Universidad de Oviedo

Master's thesis

Master's in Advanced Physics: Particles, Astrophysics, Nanophysics and Quantum Materials

Dynamic translation breaking

Sergio Iglesias Noval

Supervisor: Daniele Musso

2023

Call: July

Contents

1 Introduction

The concept of symmetry is central in Physics due to the amount of information of a theory that can be extracted just by studying the symmetries it enjoys. Among this information we can find for instance conserved quantities, whose existence is guaranteed always that the theory has a symmetry (provided that this symmetry is continuous so that we can apply Noether's theorem as we will see). This powerful relation between symmetries and conserved quantities could be itself a reason why the study of symmetries is so relevant because once we know which quantities are conserved in a system, the study of such a system becomes much easier. Moreover, symmetries provide strong constraints when it comes to formulating phenomenological theories.

Another relevant concept in Physics is the breaking of symmetries. Symmetry breaking has been studied systematically in several fields (in fact, it was first developed in the context of Condensed Matter Physics) from the past century. One of the main results of the study of symmetry breaking is Goldstone's theorem, which states that whenever there is a spontaneously broken symmetry, there must be a massless mode. This theorem is still a resourceful topic to study. In fact, its generalizations, for instance in curved spacetime, and the implication they have are a possible direction of research. More generalizations include non-relativistic theories and spacetime symmetries. Other directions of research are related with counting the number of modes that arise when a symmetry is broken or stating the interactions between these modes and other massless modes. Spontaneous symmetry breaking is a cornerstone of the Standard Model of Particle Physics because it explains the mechanism that provides mass to the particles, the Higgs mechanism. However, note that Goldstone's theorem can not be applied to this case since Higgs mechanism deals with the spontaneous breaking of gauge symmetries.

The objective of this thesis is the study of the effective theory that arises from the spontaneous symmetry breaking of translations in some specific models. The study includes the understanding of the dispersion relation of the modes that emerge when the symmetries are broken. Some of these dispersion relations are exotic and they have potential interest in directions of research, for instance the validity of renormalization in such theories. In this particular theory, the energy is proportional to the product of the transversal and longitudinal momentum. This means that the energy can attain a null value even if one component of the momentum takes a very high (but finite) value. This fact makes the limit between the regimes of high and low energy/momentum diffuse. We will obtain these results and discuss them in section 6.

Another objective is to make clear that spontaneous symmetry breaking and the associated formalism is relevant beyond High Energy Physics. In particular, we present an example that possesses a huge relevance both in the theoretical study and its applications: the superfluid. With this example we show that the methods of spontaneous symmetry breaking can be used to define phenomenological models (for instance, Ginzburg-Landau model) and we also explore the possibility to extend them into finite density (non-zero chemical potential) and finite temperature.

The structure of the thesis is as follows. The first section is devoted to the study of symmetries at its most general level. We introduce the formalism that will be used during the thesis and some notation. We give the definition of symmetry at both classical and quantum level and we state (and proof) the relevant theorems to study symmetries. In particular, at classical level we focus on Noether's theorem and its implications through some examples. During the proof of the theorem we present an example of huge relevance, scale transformations [2]. At quantum level we derive Ward identities. After that we present the different types of symmetries that we can encounter, stressing which kinds are the ones that are interesting in symmetry breaking. Then we recall some aspects of group theory that are relevant when it comes to studying symmetries.

The second section deals with the breaking of symmetries, differentiating between spontaneous and explicit breaking. For completeness, an example of explicit symmetry breaking is provided. In the third section we introduce two versions of Goldstone's theorem, one of them for classical theories and the other for quantum theories, and we provide the proof for both. In the case of the theorem for classical theories we provide two examples, one of them including the counting rule for massless bosons. We end this part by stating some possible directions of research.

The fourth section should be understood as a completeness exercise where we present an example where we turn on a chemical potential and a temperature. The study of a superfluid is included for historical reasons (spontaneous symmetry breaking was first used to deal with these kind of system among others) and to show the broad applications of these formalism and its relevance in contemporary research in fields that are not necessarily High Energy Physics.

We return to zero chemical potential and zero temperature in the fifth section, where we study the breaking of translations. The whole section is developed around the same theoretical model but with some modifications in the parameters of the theory, in particular setting the value of some of them to zero. These modifications will allow us to explore some exotic modes such as fractons, that are currently an active research field in Theoretical Physics or phonons. We will make a connection between the dispersion relation of fractons and the well-known elastic string model used to go from discrete theories to continuous theories.

2 Symmetries

Symmetry is one of the most fundamental and relevant concepts in Physics. The symmetries of a theory will allow us to get information about conserved quantities. This can be achieved via one of the most powerful theorems in Physics, Noether's theorem, but it only applies to classical theories and continuous symmetries as we will see in a few sections. Moreover, while modelling a theory, the symmetries can constrain the form of the Lagrangian since they give information about the form of the interaction and the content of matter of the theory.

