(W^{\varepsilon,p_1}) \cap L^2(Q)$ for all $1 \le j \le N$. Notice that $\psi_k\varphi_s\hat{f}$ belongs to $L^{\tilde{k}_1}(L^{k_1}(\Omega))$, $\psi_k\varphi_s\hat{g}$ belongs to $L^{\tilde{\sigma}_1}(L^{\sigma_1}(\Gamma))$. Due to step 1 and Lemma 1, $\psi_k \sum_{i=1}^N a_{ij}y\partial_{x_i}\varphi_s$ belongs to $L^\tau(W^{\varepsilon,p}) \cap L^2(Q)$ for

$1 \leq j \leq N$. Also observe that $\psi_k \sum_{i,j=1}^N a_{ij}\partial_{x_i} y \partial_{x_j}\varphi_s$ belongs to $L^2(Q)$, and $\varphi_s y \frac{\partial \psi_k}{\partial t}$ belongs to $L^\infty(Q)$. From Propositions 1 to 3, it follows that $z(\xi)$ belongs to $L^\tau(W^{1+\varepsilon,p_1}) \cap L^2(H^1)$ for all $\xi \in L^\tau(W^{1+\varepsilon,p_1}) \cap L^2(H^1)$. On the other hand, due to Proposition 2 and to Lemma 1, it follows that

$$\|z(\xi_1) - z(\xi_2)\|_{L^\tau(W^{1+\varepsilon,p_1})\cap L^2(H^1)} \leq C \sum_{i,j=1}^N \|(\bar{a}_{ij}^{sk} - a_{ij})(\partial_{x_i}\xi_1 - \partial_{x_i}\xi_2)\|_{L^\tau(W^{\varepsilon,p_1})\cap L^2(]t_{k-1},t_{k+1}[\times C_\rho^s)}$$

$$\leq C \left(\max_{i,j} \|\bar{a}_{ij}^{sk} - a_{ij}(t_k,\cdot)\|_{C^{0,\tilde{\varepsilon}}(\bar{C}_\rho^s)} + \max_{i,j}\|a_{ij}(t_k,\cdot) - a_{ij}(\cdot)\|_{C([t_{k-1},t_{k+1}];C^{0,\tilde{\varepsilon}}(\bar{C}_\rho^s))} \right)$$
$$\times \|\nabla\xi_1 - \nabla\xi_2\|_{(L^\tau(W^{\varepsilon,p_1})\cap L^2(Q))^N}$$
$$\leq C(\rho^{\hat{\varepsilon}-\tilde{\varepsilon}} + \rho)\|\nabla\xi_1 - \nabla\xi_2\|_{(L^\tau(W^{\varepsilon,p_1})\cap L^2(Q))^N},$$

for some $\tilde{\varepsilon} \in]\varepsilon, \hat{\varepsilon}[$. Therefore, for ρ small enough, the mapping $\xi \to z(\xi)$ is a contraction in $L^\tau(W^{1+\varepsilon,p_1}) \cap L^2(H^1)$. Since the solution z of the equation

$$-\int_Q z\frac{\partial\phi}{\partial t}\,dx\,dt + \int_Q \sum_{i,j=1}^N \bar{a}_{ij}^{sk}\partial_{x_i} z \partial_{x_j}\phi\,dx\,dt = T_{y_{sk}}^{ks}(v) \text{ for all } \phi \in C^1(\bar{Q}) \text{ such that } \phi(T) = 0,$$

is unique in $L^2(H^1)$ and is equal to y_{sk}, this fixed point is y_{sk}. From the equality $y = \Sigma_{k=1}^K \Sigma_{s=1}^\mu y_{sk}$, it follows that y belongs to $L^\tau(W^{1+\varepsilon,p_1})$.

Step 3. If $p = p_1$ the proof is complete. Otherwise, we set $p_2 = \frac{Np_1}{N-p_1}$ if $p_1 < N$, and $p_2 = p$ if $p_1 \geq N$. We repeat Step 2. We want to prove that the mapping $\xi \mapsto z(\xi)$ admits a fixed point in $L^\tau(W^{1+\varepsilon,p_2}) \cap L^2(H^1)$. Due to Lemma 1, if $\xi \in L^\tau(W^{1+\varepsilon,p_2}) \cap L^2(H^1)$, then $\sum_{i=1}^N(\bar{a}_{ij}^{st} - a_{ij})\partial_{x_i}\xi$ belongs to $L^\tau(W^{\varepsilon,p_2}) \cap L^2(Q)$ for all $1 \leq j \leq N$. Since y belongs to $L^\tau(W^{1+\varepsilon,p_1})$, $\psi_k \sum_{i,j=1}^N a_{ij}\partial_{x_i} y \partial_{x_j}\varphi_s$ belongs to $L^\tau(W^{\varepsilon,p_1}) \cap L^2(Q)$, and due to Sobolev embeddings, $\psi_k \sum_{i=1}^N a_{ij} y \partial_{x_i}\varphi_s$ belongs to $L^\tau(W^{\varepsilon,p_2}) \cap L^2(Q)$ for $1 \leq j \leq N$.

As before $\psi_k\varphi_s\hat{f}$ belongs to $L^{\tilde{k}_1}(L^{k_1}(\Omega))$, $\psi_k\varphi_s\hat{g}$ belongs to $L^{\tilde{\sigma}_1}(L^{\sigma_1}(\Gamma))$, and $\varphi_s y \frac{\partial\psi_k}{\partial t}$ belongs to $L^\infty(Q)$. From Propositions 1, 2 and 4, it follows that $z(\xi)$ belongs to $L^\tau(W^{1+\varepsilon,p_2}) \cap L^2(H^1)$ for all $\xi \in L^\tau(W^{1+\varepsilon,p_2}) \cap L^2(H^1)$. We conclude by proving that the mapping $\xi \mapsto z(\xi)$ is a contraction in $L^\tau(W^{1+\varepsilon,p_2}) \cap L^2(H^1)$ for the same ρ as in step 2, and that y belongs to $L^\tau(W^{1+\varepsilon,p_2})$. Repeating this argument a finite number of times, we finally prove that y belongs to $L^\tau(W^{1+\varepsilon,p})$ and that

$$\|y\|_{L^\tau(W^{1+\varepsilon,p})} \leq C(\|\hat{f}\|_{L^{\tilde{k}_1}(L^{k_1}(\Omega))} + \|\hat{g}\|_{L^{\tilde{\sigma}_1}(L^{\sigma_1}(\Gamma))}).$$

Observe that the first iteration of Step 2 (with p_1) is different from the second one. Indeed, for the first iteration we only know that $\psi_k \sum_{i,j=1}^N a_{ij}\partial_{x_i} y \partial_{x_j}\varphi_s$ belongs to $L^2(Q)$, and we use Proposition 3. For the second iteration of Step 2, we know that $\psi_k \sum_{i,j=1}^N a_{ij}\partial_{x_i} y \partial_{x_j}\varphi_s$ belongs to $L^\tau(W^{\varepsilon,p_1}) \cap L^2(Q)$, and we use Proposition 4.

