9th International Operational Modal Analysis Conference
iomac
July 3 - 6, 2022
Vancouver, Canada

LENGTH OF MODE SHAPES IN NUMERICAL AND EXPERIMENTAL MODELS

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ABSTRACT

A new concept of length of a continuous mode shape has been recently defined by the authors, which depends on the mode shape and how the volume is distributed in the structure. This concept was then extended to discrete systems by introducing the concept of a volume matrix. However, finite element programs do not provide the lengths of the mode shapes according to this new definition. Moreover, the volume matrices cannot be exported from the finite element programs.

In this paper, an approximate approach is proposed to calculate the length of numerical mode shapes from the nodal components. It has been demonstrated that the length can be estimated with a reasonable accuracy if small finite elements are used. These new techniques are illustrated by several numerical models assembled in ABAQUS.

The length of numerical mode shapes can be used to estimate the length of experimental mode shapes using the structural dynamic modification, which is valuable information to validate the modal masses estimated with the existing techniques to determine the modal masses in operational modal analysis.

Keywords: Mode shapes, Conference, Operational, Modal, Analysis

1. INTRODUCTION

A mode shape is said to be normalized to the unit length if its length is unity. In one-dimensional continuous systems, the Euclidean length squared $L_{E\psi}^2$ of a function $\boldsymbol{\psi(x)}$, also known as Euclidean norm or $L^2 - norm$, is defined as [1]:

$$L_{E\psi}^2 = \int_0^L |\psi(x)|^2 dx \tag{1}$$

In discrete systems, the length squared of the mode shape vector $\boldsymbol{\psi}$ (length of a vector in an Euclidean space) is defined as [1]:

$$L_{E\psi}^2 = \boldsymbol{\psi^T \psi} \tag{2}$$

The main inconvenience of Eq. (2) is that the length depends on the number of components of the vector.

In [2] the squared length L_ψ^2 of the mode shape $\boldsymbol{\psi}$ was defined as the average of the length squared $|\psi|^2$ of the mode shape over the volume $V$of the structure i.e.:

$$L_\psi^2 = \frac{1}{V_T} \int_{\boldsymbol{V}} |\boldsymbol{\psi}|^2 \, dV \tag{3}$$

where V_T is the total volume of the system.

Eq. (3) secures that the length definition has the same unit as the mode shape. Thus, if the mode shape is dimensionless, so is the length. Eq. (3) was naturally extended to discrete systems as:

$$L_\psi^2 = \frac{1}{V_T} \, \boldsymbol{\psi^T V \psi} \tag{4}$$

where $\boldsymbol{V}$ is the volume matrix of the system.

In continuous straight planar beams with length L, distributed mass density $\rho(x)$ and cross section with area $A(x)$, the modal mass (also denoted as generalized mass in some books of structural dynamics) corresponding to an arbitrary normalized continuous mode shape vector $\boldsymbol{\psi(x)}$, is given by [2,3,4]:

$$m_\psi = \int_0^L \rho(x) A(x) |\boldsymbol{\psi(x)}|^2 dx \tag{5}$$

A general equation to calculate the modal mass for the continuous case is given by [2]:

$$m_\psi = \int_V \rho |\boldsymbol{\psi}|^2 dV \tag{6}$$

wich can be easily extended to discrete systems as:

$$m = \boldsymbol{\psi}^T \boldsymbol{M\psi} \tag{7}$$

where $\boldsymbol{M}$ is the mass matrix.

If the mass-density ρ of a system is constant, Eq. (7) can be expressed as:

$$m_\psi = \boldsymbol{\psi^T M \psi} = \rho \, \boldsymbol{\psi^T V \psi} \tag{8}$$

where $\boldsymbol{V}$ is the volume matrix. Eq. (8) can also be formulated as:

$$m_\psi = M_T \frac{\boldsymbol{\psi}^{\boldsymbol{T}} \boldsymbol{V} \boldsymbol{\psi}}{V_T} = M_T L_\psi^2 \tag{9}$$

If the mass density ρ is not constant, Eq. (7) can be expressed as:

$$m_\psi = M_{ap} L_\psi^2 \tag{10}$$

where M_{ap} is an apparent mass.