2.1 Definition

2.1.1 Classical theories

A general transformation can be written as

$$\begin{aligned} x^\mu &\rightarrow x'^\mu(x^\mu) = x^\mu + \Delta x^\mu, \\ \phi(x) &\rightarrow \phi'(x') = \phi(x) + \Delta\phi(x). \end{aligned} \tag{2.1}$$

We say that a transformation is a symmetry if it leaves the action invariant. The mathematical condition for a transformation to be a symmetry transformation is then

$$S[\phi] = S[\phi'], \tag{2.2}$$

where $S[\phi]$ is the action in terms of the fields before the transformation is carried out and $S[\phi']$ is the action after the transformation. Note that the condition is imposed on the action and not on the Lagrangian density, meaning that the Lagrangian density can vary up to a total derivative.

If the considered transformation is infinitesimal, we can express the transformation of the fields as

$$\phi'(x') = \phi'(x + \Delta x) \approx \phi'(x) + [\partial_\mu \phi'(x)]\, \Delta x^\mu. \tag{2.3}$$

The variation of the field is defined as $\delta\phi(x) = \phi'(x') - \phi(x)$. Note that we are referring only to the variation of the field induced by the transformation itself and not by the transformation of the coordinates. With this we get the transformation at first order,

$$\phi'(x) \approx \delta\phi(x) + [\partial_\mu \phi'(x)]\, \Delta x^\mu \approx \phi(x) + \delta\phi(x) + [\partial_\mu \phi(x)]\, \Delta x^\mu, \tag{2.4}$$

where we have neglected the term $[\partial_\mu \phi'(x)]\, \Delta x^\mu$ because it is of higher order. The total field transformation is then

$$\Delta\phi(x) = \delta\phi(x) + [\partial_\mu \phi(x)]\, \Delta x^\mu. \tag{2.5}$$

The variation of the action is

$$\Delta S = S\left[\phi'\right] - S\left[\phi\right] = \int d^4x' \mathcal{L}'(x') - \int d^4x \mathcal{L}(x)\,. \tag{2.6}$$

In order to write the variation at first order in the parameter of the transformation we have to discuss the transformation of the integration measure. Given the transformation of the coordinates in (2.1), the integration measure in terms of the transformed coordinates is given by

$$d^4x' = \left| \mathrm{Det}\left(\frac{\partial x'^\alpha}{\partial x^\beta}\right)\right| = |\mathrm{Det}(\delta^\alpha_\beta + \partial_\beta \Delta x^\alpha)| d^4x. \tag{2.7}$$

Now using the relation between the determinant of a generic matrix and its trace

$$\mathrm{Det}(e^B) = e^{\mathrm{Tr}(B)}, \tag{2.8}$$

and defining $A = e^B$ we can deduce

$$\log\left[\mathrm{Det}(A)\right] = \mathrm{Tr}\left[\log(A)\right]. \tag{2.9}$$

If the matrix A can be written as $A = \mathbb{I} + \epsilon M$ where $\mathbb{I}$ is the identity matrix and $1 \gg \epsilon\ \mathrm{Max}|M_{ij}|$, then

$$\log\left[\mathrm{Det}(1+\epsilon M)\right] = \mathrm{Tr}\left[\log(1+\epsilon M)\right] = \mathrm{Tr}\left[\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n}(\epsilon M)^n\right] \simeq \epsilon \mathrm{Tr}(M), \tag{2.10}$$

which in our case gives

$$\mathrm{Det}(\delta^\alpha_\beta + \partial_\beta \Delta x^\beta) \simeq 1 + \partial_\alpha \Delta x^\alpha. \tag{2.11}$$

Inserting this approximate relation and

$$\mathcal{L}'(x') = \mathcal{L}(x) + \Delta\mathcal{L}(x) = \mathcal{L}(x) + \delta\mathcal{L}(x) + \left[\partial_\mu \mathcal{L}(x)\right]\Delta x^\mu \tag{2.12}$$

in (2.6) we finally get

$$\Delta S \simeq \int d^4x \{\delta\mathcal{L}(x) + \partial_\mu\left[\mathcal{L}(x)\Delta x^\mu\right]\}, \tag{2.13}$$

where we have defined the variation of the Lagrangian as

$$\delta\mathcal{L} = \frac{\delta\mathcal{L}}{\delta\phi}\delta\phi + \frac{\delta\mathcal{L}}{\delta(\partial_\mu\phi)}\delta(\partial_\mu\phi) = \left[\frac{\delta\mathcal{L}}{\delta\phi} - \partial_\mu\left(\frac{\delta\mathcal{L}}{\delta(\partial_\mu\phi)}\right)\right]\delta\phi + \partial_\mu\left(\frac{\delta\mathcal{L}}{\delta(\partial_\mu\phi)}\delta\phi\right), \tag{2.14}$$