Step 4. If the boundary Γ is of class $C^{1,\hat{\varepsilon}}$, by making a change of variable in the variational formulation of equation (15), the equation can be reduced to an equation similar to (15) but with a regular boundary. Due to steps 1-3, the corresponding solution belongs to $L^\tau(W^{1+\varepsilon,p})$. By making the reverse change of variable, we can prove that the solution to equation (15) satisfies (16). □

Suppose that H1 and H2 are replaced by

H1' - The boundary Γ is of class C^1.

H2' - The coefficients a_{ij} belong to $C(\bar{Q})$ and satisfy

$$m\|\xi\|^2 \leq \sum_{i,j=1}^N a_{ij}(x,t)\xi_i\xi_j \leq M\|\xi\|^2 \text{ for all } \xi \in \mathrm{R}^N \text{ and all } (x,t) \in Q$$

for some $m, M > 0$.

In this case, we can adapt the proof of Proposition 5 to establish the following result.

Proposition 6 *Suppose hypotheses H1' and H2' are satisfied. Let a be in $L^{\tilde{k}_1}(L^{k_1}(\Omega))$, b be in $L^{\tilde{\sigma}_1}(L^{\sigma_1}(\Gamma))$, $\hat{f}$ be in $L^{\tilde{k}_1}(L^{k_1}(\Omega))$ and $\hat{g}$ be in $L^{\tilde{\sigma}_1}(L^{\sigma_1}(\Gamma))$. Then the solution y in $L^2(H^1)\cap C([0,T];L^2(\Omega))$ to the equation*

$$\begin{cases} \dfrac{\partial y}{\partial t} + Ay + ay & = & \hat{f} & \text{in } Q, \\ \dfrac{\partial y}{\partial n_A} + by & = & \hat{g} & \text{on } \Sigma, \\ y(\cdot,0) & = & 0 & \text{in } \Omega, \end{cases} \tag{21}$$

satisfies the estimate

$$\|y\|_{L^\tau(W^{1,p})} \le C(\|\hat{f}\|_{L^{\tilde{k}_1}(L^{k_1}(\Omega))} + \|\hat{g}\|_{L^{\tilde{\sigma}_1}(L^{\sigma_1}(\Gamma))}), \tag{22}$$

where C only depends on Ω, T, A and an upper bound for $\|a\|_{L^{\tilde{k}_1}(L^{k_1}(\Omega))} + \|b\|_{L^{\tilde{\sigma}_1}(L^{\sigma_1}(\Gamma))}$.

Proposition 7 *Suppose hypotheses H1 and H2 are satisfied. Let a be in $L^{\tilde{k}_1}(L^{k_1}(\Omega))$, b be in $L^{\tilde{\sigma}_1}(L^{\sigma_1}(\Gamma))$, $\hat{f}$ be in $L^{\tilde{k}_1}(L^{k_1}(\Omega))$, and $\hat{g}$ be in $L^{\tilde{\sigma}_1}(L^{\sigma_1}(\Gamma))$. Then the solution y to equation (21) satisfies the estimate*

$$\|y\|_{C^{\bar{\varepsilon}}(\bar{Q})} \le C(\|\hat{f}\|_{L^{\tilde{k}_1}(L^{k_1}(\Omega))} + \|\hat{g}\|_{L^{\tilde{\sigma}_1}(L^{\sigma_1}(\Gamma))}), \tag{23}$$

where $0<\bar{\varepsilon}<\varepsilon$, C only depends on Ω, T, A, $\bar{\varepsilon}$ and an upper bound for $\|a\|_{L^{\tilde{k}_1}(L^{k_1}(\Omega))} + \|b\|_{L^{\tilde{\sigma}_1}(L^{\sigma_1}(\Gamma))}$.

Proof. For the proof we refer to Corollary 3.2 in [13]. □

Before studying the adjoint equation, consider the following equation

$$\begin{cases} -\dfrac{\partial \varphi}{\partial t} + A^*\varphi & = & \operatorname{div}\vec{\eta} & \text{in } Q, \\ \dfrac{\partial \varphi}{\partial n_{A^*}} & = & -\vec{\eta}\cdot\vec{n} & \text{on } \Sigma, \\ \varphi(\cdot,T) & = & 0 & \text{in } \Omega, \end{cases} \tag{24}$$

where $\vec{n}$ is the outward unit normal to Γ, and $\vec{\eta}$ is supposed to be regular. (As usual A^* denotes the formal adjoint of A.) By definition, a function $\varphi \in L^1(W^{1,1})$ is a solution to (24) if, and only if,

$$\int_Q (\varphi \frac{\partial y}{\partial t} + \sum_{i,j=1}^N a_{ij}\partial_{x_j}\varphi\partial_{x_i}y)\,dx\,dt = -\int_Q \vec{\eta}\cdot\nabla y\,dx\,dt \tag{25}$$

for every $y \in C^1(\bar{Q})$ such that $y(0)=0$. The variational equation (25) is still meaningful if $\vec{\eta}$ belongs to $L^r(Q)$ for some $r>1$, even if the normal trace $\vec{\eta}\cdot\vec{n}$ is not defined.

For simplicity, we still continue to write the variational equation (25) in the form (24), even if the writing $\vec{\eta}\cdot\vec{n}$ may be abusive when $\vec{\eta}$ is not regular.

In the rest of the paper $\tilde{k}_2$, k_2, $\tilde{\sigma}_2$, σ_2 and ν are exponents satisfying

$$\frac{N}{2k_2} + \frac{1}{\tilde{k}_2} \le 1, \quad \frac{N-1}{2\sigma_2} + \frac{1}{\tilde{\sigma}_2} \le \frac{1}{2}, \quad \text{and} \quad \nu \ge 2. \tag{26}$$

We also suppose that $\tilde{k}_1$, k_1, $\tilde{\sigma}_1$, and σ_1 satisfy the following additional conditions

$$\tilde{k}_1 \ge \tau,\ \tilde{\sigma}_1 \ge \tau,$$

$$k_1 \ge \frac{Np'}{Np'-N+p'} \quad \text{and} \quad \sigma_1 \ge \frac{(N-1)p'}{(N-1)p'-N+p'} \quad \text{if } p'<N.$$

In the following proposition we state regularity properties for the adjoint state. The proof is similar to the proof of Proposition 5.

