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A PHYSICAL INTERPRETATION OF THE MODAL MASS IN STRUCTURAL DYNAMICS

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ABSTRACT

The magnitude and the units of the modal mass of a mode shape is not unique but it depends on the normalization method used to define the mode shape. Moreover, the magnitude can also depend depends on the number of degrees of freedom (DOFs) used to discretize the model. Recently, a new definition of the length of a mode shape, which depends on the mode shape and how the volume is distributed in the structure, has been proposed by the authors. This definition allows a better definition of the modal mass, which is physically meaningful and does not depend on the number of DOFs of a discrete model. With this new definition, the modal mass in constant mass-density systems is equal to the product between the total mass of the structure and the length squared. This property can be used advantageously to validate the modal masses estimated with the techniques proposed by different authors to determine the modal masses in operational modal analysis.

In this paper, these new concepts are explained by analytical, numerical, and experimental examples. The model masses of an experimental steel beam structure were estimated by experimental modal analysis and validated with the equations proposed in this paper. Moreover, the modal masses and lengths of a rigid beam supported on two springs, were calculated using different sets of DOF's and different types of normalization, demonstrating that the same mass normalized mode shapes are obtained.

Keywords: Modal mass, Normalization of mode shapes, apparent mass

1. INTRODUCTION

A mode shape contains information of both the deflection shape and the length of the vector. The length (also denoted in algebra as Euclidean norm, Euclidean length or L^2norm) of an arbitrary normalized mode shape $\boldsymbol{\psi}$ is given by [1]:

$$L_{E\psi} \quad = \sqrt{\boldsymbol{\psi}^T \cdot \boldsymbol{\psi}} \tag{1}$$

A mode shape is said to be normalized to the unit length when its length L_{ψ_L} is unity. The mode shape normalized to the unit length, hereafter denoted $\boldsymbol{\psi_L}$, is related to the mode shape $\boldsymbol{\psi}$ by:

$$\boldsymbol{\psi_L} = \frac{\boldsymbol{\psi}}{\sqrt{\boldsymbol{\psi}^T \cdot \boldsymbol{\psi}}} = \frac{\boldsymbol{\psi}}{L_{E\psi}} \tag{2}$$

In structural dynamics the modal mass of a mode shape $\boldsymbol{\psi}$ is defined as [2,3]:

$$m_{\psi} = \boldsymbol{\psi^T M \psi} \tag{3}$$

Where $\boldsymbol{M}$ is the mass matrix.

A mode shape is said to be mass normalized, hereafter denoted $\boldsymbol{\phi}$**,** if the modal mass is dimensionless unity [2,5], i..e:

$$m_{\phi} = \boldsymbol{\phi^T M \phi} = 1 \tag{4}$$

The mass normalized mode shape $\boldsymbol{\phi}$ is related with the mode shapes $\boldsymbol{\psi}$ and $\boldsymbol{\psi_L}$ as:

$$\boldsymbol{\phi} = \boldsymbol{\psi} \frac{1}{\sqrt{m}} = \boldsymbol{\psi_L} \frac{1}{\sqrt{m_L}} \tag{5}$$

Where m and m_L are the modal masses of $\boldsymbol{\psi}$ and $\boldsymbol{\psi_L}$**,** respectively.

Eq. (5) can also be expressed as:

$$\boldsymbol{\phi} = \boldsymbol{\psi}\alpha = \boldsymbol{\psi_L}\alpha_L \tag{6}$$

Where α and α_L are scaling factors, related to the modal masses as:

$$\alpha = \frac{1}{\sqrt{m}} \tag{7}$$

$$\alpha_L = \frac{1}{\sqrt{m_L}} \tag{8}$$

The modal masses and the lengths of the mode shapes are related by:

$$\frac{m_{\phi} = 1}{L_{E\phi}^2} = \frac{m_{\psi_L}}{L_{E\psi_L}^2 = 1} = \frac{m_{\psi}}{L_{E\psi}^2} \tag{9}$$

whereas the scaling factors are related to the lengths as:

$$L_{E\phi}^2 = \alpha_L^2 \ = \alpha L_{E\psi}^2 \tag{10}$$

A mode shape is commonly defined with the deflection shape and the modal mass. However, from eqs. (7) to (9) it is inferred that the length of the mode shapes and the scaling factors can be used as an alternative to the modal masses.