and we have made use of the fact that the variation commutes with the partial derivative, that is $\delta(\partial_\mu \phi) = \partial_\mu(\delta\phi)$. Note that the term in square brackets vanishes if the field satisfies the equations of motion. With this, we find that the variation of the action to first order can be written as

$$\Delta S = \int d^4x \partial_\mu \left[\frac{\delta \mathcal{L}}{\delta(\partial_\mu \phi)} \delta\phi + \mathcal{L} \Delta x^\mu \right], \tag{2.15}$$

so the variation is given by the integral over the whole spacetime of the 4-divergence of

$$j^\mu \equiv \frac{\delta \mathcal{L}}{\delta(\partial_\mu \phi)} \delta\phi + \mathcal{L} \Delta x^\mu, \tag{2.16}$$

which receives the name of current. This current can also be written as

$$j^\mu \equiv \frac{\delta \mathcal{L}}{\delta(\partial_\mu \phi)} \Delta\phi + \left[\delta^\mu_\nu \mathcal{L}(x) - \frac{\delta \mathcal{L}}{\delta(\partial_\mu \phi)} \partial_\nu \phi \right] \Delta x^\nu. \tag{2.17}$$

2.1.2 Quantum theories

As in the classic formalism, we have to distinguish between symmetries of the action and symmetries of the states. To treat the symmetries of quantum theories in this section we will use the Hamiltonian formalism even if there is a more powerful formalism called path integral that we will use and explain later. If we take into account the connection between the Hamiltonian and the Lagrangian via a Legendre transformation, we can see that there is a connection between the Hamiltonian and the action. We say that a state $|\psi\rangle$ is symmetric under a unitary transformation U if

$$U |\psi\rangle = e^{i\varphi} |\psi\rangle, \tag{2.18}$$

where the exponential is a global phase that does not affect the state. This is true because experimentally we only have access to probabilities and $|\langle\phi|\psi\rangle|^2 = |\langle\phi|\psi'\rangle|^2$ where $|\psi'\rangle = e^{i\alpha}|\psi\rangle$ and $\langle\phi|$ is an arbitrary bra. Taking into account this fact, we can classify physical states in equivalence classes of vectors stating that two kets $|\psi\rangle$, $|\psi'\rangle$ are of the same class if $|\psi'\rangle = e^{i\alpha}|\psi\rangle$. We can denote this classes of equivalence as $[|\psi\rangle]$ where $|\psi\rangle$ is said to be the representative of the class. These classes of equivalence receive the name of rays. This can be regarded as a relaxation of the concept of vector because it states that global phases do not have any physical meaning.

Theorem 1. *(Wigner's theorem). Consider a group G of transformations acting on physical states as $[|\psi\rangle] \to [|\psi_g\rangle]$ and suppose that the probabilities are invariant under the action of $g \in G$ for rays $[|\psi\rangle], [|\chi\rangle]$, that is, $|[\langle\psi|][|\chi\rangle]|^2 = |[\langle\psi_g|][|\chi_g\rangle]|^2$. Under these conditions, it is possible to choose representatives $|\psi\rangle \,\forall\, [|\psi\rangle]$ such that $|\psi_g\rangle = U(g)|\psi\rangle$ where $U(g)$ is a unitary (or antiunitary) operator that is unique up to a global phase.*

Recall that an operator U is antiunitary if $\langle\psi| U^\dagger U |\chi\rangle = \langle\psi|\chi\rangle^*$. A consequence of this theorem is that transformations of rays can be seen as transformations of vectors by applying the corresponding operator $U(g)$ which only depends on the transformation.

We say that a general operator $\hat{A}$ is invariant under a transformation U if

$$\hat{U}^{\dagger}\hat{A}\hat{U} = \hat{A}. \tag{2.19}$$

This condition can be expressed in terms of the commutator as $[\hat{A}, \hat{U}] = 0$. We can check that both expressions are equivalent multiplying the commutator by the left by $\hat{U}^{\dagger}$. We say that $\hat{U}$ is a symmetry of the Hamiltonian $\hat{H}$ if $[\hat{H}, \hat{U}] = 0$

2.2 Conservation laws at a classical level

Given a classical theory, we say that a quantity is conserved if its value does not change with time. Every time there is a conservation law, the system must possess a symmetry according to Noether's theorem. For instance, in any system that is invariant under time translation the total energy is conserved as we will see.