Proposition 8 *Let a be in $L^{\tilde{k}_1}(L^{k_1}(\Omega))$, b be in $L^{\tilde{\sigma}_1}(L^{\sigma_1}(\Gamma))$, $\hat{F}$ be in $L^{\tilde{k}_2}(L^{k_2}(\Omega))$, $\vec{\eta}$ be in $(L^{\tau'}(L^{p'}))^N$, $\hat{G}$ be in $L^{\tilde{\sigma}_2}(L^{\sigma_2}(\Gamma))$ and $\hat{L}$ be in $L^{\nu}(\Omega)$. Then there exists a unique $\varphi \in L^{\tau'}(W^{1,p'}) + L^2(H^1)$ satisfying the equation*

$$\begin{cases} -\dfrac{\partial \varphi}{\partial t} + A^*\varphi + a\varphi &= \hat{F} + \operatorname{div} \vec{\eta} \quad \text{in } Q, \\ \dfrac{\partial \varphi}{\partial n_{A^*}} + b\varphi &= \hat{G} - \vec{\eta}\cdot\vec{n} \quad \text{on } \Sigma, \\ \varphi(\cdot, T) &= \hat{L} \quad \text{in } \Omega, \end{cases} \tag{27}$$

and the following estimate holds

$$\|\varphi\|_{L^{\tau'}(W^{1,p'})+L^2(H^1)} \le C(\|\eta\|_{(L^{\tau'}(L^{p'}))^N} + \|\hat{F}\|_{L^{\tilde{k}_2}(L^{k_2}(\Omega))} + \|\hat{G}\|_{L^{\tilde{\sigma}_2}(L^{\sigma_2}(\Gamma))} + \|\hat{L}\|_{L^{\nu}(\Omega)}),$$

where C depends only on Ω, T, A and an upper bound for $\|a\|_{L^{\tilde{k}_1}(L^{k_1}(\Omega))} + \|b\|_{L^{\tilde{\sigma}_1}(L^{\sigma_1}(\Gamma))}$.

Moreover, if y is the solution to equation (15), the following Green formula is satisfied

$$\begin{aligned} &\int_Q \varphi\left(\frac{\partial y}{\partial t} + Ay + ay\right)\, dx\, dt + \int_\Sigma \varphi\left(\frac{\partial y}{\partial n_A} + by\right)\, ds\, dt = \\ &\int_Q \hat{F}y\, dx\, dt - \int_Q \vec{\eta}\cdot\nabla y\, dx\, dt + \int_\Sigma \hat{G}y\, ds\, dt + \int_\Omega \hat{L}y(T)\, dx. \end{aligned} \tag{28}$$

Proof. [of Proposition 8] We first consider the case where $\hat{F} \equiv 0$, $\hat{L} \equiv 0$, and $\hat{G} \equiv 0$.

If $a \equiv 0$ and $b \equiv 0$, and if the coefficients of the operator A are regular and independent of time, the existence of $\varphi \in L^{\tau'}(W^{1,p'})$ satisfying (27) can be obtained using duality techniques, interpolation and maximal regularity results as in Vespri [17, Theorem 3.3] and references therein. The passage from regular to continuous coefficients (also depending on time) for A may be performed by localization and a fixed point theorem as in [17, Theorem 3.1].

The case $a \not\equiv 0$ and $b \not\equiv 0$ may be deduced from the previous one by using a fixed point argument. Indeed, observe that if $\xi \in L^{\tau'}(W^{1,p'})$ then $\xi \in L^{\tau'}(L^{p'^*}(\Omega))$, $\xi_{|\Sigma} \in L^{\tau'}(L^{\beta}(\Gamma))$, where $p'^* = p'N/(N-p')$ and $\beta = ((N-1)p')/(N-p')$ if $p' < N$, p'^* and β are any real in $(1, +\infty)$ if $p' \ge N$. Since $a \in L^{\tilde{k}_1}(L^{k_1}(\Omega))$, $b \in L^{\tilde{\sigma}_1}(L^{\sigma_1}(\Gamma))$, we verify that $a\xi \in L^{\tilde{r}}(L^r(\Omega))$ and $b\xi_{|\Sigma} \in L^{\tilde{s}}(L^s(\Gamma))$, where $1/\tilde{r} = 1/\tilde{k}_1 + 1/\tau'$, $1/r = 1/k_1 + 1/p'^*$, $1/\tilde{s} = 1/\tilde{\sigma}_1 + 1/\tau'$ and $1/s = 1/\sigma_1 + 1/\beta$. Using (11) and (12), it follows that

$$\frac{N}{2r} + \frac{1}{\tilde{r}} < \frac{N}{2p'} + \frac{1}{\tau'} + \frac{1}{2} \quad \text{and} \quad \frac{N-1}{2s} + \frac{1}{\tilde{s}} < \frac{N}{2p'} + \frac{1}{\tau'}.$$

Suppose that $1/k_1 \ge 1/p' - 1/p'^*$ and $1/\sigma_1 \ge 1/p' - 1/\beta$. In this case, the mapping that associates the solution φ_ξ of the equation

$$-\frac{\partial \varphi_\xi}{\partial t} + A^*\varphi_\xi = \operatorname{div} \vec{\eta} - a\xi \text{ in } Q, \quad \frac{\partial \varphi_\xi}{\partial n_{A^*}} = -\vec{\eta}\cdot\vec{n} - b\xi \text{ on } \Sigma, \quad \varphi_\xi(\cdot, T) = 0 \text{ in } \Omega,$$

with ξ is affine continuous from $L^{\tau'}(W^{1,p'})$ into itself. Using this property, we can prove that $\xi \to \varphi_\xi$ is a contraction in $L^{\tau'}(0, \bar{t}; W^{1,p'})$ for $\bar{t}$ small enough. The estimate in $L^{\tau'}(W^{1,p'})$ may next be deduced by a standard technique. If $1/k_1 < 1/p' - 1/p'^*$ or $1/\sigma_1 < 1/p' - 1/\beta$, the above fixed point method may be performed by replacing k_1 by $\min(k_1, (1/p' - 1/p'^*)^{-1})$, and σ_1 by $\min(\sigma_1, (1/p' - 1/\beta)^{-1})$.

Consider the case where $\hat{F}$, $\hat{L}$, and $\hat{G}$ are different from zero. The equation

$$-\frac{\partial \varphi}{\partial t} + A^*\varphi + a\varphi = \hat{F} \text{ in } Q, \quad \frac{\partial \varphi}{\partial n_{A^*}} + b\varphi = \hat{G} \text{ on } \Sigma, \quad \varphi(\cdot, T) = \hat{L} \text{ in } \Omega,$$

admits a unique solution φ satisfying

$$\|\varphi\|_{L^2(H^1)} \le C(\|\hat{F}\|_{L^{\tilde{k}_2}(L^{k_2}(\Omega))} + \|\hat{G}\|_{L^{\tilde{\sigma}_2}(L^{\sigma_2}(\Gamma))} + \|\hat{L}\|_{L^{\nu}(\Omega)})$$

(see [10]). The Green formula is true for regular functions y, and it follows from a denseness argument. □

3 Taylor expansion of the state and the functional.

Now we are ready to study the state equation. Let us suppose that

H5 - For every $y \in \mathrm{R}$, $f(\cdot,\cdot,y)$ is measurable on Q. For almost every $(x,t) \in Q$, $f(x,t,\cdot)$ is of class C^1 on R. The following estimates hold:

$$|f(x,t,0)| \leq M_1(x,t), \qquad C_0 \leq f'_y(x,t,y) \leq M_1(x,t)\eta(|y|),$$

where $C_0 \in \mathrm{R}$, η is an increasing function from R^+ into R^+, and $M_1 \in L^{\tilde{k}_1}(L^{k_1}(\Omega))$.