One of the problems of the length defined by Eq. (4) is that finite element programs do not provide the lengths of the mode shapes. On the other hand, the volume matrices $\boldsymbol{V}$ cannot be exported from the finite element programs either.

If the numerical model is discretized with N_V small finite elements of equal volume ΔV Eq. (4) can be approximated as:

$$L_\psi^2 \cong \frac{\Delta V \sum_{\boldsymbol{k=1}}^{\boldsymbol{N_V}} \boldsymbol{\psi_k^2}}{N_V \Delta V} = \frac{\sum_{\boldsymbol{k=1}}^{\boldsymbol{N_V}} \boldsymbol{\psi_k^2}}{N_V} = \frac{\boldsymbol{\psi}^{\boldsymbol{T}} \boldsymbol{\psi}}{N_V} \tag{11}$$

However, in numerical models, the components of the mode shapes are commonly known at the nodes of the elements, and Eq. (11) can also be approximated by means of the expression:

$$L_\psi^2 \cong \frac{\boldsymbol{\psi}^{\boldsymbol{T}} \boldsymbol{\psi}}{N} \tag{12}$$

where N is the number of nodes in the model.

According to the structural dynamic modification (SDM), the experimental mode shapes can be expressed as a linear combination of the numerical mode shapes [5,6], i.e.:

$$\boldsymbol{\psi_X} = \boldsymbol{\psi_{FE}} \boldsymbol{T} \tag{13}$$

where $\boldsymbol{T}$ is a transformation matrix.

Due to the fact that the experimental mode shapes are only known at the measured DOF’s, an approximation of matrix $\boldsymbol{T}$ can be obtained by means of the expression [5]:

$$\boldsymbol{T} = \boldsymbol{\psi_{FE_a}^+} \boldsymbol{\psi_{X_a}} \tag{14}$$

where ‘+’ indicates pseudoinverse and subindex ‘a’ indicates active or measured DOF’s. The experimental mode shapes can then be expanded to the unmeasured DOF’s by:

$$\boldsymbol{\psi_{X_d}} = \boldsymbol{\psi_{FE_d}} \boldsymbol{T} \tag{15}$$

where subindex ‘d’ indicates deleted or unmeasured.

Finally, an approximation of the squared length of the experimental mode shapes can be obtained using the expanded experimental shapes with the expression:

$$L_{\psi_X}^2 \cong \frac{\boldsymbol{\psi_X^T} \boldsymbol{\psi_X}}{N} \tag{16}$$

where it is assumed that the number of elements is the same in both the numerical and the experimental models.

2. A CANTILEVER BEAM WITH CONSTANT MASS-DENSITY

2.1. 3D numerical model

A steel cantilever beam with rectangular cross-section (4cm×5cm) and 1 meter long, was assembled in the finite element software ABAQUS [7] (see Figure 1-a). The steel was considered linear -elastic and the following material properties were assumed: mass-density $\varrho = 7850\ kg/m^3$, Young's modulus $E = 210\ GPa$, and Poisson ratio $\nu = 0.3$. The total mass of the system is $M_T = 15.7\ kg$.