2.2.1 Noether's current and continuity equations

As we have already mentioned, this theorem is valid only for continuous symmetries. Each continuous symmetry is associated with a current $j^{\mu}(t, x)$ that satisfies a continuity equation

$$\partial_{\mu} j^{\mu}(t, x) = 0. \tag{2.20}$$

This relation is the statement of Noether's theorem and we will prove it in the next subsection. Starting from the current we can obtain a conserved global quantity called Noether's charge and defined as

$$Q(t) = \int d^D x \, j^0(t, x), \tag{2.21}$$

where the integration is performed over the spatial manifold and $j^0(t, x)$ is the time component of the current. We can prove the conservation of this charge as follows

$$0 = \int d^D x \partial_{\mu} j^{\mu}(t, x) = \int d^D x \partial_t \rho(t, x) + \int d^D x \partial_i j^i(t, x) = \partial_t Q(t) + \oint dS_i j^i(t, x),$$

where we have used Gauss' theorem in the last step and we have defined $\rho(t, x) \equiv j^0(t, x)$. Considering that the current decreases to zero in the infinity, which means that the charge is contained within the whole space and there is no energy transfer through the boundaries of the space, the last term vanishes and we are left with the expression we wanted to proof. This connection between symmetries and continuity equations is stated by Noether's theorem.

The conserved charge is said to be the generator of the symmetry. Considering (2.1) and defining the variation of the field as $\delta\phi \equiv \phi'(t) - \phi(t)$, we can obtain the relation

$$\Delta\phi^i = \phi'^i(t') - \phi'^i(t') + \phi'^i(t') - \phi^i(t) \simeq \dot{\phi}'^i \Delta t + \delta\phi^i \simeq \dot{\phi}^i \Delta t + \delta\phi^i, \tag{2.22}$$

where $\dot{\phi}^i$ denotes the time derivative of ϕ^i. Taking this into account we can express the charge as

$$Q = \frac{\delta \mathcal{L}}{\delta \dot{\phi}^i} \Delta \phi^i - \left(\frac{\delta \mathcal{L}}{\delta \dot{\phi}^i} \dot{\phi}^i - \mathcal{L} \right) \Delta t. \tag{2.23}$$

In order to clarify what we understand by generator, we will identify the Hamiltonian as the generator of time evolution and afterwards, we will follow the same steps to identify the charge as the generator of a symmetry. Given a function only of the coordinates and the momentum $F(\phi^i, \pi_i)$ (note that we have excluded specifically any possible time dependence of the function), its evolution in time will be given by the Poisson bracket

$$\frac{\mathrm{d}}{\mathrm{d}t} F(\phi^i, \pi_i) = \{F, H\} = \frac{\delta F}{\delta \phi^i} \frac{\delta H}{\delta \pi_i} - \frac{\delta H}{\delta \phi^i} \frac{\delta F}{\delta \pi_i}. \tag{2.24}$$

This relation is where we identify the Hamiltonian as the generator of time evolution. Assuming now a transformation that affects only the fields, that is $\Delta t = 0$ in (2.22), we can express the charge as written in (2.23) as

$$Q = \frac{\delta \mathcal{L}}{\delta \dot{\phi}^i} \Delta \phi^i = \pi_i \Delta \phi^i, \tag{2.25}$$

were we have substituted the definition of canonical conjugate momentum. Now if we assume that the fields do not depend explicitly on the canonical conjugate momentum and compute the Poisson bracket,

$$\{\phi^i, Q\} = \Delta \phi^i, \tag{2.26}$$

from where we identify the conserved charge as the generator of the symmetry in the same sense that we say the Hamiltonian is the generator of time evolution.

2.2.2 Proof of the theorem

We want to study how the action

$$S = \int d^d x \mathcal{L}(\phi, \partial_\mu \phi), \tag{2.27}$$

is affected by a continuous transformation

$$\begin{aligned} x &\rightarrow x'(x) \\ \phi(x) &\rightarrow \phi'(x'). \end{aligned} \tag{2.28}$$

Note that the field transforms in two ways. Since it depends on coordinates and they are transformed, this will induce a transformation on the field, but the field can also by transformed directly by the transformation itself. This means that we can write the transformed field as

$$\phi(x) \rightarrow \phi'(x') = F(\phi(x)), \tag{2.29}$$

so we can express the transformed action as

$$S' = \int d^d x \left| \frac{\partial x'}{\partial x} \right| \mathcal{L} \left[F(\phi(x)), \frac{\partial x^\nu}{\partial x'^\mu} \partial_\nu F(\phi(x)) \right]. \tag{2.30}$$

As an example we can consider scale transformations. That is, we will perform the transformation

$$\begin{aligned} x' &= \lambda x \\ \phi'(\lambda x) &= \lambda^{-\Delta} \phi(x), \end{aligned} \tag{2.31}$$

where λ is the dilation factor and Δ is the scaling dimension of the field. The determinant of the Jacobian matrix is λ^d, so the action transforms into

$$S' = \lambda^d \int d^d x \mathcal{L}(\lambda^{-\Delta} \phi, \lambda^{-1-\Delta} \partial_\mu \phi). \tag{2.32}$$