The modal mass m_ϕ corresponding to the mass normalized mode shape $\boldsymbol{\phi}$ is dimensionless unity. From eq. (4) it is easily inferred that the translational components $\boldsymbol{\phi_T}$ of the mode shape $\boldsymbol{\phi}$ have the units $1/\sqrt{kg}$ in the international system, whereas the units of the rotational components $\boldsymbol{\phi_R}$ are $1/\left(m\sqrt{kg}\right)$. On the other hand, the translational components $\boldsymbol{\psi_{LT}}$ of the mode shape $\boldsymbol{\psi_L}$ are dimensionless and the modal mass m_{ψ_L} has the unit of kg. With respect to the length of mode shapes, $L^2_{E\psi}$ is dimensionless whereas $L^2_{E\phi}$ has units of $1/kg$ [6,7].

2. CONSTANT MASS DENSITY SYSTEMS

If the mass-density ρ of a system is constant, eq. (3) can be expressed as [6]:

$$m_\psi = \boldsymbol{\psi^T M \psi} = \boldsymbol{\rho}\ \boldsymbol{\psi^T V \psi} \tag{11}$$

Where **V** is the volume matrix. If the total volume of the system is denoted as V_T, eq. (11) can also be formulated as [6]:

$$m_\psi = M_T\ \frac{\boldsymbol{\psi^T V \psi}}{V_T} = M_T L^2_\psi \tag{12}$$

Where the term [6]:

$$L^2_\psi = \frac{\boldsymbol{\psi}^T\ \boldsymbol{V\psi}}{V_T} \tag{13}$$

is the length of the mode shape, which depends on the volume of the structure and on the mode shape. This new definition of length secures that the length has the same unit as the mode shape. Thus, if the mode shape is dimensionless, so is the length.

Eq. (13) involves the volume matrix $\boldsymbol{V}$ and it is different to the usual concept of Euclidean length. If a structure is discretized with small finite elements of equal volume ΔV eq. (13) can be approximated as:

$$L^2_\psi \cong \frac{\Delta V \sum_{\boldsymbol{k=1}}^{\boldsymbol{N_V}} \boldsymbol{\psi_k^2}}{N_V \Delta V} = \frac{\sum_{\boldsymbol{k=1}}^{\boldsymbol{N_V}} \boldsymbol{\psi_k^2}}{N_V} = \frac{\boldsymbol{\psi^T \psi}}{N_V} \tag{14}$$

And the length L^2_ψ can be related to the euclidean length $L^2_{E\psi}$ as:

$$L^2_\psi \cong \frac{L^2_{E\psi}}{N_V} \tag{15}$$

In finite element models, the components of the mode shapes are commonly known at the nodes of the elements, and eq. (14) can also be approximated by means of the expression:

$$L^2_\psi \cong \frac{\boldsymbol{\psi^T \psi}}{N} \tag{16}$$

Where N is the number of nodes in the model.

3. NON-CONSTANT MASS DENSITY SYSTEMS

If the structure is constituted by two parts with the two volumes, V_1 with the mass density ρ_1 , and , V_2 with the mass density ρ_2, from eq. (3) is inferred that the modal mass is given by:

$$m_\psi = M_1 \frac{\boldsymbol{\psi}^T \boldsymbol{V_1} \boldsymbol{\psi}}{V_1} + M_2 \frac{\boldsymbol{\psi}^T \boldsymbol{V_2} \boldsymbol{\psi}}{V_2} = M_1 L_{\psi_1}^2 + M_2 L_{\psi_2}^2 \tag{17}$$

Where

$$L_{\psi_1} = \frac{\boldsymbol{\psi}^T \boldsymbol{V_1} \boldsymbol{\psi}}{V_1};\ L_{\psi_2} \frac{\boldsymbol{\psi}^T \boldsymbol{V_2} \boldsymbol{\psi}}{V_2} \tag{18}$$

are the partial lengths defined over the partial volumes, $V_1$and , V_2, respectively, which are related to the total length by:

$$V_T L_\psi^2 = V_1 L_{\psi_1}^2 + V_2 L_{\psi_2}^2 \tag{19}$$

Eq.(17) can also be expressed as:

$$m_\psi = M_{ap} L_\psi^2 \tag{20}$$

Where M_{ap} is an apparent mass given by:

$$M_{ap} = \frac{M_1 L_{\psi_1}^2 + M_2 L_{\psi_2}^2}{L_\psi^2} = V_T \frac{M_1 L_{\psi_1}^2 + M_2 L_{\psi_2}^2}{V_1 L_{\psi_1}^2 + V L_{\psi_2}^2} \tag{21}$$

