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Infinite Series and Logarithmic Integrals Associated to Differentiation with Respect to Parameters of the Whittaker $\mathrm{M}_{\kappa,\mu}(x)$ Function I

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Abstract: In this paper, first derivatives of the Whittaker function $\mathrm{M}_{\kappa,\mu}(x)$ are calculated with respect to the parameters. Using the confluent hypergeometric function, these derivarives can be expressed as infinite sums of quotients of the digamma and gamma functions. Moreover, from the integral representation of $\mathrm{M}_{\kappa,\mu}(x)$ it is possible to obtain these parameter derivatives in terms of finite and infinite integrals with integrands containing elementary functions (products of algebraic, exponential, and logarithmic functions). These infinite sums and integrals can be expressed in closed form for particular values of the parameters. For this purpose, we have obtained the parameter derivative of the incomplete gamma function in closed form. As an application, reduction formulas for parameter derivatives of the confluent hypergeometric function are derived, along with finite and infinite integrals containing products of algebraic, exponential, logarithmic, and Bessel functions. Finally, reduction formulas for the Whittaker functions $\mathrm{M}_{\kappa,\mu}(x)$ and integral Whittaker functions $\mathrm{Mi}_{\kappa,\mu}(x)$ and $\mathrm{mi}_{\kappa,\mu}(x)$ are calculated.

Keywords: derivatives with respect to parameters; Whittaker functions; integral Whittaker functions; incomplete gamma functions; sums of infinite series of psi and gamma; finite and infinite logarithmic integrals and Bessel functions

MSC: 33B15; 33B20; 33C10; 33C15; 33C20; 33C50; 33E20

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Citation: Apelblat, A.; González-Santander, J.L. Infinite Series and Logarithmic Integrals Associated to Differentiation with Respect to Parameters of the Whittaker $\mathrm{M}_{\kappa,\mu}(x)$ Function I. *Axioms* **2023**, *12*, 381. https://doi.org/10.3390/axioms12040381

Academic Editors: Sevtap Sümer Eker and Juan J. Nieto

Received: 24 February 2023
Revised: 7 April 2023
Accepted: 14 April 2023
Published: 16 April 2023

1. Introduction

Introduced in 1903 by Whittaker [1], the $\mathrm{M}_{\kappa,\mu}(z)$ and $\mathrm{W}_{\kappa,\mu}(z)$ functions are defined as follows:

$$\begin{aligned} \mathrm{M}_{\kappa,\mu}(z) &= z^{\mu+1/2}e^{-z/2}\,{}_1F_1\left(\left.\begin{array}{c} \frac{1}{2}+\mu-\kappa \\ 1+2\mu \end{array}\right| z\right), \\ 2\mu &\neq -1,-2,\ldots \end{aligned} \tag{1}$$

and

$$\begin{aligned} \mathrm{W}_{\kappa,\mu}(z) &= \frac{\Gamma(-2\mu)}{\Gamma\left(\frac{1}{2}-\mu-\kappa\right)}\mathrm{M}_{\kappa,\mu}(z)+\frac{\Gamma(2\mu)}{\Gamma\left(\frac{1}{2}+\mu-\kappa\right)}\mathrm{M}_{\kappa,-\mu}(z), \\ 2\mu &\neq \pm 1,\pm 2,\ldots \end{aligned} \tag{2}$$

respectively, where $\Gamma(x)$ denotes the gamma function and $z \in \mathbb{C} \setminus (-\infty, 0]$. These functions, called Whittaker functions, are closely associated with the following *confluent hypergeometric function* (Kummer function):

$$ {}_1F_1\left(\begin{array}{c} a \\ b \end{array} \middle| z \right) = \frac{\Gamma(b)}{\Gamma(a)} \sum_{n=0}^{\infty} \frac{\Gamma(a+n)}{\Gamma(b+n)} \frac{z^n}{n!}, \tag{3} $$

where ${}_pF_q\left(\begin{array}{c} a_1, \ldots, a_p \\ b_1, \ldots, b_q \end{array} \middle| z \right)$ denotes the *generalized hypergeometric function*.

For particular values of the parameters κ and μ, the Whittaker functions can be reduced to a variety of elementary and special functions. Whittaker [1] discussed the connection between the functions defined in (1) and (2) and many other special functions, such as the modified Bessel function, the incomplete gamma functions, the parabolic cylinder function, the error functions, the logarithmic and the cosine integrals, and the generalized Hermite and Laguerre polynomials. Monographs and treatises dealing with special functions [2–10] present properties of the Whittaker functions with more or less extension.

The Whittaker functions are frequently applied in various areas of mathematical physics (see for example [11–13]), such as the well-known solution of the Schrödinger equation for the harmonic oscillator [14].

$\mathrm{M}_{\kappa,\mu}(x)$ and $\mathrm{W}_{\kappa,\mu}(x)$ are usually treated as functions of variable x with fixed values of the parameters κ and μ. However, there are other investigations which consider κ and μ as variables. For instance, Laurenzi [15] discussed methods to calculate derivatives of $\mathrm{M}_{\kappa,1/2}(x)$ and $\mathrm{W}_{\kappa,1/2}(x)$ with respect to κ when this parameter is an integer. Using the Mellin transform, Buschman [16] showed that the derivatives of the Whittaker functions with respect to the parameters for certain particular values of these parameters can be expressed in finite sums of Whittaker functions. López and Sesma [17] considered the behaviour of $\mathrm{M}_{\kappa,\mu}(x)$ as a function of κ. They derived a convergent expansion in ascending powers of κ and an asymptotic expansion in descending powers of κ. Using series of Bessel functions and Buchholz polynomials, Abad and Sesma [18] presented an algorithm for the calculation of the nth derivative of the Whittaker functions with respect to the κ parameter. Becker [19] investigated certain integrals with respect to the μ parameter. Ancarini and Gasaneo [20] presented a general case of differentiation of generalized hypergeometric functions with respect to the parameters in terms of infinite series containing the digamma function. In addition, Sofostasios and Brychkov [21] considered derivatives of hypergeometric functions and classical polynomials with respect to the parameters.

The primary focus of this research is a systematic investigation of the first derivatives of $\mathrm{M}_{\kappa,\mu}(x)$ with respect to the parameters. We primarily base our findings on two distinct methods. The first pertains to the series representation of $\mathrm{M}_{\kappa,\mu}(x)$, whereas the second pertains to the integral representations of $\mathrm{M}_{\kappa,\mu}(x)$. Regarding the first approach, direct differentiation of (1) with respect to the parameters leads to infinite sums of quotients of digamma and gamma functions. It is possible to calculate such sums in closed form for particular values of the parameters. The parameter differentiation of the integral representations of $\mathrm{M}_{\kappa,\mu}(x)$ leads to finite and infinite integrals of elementary functions, such as products of algebraic, exponential, and logarithmic functions. These integrals are similar to those investigated by Kölbig [22] and Geddes et al. [23]. As in the case of the first approach, it is possible to calculate such integrals in closed form for some particular values of the parameters.

In the Appendices, we calculate the first derivative of the incomplete gamma functions $\gamma(\nu, x)$ and $\Gamma(\nu, x)$ with respect to the parameter ν. These results are used when we calculate several of the integrals found in the second approach mentioned above. In addition, we calculate new reduction formulas of the integral Whittaker functions which we recently introduced in [24]. These are defined in a similar way as other integral functions in the mathematical literature:

$$\mathrm{Mi}_{\kappa,\mu}(x) = \int_0^x \frac{\mathrm{M}_{\kappa,\mu}(t)}{t} dt, \tag{4}$$

$$\mathrm{mi}_{\kappa,\mu}(x) = \int_x^\infty \frac{\mathrm{M}_{\kappa,\mu}(t)}{t} dt. \tag{5}$$

Finally, we include a list of reduction formulas for the Whittaker function $\mathrm{M}_{\kappa,\mu}(x)$ in the Appendices.

2. Parameter Differentiation of $\mathrm{M}_{\kappa,\mu}$ via Kummer Function ${}_1F_1$

As mentioned above, the Whittaker function $\mathrm{M}_{\kappa,\mu}(x)$ is closely related to the confluent hypergeometric function ${}_1F_1(a;b;x)$. Likewise, the parameter derivatives of $\mathrm{M}_{\kappa,\mu}(x)$ are related to the parameter derivatives of ${}_1F_1(a;b;x)$. Below, we introduce the following notation set by Ancarini and Gasaneo [20].

Definition 1. *Define the parameter derivatives of the confluent hypergeometric function as*

$$G^{(1)}\left(\begin{array}{c} a \\ b \end{array} \middle| x \right) = \frac{\partial}{\partial a} \left[{}_1F_1\left(\begin{array}{c} a \\ b \end{array} \middle| x \right) \right], \tag{6}$$

and

$$H^{(1)}\left(\begin{array}{c} a \\ b \end{array} \middle| x \right) = \frac{\partial}{\partial b} \left[{}_1F_1\left(\begin{array}{c} a \\ b \end{array} \middle| x \right) \right]. \tag{7}$$

According to (3), we have

$$\begin{aligned} G^{(1)}\left(\begin{array}{c} a \\ b \end{array} \middle| x \right) &= \frac{\Gamma(b)}{\Gamma(a)} \sum_{n=0}^{\infty} \frac{\Gamma(a+n)}{\Gamma(b+n)} [\psi(a+n) - \psi(a)] \frac{x^n}{n!}, \\ H^{(1)}\left(\begin{array}{c} a \\ b \end{array} \middle| x \right) &= -\frac{\Gamma(b)}{\Gamma(a)} \sum_{n=0}^{\infty} \frac{\Gamma(a+n)}{\Gamma(b+n)} [\psi(b+n) - \psi(b)] \frac{x^n}{n!}. \end{aligned}$$

Additionally, according to [25], we have

$$G^{(1)}\left(\begin{array}{c} a \\ b \end{array} \middle| z \right) = \frac{z}{b} \sum_{m=0}^{\infty} \frac{(a)_m \, z^m}{(b+1)_m (2)_m} \, {}_2F_2\left(\begin{array}{c} 1, a+m+1 \\ m+2, b+m+1 \end{array} \middle| z \right), \tag{8}$$

and

$$H^{(1)}\left(\begin{array}{c} a \\ b \end{array} \middle| z \right) = -\frac{a\, z}{b^2} \sum_{m=0}^{\infty} \frac{(a+1)_m (b)_m \, z^m}{[(b+1)_m]^2 (2)_m} \, {}_2F_2\left(\begin{array}{c} 1, a+m+1 \\ m+2, b+m+1 \end{array} \middle| z \right). \tag{9}$$

Because one of the integral representations of the confluent hypergeometric function is ([6], Section 6.5.1)

$$\begin{aligned} &{}_1F_1\left(\begin{array}{c} a \\ b \end{array} \middle| x \right) = \frac{\Gamma(b)}{\Gamma(a)\Gamma(b-a)} \int_0^1 e^{xt} t^{a-1} (1-t)^{b-a-1} dt \\ &\mathrm{Re}\, b > \mathrm{Re}\, a > 0, \end{aligned} \tag{10}$$

by direct differentiation of (10) with respect to parameters a and b we obtain

$$\begin{aligned} G^{(1)}\left(\begin{array}{c} a \\ b \end{array} \middle| x \right) &= [\psi(b) - \psi(a)] \, {}_1F_1\left(\begin{array}{c} a \\ b \end{array} \middle| x \right) \\ &\quad + \frac{\Gamma(b)}{\Gamma(a)\Gamma(b-a)} \int_0^1 e^{xt} t^{a-1} (1-t)^{b-a-1} \ln\left(\frac{t}{1-t} \right) dt, \end{aligned}$$

and

$$H^{(1)}\left(\begin{array}{c} a \\ b \end{array} \middle| x\right) = -[\psi(b)-\psi(b-a)]\,{}_1F_1\left(\begin{array}{c} a \\ b \end{array} \middle| x\right) + \frac{\Gamma(b)}{\Gamma(a)\Gamma(b-a)}\int_0^1 e^{xt}t^{a-1}(1-t)^{b-a-1}\ln(1-t)dt.$$

Because our main focus is the systematic investigation of the parameter derivatives of $\mathrm{M}_{\kappa,\mu}(x)$, we present these parameter derivatives as Theorems throughout the paper and the corresponding results for $G^{(1)}(a;b;x)$ and $H^{(1)}(a;b;x)$ as Corollaries. Additionally, note that all the results regarding $G^{(1)}(a;b;x)$ can be transformed according to the next Theorem.

Theorem 1. *The following transformation holds true:*

$$G^{(1)}\left(\begin{array}{c} a \\ b \end{array} \middle| x\right) = -e^x G^{(1)}\left(\begin{array}{c} b-a \\ b \end{array} \middle| -x\right).$$

Proof. Differentiate Kummer's transformation formula ([8], Equation 13.2.39) with respect to a:

$${}_1F_1\left(\begin{array}{c} a \\ b \end{array} \middle| x\right) = e^x\,{}_1F_1\left(\begin{array}{c} b-a \\ b \end{array} \middle| -x\right)$$

to obtain the desired result. □

2.1. Derivative with Respect to the First Parameter $\partial\mathrm{M}_{\kappa,\mu}(x)/\partial\kappa$

Using (1) and (3), the first derivative of $\mathrm{M}_{\kappa,\mu}(x)$ with respect to the first parameter κ is

$$\frac{\partial\mathrm{M}_{\kappa,\mu}(x)}{\partial\kappa} = \psi\left(\frac{1}{2}+\mu-\kappa\right)\mathrm{M}_{\kappa,\mu}(x) - \frac{\Gamma(1+2\mu)}{\Gamma\left(\frac{1}{2}+\mu-\kappa\right)}x^{\mu+1/2}e^{-x/2}S_1(\kappa,\mu,x), \tag{11}$$

where $\psi(x)$ denotes the *digamma function* and

$$S_1(\kappa,\mu,x) = \sum_{n=0}^{\infty}\frac{\Gamma\left(\frac{1}{2}+\mu-\kappa+n\right)}{\Gamma(1+2\mu+n)}\psi\left(\frac{1}{2}+\mu-\kappa+n\right)\frac{x^n}{n!}. \tag{12}$$

Theorem 2. *For $2\mu\notin\mathbb{Z}^-$ and for $x\in\mathbb{R}$, $x\neq 0$, the following parameter derivative formula of $\mathrm{M}_{\kappa,\mu}(x)$ holds true:*

$$\left.\frac{\partial\mathrm{M}_{\kappa,\mu}(x)}{\partial\kappa}\right|_{\kappa=-\mu-1/2} = -\frac{x^{\mu+3/2}}{2\mu+1}e^{x/2}\,{}_2F_2\left(\begin{array}{c} 1,1 \\ 2(\mu+1),2 \end{array} \middle| -x\right). \tag{13}$$

Proof. For $\kappa=-\mu-1/2$, Equation (11) becomes

$$\left.\frac{\partial\mathrm{M}_{\kappa,\mu}(x)}{\partial\kappa}\right|_{\kappa=-\mu-1/2} = x^{\mu+1/2}e^{-x/2}\left[\psi(1+2\mu)\sum_{n=0}^{\infty}\frac{x^n}{n!} - \sum_{n=0}^{\infty}\psi(2\mu+1+n)\frac{x^n}{n!}\right].$$

Apply ([26], Equation 6.2.1(60))

$$\sum_{k=0}^{\infty} \frac{t^k}{k!} \psi(k+a) = e^t \left[\psi(a) + \frac{t}{a} \, {}_2F_2\left(\begin{array}{c} 1,1 \\ a+1,2 \end{array} \middle| -t \right) \right] \tag{14}$$

to obtain (13), completing the proof. □

Corollary 1. *For $a \in \mathbb{R}$, $a \neq 0$, and for $x \in \mathbb{R}$, the following reduction formula holds true:*

$$G^{(1)}\left(\begin{array}{c} a \\ a \end{array} \middle| x \right) = \frac{x\, e^x}{a} \, {}_2F_2\left(\begin{array}{c} 1,1 \\ a+1,2 \end{array} \middle| -x \right). \tag{15}$$

Proof. Direct differentiation of (1) yields

$$\frac{\partial \mathrm{M}_{\kappa,\mu}(x)}{\partial \kappa} = -x^{\mu+1/2} e^{-x/2} G^{(1)}\left(\begin{array}{c} \frac{1}{2}+\mu-\kappa \\ 1+2\mu \end{array} \middle| x \right), \tag{16}$$

thus, by comparing (16) with $\kappa = -\mu - \frac{1}{2}$ to (13), we arrive at (15), as we wanted to prove. □

Corollary 2. *For $a \in \mathbb{R}$, $a \neq 0$ and for $x \in \mathbb{R}$, the following sum holds true:*

$$\sum_{m=0}^{\infty} \frac{\gamma(m+1,x)}{(m+a)m!} = \frac{x}{a} \, {}_2F_2\left(\begin{array}{c} 1,1 \\ a+1,2 \end{array} \middle| -x \right),$$

where $\gamma(\nu, z)$ denotes the lower incomplete gamma function (A1).