In order to prove the theorem we consider

$$\begin{aligned} x'^\mu &= x^\mu + \frac{\delta x^\mu}{\delta \omega_a}, \\ \phi'(x') &= \phi(x) + \omega_a \frac{\delta F(x)}{\delta \omega_a}, \end{aligned} \tag{2.33}$$

where ω_a is a collection of N $(N = 1, \dots, a)$ infinitesimal parameters that we will keep only to first order. We can define the generator of the transformation G_a as

$$\delta_\omega \phi \equiv \phi'(x) - \phi(x) = -i \omega_a G_a(x). \tag{2.34}$$

Its expression in terms of variations can be found from the transformation of the field and it can be written as

$$i G_a \phi = \frac{\delta x^\mu}{\delta \omega_a} \partial_\mu \phi - \frac{\delta F}{\delta \omega_a}. \tag{2.35}$$

In order to compute the variation of the action under this transformation we have to compute the determinant of the Jacobian. We will use the property $\det(1+E) \simeq 1 + \mathrm{Tr}(E)$, $E \ll 1$ proven in (2.10). With this, the Jacobian and its determinant are given by

$$\frac{\partial x'^\nu}{\partial x^\mu} = \delta^\nu_\mu + \partial_\mu \left(\omega_a \frac{\delta x^\nu}{\delta \omega_a} \right) \Rightarrow \left| \frac{\partial x'^\nu}{\partial x^\mu} \right| \approx 1 + \partial_\mu \left(\omega_a \frac{\delta x^\nu}{\delta \omega_a} \right), \tag{2.36}$$

The action after the transformation is then

$$S' = \int d^d x \left[1 + \partial_\mu \left(\omega_a \frac{\delta x^\mu}{\delta \omega_a}\right)\right] \mathcal{L}\left[\phi + \omega_b \frac{\delta F}{\delta \omega_b}, \left(\delta^\nu_\mu - \partial_\mu \left(\omega_c \frac{\delta x^\nu}{\delta \omega_c}\right)\right)\left(\partial_\nu \phi + \partial_\nu \left(\frac{\delta F}{\delta \omega_d}\right)\omega_d\right)\right], \tag{2.37}$$

where a, b, c, d are dummy indices that run through the number of infinitesimal parameters ω_a. The variation of the action, which has already been defined as $\delta S = S[\phi'] - S[\phi]$ can be expressed as

$$\delta S = -\int dx j^\mu_a \partial_\mu \omega_a, \tag{2.38}$$

where we have assumed that the equations of motion hold, which implies that the variation of the fields and the fields themselves vanish at the boundary. We also assume that the transformation in the coordinates and the fields, that is the second terms in (2.33) do not depend on the coordinates and then its derivative is zero. With all this, the current associated to the transformation is defined as

$$j^\mu_a = \left[\frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)} \partial_\nu \phi - \delta^\mu_\nu \mathcal{L}\right] \frac{\delta x^\nu}{\delta \omega_a} - \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)} \frac{\delta F}{\delta \omega_a}. \tag{2.39}$$

We are under the hypothesis that the equations of motion hold, hence the variation of the action under general transformations of the field will be zero. After integrating (2.38) by parts and using (2.39) we find a condition on the current,

$$\partial_\mu j^\mu_a = 0. \tag{2.40}$$

thus proving the theorem. Note that the current is not uniquely defined since the addition of the derivative of an anti-symmetric tensor would not affect to the result in (2.40). Another relevant observation to take into account is that this theorem is only valid for classical theories and continuous symmetries, so it can not be applied neither to the study of symmetries in quantum theories nor to the study of discrete symmetries.

2.2.3 Adding a non-symmetric term

Now we can add a term that does not have the symmetry considered, that is

$$S = S_{\text{sym}} + S_b = \int d^d x \mathcal{L}_{\text{sym}} + \int d^d x B, \tag{2.41}$$

where S_{sym} is the part of the action that has the symmetry and leads to (2.40) and S_b is the part of the action that is not invariant under the symmetry, being B the term that is not invariant. The variation of this term under the symmetry is given by

$$\delta B = \omega_a \frac{\delta B}{\delta \omega_a} = \omega_a \frac{\delta B}{\delta \phi} \frac{\delta \phi}{\delta \omega_a}, \tag{2.42}$$

where we have assumed that the field content of the theory is just a real scalar field (it can be generalized to other cases) and that the term B does not depend on derivatives of the field. The equations of motion derived from (2.41) are

$$\partial_\mu \left(\frac{\delta \mathcal{L}}{\delta \partial_\mu \phi} \right) - \frac{\delta \mathcal{L}}{\delta \phi} = \partial_\mu \frac{\delta \mathcal{L}_{\text{sym}}}{\delta \partial_\mu \phi} - \frac{\delta \mathcal{L}_{\text{sym}}}{\delta \phi} - \frac{\delta B}{\delta \phi} = 0. \tag{2.43}$$