Eqs. (17) and (21) can be generalized to systems constituted by n parts as:

$$m_\psi = \sum_{k=1}^{n} M_k L_{\psi_k}^2 \tag{22}$$

And

$$M_{ap} = \frac{\sum_{k=1}^{n} M_k L_{\psi_k}^2}{\sum_{k=1}^{n} V_k L_{\psi_k}^2} \tag{23}$$

4. A STEEL STRUCTURE

The structure consists of a vertical column (length 1.45 m) and a horizontal beam (0.615 m), both with a rectangular hollow steel section 8cm×4cm and thickness 4mm, which is fixed at the bottom of the column (see Fig. 1). The structure was weighed the total mass being $M_{T_X} = 13.24$ kg. The modal parameters were estimated with experimental modal analysis and the test setup is also shown in Fig. 1.

The structure was excited with an impact hammer applying forces in DOF's 10, 11 and 12, respectively, and the responses were measured in fifteen points using twelve accelerometers (two data sets) with a sensitivity of 100 mV/g, , using a sampling frequency of 2132 Hz. The responses were recorded with a National Instruments Compact DAQ acquisition system equipped with NI9234 acceleration modules. The modal parameters were estimated with the with the Complex Mode Indication Function (CMIF) technique [4] technique.

A model of the structure was assembled in ABAQUS [8] and meshed with shell elements S8R (8 nodes with reduced integration) using a global size of 0.005m. The following mechanical properties were considered for the steel: mass density $\varrho = 7850\ kg/m3$, Young's modulus $E = 210 \cdot 10^9\ \ N/m2$ and

Poisson ratio $\nu = 0.3$. The total mass of the model is $M_{T_{FE}} = 14.59\ kg$. The length of the numerical mode shapes was estimated with eq. (16).