Proof. According to (8) and the reduction formula ([9], Equation 7.11.1(15))

$${}_1F_1\left(\begin{array}{c} 1 \\ b \end{array} \middle| z \right) = (b-1) z^{1-b} e^z \gamma(b-1,z),$$

we have

$$\begin{aligned} G^{(1)}\left(\begin{array}{c} a \\ a \end{array} \middle| x \right) &= \frac{x}{a} \sum_{m=0}^{\infty} \frac{(a)_m \, x^m}{(a+1)_m (m+1)!} \, {}_1F_1\left(\begin{array}{c} 1 \\ m+2 \end{array} \middle| x \right) \\ &= e^x \sum_{m=0}^{\infty} \frac{\gamma(m+1,x)}{(m+a)m!}. \end{aligned} \tag{17}$$

Comparing (15) to (17) completes the proof. □

Table 1 presents explicit expressions for particular values of (13) and $x \in \mathbb{R}$, obtained with the help of the MATHEMATICA program. Note that the $\mathrm{Shi}(x)$ and $\mathrm{Chi}(x)$ functions are defined in (61) and (62), respectively.

Next, we present other reduction formula of $\partial \mathrm{M}_{\kappa,\mu}(x)/\partial\kappa$ from the result found in [15] for $x \in \mathbb{R}$:

$$\begin{aligned} &\left. \frac{\partial \mathrm{M}_{\kappa,\mu}(x)}{\partial \kappa} \right|_{\kappa=n,\mu=1/2} \\ &= [\ln|x| - \psi(n+1) - \mathrm{Ei}(x)] \, \mathrm{M}_{n,1/2}(x) + \sum_{\ell=0}^{n-1} (a_\ell + b_\ell \, e^x) \, \mathrm{M}_{\ell,1/2}(x), \end{aligned} \tag{18}$$

where $\mathrm{Ei}(x)$ denotes the exponential integral, and for $n, \ell = 1, 2, \ldots$

$$a_\ell = \frac{1}{n} \binom{n+\ell}{n-\ell} \tag{19}$$

and

$$b_\ell = \begin{cases} \dfrac{1}{n}\displaystyle\sum_{k=0}^{n-\ell-1} \frac{(\ell)_k 2^k}{(\ell+n)_k}, & \ell = 1,2,\ldots \\ 0, & \ell = 0. \end{cases} \quad (20)$$

Table 1. Derivative of $M_{\kappa,\mu}$ with respect to κ using (13).

κ	μ	$\frac{\partial \mathrm{M}_{\kappa,\mu}(x)}{\partial \kappa}$
$-\frac{3}{4}$	$\frac{1}{4}$	$-\frac{2}{3}x^{7/4}e^{x/2}\,{}_2F_2\left(1,1;\frac{5}{2},2;-x\right)$
$-\frac{1}{2}$	0	$-\sqrt{x}e^{x/2}[\gamma+\ln x+\mathrm{Shi}(x)-\mathrm{Chi}(x)]$
$-\frac{1}{4}$	$-\frac{1}{4}$	$-2x^{5/4}e^{x/2}\,{}_2F_2\left(1,1;\frac{3}{2},2;-x\right)$
$-\frac{1}{6}$	$-\frac{1}{3}$	$-3x^{7/6}e^{x/2}\,{}_2F_2\left(1,1;\frac{4}{3},2;-x\right)$
0	$\frac{1}{2}$	$e^{-x/2}[\mathrm{Shi}(x)+\mathrm{Chi}(x)-\ln x-\gamma]-e^{x/2}[\mathrm{Shi}(x)-\mathrm{Chi}(x)+\ln x+\gamma]$
$\frac{1}{6}$	$-\frac{2}{3}$	$3x^{5/6}e^{x/2}\,{}_2F_2\left(1,1;\frac{2}{3},2;-x\right)$
$\frac{1}{2}$	1	$-\frac{2}{\sqrt{x}}\left\{e^{x/2}[\gamma+1+\ln x+\mathrm{Shi}(x)-\mathrm{Chi}(x)] + e^{-x/2}(x+1)[\gamma-1+\ln x-\mathrm{Shi}(x)-\mathrm{Chi}(x)]\right\}$
1	$\frac{3}{2}$	$-\frac{3}{x}\left\{e^{-x/2}\left[\left(x^2+2x+2\right)(\ln x-\mathrm{Shi}(x)-\mathrm{Chi}(x)+\gamma)\right] + e^{x/2}[2\ln x+2\,\mathrm{Shi}(x)-2\,\mathrm{Chi}(x)+x+2\gamma+3]\right\}$

In order to calculate the finite sum provided in (20), we derive the following Lemma.

Lemma 1. *The following finite sum holds true* $\forall n,\ell = 1,2,\ldots$

$$S(n,\ell) = \sum_{k=0}^{n-\ell-1} \frac{(\ell)_k 2^k}{(\ell+n)_k} = \mathrm{Re}\left[{}_2F_1\left(\begin{array}{c} 1,\ell \\ \ell+n \end{array}\middle| 2\right)\right]. \quad (21)$$

Proof. Split the sum in two as

$$S(n,\ell) = \underbrace{\sum_{k=0}^{\infty} \frac{(\ell)_k (1)_k 2^k}{k!(\ell+n)_k}}_{S_1(n,\ell)} - \underbrace{\sum_{k=n-\ell}^{\infty} \frac{(\ell)_k (1)_k 2^k}{k!(\ell+n)_k}}_{S_2(n,\ell)},$$

where

$$S_1(n,\ell) = {}_2F_1\left(\begin{array}{c} 1,\ell \\ \ell+n \end{array}\middle| 2\right),$$

and

$$\begin{aligned} S_2(n,\ell) &= 2^{n-\ell}\sum_{s=0}^{\infty} \frac{(\ell)_{s+n-\ell}(1)_s 2^s}{s!(\ell+n)_{s+n-\ell}} \\ &= 2^{n-\ell}\frac{(\ell)_n}{(n)_n}\sum_{s=0}^{\infty} \frac{(n)_s (1)_s 2^s}{s!(2n)_s} \\ &= 2^{n-\ell}\frac{(\ell)_n}{(n)_n}\,{}_2F_1\left(\begin{array}{c} 1,n \\ 2n \end{array}\middle| 2\right). \end{aligned}$$

Take $a = 1$, $b = n$, and $z = 2$ in the quadratic transformation ([8], Equation 15.18.3)

$$\begin{aligned} & {}_2F_1\left(\begin{array}{c} a, b \\ 2b \end{array} \middle| z \right) \\ = \; & (1 - z)^{-a/2} \, {}_2F_1\left(\begin{array}{c} \frac{a}{2}, b - \frac{a}{2} \\ b + \frac{1}{2} \end{array} \middle| \frac{z^2}{4(z-1)} \right), \end{aligned}$$

to obtain

$${}_2F_1\left(\begin{array}{c} 1, n \\ 2n \end{array} \middle| 2 \right) = i \, {}_2F_1\left(\begin{array}{c} \frac{1}{2}, n - \frac{1}{2} \\ n + \frac{1}{2} \end{array} \middle| 1 \right).$$

Now, apply Gauss's summation theorem ([8], Equation 15.4.20)

$$\begin{aligned} {}_2F_1\left(\begin{array}{c} a, b \\ c \end{array} \middle| 1 \right) &= \frac{\Gamma(c)\Gamma(c-a-b)}{\Gamma(c-a)\Gamma(c-b)}, \\ \mathrm{Re}\,(c-a-b) &> 0, \end{aligned}$$

and the formula ([7], Equation 43:4:3)

$$\Gamma\left(n + \frac{1}{2}\right) = \frac{(2n-1)!!}{2^n} \sqrt{\pi},$$

to arrive at

$${}_2F_1\left(\begin{array}{c} 1, n \\ 2n \end{array} \middle| 2 \right) = i\pi \frac{(2n-1)!!}{2^n (n-1)!}.$$

Therefore, $S_2(n,\ell)$ is a pure imaginary number. Because $S(n,\ell)$ is a real number, we conclude that $S(n,\ell) = \mathrm{Re}[S_1(n,\ell)]$, as we wanted to prove. $\square$

Theorem 3. *The following reduction formula holds true for $n = 1, 2, \ldots$ and $x \in \mathbb{R}$:*

$$\begin{aligned} & \left. \frac{\partial \mathrm{M}_{\kappa,\mu}(x)}{\partial \kappa} \right|_{\kappa=n,\mu=1/2} \qquad (22) \\ = \; & \frac{2}{n} \sinh\left(\frac{x}{2}\right) + \frac{x\, e^{-x/2}}{n} \\ & \left\{ \left[\ln|x| + \gamma - H_n - \mathrm{Ei}(x)\right] L_{n-1}^{(1)}(x) \right. \\ & \left. + \sum_{\ell=1}^{n-1} \left(\frac{n+\ell}{n-\ell} - e^x \, \mathrm{Re}\left[{}_2F_1\left(\begin{array}{c} 1, \ell \\ \ell + n \end{array} \middle| 2 \right) \right] \right) \frac{L_{\ell-1}^{(1)}(x)}{\ell} \right\}, \end{aligned}$$

where $L_n^{(\alpha)}(x)$ denotes the Laguerre polynomials (A14) and $H_n = \sum_{k=1}^{n} \frac{1}{k}$ the n-th harmonic number.

Proof. From (21) and (20), we can see that

$$b_\ell = \mathrm{Re}\left[{}_2F_1\left(\begin{array}{c} 1, \ell \\ \ell + n \end{array} \middle| 2 \right) \right], \quad \ell = 1, 2, \ldots \qquad (23)$$

Additionally, according to ([8], Equation 13.18.1),

$$\mathrm{M}_{0,1/2}(x) = 2 \sinh\left(\frac{x}{2}\right). \qquad (24)$$

By performing the transformations $\kappa \to \kappa + 1$, $\kappa \to 0$, and $n \to n - 1$ in (A13), we obtain $\forall n = 1, 2, \ldots$

$$\mathrm{M}_{n,1/2}(x) = \frac{x\, e^{-x/2}}{n} L_{n-1}^{(1)}(x). \qquad (25)$$

Finally, we have the following for $n = 1, 2, \ldots$ ([27], Equation 1.3.7):

$$\psi(n+1) = -\gamma + H_n. \tag{26}$$

Now, insert (19) and (20)–(26) in (18) to arrive at (22), as we wanted to prove. □

Corollary 3. *The following reduction formula holds true for $n = 1, 2, \ldots$ and $x \in \mathbb{R}$,*

$$\begin{aligned} & G^{(1)}\left(\begin{array}{c} 1-n \\ 2 \end{array} \middle| x \right) \\ = \; & \frac{1}{n}\left\{ \frac{1-e^x}{x} - [\ln|x| + \gamma - H_n - \mathrm{Ei}(x)]\, L_{n-1}^{(1)}(x) \right. \\ & \left. - \sum_{\ell=1}^{n-1} \left(\frac{n+\ell}{n-\ell} - e^x \,\mathrm{Re}\left[{}_2F_1\left(\begin{array}{c} 1, \ell \\ \ell+n \end{array} \middle| 2 \right) \right] \right) \frac{L_{\ell-1}^{(1)}(x)}{\ell} \right\}. \end{aligned}$$

Proof. Consider (16) and (22) to arrive at the desired result. □

In Table 2, we collect particular cases of (22) for $x \in \mathbb{R}$ obtained with the help of the MATHEMATICA program.

Table 2. Derivative of $M_{\kappa,\mu}$ with respect to κ using (22).