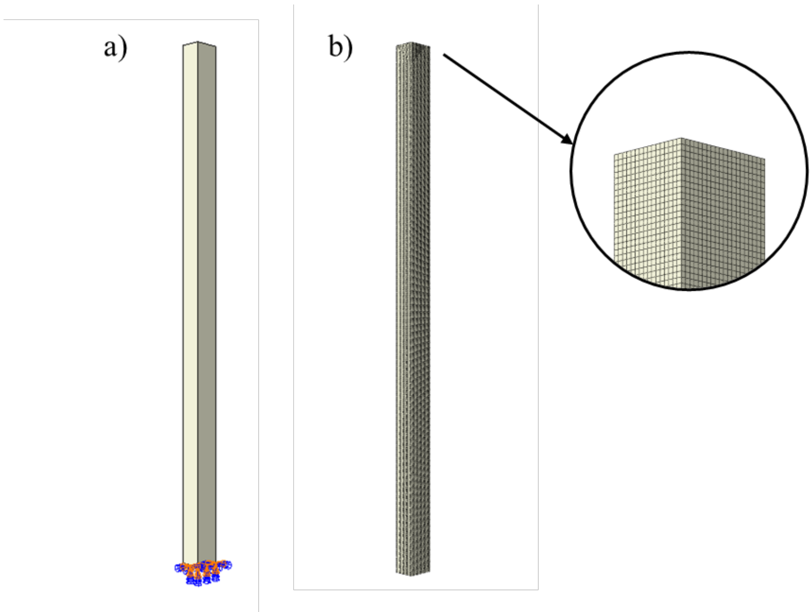

Figure 1. a) 3D steel numerical model; b) mesh with element size of 0.0025 m

Particularizing Eq. (3) to a beam with constant cross-section, the analytical length of the mode shapes can be obtained with:

$$L_\psi^2 = \frac{1}{L}\int_{\mathbf{0}}^{\boldsymbol{L}} |\boldsymbol{\psi}(\boldsymbol{x})|^{\mathbf{2}}\, dx \tag{17}$$

The analytical expressions of the bending mode shapes $\boldsymbol{\psi}(\boldsymbol{x})$ corresponding to beams with constant mass-density and constant cross-section, are reported in the literature [1,8], and for a cantilever beam are given as:

$$\psi_k = C_1\left(cosh(\beta_k x) - cos(\beta_k x) - \frac{cosh(\beta_k L) + cos(\beta_k L)}{sinh(\beta_k L) + sin(\beta_k L)}\big(sinh(\beta_k x) - sin(\beta_k x)\big)\right) \tag{18}$$

where 'k' indicates the order of the mode and the values of β_k are shown in Table 1.

Table 1. Values of $\boldsymbol{\beta_k L}$ for bending modes of a cantilever beam [1,8]

	k				
	1	2	3	4	$k > 4$
$\beta_k L$	1.8751	4.6941	7.8548	10.996	$(2k-1)\frac{\pi}{2}$

In this particular case the squared length results $L_\psi^2 = 0.25$ for all the bending modes.

The beam was meshed with quadratic twenty-node hexahedral elements (C3D20R) and with different size of the elements. The natural frequencies corresponding to the first eight modes, using an element size of 2.5 mm, are presented in Table 2, whereas the mode shapes are shown in Figures 2. The modal masses, corresponding to mode shapes normalized to the largest component equal to unity, are presented in Table 3. Where it can be observed that approximately the same modal masses are obtained for sizes less than 20 mm.

Table 2. Natural frequencies of the cantilever beam.

Mode		Finite element model C3D20R (Element size 2.5 mm)	Beam model B32 (Element size 2.5 mm)
1	1st Bending Y	33.452	33.379
2	1st Bending X	41.774	41.694
3	2nd Bending Y	208.090	207.630
4	2nd Bending X	258.800	258.290
5	3rd Bending Y	575.950	574.610
6	3rd Bending X	711.860	710.370
7	1st Torsion	720.160	719.400
8	4rd Bending Y	1110.400	1107.600