H6 - For every $y, v \in \mathrm{R}$, $g(\cdot,\cdot,y,v)$ is measurable on Σ. For all $v \in \mathrm{R}$ and almost every $(s,t) \in \Sigma$, $g(s,t,\cdot,v)$ is of class C^1 on R. For almost every $(s,t) \in \Sigma$, $g(s,t,\cdot)$ and $g'_y(s,t,\cdot)$ are continuous on R^2. The following estimates hold:

$$|g(s,t,0,v)| \leq N_1(s,t) + |v|, \qquad C_0 \leq g'_y(s,t,y,v) \leq (N_1(s,t) + |v|)\eta(|y|),$$

where $N_1 \in L^{\tilde{\sigma}_1}(L^{\sigma_1}(\Gamma))$.

H7 - The initial state w is given fixed in $L^\infty(\Omega) \cap W^{1,p}(\Omega)$.

Then we have

Theorem 1 *The mapping that links the solution y_v of equation* (1) *with v is continuous from $L^\alpha(\Sigma)$ into $L^\tau(W^{1,p}) \cap C_b(\bar{Q} \setminus \bar{\Omega} \times \{0\})$, for any $N+1 < \alpha < \infty$ such that the pair $(\sigma_1, \tilde{\sigma}_1) = (\alpha, \alpha)$ obeys one of the conditions* (7)–(10)*.*

Proof. Taking into account Proposition 5, the proof can be performed as in Casas, Raymond and Zidani [3], or Raymond and Zidani [13, 14]. □

We are now in position to establish Taylor expansions for the cost functional and the state equation. Let us state the following hypotheses on the control problem.

H8 - For every $y \in \mathrm{R}$, $F(\cdot,\cdot,y)$ is measurable on Q. For almost every $(x,t) \in Q$, $F(x,t,\cdot)$ is of class C^1 on R. The following estimates hold:

$$|F(x,t,0)| \leq M_2(x,t), \qquad |F'_y(x,t,y)| \leq M_2(x,t)\eta(|y|),$$

where $M_2 \in L^{\tilde{k}_2}(L^{k_2}(\Omega))$.

H9 - For every $y, v \in \mathrm{R}$, $G(\cdot,y,v)$ is measurable on Σ. For all $v \in \mathrm{R}$ and almost every $(s,t) \in \Sigma$, $G(s,t,\cdot,v)$ is of class C^1 on R. For almost every $(s,t) \in \Sigma$, $G(s,t,\cdot)$ and $G'_y(s,t,\cdot)$ are continuous on R^2. The following estimates hold:

$$|G(s,t,0,v)| \leq N_2(s,t) + |v|, \qquad |G'_y(s,t,y,v)| \leq (N_2(s,t) + |v|)\eta(|y|),$$

where $N_2 \in L^{\tilde{\sigma}_2}(L^{\sigma_2}(\Gamma))$.

H10 - For every $y \in \mathrm{R}$, $L(\cdot,y)$ is measurable in Ω. For almost every $x \in \Omega$, $L(x,\cdot)$ is of class C^1 in R. The following estimates hold:

$$|L(x,y)| \leq M_3(x), \quad \text{and} \quad |L'_y(x,y)| \leq M_4(x)\eta(|y|),$$

where $M_3(x) \in L^1(\Omega)$ and $M_4 \in L^\nu(\Omega)$.

H11 - V_{ad} is a bounded subset in $L^\infty(\Sigma)$ of the form

$$V_{ad} = \{v \in L^\infty(\Sigma) : \ v(s,t) \in K_\Sigma(s,t) \text{ for almost all } (s,t) \in \Sigma\},$$

where K_Σ are measurable multimapping with nonempty compact values in $\mathcal{P}(\mathrm{R})$.

For the proof of the following Lemma, see for instance [13].

Lemma 2 *For all $\rho \in (0,1)$, there exists a sequence of measurable sets $E_\rho^k \subset \Sigma$ such that*

$$\mathcal{L}^N(E_\rho^k) = \rho \mathcal{L}^N(\Sigma),$$

and

$$\lim_{k\to\infty} \frac{1}{\rho}\chi_{E_\rho^k} = 1 \text{ weak-star in } L^\infty(\Sigma). \tag{29}$$

($\mathcal{L}^N$ denotes the N-dimensional Lebesgue measure and $\chi_{E_\rho^k}$ is the characteristic function of the set E_ρ^k.)

Theorem 2 *For every $\rho \in (0,1)$, and all $v_1, v_2 \in V_{ad}$, there exists a measurable set $E_\rho \subset \Sigma$ such that*

$$\mathcal{L}^N(E_\rho) = \rho \mathcal{L}^N(\Sigma),$$

$$y_\rho = y_1 + \rho z + r_\rho \text{ with } \lim_{\rho\to 0} \frac{1}{\rho}\|r_\rho\|_{L^\tau(W^{1,p})} = 0, \tag{30}$$

and

$$J(y_\rho, v_\rho) = J(y_1, v_1) + \rho \Delta J + o(\rho), \tag{31}$$

where

$$v_\rho(s,t) = \begin{cases} v_1 & \text{in } \Sigma \setminus E_\rho \\ v_2 & \text{in } E_\rho \end{cases}, \quad y_\rho = y_{v_\rho}, \quad y_1 = y_{v_1},$$

$$\begin{cases} \dfrac{\partial z}{\partial t} + Az + f_y'(x,t,y_1)z = 0 & \text{in } Q, \\ \dfrac{\partial z}{\partial n_A} + g_y'(s,t,y_1,v_1)z = g(s,t,y_1,v_1) - g(s,t,y_1,v_2) & \text{on } \Sigma, \\ z(\cdot,0) = 0 & \text{in } \Omega \times \{0\}, \end{cases}$$

and

$$\begin{aligned} \Delta J = & \int_Q F_y'(\cdot,y_1)z\,dx\,dt + \int_\Sigma G_y'(\cdot,y_1,v_1)z ds\,dt + \int_\Omega L_y'(\cdot,y_1(\cdot,T))z(\cdot,T)\,dx \\ & + \int_\Sigma (G(s,t,y_1,v_2) - G(s,t,y_1,v_1))\,ds\,dt. \end{aligned}$$

Proof. Let us prove (30). Take a sequence $(E_\rho^k)_k$ as in Lemma 2. Set

$$v_\rho^k(s,t) = \begin{cases} v_1 & \text{in } \Sigma \setminus E_\rho^k \\ v_2 & \text{in } E_\rho^k \end{cases}, \quad y_\rho^k = y_{v_\rho^k} \text{ and } \xi_\rho^k = \frac{y_\rho^k - y_1}{\rho} - z.$$