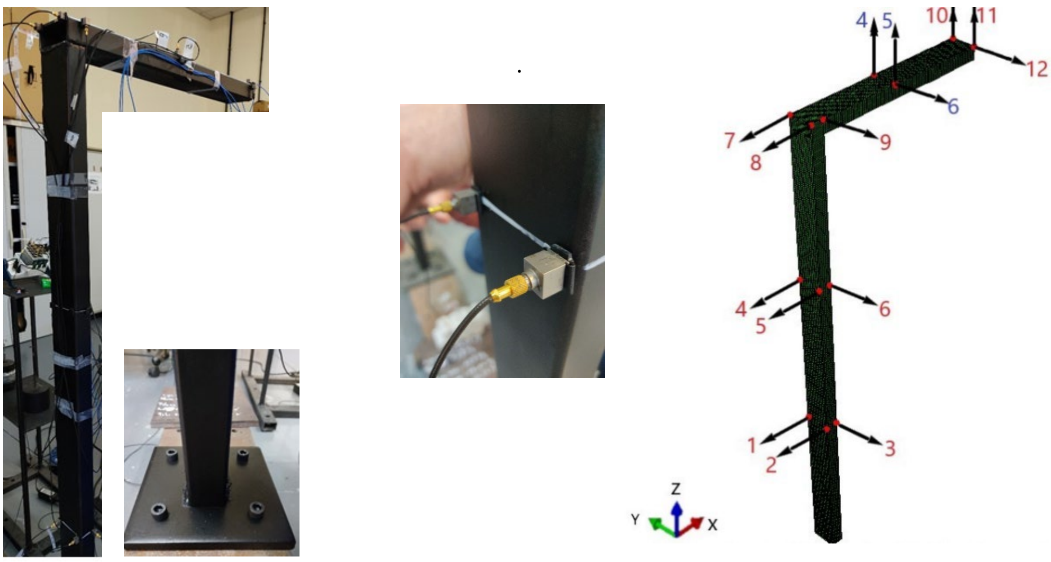


Figure 1. Steel structure and test setup

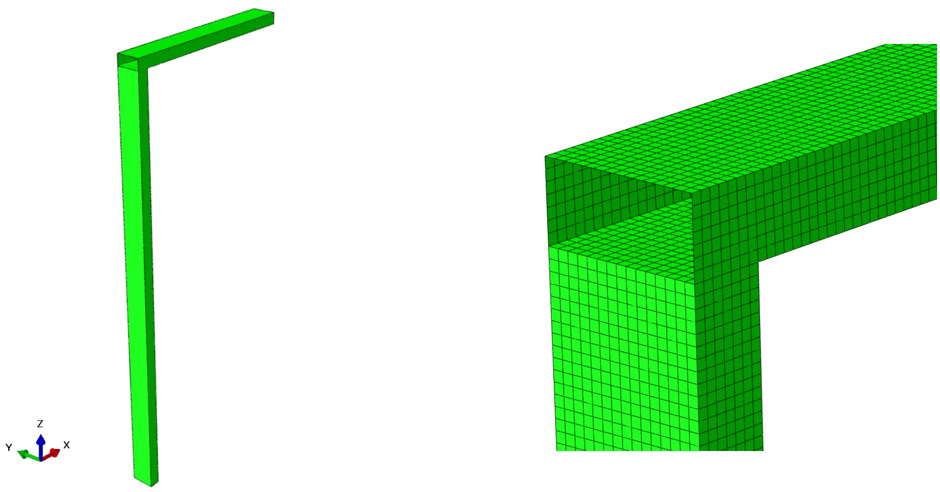


Figure 2. Numerical model meshed with shell elements.

The natural frequencies and the modal masses (mode shapes normalized to the largest component equal to unity) corresponding to the first 8 modes are shown in Table 1. The mode shapes are presented in Table 2.

An approximate transformation matrix $\boldsymbol{T}$ was obtained with the equation:

$$\boldsymbol{T} = \boldsymbol{\Phi}_{FEa}^{+} \cdot \boldsymbol{\Psi}_{Xa} \tag{24}$$

Where the subindex 'a' indicates active or measured DOF's. Then, the experimental mode shapes were expanded to the unmeasured DOF's using the numerical mode shapes, i.e:

$$\boldsymbol{\Psi}_{Xd} = \boldsymbol{\Phi}_{FEd} \cdot \boldsymbol{T} \tag{25}$$

Where subindex 'd' indicates unmeasured DOF's.

The length of the experimental mode shapes was also estimated with eq. (16) using the expanded mode shapes.

The ratio modal mass-square length is presented in Table 2. In this structure the mass-density is constant, and it can be observed that the ratio m_{FE}/L_{FE}^2 is equal for all the modes and equal to the total mass of the system ($M_{T_{FE}} = 14.59\ kg$).

Similar ratio m_X/L_X^2 has been obtained for all the experimental modes. As this ratio must be the same for all the modes, the results presented in Table 2 indicates that the modal masses have been estimated with a good accuracy. Morevoer, the ratio m_X/L_X^2 is, as expected, very close to the mass of the system $M_{T_X} = 13.24$ kg. The results of the ratio m_ψ/L_ψ^2 (Table 2) show that the modal masses were estimated with a good accuracy (error less than 1.5%), whereas modes 3 and 6 were estimated with errors of 5.5% and 8.25%, respectively.

Table 1. Numerical and experimental natural frequencies and modal masses.

Mode	Natural frequency [Hz]		Modal mass [kg]	
	Experimental	Numerical	Experimental	Numerical
1	12.53	10.938	6.25	6.90
2	20.85	18.75	3.74	3.66
3	55.74	50.781	1.47	1.71
4	55.31	54.688	1.87	2.04
5	131.98	115.625	6.57	7.12
6	198.10	180.469	5.51	5.63
7	324.78	284.572	4.93	5.42
8	502.56	465.35	4.18	5.63

Table 2. Comparison between experimental and numerical results.

Mode	**Ratio modal mass-square length**		**Error** $\frac{m_X}{L_X^2}/M_{TX}$ **(%)**
	m_X/L_X^2	m_{FE}/L_{FE}^2	
1	13.1607	14.5815	0.599
2	13.4442	14.5658	-1.542
3	12.5100	14.5152	5.513
4	13.3107	14.4991	-0.534
5	13.4068	14.5855	-1.260
6	14.3328	14.6002	-8.254
7	13.6395	14.6066	-3.017
8	13.0553	14.5295	1.395

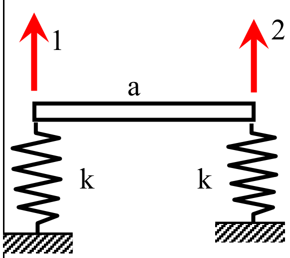
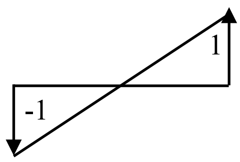
5. A RIGID BEAM ON SPRINGS. CONSTANT MASS-DENSITY

In this section the modal masses and mode shape lengths of a rigid beam supported on two springs (see Tables 3, 4 and 5) vibrating in the x-y plane (bouncing mode and pitch mode) have been calculated.