κ	μ	$\frac{\partial \mathrm{M}_{\kappa,\mu}(x)}{\partial \kappa}$
1	$\frac{1}{2}$	$x\, e^{-x/2}[\ln|x| - \mathrm{Ei}(x) + \gamma - 1] + 2\sinh\left(\frac{x}{2}\right)$
2	$\frac{1}{2}$	$\frac{1}{2} x\, e^{-x/2}\left\{(2-x)\left[\ln|x| - \mathrm{Ei}(x) + \gamma - \frac{3}{2}\right] - e^x + 3\right\} + \sinh\left(\frac{x}{2}\right)$
3	$\frac{1}{2}$	$\frac{1}{6} x\, e^{-x/2}\left[(x^2 - 6x + 6)\left(\ln|x| - \mathrm{Ei}(x) + \gamma - \frac{11}{6}\right) + (e^x - 5)(x-2) - 3e^x + 4\right] + \frac{2}{3}\sinh\left(\frac{x}{2}\right)$

2.2. Derivative with Respect to the Second Parameter $\partial \mathrm{M}_{\kappa,\mu}(x)/\partial\mu$

Using (1) and (3), the first derivative of $\mathrm{M}_{\kappa,\mu}(x)$ with respect to the parameter μ is

$$\begin{aligned} & \frac{\partial \mathrm{M}_{\kappa,\mu}(x)}{\partial \mu} \\ = \; & \left[\ln x + 2\,\psi(1+2\mu) - \psi\left(\frac{1}{2} + \mu - \kappa \right) \right] \mathrm{M}_{\kappa,\mu}(x) \\ & + x^{\mu+1/2} e^{-x/2} \frac{\Gamma(1+2\mu)}{\Gamma\left(\frac{1}{2} + \mu - \kappa\right)} [S_1(\kappa,\mu,x) - S_2(\kappa,\mu,x)], \end{aligned} \tag{27}$$

where $S_1(\kappa,\mu,x)$ is provided in (12) and the series $S_2(\kappa,\mu,x)$ is

$$S_2(\kappa,\mu,x) = 2\sum_{n=0}^{\infty} \frac{\Gamma\left(\frac{1}{2} + \mu - \kappa + n\right)}{\Gamma(1+2\mu+n)} \psi(1+2\mu+n) \frac{x^n}{n!}. \tag{28}$$

Theorem 4. *For $\mu \neq -1/2$ and $x \in \mathbb{R}$, the following parameter derivative formula of $\mathrm{M}_{\kappa,\mu}(x)$ holds true:*

$$\begin{aligned} & \left. \frac{\partial \mathrm{M}_{\kappa,\mu}(x)}{\partial \mu} \right|_{\kappa=-\mu-1/2} \\ = \; & x^{\mu+1/2} e^{x/2} \left[\ln x - \frac{x}{1+2\mu}\, {}_2F_2\left(\begin{array}{c} 1, 1 \\ 2(\mu+1), 2 \end{array} \middle| -x \right) \right]. \end{aligned} \tag{29}$$

Proof. For $\kappa = -\mu - 1/2$, we have $S_2(\kappa, \mu, x) = 2\, S_1(\kappa, \mu, x)$; therefore, (27) becomes

$$\begin{aligned}
&\left.\frac{\partial \mathrm{M}_{\kappa,\mu}(x)}{\partial \mu}\right|_{\kappa=-\mu-1/2} \\
=\ & [\ln x + \psi(1+2\mu)]\mathrm{M}_{-\mu-1/2,\mu}(x) - x^{\mu+1/2}e^{-x/2}S_1\left(-\mu - \frac{1}{2}, \mu, x\right),
\end{aligned}$$

where

$$S_1\left(-\mu - \frac{1}{2}, \mu, x\right) = \sum_{n=0}^{\infty} \psi(1+2\mu+n)\frac{x^n}{n!}.$$

Thus, using (14),

$$\begin{aligned}
&\left.\frac{\partial \mathrm{M}_{\kappa,\mu}(x)}{\partial \mu}\right|_{\kappa=-\mu-1/2} \\
=\ & [\ln x + \psi(1+2\mu)]\mathrm{M}_{-\mu-1/2,\mu}(x) \\
& -x^{\mu+1/2}e^{x/2}\left[\psi(1+2\mu) + \frac{x}{1+2\mu}\, {}_2F_2\left(\left. \begin{array}{c} 1,1 \\ 2\mu+2,2 \end{array} \right| -x \right)\right].
\end{aligned} \tag{30}$$

Because, according to (1) and (3),

$$\mathrm{M}_{-\mu-1/2,\mu}(x) = x^{\mu+1/2}e^{x/2},$$

(30) now takes the simple form provided in (29), as we wanted to prove. □

Corollary 4. *For $a \in \mathbb{R}$, $a \neq 0$, and $x \in \mathbb{R}$, the following reduction formula holds true:*

$$H^{(1)}\left(\left. \begin{array}{c} a \\ a \end{array} \right| x \right) = -\frac{x\, e^x}{a}\, {}_2F_2\left(\left. \begin{array}{c} 1,1 \\ a+1,2 \end{array} \right| -x \right). \tag{31}$$

Proof. Direct differentiation of (1) yields

$$\begin{aligned}
\frac{\partial \mathrm{M}_{\kappa,\mu}(x)}{\partial \mu} =\ & \ln x\, \mathrm{M}_{\kappa,\mu}(x) + x^{\mu+1/2}e^{-x/2} \\
& \left[G^{(1)}\left(\left. \begin{array}{c} \frac{1}{2}+\mu-\kappa \\ 1+2\mu \end{array} \right| x \right) + 2\, H^{(1)}\left(\left. \begin{array}{c} \frac{1}{2}+\mu-\kappa \\ 1+2\mu \end{array} \right| x \right)\right],
\end{aligned} \tag{32}$$

thus, comparing (32) with $\kappa = -\mu - \frac{1}{2}$ to (29) and taking into account (15), we arrive at (31), as we wanted to prove. □

Using (29), the derivative of $\mathrm{M}_{\kappa,\mu}(x)$ with respect μ can be calculated for particular values of κ and μ with $x \in \mathbb{R}$; as obtained with the help of MATHEMATICA, these are presented in Table 3.

Note that for $\mu = -1/2$, we obtain an indeterminate expression in (29). For this case, we present the following result.

Theorem 5. *The following parameter derivative formula of $\mathrm{M}_{\kappa,\mu}(x)$ holds true for $x \in \mathbb{R}$:*

$$\begin{aligned}
&\left.\frac{\partial \mathrm{M}_{\kappa,\mu}(x)}{\partial \mu}\right|_{\kappa=0} \\
=\ & 4^{\mu}\sqrt{x}\,\Gamma(1+\mu)\left\{ I_{\mu}\left(\frac{x}{2}\right)[\ln 4 + \psi(1+\mu)] + \frac{\partial I_{\mu}(x/2)}{\partial \mu} \right\},
\end{aligned} \tag{33}$$

where $I_{\nu}(x)$ denotes the modified Bessel function.

Table 3. Derivative of $M_{\kappa,\mu}$ with respect to μ using (29).

κ	μ	$\frac{\partial \mathrm{M}_{\kappa,\mu}(x)}{\partial\mu}$
$-\frac{3}{2}$	1	$\frac{1}{\sqrt{x}}\left\{e^{x/2}\left[x^2(\mathrm{Chi}(x)-\mathrm{Shi}(x)-\gamma)+\frac{3}{2}x^2-2x+1\right]+e^{-x/2}(x-1)\right\}$
-1	$\frac{1}{2}$	$x\,e^{x/2}[\mathrm{Chi}(x)-\mathrm{Shi}(x)-\gamma+1]-2\sinh\left(\frac{x}{2}\right)$
$-\frac{3}{4}$	$\frac{1}{4}$	$e^{x/2}x^{3/4}\left[\ln x-\frac{2}{3}x\,{}_2F_2\left(\begin{array}{c}1,1\\2,\frac{5}{2}\end{array}\middle\vert -x\right)\right]$
$-\frac{1}{2}$	0	$e^{x/2}\sqrt{x}[\mathrm{Chi}(x)-\mathrm{Shi}(x)-\gamma]$
$-\frac{1}{4}$	$-\frac{1}{4}$	$e^{x/2}x^{1/4}\left[\ln x-2x\,{}_2F_2\left(\begin{array}{c}1,1\\2,\frac{3}{2}\end{array}\middle\vert -x\right)\right]$
$-\frac{1}{6}$	$-\frac{1}{3}$	$e^{x/2}x^{1/6}\left[\ln x-3x\,{}_2F_2\left(\begin{array}{c}1,1\\2,\frac{4}{3}\end{array}\middle\vert -x\right)\right]$
$\frac{1}{6}$	$-\frac{2}{3}$	$e^{x/2}x^{-1/6}\left[\ln x+3x\,{}_2F_2\left(\begin{array}{c}1,1\\2,\frac{2}{3}\end{array}\middle\vert -x\right)\right]$

Proof. Differentiating with respect to μ the expression ([8], Equation 13.18.8)

$$\mathrm{M}_{0,\mu}(x)=4^{\mu}\,\Gamma(1+\mu)\sqrt{x}I_{\mu}\left(\frac{x}{2}\right), \tag{34}$$

we obtain (33), as we wanted to prove. □

The order derivative of the modified Bessel function $I_{\mu}(x)$ is provided in terms of the Meijer-G function and the generalized hypergeometric function $\forall\,\mathrm{Re}\,x>0,\mu\geq 0$ [28]:

$$\begin{aligned}\frac{\partial I_{\mu}(x)}{\partial\mu} &= -\frac{\mu\,I_{\mu}(x)}{2\sqrt{\pi}}G^{3,1}_{2,4}\left(x^2\middle\vert\begin{array}{c}\frac{1}{2},1\\0,0,\mu,-\mu\end{array}\right)\\ &\quad -\frac{K_{\mu}(x)}{\Gamma^2(\mu+1)}\left(\frac{x}{2}\right)^{2\mu}{}_2F_3\left(\begin{array}{c}\mu,\mu+\frac{1}{2}\\\mu+1,\mu+1,2\mu+1\end{array}\middle\vert x^2\right),\end{aligned} \tag{35}$$

where $K_{\nu}(x)$ is the *modified Bessel function of the second kind*, or in terms of generalized hypergeometric functions, only $\forall\,\mathrm{Re}\,x>0,\mu>0,\,\mu\notin\mathbb{Z}$ [29]:

$$\begin{aligned}&\frac{\partial I_{\mu}(x)}{\partial\mu}\\ &= I_{\mu}(x)\left[\frac{x^2}{4(1-\mu^2)}\,{}_3F_4\left(\begin{array}{c}1,1,\frac{3}{2}\\2,2,2-\mu,2+\mu\end{array}\middle\vert x^2\right)+\ln\left(\frac{x}{2}\right)-\psi(\mu)-\frac{1}{2\mu}\right]\\ &\quad -I_{-\mu}(x)\frac{\pi\csc(\pi\mu)}{2\,\Gamma^2(\mu+1)}\left(\frac{x}{2}\right)^{2\mu}{}_2F_3\left(\begin{array}{c}\mu,\mu+\frac{1}{2}\\\mu+1,\mu+1,2\mu+1\end{array}\middle\vert x^2\right).\end{aligned} \tag{36}$$

There are different expressions for the order derivatives of the Bessel functions [30,31]. This subject is summarized in [32], where more general results are presented in terms of convolution integrals, while order derivatives of Bessel functions are found for particular values of the order.

Using (33), (35), and (36), derivatives of $\mathrm{M}_{\kappa,\mu}(x)$ with respect to μ can be calculated for $x\in\mathbb{R}$; these are presented in Table 4 as obtained with the help of MATHEMATICA.

Table 4. Derivative of $M_{\kappa,\mu}$ with respect to μ using (33).

κ	μ	$\frac{\partial \mathrm{M}_{\kappa,\mu}(x)}{\partial \mu}$
0	$-\frac{1}{2}$	$[\mathrm{Chi}(x)-\gamma]\cosh\left(\frac{x}{2}\right)-\frac{2}{x}\sinh^3\left(\frac{x}{2}\right)$
0	0	$\sqrt{x}\left[(\ln 4-\gamma)I_0\left(\frac{x}{2}\right)-K_0\left(\frac{x}{2}\right)\right]$
0	$\frac{1}{4}$	$\frac{x^{3/4}}{15}\,{}_0F_1\left(;\frac{5}{4};\frac{x^2}{16}\right)\left[x^2\,{}_3F_4\left(1,1,\frac{3}{2};\frac{7}{4},2,2,\frac{9}{4};\frac{x^2}{4}\right)+15(\ln x+2)\right]$ $-\frac{2\pi x}{\Gamma\left(\frac{1}{4}\right)}I_{-\frac{1}{4}}\left(\frac{x}{2}\right)\,{}_2F_3\left(\frac{1}{4},\frac{3}{4};\frac{5}{4},\frac{5}{4},\frac{3}{2};\frac{x^2}{4}\right)$
0	$\frac{1}{3}$	$\frac{x^{5/6}}{128}\left\{{}_0F_1\left(;\frac{4}{3};\frac{x^2}{16}\right)\left[9x^2\,{}_3F_4\left(1,1,\frac{3}{2};\frac{5}{3},2,2,\frac{7}{3};\frac{x^2}{4}\right)+64(2\ln x+3)\right]\right.$ $\left.-192\,{}_0F_1\left(;\frac{2}{3};\frac{x^2}{16}\right)\,{}_2F_3\left(\frac{1}{3},\frac{5}{6};\frac{4}{3},\frac{4}{3},\frac{5}{3};\frac{x^2}{4}\right)\right\}$
0	$\frac{1}{2}$	$2[\mathrm{Chi}(x)-\gamma+2]\sinh\left(\frac{x}{2}\right)-2\,\mathrm{Shi}(x)\cosh\left(\frac{x}{2}\right)$
0	$\frac{2}{3}$	$\frac{x^{7/6}}{80}\left\{{}_0F_1\left(;\frac{5}{3};\frac{x^2}{16}\right)\left[9x^2\,{}_3F_4\left(1,1,\frac{3}{2};\frac{4}{3},2,2,\frac{8}{3};\frac{x^2}{4}\right)+80\ln x+60\right]\right.$ $\left.-60\,{}_0F_1\left(;\frac{1}{3};\frac{x^2}{16}\right)\,{}_2F_3\left(\frac{2}{3},\frac{7}{6};\frac{5}{3},\frac{5}{3},\frac{7}{3};\frac{x^2}{4}\right)\right\}$
0	$\frac{3}{4}$	$\frac{x^{5/4}}{21}\,{}_0F_1\left(;\frac{7}{4};\frac{x^2}{16}\right)\left[3x^2\,{}_3F_4\left(1,1,\frac{3}{2};\frac{5}{4},2,2,\frac{11}{4};\frac{x^2}{4}\right)+21\ln x+14\right]$ $-\frac{\pi x^2}{4\Gamma\left(\frac{7}{4}\right)}I_{-\frac{3}{4}}\left(\frac{x}{2}\right)\,{}_2F_3\left(\frac{3}{4},\frac{5}{4};\frac{7}{4},\frac{7}{4},\frac{5}{2};\frac{x^2}{4}\right)$
0	1	$4\sqrt{x}\left\{I_1\left(\frac{x}{2}\right)\left[1-\gamma+\ln 4-\frac{1}{2\sqrt{\pi}}G^{2,1}_{1,3}\left(\frac{x^2}{4};\frac{1}{2};0,0,-1\right)\right]\right.$ $\left.-K_1\left(\frac{x}{2}\right)\left[I_0^2\left(\frac{x}{2}\right)-I_1^2\left(\frac{x}{2}\right)-1\right]\right\}$
0	$\frac{3}{2}$	$\frac{4}{x}\left\{\sinh\left(\frac{x}{2}\right)[6\gamma-6\,\mathrm{Chi}(x)-3x\,\mathrm{Shi}(x)-28]\right.$ $\left.+\cosh\left(\frac{x}{2}\right)[(3\,\mathrm{Chi}(x)+8-3\gamma)x+6\,\mathrm{Shi}(x)]\right\}$
0	2	$32\sqrt{x}\left\{I_2\left(\frac{x}{2}\right)\left[\frac{3}{2}-\gamma+\ln 4-\frac{1}{\sqrt{\pi}}G^{3,1}_{2,4}\left(\frac{x^2}{4};\frac{1}{2},1;0,0,2,-2\right)\right]\right.$ $\left.+K_2\left(\frac{x}{2}\right)\left[2\,{}_1F_2\left(\frac{1}{2};1,3;\frac{x^2}{4}\right)-{}_2F_3\left(\frac{1}{2},2;1,1,3;\frac{x^2}{4}\right)-1\right]\right\}$