The function ξ_ρ^k satisfies the equation

$$\begin{cases} \dfrac{\partial \xi_\rho^k}{\partial t} + A\xi_\rho^k + a_\rho^k \xi_\rho^k = f_\rho^k & \text{in } Q, \\ \dfrac{\partial \xi_\rho^k}{\partial n_A} + b_\rho^k \xi_\rho^k = g_\rho^k + h_\rho^k & \text{on } \Sigma, \\ \xi_\rho^k(\cdot,0) = 0 & \text{in } \Omega, \end{cases}$$

with

$$a_\rho^k(x,t) = \int_0^1 f_y'(x,t,(y_1 + \theta(y_\rho^k - y_1)))\,d\theta,$$

$$f_\rho^k = (f_y'(x,t,y_1) - a_\rho^k)z,$$

$$b_\rho^k(s,t) = \int_0^1 g_y'(s,t,(y_1 + \theta(y_\rho^k - y_1)),v_\rho^k)\,d\theta,$$

$$g_\rho^k = (g'_y(s,t,y_1,v_1) - b_\rho^k)z,$$

and

$$h_\rho^k = (1 - \frac{1}{\rho}\chi_{E_\rho^k})(g(s,t,y_1,v_2) - g(s,t,y_1,v_1)).$$

We denote by $\xi_\rho^{k,1}$ the solution to

$$\begin{cases} \dfrac{\partial \xi_\rho^{k,1}}{\partial t} + A\xi_\rho^{k,1} + a_\rho^k \xi_\rho^{k,1} = f_\rho^k & \text{in } Q, \\ \dfrac{\partial \xi_\rho^{k,1}}{\partial n_A} + b_\rho^k \xi_\rho^{k,1} = g_\rho^k & \text{on } \Sigma, \\ \xi_\rho^{k,1}(\cdot,0) = 0 & \text{in } \Omega, \end{cases}$$

by $\xi_\rho^{k,2}$ the solution to

$$\begin{cases} \dfrac{\partial \xi_\rho^{k,2}}{\partial t} + A\xi_\rho^{k,2} + a_\rho^k \xi_\rho^{k,2} = 0 & \text{in } Q, \\ \dfrac{\partial \xi_\rho^{k,2}}{\partial n_A} + b_\rho^k \xi_\rho^{k,2} = h_\rho^k & \text{on } \Sigma, \\ \xi_\rho^{k,2}(\cdot,0) = 0 & \text{in } \Omega, \end{cases} \tag{32}$$

and by ζ_ρ^k the solution to

$$\begin{cases} \dfrac{\partial \zeta_\rho^k}{\partial t} + A\zeta_\rho^k + a\zeta_\rho^k = 0 & \text{in } Q, \\ \dfrac{\partial \zeta_\rho^k}{\partial n_A} + b\zeta_\rho^k = h_\rho^k & \text{on } \Sigma, \\ \zeta_\rho^k(\cdot,0) = 0 & \text{in } \Omega, \end{cases} \tag{33}$$

where $a(x,t) = f'_y(x,t,y_1(x,t))$, and $b(s,t) = g'_y(s,t,y_1(s,t),v_1(s,t))$. From (32) and (33) it follows that:

$$\begin{cases} \dfrac{\partial(\xi_\rho^{k,2} - \zeta_\rho^k)}{\partial t} + A(\xi_\rho^{k,2} - \zeta_\rho^k) + a_\rho^k(\xi_\rho^{k,2} - \zeta_\rho^k) = (a - a_\rho^k)\zeta_\rho^k & \text{in } Q, \\ \dfrac{\partial(\xi_\rho^{k,2} - \zeta_\rho^k)}{\partial n_A} + b_\rho^k(\xi_\rho^{k,2} - \zeta_\rho^k) = (b - b_\rho^k)\zeta_\rho^k & \text{on } \Sigma, \\ (\xi_\rho^{k,2} - \zeta_\rho^k)(\cdot,0) = 0 & \text{in } \Omega. \end{cases}$$

Due to Proposition 7 ζ_ρ^k belongs to $C^{\bar{\varepsilon}}(\bar{Q})$. From Proposition 6, it follows that $\xi_\rho^{k,1}$, $\xi_\rho^{k,2}$ and ζ_ρ^k belong to $L^\tau(W^{1,p})$ and the following estimates hold:

$$\|\xi_\rho^{k,2} - \zeta_\rho^k\|_{L^\tau(W^{1,p})} \le C_1\Big(\|a - a_\rho^k\|_{L^{\tilde{k}_1}(L^{k_1}(\Omega))} + \|b - b_\rho^k\|_{L^{\tilde{\sigma}_1}(L^{\sigma_1}(\Gamma))}\Big)\|\zeta_\rho^k\|_{C(\bar{Q})}, \tag{34}$$

$$\|\xi_\rho^{k,1}\|_{L^\tau(W^{1,p})} \le C_2(\|f_\rho^k\|_{L^{\tilde{k}_1}(L^{k_1}(\Omega))} + \|g_\rho^k\|_{L^{\tilde{\sigma}_1}(L^{\sigma_1}(\Gamma))}), \tag{35}$$

where the constants C_1 and C_2 do not depend on k.

The operator $\mathcal{T}$ which associates ζ, the solution in $L^\tau(W^{1+\varepsilon,p}) \cap W^{1,\tau}((W^{1,p'})')$ of

$$\begin{cases} \dfrac{\partial \zeta}{\partial t} + A\zeta + a\zeta = 0 & \text{in } Q, \\ \dfrac{\partial \zeta}{\partial n_A} + b\zeta = h & \text{on } \Sigma, \\ \zeta(\cdot,0) = 0 & \text{in } \Omega, \end{cases} \tag{36}$$

with h, is continuous from $L^{\tilde{\sigma}_1}(L^{\sigma_1}(\Gamma))$ into $L^\tau(W^{1+\varepsilon,p}) \cap W^{1,\tau}((W^{1,p'})')$. The continuity into $L^\tau(W^{1+\varepsilon,p})$ follows from Proposition 5. With equation (36) we prove that ζ belongs to $W^{1,\tau}((W^{1,p'})')$, and the corresponding estimate follows from the estimate in $L^\tau(W^{1+\varepsilon,p})$. Since the imbedding from $W^{1+\varepsilon,p}(\Omega)$ into $W^{1,p}(\Omega)$ is compact, (see Grisvard [8]), then the imbedding from $L^\tau(W^{1+\varepsilon,p}) \cap W^{1,\tau}((W^{1,p'})')$ into $L^\tau(W^{1,p})$ is compact (see Simon, [15, Corollary 4]). Thus $\mathcal{T}$ may be considered as a compact operator from $L^{\tilde{\sigma}_1}(L^{\sigma_1}(\Gamma))$ into