Now, the variation of the invariant part with respect to the parameter of the transformation vanishes, that is,

$$\delta S_{\text{sym}} = \int d^d x \left[\frac{\delta \mathcal{L}_{\text{sym}}}{\delta \partial_\mu \phi} \delta \partial_\mu \phi + \frac{\delta \mathcal{L}_{\text{sym}}}{\delta \phi} \delta \phi \right] = \int d^d x \left[\frac{\delta \mathcal{L}_{\text{sym}}}{\delta \partial_\mu \phi} \omega_a \partial_\mu \frac{\delta \phi}{\delta \omega_a} + \frac{\delta \mathcal{L}_{\text{sym}}}{\delta \phi} \omega_a \frac{\delta \phi}{\delta \omega_a} \right] = 0. \tag{2.44}$$

Integrating by parts and introducing the expression of the conserved current we get

$$\delta S_{\text{sym}} = - \int d^d x \left[\omega_a \left(\partial_\mu j^\mu + \frac{\delta B}{\delta \omega} \right) \right] = 0. \tag{2.45}$$

Since the parameters ω_a can have any arbitrary dependence on the coordinates and the integral must be zero, we obtain that the expression for divergence of the current is

$$\partial_\mu j^\mu = - \frac{\delta B}{\delta \omega_a}. \tag{2.46}$$

2.2.4 An example of Noether's currents and charge: spacetime translations

We are considering a transformation on the coordinates. This transformation can be expressed as

$$x^\mu \to x'^\mu = x^\mu + \alpha^\mu, \tag{2.47}$$

which is a particular case of (2.1) if $\delta\phi = 0$ under translations. As we have already mentioned, the transformation on the coordinates induces a transformation on the fields,

$$\phi(x^\mu) \to \phi(x'^\mu) = \phi(x^\mu) + \alpha^\nu \partial_\nu \phi(x^\mu) + \mathcal{O}(\alpha^2). \tag{2.48}$$

The variation of the Lagrangian under this symmetry transformation is given by

$$\Delta_\mu \mathcal{L} = \frac{\partial \mathcal{L}}{\partial \phi} \Delta_\mu \phi + \frac{\partial \mathcal{L}}{\partial(\partial_\nu \phi)} \Delta_\mu (\partial_\mu \phi) = \frac{\partial \mathcal{L}}{\partial \phi} \partial_\mu \phi + \frac{\partial \mathcal{L}}{\partial(\partial_\nu \phi)} \partial_\mu \partial_\nu \phi = \partial_\mu \mathcal{L}, \tag{2.49}$$

where we have defined $\Delta_\mu \phi = \partial_\mu \phi$. The variation of the Lagrangian can be expressed as $\delta \mathcal{L} = \mathcal{L}' - \mathcal{L} = \partial_\mu (\alpha^\mu \mathcal{L}) = \partial_\mu K^\mu$ where we have made use of the relation $\alpha^\mu \Delta_\mu \mathcal{L} = \delta \mathcal{L}$. Recalling the definition of the canonical conjugate momenta we can express the conserved current as

$$j^\mu_\nu = \pi^\mu \Delta_\nu \phi - \delta^\mu_\nu \mathcal{L} = \pi^\mu \partial_\nu \phi - \delta^\mu_\nu \mathcal{L}. \tag{2.50}$$

The conserved current in this case is a very important quantity in Physics and it receives the name of energy-momentum tensor or stress-energy tensor and it is represented by $T^{\mu\nu}$. It can be rewritten as

$$T^{\mu\nu} = \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)} \partial^\nu \phi - \delta^{\mu\nu} \mathcal{L}. \tag{2.51}$$

The charge can be obtained as the integral over the spatial manifold of the temporal component of the current. In this case the current is a tensor, so if we fix one of its indices to be the temporal component, we will have another free index. This means that the charge will be a vector of $D+1$ components with these components given by

$$Q_0 = \int d^D x j_0^0 = \int d^D x (\pi^0 \partial_0 \phi - \mathcal{L}) = \int d^D x \mathcal{H} = H, \tag{2.52}$$

$$Q_i = \int d^D x\, j_i^0 = \int d^D x\, \pi^0 \partial_i \phi. \tag{2.53}$$

The conserved charge associated with time translations (2.52) is the Hamiltonian, which is the total energy operator. In the case of space translations (2.53), the conserved charge is the total momentum associated to the field ϕ.