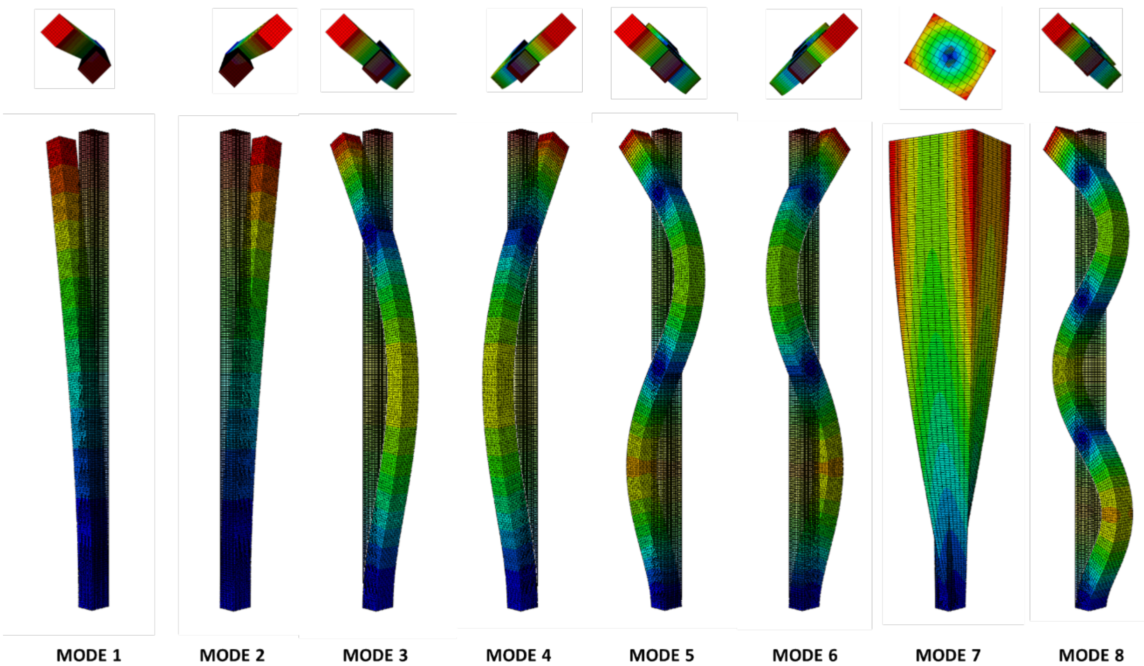

Figure 2. First 8 mode shapes.

The length of the mode shapes was calculated with Eq. (12) and they are presented in Table 4. As expected, the accuracy obtained with Eq. (12) increases as increasing the number of the elements in the model. Moreover, the error is larger for the higher modes. All the lengths obtained with the numerical model approaches the analytical values as decreasing the size of the elements, but the rate of convergence is slower for the torsional mode.

Table 3. Modal masses obtained from the numerical model.

	Mode	Element size (mm)					Analytical
		2.5	5	10	20	30	
1	1st Bending Y	3.9267	3.9265	3.9262	3.9253	3.9221	3.925
2	1st Bending X	3.9305	3.9304	3.9301	3.9294	3.9285	3.925
3	2nd Bending Y	3.9667	3.9665	3.9661	3.9650	3.9592	3.925
4	2nd Bending X	3.9927	3.9925	3.9922	3.9914	3.9902	3.925
5	3rd Bending Y	4.0317	4.0315	4.0311	4.0296	4.0200	3.925
6	3rd Bending X	4.0927	4.0925	4.0921	4.0912	4.0897	3.925
7	1st Torsion	4.2839	4.2838	4.2835	4.2884	4.2819	3.925
8	4rd Bending Y	4.1246	4.1244	4.1239	4.1211	4.1073	3.925

Table 4. Values $\boldsymbol{L_\psi^2}$ using Eq. (12).