The beam has constant density ρ, length a, total mass M, total volume $V = V/\rho$, and inertia $J = Ma^2/12$ with respect to de center of gravity of the beam.

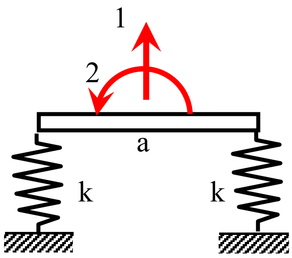
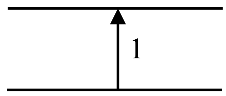
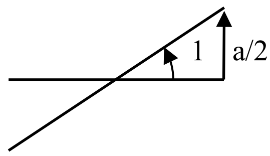
In Table 3 the system is modelled with two translational DOF's and the mode shapes are normalized to the largest component equal to unity. All the components of the mode shapes are dimensionless and the lengths of the mode shapes are also dimensionless. The modal masses of both modes are given in kg. As the density is constant, the modal mass is equal to the product between the total mass of the structure and the length squared.

Table 3. Rigid beam on two springs. Two traslational DOF's. Mode shapes normalized to the largest traslational component equal to unity.

	MODE 1	MODE 2
		
NORMALIZATION	Largest component equal to unity	Largest component equal to unity
MODE SHAPES	$\boldsymbol{\psi_1} = \begin{Bmatrix} \mathbf{1} \\ \mathbf{1} \end{Bmatrix} \begin{bmatrix} m/m \\ m/m \end{bmatrix}$	$\boldsymbol{\psi_2} = \begin{Bmatrix} -\mathbf{1} \\ \mathbf{1} \end{Bmatrix} \begin{bmatrix} m/m \\ m/m \end{bmatrix}$
MASS MATRIX $\boldsymbol{M}$	$\begin{bmatrix} \frac{M}{3} & \frac{M}{6} \\ \frac{M}{6} & \frac{M}{3} \end{bmatrix}$	
VOLUME MATRIX $\boldsymbol{V}$	$\begin{bmatrix} \frac{V}{3} & \frac{V}{6} \\ \frac{V}{6} & \frac{V}{3} \end{bmatrix}$	
LENGTH OF MODE SHAPES $L^2 = \frac{1}{V} \boldsymbol{\psi^T V \psi}$	$L_1^2 = 1$	$L_2^2 = \frac{1}{3}$
MODAL MASS $m = \boldsymbol{\psi}^T \boldsymbol{M\psi}$	$m_1 = M\ (kg)$	$m_2 = \frac{M}{3}\ (kg)$

In Table 4 the system is modelled with a translational and a rotational DOF's, The first mode is normalized to the largest translational component equal to unity, and the second mode with largest rotational component equat to unity. The length of the first mode shape is dimensionless and that corresponding to the second mode has the units of m^2. The modal mass of the first mode is a mass (kg) and that of the second mode is a modal inertia (kgm^2) .

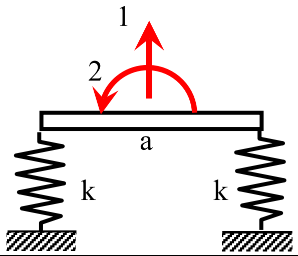
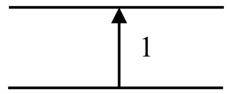
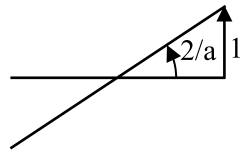
Table 4. Rigid beam on two springs. Two traslational DOF's. Mode shapes normalized to the largest traslational component equal to unity.