$L^\tau(W^{1,p})$. Observe that $\mathcal{T}$ is also a continuous linear operator from $L^{\tilde\sigma_1}(L^{\sigma_1}(\Gamma))$ into $C^{\bar\varepsilon}(\overline{Q})$. Thus $\mathcal{T}$ is a compact operator from $L^{\tilde\sigma_1}(L^{\sigma_1}(\Gamma))$ into $C(\bar Q)$. From (29) it follows that

$$\lim_{k\to\infty} h^k_\rho = 0 \text{ weakly in } L^{\tilde\sigma_1}(L^{\sigma_1}(\Gamma)),$$

and hence

$$\lim_{k\to\infty}(\|\zeta^k_\rho\|_{L^\tau(W^{1,p})} + \|\zeta^k_\rho\|_{C(\bar Q)}) = 0.$$

Thus, for all $\rho\in(0,1)$, there exists a $k(\rho)$ such that

$$\|\zeta^{k(\rho)}_\rho\|_{L^\tau(W^{1,p})} + \|\zeta^{k(\rho)}_\rho\|_{C(\bar Q)} \le \rho. \tag{37}$$

Notice that

$$\lim_{\rho\to 0} v^{k(\rho)}_\rho = v_1 \text{ in } L^\alpha(\Sigma) \text{ for any } \alpha<\infty.$$

Therefore, due to Theorem 1, we have

$$\lim_{\rho\to 0} y^{k(\rho)}_\rho = y_1 \text{ in } C_b(\bar Q\setminus\bar\Omega\times\{0\}). \tag{38}$$

Relation (38) implies that

$$\lim_{\rho\to 0} f^{k(\rho)}_\rho = 0 \text{ in } L^{\tilde k_1}(L^{k_1}(\Omega)),\quad \lim_{\rho\to 0} g^{k(\rho)}_\rho = 0 \text{ in } L^{\tilde\sigma_1}(L^{\sigma_1}(\Gamma)), \tag{39}$$

and

$$\lim_{\rho\to 0}(a-a^{k(\rho)}_\rho) = 0 \text{ in } L^{\tilde k_1}(L^{k_1}(\Omega)),\quad \lim_{\rho\to 0}(b-b^{k(\rho)}_\rho) = 0 \text{ in } L^{\tilde\sigma_1}(L^{\sigma_1}(\Gamma)). \tag{40}$$

With (34), (35), (37), (39) and (40), we obtain

$$\lim_{\rho\to 0}\|\xi^{k(\rho)}_\rho\|_{L^\tau(W^{1,p})} = 0. \tag{41}$$

Set $E_\rho = E^{k(\rho)}_\rho$, we have $r_\rho = \rho\xi^{k(\rho)}_\rho$. Thus (30) follows from (41). Due to (29) and (30), we can verify (31). □

4 Main Result.

We define the boundary Hamiltonian function by

$$H_\Sigma(s,t,y,v,\varphi,\nu) = \nu G(s,t,y,v) - \varphi g(s,t,y,v)$$

for every $(s,t,y,v,\varphi,\nu)\in\Gamma\times[0,T]\times\mathrm{R}^4$. The main result of this paper is the Pontryagin principle stated in the following theorem.

Theorem 3 *Assume that H1–H11 are satisfied. If $(\bar y,\bar v)$ is a solution to the control problem (P), then there exists $\bar\varphi\in L^{\tau'}(W^{1,p'})$, $\bar\nu\in\mathrm{R}^+$, and $\vec f\in(L^{\tau'}(L^{p'}))^N$, such that*

$$(\vec f,\bar\nu) \neq (0,0), \tag{42}$$

$$\int_Q (z-\nabla_x\bar y)\vec f \le 0 \quad \textit{for all } z\in C, \tag{43}$$

$$\begin{cases} -\dfrac{\partial\bar\varphi}{\partial t} + A^*\bar\varphi + f'_y(x,t,\bar y)\bar\varphi = \bar\nu F'_y(x,t,\bar y) + \operatorname{div}\vec f & \textit{in } Q,\\ \dfrac{\partial\bar\varphi}{\partial n_{A^*}} + g'_y(s,t,\bar y,\bar v)\bar\varphi = \bar\nu G'_y(s,t,\bar y,\bar v) - \vec f\cdot\vec n & \textit{on } \Sigma,\\ \bar\varphi(\cdot,T) = \bar\nu L'_y(x,\bar y(T)) & \textit{in } \Omega, \end{cases} \tag{44}$$

and

$$H_\Sigma(s,t,\bar y(s,t),\bar v(s,t),\bar\varphi(s,t),\bar\nu) = \min_{v\in K_\Sigma(s,t)} H_\Sigma(s,t,\bar y(s,t),v,\bar\varphi(s,t),\bar\nu) \tag{45}$$

for almost all (s,t) in Σ.

Proof. We define Ekeland's distance on the space V_{ad}:

$$d_E(v_1, v_2) = \mathcal{L}^N\{(s,t) : \ v_1(s,t) \neq v_2(s,t)\}.$$

The space (V_{ad}, d_E) is a complete metric space, and convergence in (V_{ad}, d_E) implies convergence in $L^\alpha(\Sigma)$ for any $\alpha < \infty$. Consider the penalized functional

$$J_n(y,v) = \left\{ \left[\left(J(y,v) - J(\bar{y},\bar{v}) + \frac{1}{n^2} \right)^+ \right]^2 + d_C(\nabla_x y)^2 \right\}^{1/2},$$

where $d_C(\cdot)$ is the distance in $(L^\tau(L^p))^N$ to the set C defined by

$$d_C(z) = \inf_{\varphi \in C} \|z - \varphi\|_{(L^\tau(L^p))^N}.$$

The functional $d_C(\cdot)$ is Lipschitz, convex and Gâteaux-differentiable at every $z \notin C$, and at those points $\|\nabla d_C(z)\|_{(L^{\tau'}(L^{p'}))^N} = 1$. Consider the problem

$$(P_n) : \ \min_{v \in V_{ad}} J_n(y_v, v).$$

With such a choice, $(\bar{y},\bar{v})$ is a $\frac{1}{n^2}$-solution of (P_n). Theorem 1 and assumptions H8–H10 imply that $J_n(y_v, v)$ is continuous for Ekeland's metric. Thus, due to Ekeland's variational principle there exists $v_n \in V_{ad}$ such that

$$d_E(v_n, \bar{v}) \leq \frac{1}{n} \quad \text{and} \quad J_n(y_n, v_n) \leq J_n(y_v, v) + \frac{1}{n} d_E(v, v_n) \quad \text{for all } v \in V_{ad}, \tag{46}$$

where $y_n = y_{v_n}$.