3. Parameter Differentiation of $\mathrm{M}_{\kappa,\mu}$ via Integral Representations

3.1. Derivative with Respect to the First Parameter $\partial \mathrm{M}_{\kappa,\mu}(x)/\partial\kappa$

Integral representations of $\mathrm{M}_{\kappa,\mu}(x)$ can be obtained via integral representations of confluent hypergeometric functions ([6], Section 7.4.1); thus,

$$\begin{aligned}
&\mathrm{M}_{\kappa,\mu}(x)\\
=\;& \frac{x^{\mu+1/2}e^{-x/2}}{\mathrm{B}\left(\mu+\kappa+\frac{1}{2},\mu-\kappa+\frac{1}{2}\right)}\int_0^1 e^{xt}t^{\mu-\kappa-1/2}(1-t)^{\mu+\kappa-1/2}dt \qquad (37)\\
=\;& \frac{x^{\mu+1/2}e^{x/2}}{\mathrm{B}\left(\mu+\kappa+\frac{1}{2},\mu-\kappa+\frac{1}{2}\right)}\int_0^1 e^{-xt}t^{\mu+\kappa-1/2}(1-t)^{\mu-\kappa-1/2}dt \qquad (38)\\
&\mathrm{Re}\left(\mu\pm\kappa+\frac{1}{2}\right)>0,
\end{aligned}$$

where

$$\mathrm{B}(a,b)=\frac{\Gamma(a)\Gamma(b)}{\Gamma(a+b)} \tag{39}$$

denotes the beta function. In order to calculate the first derivative of $\mathrm{M}_{\kappa,\mu}(x)$ with respect to parameter κ, we introduce the following finite logarithmic integrals.

Definition 2.

$$I_1(\kappa,\mu;x) = \int_0^1 e^{xt} t^{\mu-\kappa-1/2}(1-t)^{\mu+\kappa-1/2} \ln\left(\frac{1-t}{t}\right) dt, \tag{40}$$

$$I_2(\kappa,\mu;x) = \int_0^1 e^{-xt} t^{\mu+\kappa-1/2}(1-t)^{\mu-\kappa-1/2} \ln\left(\frac{t}{1-t}\right) dt. \tag{41}$$

Differentiation of (37) and (38) with respect to parameter κ yields, respectively,

$$\begin{aligned} \frac{\partial \mathrm{M}_{\kappa,\mu}(x)}{\partial \kappa} &= \left[\psi\left(\mu-\kappa+\frac{1}{2}\right) - \psi\left(\mu+\kappa+\frac{1}{2}\right)\right]\mathrm{M}_{\kappa,\mu}(x) \\ &\quad + \frac{x^{\mu+1/2}e^{-x/2}}{\mathrm{B}\left(\mu+\kappa+\frac{1}{2},\mu-\kappa+\frac{1}{2}\right)} I_1(\kappa,\mu;x) \qquad (42) \\ &= \left[\psi\left(\mu-\kappa+\frac{1}{2}\right) - \psi\left(\mu+\kappa+\frac{1}{2}\right)\right]\mathrm{M}_{\kappa,\mu}(x) \qquad (43) \\ &\quad + \frac{x^{\mu+1/2}e^{x/2}}{\mathrm{B}\left(\mu+\kappa+\frac{1}{2},\mu-\kappa+\frac{1}{2}\right)} I_2(\kappa,\mu;x), \end{aligned}$$

Note that from (42) and (43) we have

$$I_2(\kappa,\mu;x) = e^{-x} I_1(\kappa,\mu;x). \tag{44}$$

Likewise, we can depart from other integral respresentations of $\mathrm{M}_{\kappa,\mu}(x)$ ([6], Section 7.4.1) (note that there are several typos in this reference regarding these integral representations) to obtain

$$\begin{aligned} \mathrm{M}_{\kappa,\mu}(x) &= \frac{2^{-2\mu}\, x^{\mu+1/2}}{\mathrm{B}\left(\mu+\kappa+\frac{1}{2},\mu-\kappa+\frac{1}{2}\right)} \qquad (45) \\ &\quad \int_{-1}^1 e^{xt/2}(1+t)^{\mu-\kappa-1/2}(1-t)^{\mu+\kappa-1/2} dt \\ &= \frac{2^{-2\mu}\, x^{\mu+1/2}}{\mathrm{B}\left(\mu+\kappa+\frac{1}{2},\mu-\kappa+\frac{1}{2}\right)} \qquad (46) \\ &\quad \int_{-1}^1 e^{-xt/2}(1+t)^{\mu+\kappa-1/2}(1-t)^{\mu-\kappa-1/2} dt \\ &\quad \mathrm{Re}\left(\mu\pm\kappa+\frac{1}{2}\right) > 0, \end{aligned}$$

and consequently, we have

$$\begin{aligned} \frac{\partial \mathrm{M}_{\kappa,\mu}(x)}{\partial \kappa} &= \left[\psi\left(\mu-\kappa+\frac{1}{2}\right) - \psi\left(\mu+\kappa+\frac{1}{2}\right)\right]\mathrm{M}_{\kappa,\mu}(x) \qquad (47) \\ &\quad + \frac{2^{-2\mu}\, x^{\mu+1/2}}{\mathrm{B}\left(\mu+\kappa+\frac{1}{2},\mu-\kappa+\frac{1}{2}\right)} I_3(\kappa,\mu;x) \\ &= \left[\psi\left(\mu-\kappa+\frac{1}{2}\right) - \psi\left(\mu+\kappa+\frac{1}{2}\right)\right]\mathrm{M}_{\kappa,\mu}(x) \qquad (48) \\ &\quad + \frac{2^{-2\mu}\, x^{\mu+1/2}}{\mathrm{B}\left(\mu+\kappa+\frac{1}{2},\mu-\kappa+\frac{1}{2}\right)} I_4(\kappa,\mu;x), \end{aligned}$$

where we have defined the following logarithmic integrals.

Definition 3.

$$\begin{aligned} I_3(\kappa,\mu;x) &= \int_{-1}^{1} e^{xt/2}(1+t)^{\mu-\kappa-1/2}(1-t)^{\mu+\kappa-1/2}\ln\left(\frac{1-t}{1+t}\right)dt, \qquad (49)\\ I_4(\kappa,\mu;x) &= \int_{-1}^{1} e^{-xt/2}(1+t)^{\mu+\kappa-1/2}(1-t)^{\mu-\kappa-1/2}\ln\left(\frac{1+t}{1-t}\right)dt. \qquad (50) \end{aligned}$$

Note that from (47) and (48), we have

$$I_3(\kappa,\mu;x) = I_4(\kappa,\mu;x) = 2^{2\mu}e^{-x/2}I_1(\kappa,\mu;x). \qquad (51)$$

Because $I_2(\kappa,\mu;x)$, $I_3(\kappa,\mu;x)$, and $I_4(\kappa,\mu;x)$ are reduced to the calculation of $I_1(\kappa,\mu;x)$, we next calculate the latter integral.

Theorem 6. *The following integral holds true for $x \in \mathbb{R}$:*

$$\begin{aligned} & I_1(\kappa,\mu;x) \qquad (52)\\ = \; & \mathrm{B}\left(\mu+\kappa+\frac{1}{2},\mu-\kappa+\frac{1}{2}\right)\\ & \left\{\left[\psi\left(\frac{1}{2}+\mu+\kappa\right)-\psi\left(\frac{1}{2}+\mu-\kappa\right)\right]{}_1F_1\left(\begin{array}{c}\frac{1}{2}+\mu-\kappa\\ 1+2\mu\end{array}\middle| x\right)\right.\\ & \left. - G^{(1)}\left(\begin{array}{c}\frac{1}{2}+\mu-\kappa\\ 1+2\mu\end{array}\middle| x\right)\right\}. \end{aligned}$$

Proof. Comparing (42) to (16) and taking into account (1), we arrive at (52), as we wanted to prove. □

Corollary 5. *For $\kappa = 0$, Equation (52) is reduced to*

$$I_1(0,\mu;x) = -\mathrm{B}\left(\mu+\frac{1}{2},\mu+\frac{1}{2}\right)G^{(1)}\left(\begin{array}{c}\frac{1}{2}+\mu\\ 1+2\mu\end{array}\middle| x\right). \qquad (53)$$

Theorem 7. *For $\ell \in \mathbb{Z}$ and $m = 0,1,2,\ldots$, with $m \geq \ell$, the following integral holds true for $x \in \mathbb{R}$:*

$$I_1\left(\frac{\ell}{2}, m+\frac{1-\ell}{2}; x\right) = e^x\mathcal{F}(-\ell, m-\ell, -x) - \mathcal{F}(\ell, m, x), \qquad (54)$$

where

$$\begin{aligned} & \mathcal{F}(s,k,z) \qquad (55)\\ = \; & \sum_{n=0}^{k}(-1)^n\binom{k}{n}\frac{d^{n+k-s}}{dz^{n+k-s}}\left[\frac{\ln z-\mathrm{Chi}(z)-\mathrm{Shi}(z)+\gamma}{z}\right], \end{aligned}$$

and the functions $\mathrm{Shi}(z)$ *and* $\mathrm{Chi}(z)$ *denote the hyperbolic sine and cosine integrals.*

Proof. From the definition of $I_1(\kappa,\mu;x)$ provided in (40), we have

$$\begin{aligned} I_1(\kappa,\mu;x) &= \int_0^1 e^{xt}t^{\mu-\kappa-1/2}(1-t)^{\mu+\kappa-1/2}\ln(1-t)dt\\ &\quad - \int_0^1 e^{xt}t^{\mu-\kappa-1/2}(1-t)^{\mu+\kappa-1/2}\ln t\, dt. \end{aligned}$$

We can change the variables $\tau = 1-t$ in the first integral above to arrive at

$$I_1(\kappa,\mu;x) = e^x\mathcal{I}_1(-\kappa,\mu;-x) - \mathcal{I}_1(\kappa,\mu;x), \qquad (56)$$

where we have set

$$\mathcal{I}_1(\kappa,\mu;x) = \int_0^1 e^{xt} t^{\mu-\kappa-1/2} (1-t)^{\mu+\kappa-1/2} \ln t \, dt. \tag{57}$$

Taking into account the binomial theorem and the integral (A9) calculated in Appendix A, i.e.,

$$\int_0^1 e^{xt} t^m \ln t \, dt = \frac{-1}{(m+1)^2} \, {}_2F_2\left(\left. \begin{array}{c} m+1, m+1 \\ m+2, m+2 \end{array} \right| x \right),$$

we can calculate

$$\begin{aligned} & \mathcal{I}_1\left(\frac{\ell}{2}, m + \frac{1-\ell}{2}; x\right) \\ = & \int_0^1 e^{xt} t^{m-\ell} (1-t)^m \ln t \, dt \\ = & \sum_{n=0}^{m} \binom{m}{n} (-1)^n \int_0^1 e^{xt} t^{m+n-\ell} \ln t \, dt \\ = & \sum_{n=0}^{m} \binom{m}{n} \frac{(-1)^{n+1}}{(n+m-\ell+1)^2} \, {}_2F_2\left(\left. \begin{array}{c} n+m-\ell+1, n+m-\ell+1 \\ n+m-\ell+2, n+m-\ell+2 \end{array} \right| x \right). \end{aligned} \tag{58}$$

Now, we can apply the differentiation formula ([8], Equation 16.3.1)

$$\frac{d^n}{dz^n} \, {}_pF_q\left(\left. \begin{array}{c} a_1, \ldots, a_p \\ b_1, \ldots, b_q \end{array} \right| z \right) = \frac{(a_1)_n \cdots (a_p)_n}{(b_1)_n \cdots (b_q)_n} \, {}_pF_q\left(\left. \begin{array}{c} a_1+n, \ldots, a_p+n \\ b_1+n, \ldots, b_q+n \end{array} \right| z \right),$$

to obtain

$$\mathcal{I}_1\left(\frac{\ell}{2}, m + \frac{1-\ell}{2}; x\right) = \sum_{n=0}^{m} \binom{m}{n} (-1)^{n+1} \frac{d^{n+m-\ell}}{dx^{n+m-\ell}} \, {}_2F_2\left(\left. \begin{array}{c} 1,1 \\ 2,2 \end{array} \right| x \right). \tag{59}$$

According to ([9], Equation 7.12.2(67)), we have

$${}_2F_2\left(\left. \begin{array}{c} 1,1 \\ 2,2 \end{array} \right| x \right) = \frac{\mathrm{Ei}(x) - \ln(-x) - \gamma}{x}, \tag{60}$$

In order to obtain similar expressions to those obtained in Table 1, we can derive an alternative form of (60). Indeed, from the definition of the *hyperbolic sine and cosine integrals* ([8], Equations 6.2.15–6.2.16), $\forall z \in \mathbb{C}$,

$$\mathrm{Shi}(z) = \int_0^z \frac{\sinh t}{t} dt \tag{61}$$

$$\mathrm{Chi}(z) = \gamma + \ln z + \int_0^z \frac{\cosh t - 1}{t} dt, \tag{62}$$

it is easy to prove that

$$\mathrm{Shi}(-z) = -\mathrm{Shi}(z), \tag{63}$$

$$\mathrm{Chi}(-z) = \mathrm{Chi}(z) - \ln z + \ln(-z). \tag{64}$$

Additionally, from the definition of a *complementary exponential integral* ([8], Equation 6.2.3)

$$\mathrm{Ein}(z) = \int_0^z \frac{1-e^{-t}}{t} dt$$

and the property $\forall x > 0$ ([8], Equation 6.2.7)

$$\mathrm{Ei}(-x) = -\mathrm{Ein}(x) + \ln x + \gamma,$$

it is easy to prove that

$$\mathrm{Ei}(-x) = \mathrm{Chi}(x) - \mathrm{Shi}(x),$$

thus, taking into account (63) and (64), we have

$$\mathrm{Ei}(x) = \mathrm{Chi}(x) - \ln x + \ln(-x) + \mathrm{Shi}(x). \tag{65}$$

We can insert (65) in (60) to obtain

$$ {}_2F_2\left(\begin{array}{c} 1,1 \\ 2,2 \end{array} \middle| x \right) = \frac{\mathrm{Chi}(x) - \ln x + \mathrm{Shi}(x) - \gamma}{x}. \tag{66}$$

Finally, by substituting (66) in (59) while taking into account (55), we arrive at

$$\begin{aligned} &\mathcal{I}_1\left(\frac{\ell}{2}, m + \frac{1-\ell}{2}; x\right) \\ = &\sum_{n=0}^{m} \binom{m}{n} (-1)^{n+1} \frac{d^{n+m-\ell}}{dx^{n+m-\ell}} \left[\frac{\mathrm{Chi}(x) - \ln x + \mathrm{Shi}(x) - \gamma}{x}\right] \\ = &\mathcal{F}(\ell, m, x). \end{aligned}$$

Similarly, we can calculate

$$\mathcal{I}_1\left(-\frac{\ell}{2}, m + \frac{1-\ell}{2}; -x\right) = \mathcal{F}(-\ell, m-\ell, -x). \tag{67}$$

Finally, according to (56), we arrive at (54), as we wanted to prove. □

Table 5 shows the integral $I_1(\kappa, \mu; x)$ for $x \in \mathbb{R}$ and particular values of the parameters κ and/or μ obtained from (52) and (54) with the aid of MATHEMATICA program.