	MODE 1 	MODE 2 
NORMALIZATION	Largest traslational component equal to unity	Largest rotational component equal to unity
MODE SHAPES	$\boldsymbol{\psi_1} = \begin{Bmatrix}1\\0\end{Bmatrix} \begin{bmatrix}m/m\\rd/m\end{bmatrix}$	$\boldsymbol{\psi_2} = \begin{Bmatrix}0\\1\end{Bmatrix} \begin{bmatrix}m/rd\\rd/rd\end{bmatrix}$
MASS MATRIX $\boldsymbol{M}$	$\begin{bmatrix}M & 0\\0 & J\end{bmatrix}$	$J = \frac{Ma^2}{12}$
VOLUME MATRIX $\boldsymbol{V}$	$\begin{bmatrix}V & 0\\0 & \frac{Va^2}{12}\end{bmatrix}$	
LENGTH OF MODE SHAPES $L^2 = \frac{1}{V}\boldsymbol{\psi}^T\boldsymbol{V\psi}$	$L_1^2 = 1$	$L_2^2 = \frac{a^2}{12}\ (m^2)$
MODAL MASS $m = \boldsymbol{\psi}^T\boldsymbol{M\psi}$	$m_1 = M\ \ (kg)$	$m_2 = J = \frac{Ma^2}{12}\,(kgm^2)$

In Table 5 the system is also modelled with a translational and a rotational DOF's. Both modes are normalized to the largest translational component equal to unity so the rotational components have the units $\frac{rd}{m}$. The lengths are dimensionless and the modal masses are given in kg and they are equal to those obtained Table 3.

It can be checked that multiplying the mode shapes by the term $\frac{1}{\sqrt{m}}$, the same mass normalized mode shapes are obtained with the results presented in Tables 3 to 5.

Table 5. Rigid beam on two springs. Two traslational DOF's. Mode shapes normalized to the largest traslational component equal to unity.

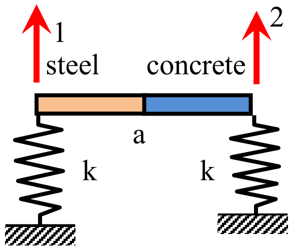
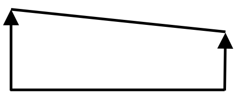
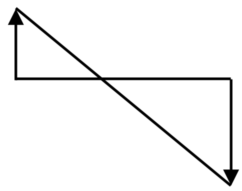
	MODE 1 	MODE 2 
NORMALIZATION	Largest traslational component equal to unity	Largest traslational component equat to unity
MODE SHAPES	$\boldsymbol{\psi_1} = \begin{Bmatrix} 1 \\ 0 \end{Bmatrix} \begin{bmatrix} m/m \\ rd/m \end{bmatrix}$	$\boldsymbol{\psi_2} = \begin{Bmatrix} 0 \\ 2/a \end{Bmatrix} \begin{bmatrix} m/m \\ rd/m \end{bmatrix}$
MASS MATRIX $\boldsymbol{M}$	$\begin{bmatrix} M & 0 \\ 0 & J \end{bmatrix} \quad J = \frac{Ma^2}{12}$	
VOLUME MATRIX $\boldsymbol{V}$	$\begin{bmatrix} V & 0 \\ 0 & \frac{Va^2}{12} \end{bmatrix}$	
LENGTH OF MODE SHAPES $L^2 = \frac{1}{V}\boldsymbol{\psi^T V \psi}$	$L_1^2 = 1$	$L_2^2 = \frac{1}{3}$
MODAL MASS $m = \boldsymbol{\psi^T M \psi}$	$m_1 = M\ (kg)$	$m_2 = \frac{M}{3}\ (kg)$

6. A RIGID BEAM ON SPRINGS. TWO DIFFERENT MATERIALS

The system shown in Table 6 also consists of a rigid beam supported on two springs, but the beam is made of steel (left half) and concrete (right half), i.e. the mass-density is not constant. The following geometrical and mechanical properties were considered: $total\ length\ of\ the\ beam\ a = 1m,$ $square\ section\ 0.1 \times 0.1\ m^2,\ \ k = \frac{10^6 N}{m}, E_s = 210\ Gpa, \nu_s = 0.3, \rho_s = 7850\frac{kg}{m^3}, E_c = 25\ Gpa,$ $\nu_c = 0.2\ , \rho_c = 2400\frac{kg}{m^3}$, where subindexes 's' and 'c' indicate steel and concrete, respectively.

The system is modelled with two translational DOF's and the mode shapes were mass normalized, i.e. the components have the units $1/\sqrt{kg}$, and the modal masses are dimensionless unity. The lengths and the partial lengths of the mode shapes were calculated with eqs. (13) and (18). In this case, an apparent mass, different for each mode, is obtained.

Table 6. Rigid beam on two springs. Two traslational DOF's. Mode shapes normalized to the largest traslational component equal to unity.