	Mode	Element size (mm)					Analytical
		2.5	5	10	20	30	
1	1st Bending Y	0.2506	0.2510	0.2519	0.2537	0.2552	0.25
2	1st Bending X	0.2508	0.2513	0.2522	0.2539	0.2556	0.25
3	2nd Bending Y	0.2532	0.2538	0.2549	0.2570	0.2592	0.25
4	2nd Bending X	0.2549	0.2555	0.2566	0.2587	0.2609	0.25
5	3rd Bending Y	0.2575	0.2582	0.2596	0.2623	0.2654	0.25
6	3rd Bending X	0.2614	0.2622	0.2636	0.2662	0.2692	0.25
7	1st Torsion	0.2953	0.3177	0.3626	0.4354	0.5568	---
8	4rd Bending Y	0.2636	0.2645	0.2664	0.2699	0.2741	0.25

Due to the fact that the mass-density of the system is constant, the ratio $\boldsymbol{m_\psi/L_\psi^2}$ is the same for all the modes and equal to the total mass, i.e. $\frac{m_\psi}{L_\psi^2} = M_T \ = 15.7\ kg$. This ratio is shown in Table 5 and as expected, the numerical ratios $\frac{m_\psi}{L_\psi^2}$ approaches the analytical values for small size of the elements.

Table 5. Ratio $\boldsymbol{m_\psi/L_\psi^2}$.

	Mode	Element size (mm)					Analytical
		2.5	5	10	20	30	
1	1st Bending Y	15.669	15.643	15.586	15.472	15.369	15.7
2	1st Bending X	15.672	15.640	15.583	15.476	15.370	15.7
3	2nd Bending Y	15.666	15.628	15.559	15.428	15.275	15.7
4	2nd Bending X	15.664	15.626	15.558	15.429	15.294	15.7
5	3rd Bending Y	15.657	15.614	15.528	15.363	15.147	15.7
6	3rd Bending X	15.657	15.608	15.524	15.369	15.192	15.7
7	1st Torsion	14.507	13.484	11.813	9.849	7.690	15.7
8	4rd Bending Y	15.647	15.593	15.480	15.269	14.985	15.7

A better estimation of the lengths can be obtained with a linear extrapolation of the results obtained with two different sizes or fitting the results corresponding to several models meshed with different size of elements. If the lengths of Table 4 obtained with sizes 5 and 10 mm are extrapolated with a straight line to zero size, the results presented in Table 6 are obtained, where it can be observed that the total mass M_T is estimated with an error less than 0.05%. The results obtained fitting all the results of Table 6 with a straight line are also shown in the same table, achieving again a good accuracy. The extrapolation of the results corresponding to the torsional mode are shown in Figure 3.

Table 6. Length L_ψ^2 and ratio m_ψ/L_ψ^2 by extrapolation.

	Mode	Sizes 5 mm and 10 mm		Linear fit of all values in Table 4	
		L_ψ^2	m_ψ/L_ψ^2	L_ψ^2	m_ψ/L_ψ^2
1	1st Bending Y	0.2501	15.7005	0.2502	15.6927
2	1st Bending X	0.2504	15.6969	0.2504	15.6941
3	2nd Bending Y	0.2527	15.6973	0.2527	15.6952
4	2nd Bending X	0.2544	15.6946	0.2544	15.6929
5	3rd Bending Y	0.2568	15.6998	0.2568	15.7004
6	3rd Bending X	0.2608	15.6929	0.2607	15.6939
7	1st Torsion	0.2728	15.7034	0.2690	15.9262
8	4rd Bending Y	0.2626	15.7068	0.2626	15.7046

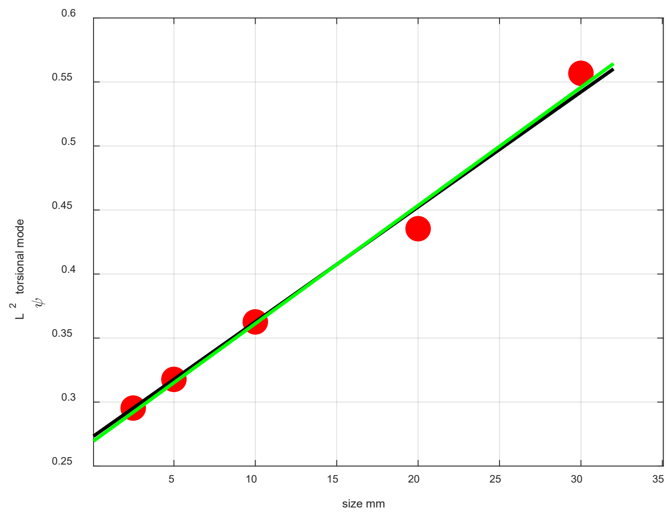

Figure 3. Calculation of the squared length of the torsional mode. Red circles: data from Table 4. Black line: fit using data of 5 and 10 mm. Green line: Linear fit using all the results.