Table 5. Integral $I_1(\kappa, \mu; x)$ for particular values of κ and μ.

κ	μ	$I_1(\kappa, \mu; x)$
$-\frac{1}{2}$	1	$\frac{1}{x^2}\{e^x(1-x)[\ln x + \gamma + \mathrm{Shi}(x) - \mathrm{Chi}(x)] + \ln x + \gamma - \mathrm{Chi}(x) - \mathrm{Shi}(x)\}$
$-\frac{1}{2}$	μ	$-\frac{\sqrt{\pi}}{2}\Gamma(\mu)\left\{\frac{e^{x/2}x^{1/2-\mu}}{\mu}\left[I_{\mu-1/2}\left(\frac{x}{2}\right) + I_{\mu+1/2}\left(\frac{x}{2}\right)\right] + \frac{2^{1-2\mu}}{\Gamma\left(\mu+\frac{1}{2}\right)} G^{(1)}(\mu+1; 2\mu+1; x)\right\}$
$\frac{1}{2}$	1	$\frac{1}{x^2}\{(x + e^x + 1)[\mathrm{Chi}(x) - \ln x - \gamma] + (x - e^x + 1)\mathrm{Shi}(x)\}$
$\frac{1}{2}$	μ	$\frac{\sqrt{\pi}}{2}\Gamma(\mu)\left\{\frac{e^{x/2}x^{1/2-\mu}}{\mu}\left[I_{\mu-1/2}\left(\frac{x}{2}\right) - I_{\mu+1/2}\left(\frac{x}{2}\right)\right] - \frac{2^{1-2\mu}}{\Gamma\left(\mu+\frac{1}{2}\right)} G^{(1)}(\mu; 2\mu+1; x)\right\}$
1	μ	$\Gamma\left(\mu-\frac{1}{2}\right)\left\{\frac{4\sqrt{\pi}\mu\, e^{x/2}x^{-\mu}}{4\mu^2-1}\left[(2\mu - x + 1) I_\mu\left(\frac{x}{2}\right) + x\, I_{\mu+1}\left(\frac{x}{2}\right)\right] - \frac{\Gamma\left(\mu+\frac{2}{3}\right)}{\Gamma(2\mu+1)} G^{(1)}\left(\mu - \frac{1}{2}; 2\mu+1; x\right)\right\}$
κ	0	$\pi\sec(\pi\kappa)\left[\pi\tan(\pi\kappa) L_{\kappa-1/2}(x) - G^{(1)}\left(\frac{1}{2}-\kappa; 1; x\right)\right]$
κ	$\frac{1}{2}$	$-\pi\csc(\pi\kappa)\left\{[\pi\kappa\cot(\pi\kappa) - 1]\, {}_1F_1(1-\kappa; 2; x) + \kappa\, G^{(1)}(1-\kappa; 2; x)\right\}$
κ	κ	$\sqrt{\pi}\frac{\Gamma\left(2\kappa+\frac{1}{2}\right)}{\Gamma(2\kappa+1)}\left\{[H_{2\kappa-1/2} + 2\ln 2]\, {}_1F_1\left(\frac{1}{2}; 2\kappa+1; x\right) - G^{(1)}\left(\frac{1}{2}; 2\kappa+1; x\right)\right\}$
$\frac{1}{4}$	$\frac{1}{4}$	$\frac{4e^x \ln 2}{\sqrt{x}} F(\sqrt{x}) - 2G^{(1)}\left(\frac{1}{2}; \frac{3}{2}; x\right)$

Theorem 8. *For $\ell \in \mathbb{Z}$ and $m = 0, 1, 2, \ldots$, with $m \geq \ell$, the following reduction formula holds true for $x \in \mathbb{R}$:*

$$
\begin{aligned}
& \mathrm{M}_{\ell/2,m+(1-\ell)/2}(x) \\
= \; & (2m-\ell+1)\binom{2m-\ell}{m}(-1)^{m-\ell}x^{\ell/2-m} \\
& \left[e^{x/2}\mathcal{P}(-\ell,m-\ell,-x)-e^{-x/2}\mathcal{P}(\ell,m,x)\right],
\end{aligned}
\tag{68}
$$

where we have set the polynomials:

$$
\mathcal{P}(s,k,z)=\sum_{n=0}^{k}\binom{k}{n}(2k-s-n)!\,z^{n}. \tag{69}
$$

Proof. According to the definition of $\mathrm{M}_{\kappa,\mu}(x)$ (1), we have

$$
\mathrm{M}_{\ell/2,m+(1-\ell)/2}(x)=x^{m+1-\ell/2}e^{-x/2}\,{}_1F_1\left(\begin{array}{c} m+1-\ell \\ 2(m+1)-\ell \end{array}\middle| x\right). \tag{70}
$$

Applying the property ([7], Equation 18:5:1)

$$
(-x)_n=(-1)^n(x-n+1)_n
$$

and the reduction formula ([9], Equation 7.11.1(12))

$$
\begin{aligned}
& {}_1F_1\left(\begin{array}{c} n \\ m \end{array}\middle| z\right)=\frac{(m-2)!(1-m)_n}{(n-1)!}\,z^{1-m} \\
& \left\{\sum_{k=0}^{m-n-1}\frac{(1+n-m)_k}{k!(2-m)_k}\,z^k-e^z\sum_{k=0}^{n-1}\frac{(1-n)_k}{k!(2-m)_k}(-z)^k\right\},
\end{aligned}
$$

where $n,m=1,2,\ldots$ and $m>n$, after some algebra we arrive at

$$
\begin{aligned}
& {}_1F_1\left(\begin{array}{c} m+1-\ell \\ 2(m+1)-\ell \end{array}\middle| x\right) \\
= \; & (2m-\ell+1)\binom{2m-\ell}{m}(-1)^{m+1-\ell}x^{\ell-2m} \\
& \left\{\sum_{k=0}^{m}\binom{m}{k}(2m-\ell-k)!x^k-e^x\sum_{k=0}^{m-\ell}\binom{m-\ell}{k}(2m-\ell-k)!(-x)^k\right\}.
\end{aligned}
\tag{71}
$$

We can now insert (71) in (70) to obtain (68), as we wanted to prove. □

In addition to (68), other reduction formulas for the Whittaker function $\mathrm{M}_{\kappa,\mu}(x)$ are presented in Appendix C. A large list of reduction formulas for $\mathrm{M}_{\kappa,\mu}(x)$ is available in [24] and in other monographs dealing with the special functions [2–10,26].

Theorem 9. *For $\ell\in\mathbb{Z}$ and $m=0,1,2,\ldots$, with $m\geq\ell$, the following reduction formula holds true for $x\in\mathbb{R}$:*

$$
\begin{aligned}
& \left.\frac{\partial \mathrm{M}_{\kappa,\mu}(x)}{\partial\kappa}\right|_{\kappa=\ell/2,\mu=m+(1-\ell)/2} \\
= \; & (2m-\ell+1)\binom{2m-\ell}{m}x^{\ell/2-m}e^{-x/2} \\
& \left\{(-1)^{m-\ell}(H_{m-\ell}-H_m)[e^x\mathcal{P}(-\ell,m-\ell,-x)-\mathcal{P}(\ell,m,x)]\right. \\
& \left.+x^{2m+1-\ell}[e^x\mathcal{F}(-\ell,m-\ell,-x)-\mathcal{F}(\ell,m,x)]\right\}.
\end{aligned}
\tag{72}
$$

Proof. According to (42), we have

$$\begin{aligned}&\left.\frac{\partial \mathrm{M}_{\kappa,\mu}(x)}{\partial\kappa}\right|_{\kappa=\ell/2,\mu=m+(1-\ell)/2}\\=\;&[\psi(m-\ell+1)-\psi(m+1)]\mathrm{M}_{\ell/2,m+(1-\ell)/2}(x)\\&+\frac{x^{m+1+\ell/2}e^{-x/2}}{\mathrm{B}(m+1,m-\ell+1)}I_1\left(\frac{\ell}{2},m+\frac{1-\ell}{2};x\right).\end{aligned}$$

Now, we can apply (39) and the property (26) to obtain

$$\begin{aligned}&\left.\frac{\partial \mathrm{M}_{\kappa,\mu}(x)}{\partial\kappa}\right|_{\kappa=\ell/2,\mu=m+(1-\ell)/2}\\=\;&(H_{m-\ell}-H_m)\mathrm{M}_{\ell/2,m+(1-\ell)/2}(x)\\&+(2m-\ell+1)\binom{2m-\ell}{m}x^{m+1-\ell/2}e^{-x/2}I_1\left(\frac{\ell}{2},m+\frac{1-\ell}{2};x\right).\end{aligned}$$

Finally, by applying the results provided in (54) and (68), we arrive at (72), as we wanted to prove. □

Corollary 6. *For $\ell\in\mathbb{Z}$ and $m=0,1,2,\ldots$, with $m\geq\ell$, the following reduction formula holds true for $x\in\mathbb{R}$:*

$$\begin{aligned}&G^{(1)}\left(\begin{array}{c}m+1-\ell\\2(m+1)-\ell\end{array}\middle|x\right)\\=\;&(2m-\ell+1)\binom{2m-\ell}{m}\\&\Big\{(-1)^{m-\ell}x^{\ell-2m-1}(H_{m-\ell}-H_m)[\mathcal{P}(\ell,m,x)-e^x\mathcal{P}(-\ell,m-\ell,-x)]\\&+\mathcal{F}(\ell,m,x)-e^x\mathcal{F}(-\ell,m-\ell,-x)\Big\}.\end{aligned}\tag{73}$$

Proof. Set (16) for $\kappa=\frac{\ell}{2}$ and $\mu=m+\frac{1-\ell}{2}$ and compare the result to (72). □

Table 6 shows the first derivative of $\mathrm{M}_{\kappa,\mu}(x)$ with respect to the κ parameter for particular values of κ and μ and for $x\in\mathbb{R}$, which are calculated from (72) and are not contained in Table 1.

Table 6. Derivative of $M_{\kappa,\mu}$ with respect to κ using (72).

κ	μ	$\frac{\partial \mathrm{M}_{\kappa,\mu}(x)}{\partial\kappa}$
$-\frac{3}{2}$	2	$-\frac{4}{x^{3/2}}\Big\{e^{x/2}\Big[\big(x^3-3x^2+6x-6\big)(\mathrm{Shi}(x)-\mathrm{Chi}(x)+\ln x+\gamma)-\frac{11}{6}x^3+\frac{15}{2}x^2-15x+11\Big]$ $+e^{-x/2}\big[6(\mathrm{Chi}(x)+\mathrm{Shi}(x)-\ln x-\gamma)-x^2+4x-11\big]\Big\}$
-1	$\frac{3}{2}$	$\frac{3}{2x}\Big\{e^{x/2}\big[\big(2x^2-4x+4\big)(\mathrm{Chi}(x)-\mathrm{Shi}(x)-\ln x-\gamma)+3x^2-8x+6\big]$ $+2\,e^{-x/2}[2\,\mathrm{Chi}(x)+2\,\mathrm{Shi}(x)+x-2\ln x-2\gamma-3]\Big\}$
$-\frac{1}{2}$	1	$\frac{2}{\sqrt{x}}\Big\{e^{x/2}(x-1)(\mathrm{Chi}(x)-\mathrm{Shi}(x)-\ln x-\gamma+1)$ $+e^{-x/2}(\ln x-\mathrm{Chi}(x)-\mathrm{Shi}(x)+\gamma+1)\Big\}$
$-\frac{1}{2}$	2	$\frac{6}{x^{3/2}}\Big\{e^{x/2}\big[\big(x^2-4x+6\big)(2\,\mathrm{Chi}(x)-2\,\mathrm{Shi}(x)-2\ln x-2\gamma+3)-12\big]$ $+e^{-x/2}[6(x-1)-4(x+3)(\ln x-\mathrm{Chi}(x)-\mathrm{Shi}(x)+\gamma)]\Big\}$
0	$\frac{3}{2}$	$\frac{6}{x}\Big\{e^{x/2}[(x-2)(\mathrm{Chi}(x)-\mathrm{Shi}(x)-\ln x-\gamma)+x]$ $+e^{-x/2}[(x+2)(\ln x-\mathrm{Chi}(x)-\mathrm{Shi}(x)+\gamma)-x]\Big\}$

Table 6. *Cont.*

κ	μ	$\frac{\partial \mathrm{M}_{\kappa,\mu}(x)}{\partial \kappa}$
$\frac{1}{2}$	2	$\frac{6}{x^{3/2}}\Big\{e^{x/2}[6(x+1)-4(x-3)(\ln x+\mathrm{Shi}(x)-\mathrm{Chi}(x)+\gamma)]$ $+e^{-x/2}\left[\left(x^2+4x+6\right)(2\ln x-2\,\mathrm{Chi}(x)-2\,\mathrm{Shi}(x)+2\gamma-3)+12\right]\Big\}$

3.2. Application to the Calculation of Infinite Integrals

Additional integral representations of the Whittaker function $\mathrm{M}_{\kappa,\mu}(x)$ in terms of Bessel functions ([6], Section 6.5.1) are known:

$$\begin{aligned}
&\mathrm{M}_{\kappa,\mu}(x) \\
&= \frac{\Gamma(1+2\mu)\,x^{1/2}e^{-x/2}}{\Gamma\left(\mu-\kappa+\frac{1}{2}\right)}\int_0^\infty e^{-t}t^{-\kappa-1/2}I_{2\mu}\left(2\sqrt{xt}\right)dt \quad (74)\\
&= \frac{\Gamma(1+2\mu)\,x^{1/2}e^{x/2}}{\Gamma\left(\mu+\kappa+\frac{1}{2}\right)}\int_0^\infty e^{-t}t^{\kappa-1/2}J_{2\mu}\left(2\sqrt{xt}\right)dt \quad (75)\\
&\mathrm{Re}\left(-\frac{1}{2}-\mu+\kappa\right)>0.
\end{aligned}$$

Let us next introduce the following infinite logarithmic integrals.