2.3 Conservation laws at a quantum level

The evolution of any time-independent quantum system, that is any system whose Hamiltonian does not depend on time, is given by its time-evolution operator,

$$\mathcal{U}(t) = e^{iHt}, \tag{2.54}$$

where H is the Hamiltonian of the theory. We have already seen that symmetry transformations commute with the Hamiltonian, so they will also commute with the time-evolution operator. Since symmetry transformations are a special case of transformations and all transformations are unitary (or anti-unitary) we can write them as

$$U = e^{iQ}, \tag{2.55}$$

for some observable Q. This exponential can be expressed as a power series of the operator,

$$e^{iQ} = \sum_{n=0}^{\infty} \frac{1}{n!} (iQ)^n, \tag{2.56}$$

so clearly if Q commutes with the time-evolution operator, U commutes with $\mathcal{U}(t)$. Unitary symmetries U correspond to observables Q. This observables are the conserved quantities. The expectation value of these observables with respect to a general state $|\psi\rangle$ is time independent,

$$\langle\psi(t)| Q |\psi(t)\rangle = \left(\langle\psi| \mathcal{U}^\dagger(t)\right) Q \left(\mathcal{U}(t) |\psi\rangle\right) = \langle\psi| Q |\psi\rangle , \tag{2.57}$$

where we have taken into account that Q commutes with $\mathcal{U}(t)$ and that $\mathcal{U}^\dagger(t)\mathcal{U}(t) = \mathbb{I}$. Eq. (2.57) can be understood as the conservation law of the observable Q. We can also see that the observable Q is conserved in Heisenberg picture. Heisenberg equation in natural units reads

$$\frac{\mathrm{d}}{\mathrm{d}t} Q_H(t) = i \left[H_H, Q_H(t)\right] , \tag{2.58}$$

where $Q_H(t)$ indicates that we are working in Heisenberg picture. The same equation holds if we write expected values,

$$\frac{\mathrm{d}}{\mathrm{d}t} \langle Q_H(t)\rangle = i \langle [H_H, Q_H(t)]\rangle , \tag{2.59}$$

and since Q is a symmetry, it commutes with the Hamiltonian, so the right hand side is identically zero, thus obtaining the conservation law for Q,

$$\frac{\mathrm{d}}{\mathrm{d}t} \langle Q_H(t)\rangle = 0. \tag{2.60}$$

2.3.1 Ward identities

Even if Noether's theorem is only valid classically and it can not be applied to the study of symmetries in quantum theories, classical symmetries give constraints to correlation functions. If the path integration measure does not have this symmetry, we will say that there is an anomaly.

Ward identities are a family (in fact, an infinite family) of relations between correlators due to the symmetries. In the path integral formalism, the expected value of an operator X is

$$\langle X\rangle = \int \mathcal{D}\phi X e^{iS}. \tag{2.61}$$

Assuming that there are no anomalies and that the transformation is a symmetry, we have

$$\int \mathcal{D}\phi \, i\delta S e^{iS} = i\langle\delta S\rangle = 0. \tag{2.62}$$

If we consider now that the parameter $\omega(x)$ depends on spacetime the total variation of the action can be written as

$$\delta S = \delta S_{\text{sym}} + \delta S_b = -\int d^d x j^\mu \partial_\mu \omega + \int d^d x \, \omega \frac{\delta B}{\delta\omega}. \tag{2.63}$$

Integrating by parts we obtain

$$\delta S = \int d^d x \omega \partial_\mu j^\mu + \int d^d x \omega \frac{\delta B}{\delta \omega}. \tag{2.64}$$

Substituting this expression into (2.62) we obtain one type of Ward identity

$$\langle \partial_\mu j^\mu \rangle = -\Big\langle \frac{\delta B}{\delta \omega} \Big\rangle. \tag{2.65}$$

In particular, this identity can be regarded as the quantum interpretation of the relation (2.46). We have been able to do this because we have split the action into a symmetric and a non-symmetric part. This indicates that the analysis of a transformation is relevant even if this transformation is not a symmetry of the whole action.

2.3.2 Some remarks on generators

We have already studied generators in classical theories. Now we study the same concept in quantum theories. We define the canonical conjugate momentum $\pi_a(x)$ as

$$\pi_a(x) = \frac{\partial \mathcal{L}}{\partial(\partial_t \phi_a)}. \tag{2.66}$$

For quantum fields, the canonical momentum and the field verify the canonical commutation rules $[\phi_a(x), \pi_b(y)] = i\delta_{ab}\delta(x-y)$. If we assume now that the Lagrangian is invariant[1] under the transformation whose associated conserved charge is Q, we can write the commutator between the charge and the field as

$$[i\alpha Q, \phi_a(x)] = i\alpha \int d^D y \, [\pi_b(y)\Delta\phi_b(y), \phi_a(x)] = i\alpha \int d^D y \, [\pi_b(y), \phi_a(x)] \, \Delta\phi_b(y) = \delta\phi_a(x), \tag{2.67}$$

where we have defined $\alpha\Delta\phi_a(x) = \delta\phi_a(x)$ and integrated over the spatial manifold. Note that the commutator of the charge and the field gives the variation of the field. The conserved charge is referred to as the generator of the symmetry due to this relation. Note that this identification is analogous to the one made in classical theories changing the commutator by a Poisson bracket. We can write the transformation of the fields in terms of the generator by expanding the transformation to first order in the parameter,

$$\phi_a(x) \to \phi'_a(x) \approx \phi_a(x) + i\alpha[Q, \phi_a(x)] = e^{i\alpha Q}\phi_a(x)e^{-i\alpha Q} + \mathcal{O}(\alpha^2). \tag{2.68}$$

It can be shown that this result is also valid in the case where the Lagrangian is not invariant under the transformation. However, recall that the Lagrangian can vary only up to a term containing a total derivative.