2.2. A 3D beam model

The cantilever beam was also meshed with quadratic beam elements B32 with a length of 2.5 mm (801 nodes) and 1.25 mm (1601 nodes), obtaining very similar modal parameters. The natural frequencies are shown in Table 2 and the modal masses (mode shapes normalized to the largest translational component equal to unity) are shown in Table 7. The length of the mode shapes was estimated with Eq. (12) using only the translational components. The ratio m_ψ/L_ψ^2 is obtained with an error less than 1.6% for all the modes. Thus, beam models can be used successfully to estimate the length of the mode shapes with low computational cost.

With this model, all the translational components of the torsional mode are zero, and ABAQUS normalize this mode shape to the largest rotation equal to unity. This means that the squared length of the torsional mode (L_θ^2) is dimensionless and the modal mass is given in units of $kg \cdot m^2$, which can also be obtained analytically with the expression:

$$m_\theta = I_M L_\theta^2 \tag{19}$$

Where I_M is the mass moment inertia of the structure with respect to the longitudinal axes of the beam, which for a rectangular section of dimensions $a \times b$ is given by:

$$I_M = M_T \frac{(a^2 + b^2)}{12} \tag{20}$$

For this beam, Eq. (20) gives I_M=0.0054 kgm^2

From the finite element model, it has been obtained that $L_\theta^2 = 0.5$ and $m_\theta = 2.682 \times 10^{-3}\ kgm^2$, which gives a ratio $\frac{m_\theta}{L_\theta^2} = 0.0054\ kgm^2$.

The modal masses and the squared lengths of the torsional mode, obtained with the 3D model and with the beam model, are related by the expression:

$$M_T = \frac{m}{L^2} = \frac{m_\theta}{L_\theta^2} \frac{1}{\frac{(a^2 + b^2)}{12}} \tag{21}$$

Table 7. Length L_ψ^2, modal mass m_ψ and ratio m_ψ / L_ψ^2 for the bending modes of the beam model.

Mode		Beam model B32 2.5 mm (801 nodes)			Beam model B32 1.25 mm (1601 nodes)		
		L_ψ^2 $[-]$	m_ψ $[kg]$	m_ψ/L_ψ^2 $[kg]$	L_ψ^2 $[-]$	m_ψ $[kg]$	m_ψ/L_ψ^2 $[kg]$
1	1st Bending Y	0.2505	3.9309	15.6922	0.2504	3.9309	15.6985
2	1st Bending X	0.2507	3.9342	15.6929	0.2505	3.9342	15.7054
3	2nd Bending Y	0.2522	3.9710	15.7454	0.2520	3.9710	15.7579
4	2nd Bending X	0.2532	3.9967	15.7848	0.2531	3.9967	15.7910
5	3rd Bending Y	0.2549	4.0369	15.8372	0.2548	4.0369	15.8434
6	3rd Bending X	0.2575	4.0978	15.9138	0.2573	4.0978	15.9262
7	1st Torsion	--	--	---	---	---	---
8	4rd Bending Y	0.2590	4.1317	15.9525	0.2588	4.1317	15.9648

3. A CANTILEVER BEAM MADE OF STEEL AND CONCRETE

A three-dimensional cantilever beam with the same dimensions as those used in section 2, and made of steel and concrete is considered in this section (see Figure 4a). The encastre boundary condition is placed at the end of the steel part. The following material properties were assumed for the steel: mass-density $\varrho = 7850\ kg/m^3$, Young's modulus $E = 210\ GPa$, and Poisson ratio $\nu = 0.3$. The material properties assumed for the concrete were following: mass-density $\varrho = 2400\ kg/m^3$, Young's modulus $E = 20\ GPa$, and Poisson ratio $\nu = 0.18$.