Definition 4.

$$\mathcal{H}_1(\kappa,\mu;x) = \int_0^\infty e^{-t}t^{-\kappa-1/2}I_{2\mu}\left(2\sqrt{xt}\right)\ln t\,dt, \quad (76)$$

$$\mathcal{H}_2(\kappa,\mu;x) = \int_0^\infty e^{-t}t^{\kappa-1/2}J_{2\mu}\left(2\sqrt{xt}\right)\ln t\,dt. \quad (77)$$

Differentiation of (74) and (75) with respect to the κ parameter respectively yields

$$\begin{aligned}
&\frac{\partial \mathrm{M}_{\kappa,\mu}(x)}{\partial \kappa} \\
&= \psi\left(\mu-\kappa+\frac{1}{2}\right)\mathrm{M}_{\kappa,\mu}(x)-\frac{\Gamma(1+2\mu)\,x^{1/2}e^{-x/2}}{\Gamma\left(\mu-\kappa+\frac{1}{2}\right)}\mathcal{H}_1(\kappa,\mu;x) \quad (78)\\
&= -\psi\left(\mu+\kappa+\frac{1}{2}\right)\mathrm{M}_{\kappa,\mu}(x)+\frac{\Gamma(1+2\mu)\,x^{1/2}e^{x/2}}{\Gamma\left(\mu+\kappa+\frac{1}{2}\right)}\mathcal{H}_2(\kappa,\mu;x). \quad (79)
\end{aligned}$$

Note that from (42) and (78) we have

$$\begin{aligned}
&\mathcal{H}_1(\kappa,\mu;x) \quad (80)\\
&= \frac{\Gamma\left(\mu-\kappa+\frac{1}{2}\right)\psi\left(\mu+\kappa+\frac{1}{2}\right)}{\Gamma(1+2\mu)\sqrt{x}e^{x/2}}\mathrm{M}_{\kappa,\mu}(x)-\frac{x^\mu\,I_1(\kappa,\mu;x)}{\Gamma\left(\mu+\kappa+\frac{1}{2}\right)},
\end{aligned}$$

while from (42) and (79) we have

$$\begin{aligned}
&\mathcal{H}_2(\kappa,\mu;x) \quad (81)\\
&= \frac{\Gamma\left(\mu+\kappa+\frac{1}{2}\right)\psi\left(\mu-\kappa+\frac{1}{2}\right)}{\Gamma(1+2\mu)\sqrt{x}e^{x/2}}\mathrm{M}_{\kappa,\mu}(x)+\frac{e^{-x}x^\mu\,I_1(\kappa,\mu;x)}{\Gamma\left(\mu-\kappa+\frac{1}{2}\right)}.
\end{aligned}$$

Corollary 7. *For $\ell \in \mathbb{Z}$ and $m = 0, 1, 2, \ldots$, with $m \geq \ell$, the following infinite integrals holds true for $x \in \mathbb{R}$:*

$$\begin{aligned} & \int_0^\infty \frac{e^{-t} \ln t}{t^{(1+\ell)/2}} \, I_{2m+1-\ell}\left(2\sqrt{xt}\right) dt \\ = \; & \mathcal{H}_1\left(\frac{\ell}{2}, m + \frac{1-\ell}{2}; x\right) \\ = \; & \frac{1}{m!}\Big\{(-1)^{m-\ell}(H_m - \gamma)x^{-m+(\ell-1)/2}[e^x \mathcal{P}(-\ell, m-\ell, -x) - \mathcal{P}(\ell, m, x)] \\ & -x^{m+(1-\ell)/2}[e^x \mathcal{F}(-\ell, m-\ell, -x) - \mathcal{F}(\ell, m, x)]\Big\}. \end{aligned} \tag{82}$$

and

$$\begin{aligned} & \int_0^\infty \frac{e^{-t} \ln t}{t^{(1-\ell)/2}} J_{2m+1-\ell}\left(2\sqrt{xt}\right) dt \\ = \; & \mathcal{H}_2\left(\frac{\ell}{2}, m + \frac{1-\ell}{2}; x\right) \\ = \; & \frac{1}{(m-\ell)!}\Big\{(-1)^{m-\ell}(H_{m-\ell} - \gamma)x^{-m+(\ell-1)/2}\left[\mathcal{P}(-\ell, m-\ell, -x) - e^{-x} \mathcal{P}(\ell, m, x)\right] \\ & +x^{m+(1-\ell)/2}\left[\mathcal{F}(-\ell, m-\ell, -x) - e^{-x} \mathcal{F}(\ell, m, x)\right]\Big\}. \end{aligned} \tag{83}$$

Proof. Substitute the results provided in (54) and (68) into (80) and (81) and apply (26). □

3.3. Derivative with Respect to the Second Parameter $\partial \mathrm{M}_{\kappa,\mu}(x)/\partial\mu$

In order to calculate the first derivative of $\mathrm{M}_{\kappa,\mu}(x)$ with respect to parameter μ, we introduce the following finite logarithmic integrals.

Definition 5.

$$J_1(\kappa, \mu; x) = \int_0^1 e^{xt} t^{\mu-\kappa-1/2} (1-t)^{\mu+\kappa-1/2} \ln[t(1-t)] dt, \tag{84}$$

$$J_2(\kappa, \mu; x) = \int_0^1 e^{-xt} t^{\mu+\kappa-1/2} (1-t)^{\mu-\kappa-1/2} \ln[t(1-t)] dt, \tag{85}$$

$$J_3(\kappa, \mu; x) = \int_{-1}^1 e^{xt/2} (1+t)^{\mu-\kappa-1/2} (1-t)^{\mu+\kappa-1/2} \ln\left(1-t^2\right) dt, \tag{86}$$

$$J_4(\kappa, \mu; x) = \int_{-1}^1 e^{-xt/2} (1+t)^{\mu+\kappa-1/2} (1-t)^{\mu-\kappa-1/2} \ln\left(1-t^2\right) dt. \tag{87}$$

Differentiation of (37) and (38) with respect to the μ parameter provides us with

$$\begin{aligned} & \frac{\partial \mathrm{M}_{\kappa,\mu}(x)}{\partial \mu} \\ = \; & \left[\ln x - \psi\left(\mu - \kappa + \frac{1}{2}\right) - \psi\left(\mu + \kappa + \frac{1}{2}\right) + 2\,\psi(2\mu+1)\right] \mathrm{M}_{\kappa,\mu}(x) \\ & + \frac{x^{\mu+1/2} e^{-x/2}}{\mathrm{B}\left(\mu + \kappa + \frac{1}{2}, \mu - \kappa + \frac{1}{2}\right)} J_1(\kappa, \mu; x) \end{aligned} \tag{88}$$

$$\begin{aligned} = \; & \left[\ln x - \psi\left(\mu - \kappa + \frac{1}{2}\right) - \psi\left(\mu + \kappa + \frac{1}{2}\right) + 2\,\psi(2\mu+1)\right] \mathrm{M}_{\kappa,\mu}(x) \\ & + \frac{x^{\mu+1/2} e^{x/2}}{\mathrm{B}\left(\mu + \kappa + \frac{1}{2}, \mu - \kappa + \frac{1}{2}\right)} J_2(\kappa, \mu; x). \end{aligned} \tag{89}$$

For the other integral representations provided in (45) and (46), we have

$$
\begin{aligned}
&\frac{\partial \mathrm{M}_{\kappa,\mu}(x)}{\partial \mu} \\
= \; & \left[\ln(x/4) - \psi\left(\mu - \kappa + \frac{1}{2}\right) - \psi\left(\mu + \kappa + \frac{1}{2}\right) + 2\,\psi(2\mu+1)\right]\mathrm{M}_{\kappa,\mu}(x) \\
&+ \frac{2^{-2\mu}\, x^{\mu+1/2}}{\mathrm{B}\left(\mu+\kappa+\frac{1}{2}, \mu-\kappa+\frac{1}{2}\right)} J_3(\kappa,\mu;x) \qquad (90) \\
= \; & \left[\ln(x/4) - \psi\left(\mu - \kappa + \frac{1}{2}\right) - \psi\left(\mu + \kappa + \frac{1}{2}\right) + 2\,\psi(2\mu+1)\right]\mathrm{M}_{\kappa,\mu}(x) \\
&+ \frac{2^{-2\mu}\, x^{\mu+1/2}}{\mathrm{B}\left(\mu+\kappa+\frac{1}{2}, \mu-\kappa+\frac{1}{2}\right)} J_4(\kappa,\mu;x). \qquad (91)
\end{aligned}
$$

From (88)–(91), we obtain the following interrelationships:

$$
\begin{aligned}
J_2(\kappa,\mu;x) &= e^{-x} J_1(\kappa,\mu;x), \\
J_3(\kappa,\mu;x) &= 2^{2\mu}\left[e^{-x/2} J_1(\kappa,\mu;x) + \frac{\ln 4}{x^{\mu+1/2}} \mathrm{B}\left(\mu+\kappa+\frac{1}{2}, \mu-\kappa+\frac{1}{2}\right)\mathrm{M}_{\kappa,\mu}(x)\right], \\
J_4(\kappa,\mu;x) &= J_3(\kappa,\mu;x).
\end{aligned}
$$

Because $J_2(\kappa,\mu;x)$, $J_3(\kappa,\mu;x)$, and $J_4(\kappa,\mu;x)$ are reduced to the calculation of $J_1(\kappa,\mu;x)$, we next calculate the latter integral.

Theorem 10. *According to the notation introduced in (6) and (7), the following integral holds true:*

$$
\begin{aligned}
&J_1(\kappa,\mu;x) \qquad (92) \\
= \; & \mathrm{B}\left(\mu+\kappa+\frac{1}{2}, \mu-\kappa+\frac{1}{2}\right) \\
&\left\{\left[\psi\left(\frac{1}{2}+\mu+\kappa\right) + \psi\left(\frac{1}{2}+\mu-\kappa\right) - 2\,\psi(2\mu+1)\right] {}_1F_1\left(\left.\begin{array}{c}\frac{1}{2}+\mu-\kappa \\ 1+2\mu\end{array}\right| x\right)\right. \\
&\left. + G^{(1)}\left(\left.\begin{array}{c}\frac{1}{2}+\mu-\kappa \\ 1+2\mu\end{array}\right| x\right) + 2\,H^{(1)}\left(\left.\begin{array}{c}\frac{1}{2}+\mu-\kappa \\ 1+2\mu\end{array}\right| x\right)\right\}.
\end{aligned}
$$

Proof. Comparing (88) to (32) while taking into account (1), we arrive at (92), as we wanted to prove. □

Theorem 11. *For $\ell \in \mathbb{Z}$ and $m = 0, 1, 2, \ldots$, with $m \geq \ell$, the following integral holds true for $x \in \mathbb{R}$:*

$$
J_1\left(\frac{\ell}{2}, m + \frac{1-\ell}{2}; x\right) = e^x \mathcal{F}(-\ell, m-\ell, -x) + \mathcal{F}(\ell, m, x). \qquad (93)
$$

Proof. From the definition of $J_1(\kappa,\mu;x)$ provided in (84), we have

$$
\begin{aligned}
J_1(\kappa,\mu;x) &= \int_0^1 e^{xt} t^{\mu-\kappa-1/2} (1-t)^{\mu+\kappa-1/2} \ln t \, dt \\
&\quad + \int_0^1 e^{xt} t^{\mu-\kappa-1/2} (1-t)^{\mu+\kappa-1/2} \ln(1-t) \, dt.
\end{aligned}
$$

By performing a change of variables $\tau = 1 - t$ in the second integral above, we arrive at

$$
J_1(\kappa,\mu;x) = e^x \mathcal{I}_1(-\kappa,\mu;-x) + \mathcal{I}_1(\kappa,\mu;x), \qquad (94)
$$

where we follow the notation in (57) for the integral $\mathcal{I}_1(\kappa,\mu;x)$. According to the results obtained in (58) and (67), we arrive at (93), as we wanted to prove. □

Theorem 12. *For $\ell \in \mathbb{Z}$ and $m = 0,1,2,\ldots$, with $m \geq \ell$, the following reduction formula holds true for $x \in \mathbb{R}$:*

$$\begin{aligned}
&\left.\frac{\partial \mathrm{M}_{\kappa,\mu}(x)}{\partial \mu}\right|_{\kappa=\ell/2,\mu=m+(1-\ell)/2} \qquad (95)\\
=\;& (2m-\ell+1)\binom{2m-\ell}{m} x^{\ell/2-m} e^{-x/2} \\
&\Big\{(-1)^{m-\ell}(\ln x + 2\,H_{2m-\ell+1} - H_{m-\ell} - H_m)[e^x \mathcal{P}(-\ell, m-\ell, -x) - \mathcal{P}(\ell, m, x)] \\
&+ x^{2m+1-\ell}[e^x \mathcal{F}(-\ell, m-\ell, -x) + \mathcal{F}(\ell, m, x)]\Big\}.
\end{aligned}$$

Proof. Insert (68) and (93) into (88) and apply (26). □

Table 7 shows the first derivative of $\mathrm{M}_{\kappa,\mu}(x)$ with respect to the μ parameter for particular values of κ and μ and for $x \in \mathbb{R}$, which are calculated from (95) and are not contained in Tables 3 and 4.

Corollary 8. *For $\ell \in \mathbb{Z}$ and $m = 0,1,2,\ldots$, with $m \geq \ell$, the following reduction formula holds true for $x \in \mathbb{R}$:*

$$\begin{aligned}
& H^{(1)}\left(\begin{array}{c} m+1-\ell \\ 2(m+1)-\ell \end{array}\middle| x\right) \qquad (96)\\
=\;& (2m-\ell+1)\binom{2m-\ell}{m} \\
&\Big\{(-1)^{m-\ell} x^{\ell-2m-1}(H_{2m-\ell+1} - H_m)[e^x \mathcal{P}(-\ell, m-\ell, -x) - \mathcal{P}(\ell, m, x)] \\
&+ e^x \mathcal{F}(-\ell, m-\ell, -x)\Big\}.
\end{aligned}$$

Proof. Take $\kappa = \frac{\ell}{2}$ and $\mu = m + \frac{1-\ell}{2}$ in (32), and substitute the results provided in (68), (73), and (95). After simplification, we arrive at (96), as we wanted to prove. □

Table 7. Derivative of $M_{\kappa,\mu}$ with respect to μ using (95).