	MODE 1	MODE 2
		
NORMALIZATION	Mass normalization $m_\phi = 1$	Mass normalization $m_\phi = 1$
MODE SHAPES	$\boldsymbol{\phi_1} = \begin{Bmatrix} 0.1829 \\ 0.05421 \end{Bmatrix} \begin{bmatrix} 1/\sqrt{kg} \\ 1/\sqrt{kg} \end{bmatrix}$	$\boldsymbol{\phi_2} = \begin{Bmatrix} 0.1599 \\ -0.3662 \end{Bmatrix} \begin{bmatrix} 1/\sqrt{kg} \\ 1/\sqrt{kg} \end{bmatrix}$
MASS MATRIX $\boldsymbol{M}$ $(\boldsymbol{kg})$	$\begin{bmatrix} \frac{7}{12}M_s + \frac{1}{12}M_c & \frac{1}{6}M_s + \frac{1}{6}M_c \\ \frac{1}{6}M_s + \frac{1}{6}M_c & \frac{1}{12}M_s + \frac{7}{12}M_c \end{bmatrix} = \begin{bmatrix} 23.8958 & 8.5417 \\ 8.5417 & 10.2708 \end{bmatrix}$ $M_s = 39.25\ kg, M_c = 12\ kg$	
VOLUME MATRIX $\boldsymbol{V}$ $(\boldsymbol{m^3})$	$\begin{bmatrix} \frac{7}{12}V_s + \frac{1}{12}V_c & \frac{1}{6}V_s + \frac{1}{6}V_c \\ \frac{1}{6}V_s + \frac{1}{6}V_c & \frac{1}{12}V_s + \frac{7}{12}V_c \end{bmatrix} = \begin{bmatrix} 0.0033 & 0.0017 \\ 0.0017 & 0.0033 \end{bmatrix}$	
LENGTH OF MODE SHAPES $L^2 = \frac{1}{V}\ \boldsymbol{\psi^T V \psi}\ \left(\frac{1}{kg}\right)$ $L_s^2 = \frac{1}{V_s}\ \boldsymbol{\psi_s^T V_s \psi_s}\left(\frac{1}{kg}\right)$ $L_c^2 = \frac{1}{V_c}\ \boldsymbol{\psi_c^T V_c \psi_c}\ \left(\frac{1}{kg}\right)$	$L_1^2 = 0.0154$ $L_{1s}^2 = 0.0231$ $L_{1c}^2 = 0.0078$	$L_2^2 = 0.0337$ $L_{2s}^2 = 0.0066$ $L_{2c}^2 = 0.0608$
APPARENT MASS $M_{ap} = \dfrac{M_s L_{\psi_s}^2 + M_c L_{\psi_c}^2}{L_\psi^2}$	$M_{ap1} = 64.717\ (kg)$	$M_{ap1} = 29.315(kg)$
MODAL MASS $m = \boldsymbol{\psi}^T \boldsymbol{M \psi}$ $m = M_s L_{\psi_s}^2 + M_c L_{\psi_c}^2$	$m_1 = 1$	$m_2 = 1$

7. CONCLUSIONS

When the mass density of a structure is constant, the modal mass is always equal to the product between the total mass of the structure and the length squared ($m_\psi = M_T L_\psi^2$). If the mass density is not constant, the modal mass is equal to the product between an apparent mass (which is different for each mode) and the length squared. The modal mass of a mode shape normalized to a displacement equal to unity, is given in kg. On the other hand, if the normalization is to a rotation equal to unity, then the modal mass is given in kgm^2, i.e. it is a modal inertia.

In constant mass density systems, the ratio m_ψ / L_ψ^2 is constant for all the modes and equal to the total mass of the system. This property has been used in an experimental steel beam structure, to validate the modal masses estimated by experimental modal analysis. The experimental mode shapes were expanded to the unmeasured DOF's using a numerical model and the squared lengths estimated with eq. (16). The results of the ratio m_ψ / L_ψ^2 (Table 2) show that the modal masses were estimated with a good accuracy (error less than 1.5%), except modes 3 and 6, which were obtained with errors of 5.5% and 8.25%, respectively.

The modal masses and lengths of a rigid beam supported on two springs, vibrating in the x-y plane, were calculated using different sets of DOF's and different types of normalization. It has been demonstrated that all the models provide the same mass normalized mode shapes, independently of the type of normalization and the set of DOF's considered to define the system.

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