Let v be in V_{ad}. Due to Theorem 2, for every $\rho \in (0,1)$, there exists a measurable set $E_\rho \subset \Sigma$ such that

$$\mathcal{L}^N(E_\rho) = \rho \mathcal{L}^N(\Sigma), \tag{47}$$

$$y_\rho = y_n + \rho z_n + r_\rho \text{ with } \lim_{\rho \to 0} \frac{1}{\rho} \|r_\rho\|_{L^\tau(W^{1,p})} = 0, \tag{48}$$

and

$$J(y_\rho, v_\rho) = J(y_n, v_n) + \rho \Delta J^N + o(\rho), \tag{49}$$

where

$$v_\rho(s,t) = \begin{cases} v_n & \text{in } \Sigma \setminus E_\rho \\ v & \text{in } E_\rho \end{cases}, \quad y_\rho = y_{v_\rho},$$

$$\begin{cases} \dfrac{\partial z_n}{\partial t} + A z_n + f'_y(x,t,y_n) z_n = 0 & \text{in } Q, \\ \dfrac{\partial z_n}{\partial n_A} + g'_y(s,t,y_n,v_n) z_n = g(s,t,y_n,v_n) - g(s,t,y_n,v) & \text{on } \Sigma, \\ z_n(\cdot,0) = 0 & \text{in } \Omega, \end{cases}$$

and

$$\begin{aligned} \Delta J^N = & \int_Q F'_y(\cdot,y_n) z_n dx\, dt + \int_\Sigma G'_y(\cdot,y_n,v_n) z_n ds\, dt + \int_\Omega L'_y(\cdot,y_n(\cdot,T)) z_n(\cdot,T)\, dx \\ & + \int_\Sigma (G(\cdot,y_n,v) - G(\cdot,y_n,v_n))\, ds\, dt. \end{aligned}$$

Relations (46) and (47) imply that

$$\frac{J_n(y_n,v_n) - J_n(y_\rho,v_\rho)}{\rho} \leq \frac{1}{n} \mathcal{L}^N(\Sigma). \tag{50}$$

We have

$$\frac{J_n(y_n,v_n)-J_n(y_\rho,v_\rho)}{\rho} = \frac{J_n^2(y_n,v_n)-J_n^2(y_\rho,v_\rho)}{\rho\,(J_n(y_n,v_n)+J_n(y_\rho,v_\rho))}$$

$$= \frac{\left[\left(J(y_n,v_n)-J(\bar y,\bar v)+\frac{1}{n^2}\right)^+\right]^2-\left[\left(J(y_\rho,v_\rho)-J(\bar y,\bar v)+\frac{1}{n^2}\right)^+\right]^2}{\rho\,(J_n(y_n,v_n)+J_n(y_\rho,v_\rho))}+\frac{d_C(\nabla y_n)^2-d_C(\nabla y_\rho)^2}{\rho\,(J_n(y_n,v_n)+J_n(y_\rho,v_\rho))}.$$

From (49) it follows that

$$\lim_{\rho\to 0}\frac{\left[\left(J(y_n,v_n)-J(\bar y,\bar v)+\frac{1}{n^2}\right)^+\right]^2-\left[\left(J(y_\rho,v_\rho)-J(\bar y,\bar v)+\frac{1}{n^2}\right)^+\right]^2}{\rho\,(J_n(y_n,v_n)+J_n(y_\rho,v_\rho))}=-\nu_n\Delta J^N, \tag{51}$$

with

$$\nu_n=\frac{\left(J(y_n,v_n)-J(\bar y,\bar v)+\frac{1}{n^2}\right)^+}{J_n(y_n,v_n)}.$$

With (48), and the properties of the distance function $d_C(\cdot)$, we can write

$$\lim_{\rho\to 0}\frac{d_C(\nabla y_n)^2-d_C(\nabla y_\rho)^2}{\rho\,(J_n(y_n,v_n)+J_n(y_\rho,v_\rho))}=\lim_{\rho\to 0}\frac{d_C(\nabla y_n)-d_C(\nabla y_\rho)}{\rho}\frac{d_C(\nabla y_n)+d_C(\nabla y_\rho)}{(J_n(y_n,v_n)+J_n(y_\rho,v_\rho))}=\int_Q \vec f_n\cdot\nabla z_n\,dx\,dt, \tag{52}$$

where

$$\vec f_n=\begin{cases}\dfrac{d_C(\nabla y_n)}{J_n(y_n,v_n)}\nabla d_C(\nabla y_n) & \text{if } \nabla y_n\notin C,\\ 0 & \text{else.}\end{cases}$$

In order to derive an approximate Pontryagin's principle we introduce the approximate adjoint equation. Due to hypotheses H3, H4, H8, H9, H10, and to the regularity result in Proposition 8, there exists a unique φ_n satisfying

$$\begin{aligned}-\frac{\partial\varphi_n}{\partial t}+A^*\varphi_n+f'_y(x,t,y_n)\varphi_n &= \nu_nF'_y(x,t,y_n)+\operatorname{div}\vec f_n && \text{in } Q,\\ \frac{\partial\varphi_n}{\partial n_{A^*}}+g'_y(s,t,y_n,v_n)\varphi_n &= \nu_nG'_y(s,t,y_n,v_n)-\vec f_n\cdot\vec n && \text{on } \Sigma,\\ \varphi_n(\cdot,T) &= \nu_nL'_y(\cdot,y_n(T)) && \text{in } \Omega.\end{aligned}$$

With Green formula (28) in Proposition 8 we have

$$\begin{aligned}&\int_Q \nu_nF'_y(x,t,y_n)z_n\,dx\,dt-\int_Q \vec f\cdot\nabla z_n\,dx\,dt+\int_\Sigma \nu_nG'_y(s,t,y_n,v_n)\,ds\,dt+\int_\Omega \nu_nL'_y(x,y_n(T))\,dx\\ &=\int_Q\varphi_n\left(\frac{\partial z_n}{\partial t}+Az_n+f'_y(x,t,y_n)z_n\right)\,dx\;dt+\int_\Sigma\varphi_n\left(\frac{\partial z_n}{\partial n_A}+g'_y(s,t,y_n,v_n)z_n\right)\,ds\;dt\\ &=\int_\Sigma\varphi_n\left(g(s,t,y_n,v_n)-g(s,t,y_n,v)\right)\,ds\;dt.\end{aligned}$$

By passing to the limit when ρ tends to zero in (50), with (51), (52) and the previous Green formula, we obtain the following approximate Pontryagin's principle:

$$\int_\Sigma\left(\nu_nG(s,t,y_n,v_n)-\varphi_ng(s,t,y_n,v_n)\right)\,ds\;dt\le\int_\Sigma\left(\nu_nG(s,t,y_n,v)-\varphi_ng(s,t,y_n,v)\right)\,ds\;dt+\frac{1}{n}\mathcal{L}^N(\Sigma)\quad\text{for all } v\in V_{ad}. \tag{53}$$

Notice that $\nu_n^2+\|\vec f_n\|^2_{(L^{\tau'}(L^{p'}))^N}=1$. Thus there exist subsequences, still indexed by n, such that $(\nu_n)_n$ converges to ν, and $(\vec f_n)_n$ converges weakly to $\vec f$ in $(L^{\tau'}(L^{p'}))^N$. If $\nu>0$ then (42) is satisfied. Otherwise, using that

$\lim_{n\to\infty} \|\vec{f}_n\|^2_{(L^{\tau'}(L^{p'}))^N} = 1$, and that the interior of C is nonempty, we can prove that $\vec{f} \neq 0$ in a standard way (see [13], for instance).