κ	μ	$\frac{\partial \mathrm{M}_{\kappa,\mu}(x)}{\partial \mu}$
$-\frac{3}{2}$	2	$\frac{4}{x^{3/2}}\Big\{e^{x/2}\Big[\big(x^3-3x^2+6x-6\big)(\mathrm{Chi}(x)-\mathrm{Shi}(x)-\gamma)+\frac{7}{3}x^3-11x^2+28x-36\Big]$ $+e^{-x/2}\big[6(\mathrm{Chi}(x)+\mathrm{Shi}(x)-\gamma)+x^2-4x+36\big]\Big\}$
-1	$\frac{3}{2}$	$\frac{1}{x}\Big\{e^{x/2}\Big[3\big(x^2-2x+2\big)(\mathrm{Chi}(x)-\mathrm{Shi}(x)-\gamma)+\frac{13}{2}x^2-22x+31\Big]$ $+e^{-x/2}[3(x-2\,\mathrm{Chi}(x)-2\,\mathrm{Shi}(x)+2\gamma)-31]\Big\}$
$-\frac{1}{2}$	1	$\frac{2}{\sqrt{x}}\Big\{e^{x/2}[(x-1)(\mathrm{Chi}(x)-\mathrm{Shi}(x)-\gamma+2)-2]$ $+e^{-x/2}(\mathrm{Chi}(x)+\mathrm{Shi}(x)-\gamma+4)\Big\}$
$-\frac{1}{2}$	2	$\frac{8}{x^{3/2}}\Big\{e^{x/2}\Big[3\Big(\frac{1}{2}x^2-2x+3\Big)(\mathrm{Chi}(x)-\mathrm{Shi}(x)-\gamma)+4x^2-22x+48\Big]$ $-e^{-x/2}[3(x+3)(\mathrm{Chi}(x)+\mathrm{Shi}(x)-\gamma)+8(x+6)]\Big\}$
$\frac{1}{2}$	1	$\frac{2}{\sqrt{x}}\Big\{e^{x/2}(\mathrm{Chi}(x)-\mathrm{Shi}(x)-\gamma+4)$ $-e^{-x/2}[(x+1)(\mathrm{Chi}(x)+\mathrm{Shi}(x)-\gamma+2)+2]\Big\}$
$\frac{1}{2}$	2	$\frac{4}{x^{3/2}}\Big\{e^{x/2}[6(x-3)(\mathrm{Chi}(x)-\mathrm{Shi}(x)-\gamma)+16(x-6)]$ $+e^{-x/2}\big[3\big(x^2+4x+6\big)(\mathrm{Chi}(x)+\mathrm{Shi}(x)-\gamma)+8x^2+44x+96\big]\Big\}$

3.4. Application to the Calculation of Finite Integrals

Theorem 13. *For $\mu \geq 0$ and $x \in \mathbb{R}$, the following finite integral holds true:*

$$\begin{aligned} & \int_0^1 e^{xt}[t(1-t)]^{\mu-1/2} \ln[t(1-t)]dt \\ = \; & J_1(0,\mu;x) \\ = \; & \mathrm{B}\left(\mu+\frac{1}{2},\mu+\frac{1}{2}\right)\left(\frac{4}{|x|}\right)^{\mu} e^{x/2}\Gamma(1+\mu) \\ & \left\{ I_\mu\left(\frac{|x|}{2}\right)\left[\psi\left(\mu+\frac{1}{2}\right)-\ln|x|\right]+\frac{\partial I_\mu(|x|/2)}{\partial\mu}\right\}, \end{aligned} \tag{97}$$

where $\partial I_\mu(x)/\partial\mu$ is provided by (35) or (36).

Proof. First, consider that $x > 0$. Take $\kappa = 0$ in (88) and substitute (34) to arrive at

$$\begin{aligned} & \left.\frac{\partial \mathrm{M}_{\kappa,\mu}(x)}{\partial\mu}\right|_{\kappa=0} \\ = \; & 4^\mu\,\Gamma(1+\mu)\sqrt{x} I_\mu\left(\frac{x}{2}\right)\left[\ln x - 2\,\psi\left(\mu+\frac{1}{2}\right)+2\,\psi(2\mu+1)\right] \\ & + \frac{x^{\mu+1/2}e^{-x/2}}{\mathrm{B}\left(\mu+\frac{1}{2},\mu+\frac{1}{2}\right)} J_1(0,\mu;x) \end{aligned} \tag{98}$$

Next, equate (98) to the expression provided in (33), and solve for $J_1(0,\mu;x)$ to obtain

$$\begin{aligned} & J_1(0,\mu;x) \\ = \; & \mathrm{B}\left(\mu+\frac{1}{2},\mu+\frac{1}{2}\right)\left(\frac{4}{x}\right)^{\mu} e^{x/2}\Gamma(1+\mu) \\ = \; & \left\{ I_\mu\left(\frac{x}{2}\right)\left[\ln\left(\frac{4}{x}\right)+\psi(1+\mu)+2\psi\left(\mu+\frac{1}{2}\right)-2\psi(2\mu+1)\right]+\frac{\partial I_\mu(x/2)}{\partial\mu}\right\}. \end{aligned} \tag{99}$$

Now, apply the property ([8], Equation 5.5.8)

$$\psi(2z) = \frac{1}{2}\left[\psi(z)+\psi\left(z+\frac{1}{2}\right)\right]+\ln 2$$

for $z = \mu + \frac{1}{2}$ to simplify (99) as

$$\begin{aligned} & J_1(0,\mu;x) \\ = \; & \mathrm{B}\left(\mu+\frac{1}{2},\mu+\frac{1}{2}\right)\left(\frac{4}{x}\right)^{\mu} e^{x/2}\Gamma(1+\mu) \\ = \; & \left\{ I_\mu\left(\frac{x}{2}\right)\left[\psi\left(\mu+\frac{1}{2}\right)-\ln x\right]+\frac{\partial I_\mu(x/2)}{\partial\mu}\right\}, \end{aligned} \tag{100}$$

where (100) holds true for $x > 0$. Finally, note that by performing the change of variables $\tau = 1 - t$ in (84) we obtain the reflection formula

$$J_1(0,\mu;x) = e^x J_1(0,\mu;-x), \tag{101}$$

thus, from (100) and (101) we arrive at (97), as we wanted to prove. □

Theorem 14. *For $\mu \geq 0$ and $x \in \mathbb{R}$, the following finite integral holds true:*

$$
\begin{aligned}
&\int_{-1}^{1} e^{xt/2}[t(1-t)]^{\mu-1/2}\ln[t(1-t)]dt \\
=\ & J_3(0,\mu;x) \\
=\ & \mathrm{B}\left(\mu+\frac{1}{2},\mu+\frac{1}{2}\right)\Gamma(1+\mu)\left(\frac{16}{|x|}\right)^{\mu} \\
& \left\{I_\mu\left(\frac{|x|}{2}\right)\left[\psi\left(\mu+\frac{1}{2}\right)+\ln\left(\frac{4}{|x|}\right)\right]+\frac{\partial I_\mu(|x|/2)}{\partial\mu}\right\},
\end{aligned}
\tag{102}
$$

where $\partial I_\mu(x)/\partial\mu$ is provided by (35) or (36).

Proof. Consider $x>0$. Take $\kappa=0$ in (90) and susbtitute (34) to obtain

$$
\begin{aligned}
& J_3(0,\mu;x) \\
=\ & 2^{2\mu}e^{-x/2}J_1(0,\mu;x) \\
& +2^{4\mu}\frac{\ln 4}{x^\mu}\mathrm{B}\left(\mu+\frac{1}{2},\mu+\frac{1}{2}\right)\ \Gamma(1+\mu)I_\mu\left(\frac{x}{2}\right).
\end{aligned}
\tag{103}
$$

Now, insert in (103) the result in (100) and simplify to obtain the following for $x>0$:

$$
\begin{aligned}
& J_3(0,\mu;x) \\
=\ & \mathrm{B}\left(\mu+\frac{1}{2},\mu+\frac{1}{2}\right)\Gamma(1+\mu)\left(\frac{16}{x}\right)^{\mu} \\
& \left\{I_\mu\left(\frac{x}{2}\right)\left[\psi\left(\mu+\frac{1}{2}\right)+\ln\left(\frac{4}{x}\right)\right]+\frac{\partial I_\mu(x/2)}{\partial\mu}\right\}.
\end{aligned}
\tag{104}
$$

Finally, note that by performing the change of variables $\tau=-t$ in (86) we obtain the reflection formula

$$J_3(0,\mu;x)=J_3(0,\mu;-x), \tag{105}$$

thus, from (104) and (105) we arrive at (102), as we wanted to prove. □

Table 8 shows the integral $J_1(\kappa,\mu;x)$ for particular values of the parameters κ and μ and for $x\in\mathbb{R}$ obtained from (92), (93), and (97) with the aid of the MATHEMATICA program.

Table 8. Integral $J_1(\kappa,\mu;x)$ for particular values of κ and μ.

κ	μ	$J_1(\kappa,\mu;x)$
-1	0	$\pi\left\{2\,e^{x/2}(\ln 4-2)\left[(x+1)I_0\left(\frac{x}{2}\right)+x\,I_1\left(\frac{x}{2}\right)\right]-G^{(1)}\left(\frac{3}{2};1;x\right)-2\,H^{(1)}\left(\frac{3}{2};1;x\right)\right\}$
$-\frac{1}{2}$	1	$x^{-2}\{e^x[(x-1)(\mathrm{Chi}(x)-\mathrm{Shi}(x)-\ln x-\gamma)-2]+\mathrm{Chi}(x)+\mathrm{Shi}(x)-\ln x-\gamma+2\}$
$-\frac{1}{3}$	0	$2\pi\left\{G^{(1)}\left(\frac{5}{6};1;x\right)+2\,H^{(1)}\left(\frac{5}{6};1;x\right)-\ln(432)L_{-5/6}(x)\right\}$
0	0	$-\pi\,e^{x/2}\left\{K_0\left(\frac{\lvert x\rvert}{2}\right)+[\ln(4\lvert x\rvert)+\gamma]I_0\left(\frac{\lvert x\rvert}{2}\right)\right\}$
0	$\frac{1}{2}$	$x^{-1}\{e^x[\mathrm{Chi}(x)-\mathrm{Shi}(x)-\ln x-\gamma]-\mathrm{Chi}(x)-\mathrm{Shi}(x)+\ln x+\gamma\}$
0	1	$\left\{I_1\left(\frac{\lvert x\rvert}{2}\right)\left[I_1\left(\frac{\lvert x\rvert}{2}\right)K_1\left(\frac{\lvert x\rvert}{2}\right)-\ln(4\lvert x\rvert)-\gamma+2-\frac{1}{2\sqrt{\pi}}G_{1,3}^{2,1}\left(\frac{x^2}{4};1/2;0,0,-1\right)\right]+K_1\left(\frac{\lvert x\rvert}{2}\right)\left[1-I_0^2\left(\frac{\lvert x\rvert}{2}\right)\right]\right\}\frac{\pi}{2\lvert x\rvert}e^{x/2}$
$\frac{1}{3}$	0	$2\pi\left\{G^{(1)}\left(\frac{1}{6};1;x\right)+2\,H^{(1)}\left(\frac{1}{6};1;x\right)-\ln(432)L_{-1/6}(x)\right\}$
$\frac{1}{2}$	$\frac{1}{2}$	$\frac{\pi}{2}\left\{G^{(1)}\left(\frac{1}{2};2;x\right)+2\,H^{(1)}\left(\frac{1}{2};2;x\right)-2\,e^{x/2}\ln 4\left[I_0\left(\frac{x}{2}\right)-\ I_1\left(\frac{x}{2}\right)\right]\right\}$
$\frac{1}{2}$	1	$x^{-2}\{e^x[\mathrm{Chi}(x)-\mathrm{Shi}(x)-\ln x-\gamma+2]-(x+1)[\mathrm{Chi}(x)+\mathrm{Shi}(x)-\ln x-\gamma]-2\}$

4. Conclusions

The Whittaker function $\mathrm{M}_{\kappa,\mu}(x)$ is defined in terms of the Kummer confluent hypergeometric function; hence, its derivative with respect to the parameters κ and μ can be expressed as infinite sums of quotients of the digamma and gamma functions. In addition, parameter differentiation of the integral representations of $\mathrm{M}_{\kappa,\mu}(x)$ leads to finite and infinite integrals of elementary functions. These sums and integrals have been calculated for particular values of the parameters κ and μ in closed form. As an application of these results, we have obtained several reduction formulas for the derivatives of the confluent Kummer function with respect to the parameters, i.e., $G^{(1)}(a,b;x)$ and $H^{(1)}(a,b;x)$. Additionally, we have calculated finite integrals containing a combination of the exponential, logarithmic, and algebraic functions, as well as several infinite integrals involving the exponential, logarithmic, algebraic, and Bessel functions. It is worth noting that all the results presented in this paper have been checked both numerically and symbolically with the MATHEMATICA program.

In Appendix A, we obtain the first derivative of the incomplete gamma functions in closed form. These results allow us to calculate a finite logarithmic integral, which is used to calculate one of the integrals appearing in the body of the paper.

In Appendix B, we calculate new reduction formulas for the integral Whittaker functions $\mathrm{Mi}_{\kappa,\mu}(x)$ and $\mathrm{mi}_{\kappa,\mu}(x)$ from two reduction formulas of the Whittaker function $\mathrm{M}_{\kappa,\mu}(x)$. One of the latter seems to have not been previously reported in the literature.

Finally, in Appendix C, we collect a number of reduction formulas for the Whittaker function $\mathrm{M}_{\kappa,\mu}(x)$.

Author Contributions: Conceptualization, A.A. and J.L.G.-S.; Methodology, A.A. and J.L.G.-S.; Resources, A.A.; Writing—original draft, A.A. and J.L.G.-S.; Writing—review and editing, A.A. and J.L.G.-S. All authors have read and agreed to the published version of the manuscript.

Funding: This research received no external funding.

Institutional Review Board Statement: Not applicable.

Informed Consent Statement: Not applicable.

Data Availability Statement: Not applicable.

Acknowledgments: We are grateful to Francesco Mainardi from the Department of Physics and Astronomy, University of Bologna, Bologna, Italy, for his kind encouragement and interest in our work.

Conflicts of Interest: The authors declare no conflict of interest.