The beam was meshed with twenty-node hexahedral elements (C3D20R) with an approximate global size of 0.0025m (see Fig. 4b). The natural frequencies and modal masses (mode shapes normalized to the largest component equal to unity) corresponding to the first eight modes are presented in Table 8. The total mass of the beam is $M_T = 10.25\ kg,$ distributed as $M_s = 7.85\ kg$ and $M_c = 2.4\ k$g, where subindexes 's' and 'c' indicate steel and concrete respectively.

The partial and the total squared lengths estimated with Eq. (12) are shown in Table 8. The partial lengths and the total length are related by the equation [2]:

$$L_\psi^2 = \frac{V_s L_{\psi_s}^2 + V_c L_{\psi_c}^2}{V_T} \tag{1}$$

In this case $V_s = V_c = V_T/2$ and Eq. (22) leads to:

$$L_\psi^2 = \frac{L_{\psi_s}^2 + L_{\psi_c}^2}{2} \tag{2}$$

The lengths of the mode shapes were calculated with models using size elements of 5 and 10 mm, and then extrapolated to zero size (see Table 8).

On the other hand the apparent mass is given by:

$$M_{ap} = V_T \frac{\rho_s L_{\psi_s}^2 + \rho_c L_{\psi_c}^2}{L_{\psi_s}^2 + L_{\psi_c}^2} \tag{3}$$

With respect to the modal masses of the structure, they can be calculated as the sum of the contributions of the steel and the concrete parts by:

$$m = M_s L_{\psi_s}^2 + M_c L_{\psi_c}^2 \tag{4}$$

The modal masses and the apparent masses calculated with Eqs. (25) and (24) are shown in Table 9.

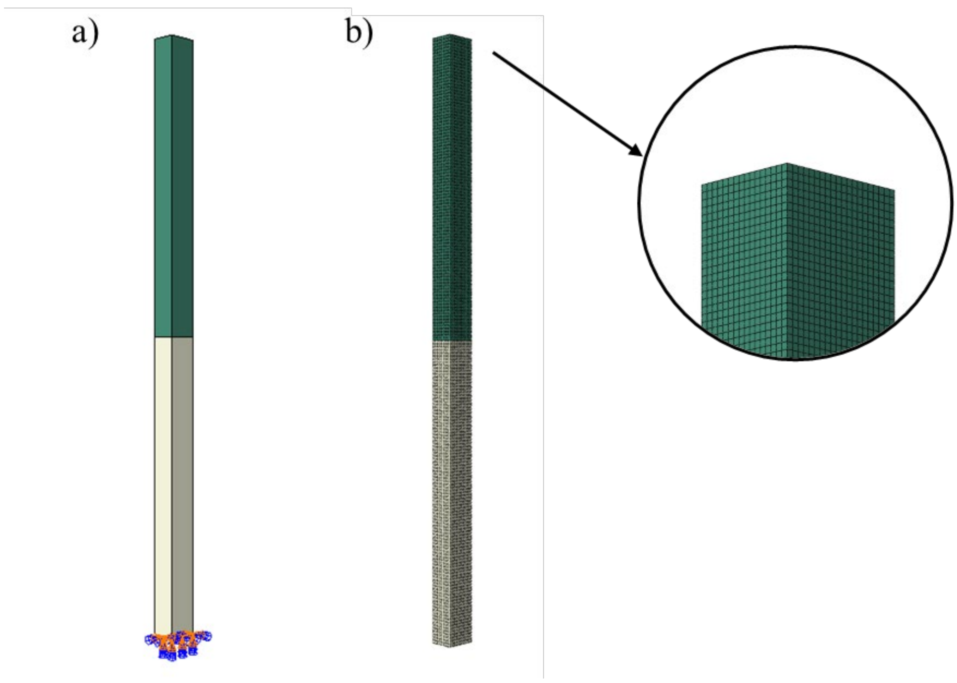

Figure 4. a) 3D concrete-steel model; b) mesh with element size of 0.0025 m

Table 8. Natural frequencies, modal masses and length of the concrete-steel the cantilever beam.