Condition (43) is fulfilled due to the definition of the subdifferential of the convex functional $d_C(\cdot)$.

With (46), we can prove that $(y_n)_n$ converges to $\bar{y}$ in $C_b(\bar{Q} \setminus \bar{\Omega} \times \{0\})$. With assumptions H3–H10 and with Proposition 8, we can prove that $(\varphi_n)_n$ converges in $L^{\tau'}(W^{1,p'}) + L^2(H^1)$ to the solution $\bar{\varphi}$ of (44).

Taking the previous convergence results for $(y_n)_n$, $(v_n)_n$, $(\varphi_n)_n$, $(\nu_n)_n$ into account, we can pass to the limit in (53) when n tends to infinity, and we obtain a Pontryagin principle in integral form.

$$\int_\Sigma (\bar{\nu} G(s,t,\bar{y},\bar{v}) - \bar{\varphi} g(s,t,\bar{y},\bar{v}))\ ds\ dt \leq \int_\Sigma (\bar{\nu} G(s,t,\bar{y},v) - \bar{\varphi} g(s,t,\bar{y},v))\ ds\ dt \quad \text{for all } v \in V_{ad}.$$

Pointwise Pontryagin's principle can now be deduced as in Raymond and Zidani [13, p. 1875]. The proof is complete. □

5 Some extensions and examples.

In this paper we have only treated of bounded boundary controls. The treatment of unbounded controls can also be done as in [13], but this implies some technical difficulties. We refer to [13] for such extensions. All the results could be performed for distributed controls, with no important changes in the proofs.

To illustrate these remarks, consider the control problem corresponding to:

- the state equation:

$$\begin{cases} \dfrac{\partial y}{\partial t} + Ay + f(x,t,y,u) = 0 & \text{in } Q, \\ \dfrac{\partial y}{\partial n_A} + g(s,t,y,v) = 0 & \text{on } \Sigma, \\ y(\cdot,0) = w & \text{in } \Omega, \end{cases} \tag{54}$$

  with $u \in U_{ad} \subset L^q(Q)$, $v \in V_{ad} \subset L^\sigma(\Sigma)$, $q > N/2+1$ and $\sigma > N+1$. We suppose in addition that the pair $(\tilde{k}_1, k_1) = (q,q)$ obeys one of the conditions (3)–(6), and the pair $(\tilde{\sigma}_1, \sigma_1) = (\sigma,\sigma)$ obeys one of the conditions (7)–(10). The control sets U_{ad} and V_{ad} are defined by

$$U_{ad} = \{u \in L^q(\Sigma) :\ u(x,t) \in K_Q(x,t) \text{ for almost all } (x,t) \in Q\},$$

$$V_{ad} = \{v \in L^\sigma(\Sigma) :\ v(s,t) \in K_\Sigma(s,t) \text{ for almost all } (s,t) \in \Sigma\},$$

  where K_Q and K_Σ are measurable multimapping with nonempty compact values in $\mathcal{P}(\mathrm{R})$.

- the cost functional:

$$J(y_{uv}, v) = \int_0^T \int_\Omega F(x,t,y_{uv},u)\ dx\ dt + \int_0^T \int_\Gamma G(s,t,y_{uv},v)\ ds\ dt + \int_\Omega L(x, y_{uv}(x,T))\ dx, \tag{55}$$

- the state constraint:

$$\int_0^T \left(\int_\Omega |\nabla_x y - g_d|^p dx \right)^{\tau/p} dt \leq \delta, \tag{56}$$

  where g_d is a given function in $(L^\tau(L^p))^N$, and $\delta > 0$.

We define the distributed and the boundary Hamiltonian function by

$$H_Q(x,t,y,u,\varphi,\nu) = \nu F(x,t,y,u) - \varphi f(x,t,y,u)$$

for every $(x,t,y,u,\varphi,\nu) \in \Omega \times [0,T] \times \mathrm{R}^4$,

$$H_\Sigma(s,t,y,v,\varphi,\nu) = \nu G(s,t,y,v) - \varphi g(s,t,y,v)$$

for every $(s,t,y,v,\varphi,\nu) \in \Gamma \times [0,T] \times \mathrm{R}^4$. With the obvious modifications of assumptions on f, g, F and G, we can prove the following result.

Theorem 4 *If* $(\bar{y}, \bar{u}, \bar{v})$ *is a solution to the control problem, then there exists* $\bar{\varphi} \in L^{\tau'}(W^{1,p'})$*,* $\bar{\nu} \in \mathrm{R}^+$*,* $\bar{\mu} \in \mathrm{R}^+$ *such that*

$$(\bar{\nu}, \bar{\mu}) \neq (0,0), \tag{57}$$

$$\bar{\mu}\left(\int_0^T \left(|\nabla_x \bar{y} - g_d|^p dx)^{\tau/p}\right) dt - \delta\right) = 0, \tag{58}$$

$$\left\{\begin{array}{rcll} -\dfrac{\partial\bar{\varphi}}{\partial t} + A\bar{\varphi} + f'_y(x,t,\bar{y},\bar{u})\bar{\varphi} &=& \bar{\nu}F'_y(x,t,\bar{y},\bar{u}) + \bar{\mu}\,\mathrm{div}\,\vec{f} & \textit{in } Q, \\ \dfrac{\partial\bar{\varphi}}{\partial n_A} + g'_y(s,t,\bar{y},\bar{v})\bar{\varphi} &=& \bar{\nu}G'_y(s,t,\bar{y},\bar{v}) - \bar{\mu}\vec{f}\cdot\vec{n} & \textit{on } \Sigma, \\ \bar{\varphi}(\cdot,T) &=& \bar{\nu}L'_y(x,\bar{y}(T)) & \textit{in } \Omega, \end{array}\right. \tag{59}$$

where $\vec{f} = 0$ *if* $\nabla_x \bar{y} = g_d$ *and*

$$\vec{f} = \left(\int_\Omega |\nabla_x \bar{y} - g_d|^p\, dx\right)^{\frac{\tau}{p}-1} \left(|\nabla_x \bar{y} - g_d|^{p-2}(\nabla_x \bar{y} - g_d)\right) \quad \textit{otherwise},$$

$$H_Q(x,t,\bar{y}(x,t),\bar{u}(x,t),\bar{\varphi}(x,t),\bar{\nu}) = \min_{u \in K_Q(x,t)} H_Q(x,t,\bar{y}(x,t),u,\bar{\varphi}(x,t),\bar{\nu})$$

for almost all (x,t) *in* Q*, and*

$$H_\Sigma(s,t,\bar{y}(s,t),\bar{v}(s,t),\bar{\varphi}(s,t),\bar{\nu}) = \min_{v \in K_\Sigma(s,t)} H_\Sigma(s,t,\bar{y}(s,t),v,\bar{\varphi}(s,t),\bar{\nu})$$

for almost all (s,t) *in* Σ*.*

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