Appendix A. Parameter Differentiation of the Incomplete Gamma Functions

Definition A1. *The lower incomplete gamma function is defined as follows [7]:*

$$\gamma(\nu,x) = \int_0^x t^{\nu-1} e^{-t} dt. \tag{A1}$$

Definition A2. *The upper incomplete gamma function is defined as follows ([7], Equation 45:3:2)*

$$\Gamma(\nu,x) = \int_x^\infty t^{\nu-1} e^{-t} dt. \tag{A2}$$

The relation between both functions is

$$\Gamma(\nu) = \gamma(\nu,x) + \Gamma(\nu,x). \tag{A3}$$

The lower incomplete gamma function has the following series expansion ([7], Equation 45:6:1):

$$\gamma(\nu,x) = e^{-x} \sum_{k=0}^{\infty} \frac{x^{k+\nu}}{(\nu)_{k+1}}. \tag{A4}$$

In addition, the following integral representations in terms of infinite integrals hold true ([8], Equations 8.6.3 and 8.6.7) for $\operatorname{Re} z > 0$:

$$\begin{aligned}\gamma(\nu,z) &= z^{\nu}\int_0^{\infty}\exp\left(-\nu t - z\,e^{-t}\right)dt,\\ \Gamma(\nu,z) &= z^{\nu}\int_0^{\infty}\exp\left(\nu t - z\,e^{-t}\right)dt.\end{aligned}$$

From (A1), the derivative of the lower incomplete gamma function with respect to the order ν has the following integral representation:

$$\frac{\partial\gamma(\nu,x)}{\partial\nu} = \int_0^x t^{\nu-1}e^{-t}\ln t\,dt \tag{A5}$$

Theorem A1. *The parameter derivative of the lower incomplete gamma function is*

$$\frac{\partial\gamma(\nu,x)}{\partial\nu} = \gamma(\nu,x)\ln x - \frac{x^{\nu}}{\nu^2}\,{}_2F_2\left(\left.\begin{array}{c}\nu,\nu\\ \nu+1,\nu+1\end{array}\right| -x\right). \tag{A6}$$

Proof. According to (A1) and (A4), the derivative of the lower incomplete gamma function with respect to the parameter ν is

$$\begin{aligned}\frac{\partial\gamma(\nu,x)}{\partial\nu} &= e^{-x}\sum_{k=0}^{\infty}\frac{x^{k+\nu}[\ln x+\psi(\nu)-\psi(k+1+\nu)]}{(\nu)_{k+1}}\\ &= [\ln x+\psi(\nu)]\gamma(\nu,x) - e^{-x}\sum_{k=0}^{\infty}\frac{x^{k+\nu-1}}{(\nu)_k}\psi(k+\nu).\end{aligned}$$

Now, we apply the sum formula ([26], Equation 6.2.1(63))

$$\begin{aligned}&\sum_{k=0}^{\infty}\frac{t^k}{(a)_k}\psi(k+a)\\ =\;& \psi(a)+e^t\left[t^{1-a}\psi(a)\,\gamma(a,t)+\frac{t}{a^2}\,{}_2F_2\left(\left.\begin{array}{c}a,a\\ a+1,a+1\end{array}\right| -t\right)\right],\end{aligned}$$

to arrive at (A6), as we wanted to prove. □

Theorem A2. *The parameter derivative of the upper incomplete gamma function is*

$$\begin{aligned}&\frac{\partial\Gamma(\nu,x)}{\partial\nu}\\ =\;& \Gamma(\nu)\psi(\nu)-\gamma(\nu,x)\ln x+\frac{x^{\nu}}{\nu^2}\,{}_2F_2\left(\left.\begin{array}{c}\nu,\nu\\ \nu+1,\nu+1\end{array}\right| -x\right).\end{aligned} \tag{A7}$$

Proof. Differentiate (A3) with respect to the parameter ν and apply the result provided in (A6). □

Corollary A1. *From (A5) and (A6), we can calculate the following integral:*

$$\int_0^x t^{\nu-1}e^{-t}\ln t\,dt = \gamma(\nu,x)\ln x - \frac{x^{\nu}}{\nu^2}\,{}_2F_2\left(\left.\begin{array}{c}\nu,\nu\\ \nu+1,\nu+1\end{array}\right| -x\right). \tag{A8}$$

Corollary A2. *The following integral holds true for $x \in \mathbb{R}$:*

$$\int_0^1 e^{xt}t^{\nu-1}\ln t\,dt = -\frac{1}{\nu^2}\,{}_2F_2\left(\left.\begin{array}{c}\nu,\nu\\ \nu+1,\nu+1\end{array}\right| x\right). \tag{A9}$$

Proof. Perform the change of variables $t = z\,\tau$ in the integral provided in (A8), split the result in two integrals, and apply the change of variables $t = x\,\tau$ again to the first integral:

$$\begin{aligned}\int_0^x t^{\nu-1} e^{-t} \ln t\, dt &= x^\nu \left[\ln x \int_0^1 \tau^{\nu-1} e^{-x\tau}\, d\tau + \int_0^1 t^{\nu-1} e^{-x\tau} \ln \tau\, d\tau\right] \\ &= \ln x \underbrace{\int_0^x t^{\nu-1} e^{-t}\, dt}_{\gamma(\nu,x)} + x^\nu \int_0^1 \tau^{\nu-1} e^{-x\tau} \ln \tau\, d\tau. \end{aligned} \tag{A10}$$

Comparing (A8) to (A10), we obtain (A9), as we wanted to prove. □

Corollary A3. *According to the notation provided in (7), the following reduction formula holds true for $x \in \mathbb{R}$:*

$$H^{(1)}\left(\begin{array}{c} 1 \\ b \end{array}\middle| x\right) = -\frac{x\, e^x}{b^2}\, {}_2F_2\left(\begin{array}{c} b, b \\ b+1, b+1 \end{array}\middle| -x\right). \tag{A11}$$

Proof. Knowing that ([7], Equation 47:4:6)

$$ {}_1F_1\left(\begin{array}{c} 1 \\ b \end{array}\middle| z\right) = 1 + z^{1-b} e^z \gamma(b,z)$$

and applying (A6), we can calculate (A11), as we wanted to prove. □

Appendix B. Reduction Formulas for Integral Whittaker Functions $\mathrm{Mi}_{\kappa,\mu}$ and $\mathrm{mi}_{\kappa,\mu}$

In [24], we found reduction formulas for the integral Whittaker function $\mathrm{Mi}_{\kappa,\mu}(x)$. Next, we derive new reduction formulas for $\mathrm{Mi}_{\kappa,\mu}(x)$ and $\mathrm{mi}_{\kappa,\mu}(x)$ from reduction formulas of the Whittaker function $\mathrm{M}_{\kappa,\mu}(x)$.

Theorem A3. *The following reduction formula holds true for $x \in \mathbb{R}$, $n = 0, 1, 2, \ldots$ and $\kappa > 0$:*

$$\mathrm{Mi}_{\kappa+n,\kappa-1/2}(x) = 2^\kappa \sum_{m=0}^{n} \binom{n}{m} \frac{(-2)^m}{(2\kappa)_m} \gamma(\kappa+m, x/2), \tag{A12}$$

where $\gamma(\nu, z)$ denotes the lower incomplete gamma function.

Proof. Next, we can apply to the definition of the Whittaker function (1) the following reduction formula ([9], Equation 7.11.1(17)):

$$ {}_1F_1\left(\begin{array}{c} -n \\ b \end{array}\middle| z\right) = \frac{n!}{(b)_n} L_n^{(b-1)}(z)$$

from which we obtain ([8], Equation 13.18.17)

$$\mathrm{M}_{\kappa+n,\kappa-1/2}(x) = \frac{n!\, e^{-x/2} x^\kappa}{(2\kappa)_n} L_n^{(2\kappa-1)}(x), \tag{A13}$$

where ([27], Equation 4.17.2)

$$L_n^{(\alpha)}(x) = \sum_{m=0}^{n} \frac{\Gamma(n+\alpha+1)}{\Gamma(m+\alpha+1)} \frac{(-x)^m}{m!(n-m)!} \tag{A14}$$

denotes the Laguerre polynomials. We can now insert (A14) in (A13) and integrate term by term according to the definition of the integral Whittaker function (4) to obtain

$$
\begin{aligned}
&\mathrm{Mi}_{\kappa+n,\kappa-1/2}(x) \\
= &\sum_{m=0}^{n} \binom{n}{m} \frac{(-1)^m}{(2\kappa)_m} \int_0^x e^{-t/2} t^{\kappa+m-1} dt.
\end{aligned}
$$

Finally, taking into account the definition of the lower incomplete gamma function (A1), we can simplify the result to arrive at (A12), as we wanted to prove. □

Remark A1. *Taking $n = 0$ in (A12), we recover the formula provided in [24].*

Theorem A4. *The following reduction formula holds true for $x > 0$, $n = 0, 1, 2, \ldots$ and $\kappa \in \mathbb{R}$:*

$$
\mathrm{mi}_{\kappa+n,\kappa-1/2}(x) = 2^{\kappa} \sum_{m=0}^{n} \binom{n}{m} \frac{(-2)^m}{(2\kappa)_m} \Gamma(\kappa + m, x/2), \tag{A15}
$$

where $\Gamma(\nu, z)$ denotes the upper incomplete gamma function.

Proof. Following similar steps as in the previous theorem, here we instead consider the definition of the upper incomplete gamma function (A2). □

Theorem A5. *The following reduction formula holds true for $x \in \mathbb{R}$, $n = 0, 1, 2, \ldots$, and $\kappa > 0$:*

$$
\mathrm{Mi}_{-\kappa-n,\kappa-1/2}(x) = (-1)^{-\mathrm{sign}(x)\kappa} 2^{\kappa} \sum_{m=0}^{n} \binom{n}{m} \frac{(-2)^m}{(2\kappa)_m} \gamma(\kappa + m, -x/2). \tag{A16}
$$

Proof. From the property for $x > 0$ ([7], Equation 48:13:3)

$$
\mathrm{M}_{\kappa,\mu}(-x) = (-1)^{\mu+1/2} \mathrm{M}_{-\kappa,\mu}(x),
$$

for $x \in \mathbb{R}$ we have

$$
\mathrm{M}_{-\kappa,\mu}(x) = (-1)^{-\mathrm{sign}(x)(\mu+1/2)} \mathrm{M}_{\kappa,\mu}(-x), \tag{A17}
$$

We can apply (A17) to (A13) to obtain

$$
\mathrm{M}_{-\kappa-n,-\kappa-1/2}(x) = (-1)^{-\mathrm{sign}(x)\,\kappa} \frac{n!\, e^{x/2} (-x)^{\kappa}}{(2\kappa)_n} L_n^{(2\kappa-1)}(-x). \tag{A18}
$$

Now, by inserting (A14) in (A13) and integrating term by term according to the definition of the integral Whittaker function (4), we obtain

$$
\begin{aligned}
&\mathrm{Mi}_{-\kappa-n,\kappa-1/2}(x) \\
= \; &(-1)^{-\mathrm{sign}(x)\,\kappa} \sum_{m=0}^{n} \frac{1}{(2\kappa)_m} \binom{n}{m} \int_0^x e^{t/2} t^{m-1} (-t)^{\kappa} dt.
\end{aligned}
$$

Finally, takeing into account the definition of the lower incomplete gamma function (A1) and simplifying the result, we arrive at (A16), as we wanted to prove. □

Remark A2. *It is worth noting here that we could not locate the reduction Formula (A18) in the existing literature.*

Appendix C. Reduction Formulas for the Whittaker Function $\mathrm{M}_{\kappa,\mu}(x)$

For the convenience of readers, reduction formulas for the Whittaker function $\mathrm{M}_{\kappa,\mu}(x)$ are presented in their explicit forms in Table A1 for $x \in \mathbb{R}$.

Table A1. Whittaker function $M_{\kappa,\mu}(x)$ for particular values of κ and μ.

κ	μ	$\mathbf{M}_{\kappa,\mu}(x)$
$-\frac{1}{4}$	$\frac{1}{4}$	$\frac{\sqrt{\pi}}{2} e^{x/2} x^{1/4} \mathrm{erf}\left(\sqrt{x}\right)$
$-\frac{1}{2}$	$\frac{1}{2}$	$x\left[I_0\left(\frac{x}{2}\right) + I_1\left(\frac{x}{2}\right)\right]$
$-\frac{1}{2}$	$\frac{1}{6}$	$2^{-2/3} x\, \Gamma\left(\frac{2}{3}\right)\left[I_{-1/3}\left(\frac{x}{2}\right) + I_{2/3}\left(\frac{x}{2}\right)\right]$
$-\frac{1}{2}$	1	$x^{-1/2} e^{-x/2}\left[2\, e^x (x-1) + 2\right]$
0	0	$\sqrt{x}\, I_0\left(\frac{x}{2}\right)$
0	$\frac{1}{2}$	$2\sinh\left(\frac{x}{2}\right)$
0	1	$4\sqrt{x}\, I_1\left(\frac{x}{2}\right)$
0	$\frac{3}{2}$	$12\left[\cosh\left(\frac{x}{2}\right) - \frac{2}{x}\sinh\left(\frac{x}{2}\right)\right]$
0	$\frac{5}{2}$	$120\, x^{-2}\left[\left(x^2+12\right)\sinh\left(\frac{x}{2}\right) - 6\, x\cosh\left(\frac{x}{2}\right)\right]$
$\frac{1}{6}$	0	$\sqrt{x} e^{-x/2} L_{-1/3}(x)$
$\frac{1}{4}$	$-\frac{1}{4}$	$x^{1/4} e^{-x/2}$
$\frac{1}{4}$	$\frac{1}{4}$	$x^{1/4} e^{x/2} F\left(\sqrt{x}\right)$
$\frac{1}{3}$	0	$\sqrt{x} e^{-x/2} L_{-1/6}(x)$
$\frac{1}{2}$	$\frac{1}{6}$	$2^{-2/3} x\, \Gamma\left(\frac{2}{3}\right)\left[I_{-1/3}\left(\frac{x}{2}\right) - I_{2/3}\left(\frac{x}{2}\right)\right]$
$\frac{1}{2}$	$\frac{1}{4}$	$2^{-1/2} x\, \Gamma\left(\frac{3}{4}\right)\left[I_{-1/4}\left(\frac{x}{2}\right) - I_{3/4}\left(\frac{x}{2}\right)\right]$
$\frac{1}{2}$	$\frac{1}{2}$	$x\left[I_0\left(\frac{x}{2}\right) - I_1\left(\frac{x}{2}\right)\right]$
$\frac{1}{2}$	1	$2\, x^{-1/2} e^{-x/2}\left(e^x - x - 1\right)$
$\frac{1}{2}$	2	$12\, x^{-3/2} e^{-x/2}\left[2\, e^x (x-3) + x^2 + 4x + 6\right]$
1	$-\frac{3}{2}$	$e^{-x/2}\left(\frac{x}{2} + 1 + \frac{1}{x}\right)$
1	1	$\frac{4}{3}\sqrt{x}\left[x\, I_0\left(\frac{x}{2}\right) - (x+1) I_1\left(\frac{x}{2}\right)\right]$
1	$\frac{3}{2}$	$x^{-1} e^{-x/2}\left(6\, e^x - 3x^2 - 6\, x - 6\right)$
1	2	$\frac{32}{5} x^{-1/2}\left[\left(x^2+4x+12\right) I_1\left(\frac{x}{2}\right) - \left(x^2+3x\right) I_0\left(\frac{x}{2}\right)\right]$
2	2	$\frac{32}{35} x^{-1/2}\left[x\left(2x^2+2x+3\right) I_0\left(\frac{x}{2}\right) - 2\left(x^3+2x^2+4x+6\right) I_1\left(\frac{x}{2}\right)\right]$

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