	Mode	Natural frequencies [Hz]	Modal mass	Length		
				Steel $L^2_{\psi_s}$	Concrete $L^2_{\psi_C}$	Total L^2_{ψ}
1	1st Bending Y	47.81	0.93	0.0089	0.3583	0.1836
2	1st Bending X	59.66	0.93	0.0089	0.3583	0.1836
3	2nd Bending Y	148.44	1.32	0.0879	0.2625	0.1752
4	2nd Bending X	184.83	1.32	0.0892	0.2625	0.1758
5	3rd Bending Y	440.37	0.90	0.0382	0.2542	0.1462
6	3rd Bending X	542.98	0.91	0.0382	0.2583	0.1483
7	1st Torsion	743.98	0.79	0.0073	0.3076	0.1705
8	3rd Bending Y	798.21	1.56	0.1108	0.2917	0.2012

Table 9. Contribution of the steel and concrete parts to the modal mass. Apparent mass.

	Mode	Modal mass			Apparent mass
		Steel $M_{Ts}L^2_s$	**Concrete** $M_{Tc}L^2_c$	$M_{Ts}L^2_s + M_{Tc}L^2_c$	
1	1st Bending Y	0.066	0.863	0.929	5.05
2	1st Bending X	0.066	0.864	0.930	5.05
3	2nd Bending Y	0.692	0.630	1.322	7.54
4	2nd Bending X	0.697	0.633	1.330	7.54
5	3rd Bending Y	0.295	0.606	0.901	6.21
6	3rd Bending X	0.300	0.617	0.917	6.21
7	1st Torsion	0.057	0.74	0.796	5.04
8	3rd Bending Y	0.865	0.698	1.564	7.79

4. CONCLUSIONS

In constant mass-density systems, the modal mass is equal to the product between the total mass of the structure and the length squared. If the mass-density is not constant, the modal mass is equal to the product between an apparent mass (different for each mode) and the length squared. The length of the mode shapes can be useful to validate experimental modal masses and to know how the mass is distributed in the structure. The experimental mode shapes can be expanded to the unmeasured DOF's using a numerical model, and the length can be estimated using eq.(16). Alternatively, the length of the experimental mode shapes can be estimated using the transformation matrix **T** and the length of the numerical mode shapes,.

In this paper, the accuracy obtained in the length of the numerical mode shapes of two cantilever structures, one with constant mass-density and the second one made of steel and concrete, is analyzed. The length of numerical mode shapes can be estimated with a good accuracy using eq.(12) if small 3D elements of equal size are used to mesh the model.

Both cantilever models were meshed with 3D elements C3D20R, and the lengths of the mode shapes were estimated with eq. (12). The results corresponding to models of different size, were linearly extrapolated to zero size, allowing to estimate the length of all the modes with a very good accuracy.

The structures were also meshed with beam elements B32 and the length of the mode shapes were calculated with eq.(12) considering only the translational DOF's (the contribution of the rotations can be neglected if the elements are small). It has been demonstrated that beam models can be used successfully to estimate the length of the mode shapes with low computational cost. Beam elements can also be used to calculate the length of the torsional modes, considering only the rotational DOF's.

ACKNOWLEDGEMENTS

The financing support given by the Spanish Ministry of Education through the project MCI-20-PID2019-105593GB-I00/AEI/10.13039/501100011033 is gratefully appreciated.

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