[1]Note that now the Lagrangian is invariant and not the action. This means that after the transformation there is not any new term that is a total derivative.

2.4 Types of symmetries

Until this point, we only have made a distinction between symmetries regarding if they depended on a continuous parameter or not (that is, between continuous and discrete symmetries). From this point, we will need to distinguish the type of symmetry that the system has.

We have already seen the difference between continuous and discrete symmetries. We say that a symmetry is continuous if it depends on a parameter that takes a continuous value. For instance, rotational symmetry is characterized by only one parameter, the rotation angle. This angle can take any value from 0 to 2π rad, hence this symmetry is continuous. If a symmetry is discrete, the parameter can only take discrete values. An example of discrete symmetry is parity, that changes the sign of the spatial components of a 4-vector.

We say that a symmetry is global if it depends on a parameter that is not a function of space-time. An example of global symmetry would be a translation in space by a constant amount $x^\mu \to x'^\mu = x^\mu + \alpha^\mu$, where α^μ is a constant. Global symmetries are symmetries that have a current associated, unlike local symmetries (also called gauge symmetries/gauge redundancies), which do not have it. We will deepen in the description of local symmetries in the next section, but for now is enough with taking into account that they are not real symmetries in the sense that they do not have an associated current and they arise from the way the system is defined.

A symmetry is said to be internal if its generators commute with the generators of Poincaré group, that is translations, rotations and boosts. In (2.1) an internal symmetry would have $\Delta x^\mu = 0$. An example of internal symmetry is $SO(2)$. The opposite case of an internal symmetry is an external or space-time symmetry. These are symmetries that transform the coordinates too and not only the fields. Among these kind of symmetries we find rotations and boosts.

2.5 Symmetry groups and Lie algebra

Before discussing gauge symmetries we recall some relevant topics of the mathematical formalism used to study symmetries in general. This formalism is group theory and it is quite obvious that it has to be if we take a look at the following properties of symmetries:

- The consecutive action of two symmetry transformations is a symmetry transformation that can be regarded as the ordered product of the two previous transformations.
- There exists an identity transformation $\mathcal{I}$.
- We can represent each transformation by an operator U and for each operator U there exists an inverse operator $U^\dagger$ such that $UU^\dagger = U^\dagger U = \mathcal{I}$.

These three properties define the mathematical structure of a group. Groups can be classified in many ways. For instance, there are continuous and discrete groups, defined in a similar way as we defined continuous and discrete transformations. Another relevant distinction can be made regarding the number of elements contained in the group. Groups can be finite or not if they have a finite number of elements or not, respectively. A group G is said to be Abelian if all its elements commute between them, that is $U_1U_2 = U_2U_1, \forall U_1, U_2 \in G$. If the elements in the group do not commute, the group is said to be non-Abelian.

2.5.1 Lie groups

In order to study continuous symmetry transformations, we will usually require the transformation to be continuous and differentiable in the parameters. Such groups receive the name of Lie groups. We define the dimension of Lie groups as the number of parameters needed to parametrise the full set of continuous transformations.

Lie groups contain elements close to the identity, so we can expand these transformations as a power series. Consider a continuous transformation $U_\alpha = \exp(i\alpha_a T_a)$ where T_a is a set of Hermitian operators and α_a is a set of small parameters. This transformation can be expressed as

$$U_\alpha = \mathcal{I} + i\alpha_a T_a + \mathcal{O}(\alpha^2). \tag{2.69}$$

The operators T_a are called the generators of the Lie groups. Given a Lie group, we describe its associated Lie algebra as

$$[T_a, T_b] = i \sum_c f_{abc} T_c, \tag{2.70}$$

where T_i are the generators of the Lie group and f_{abc} are real numbers called structure constants. Note that if the Lie group is Abelian this constants will be zero.

2.5.2 Gauge symmetries

Unlike global symmetries, local symmetries depend on the point of spacetime where we apply them. This is done by making the parameter of the transformation dependent on the spacetime point.

We consider a field theory that is invariant under a Poincaré symmetry and a global internal symmetry whose symmetry group is G_{int}, where the elements of this group do not depend on spacetime. Now, if we make the elements dependent on the spacetime, the field will transform in a covariant way,

$$\phi(x) \to U(x)\phi(x), \tag{2.71}$$

where $U(x) \in G_{\text{int}}$, but the derivative will not transform in this way. Indeed, the transformation of the derivative is

$$\partial_\mu \phi(x) \neq U(x)\partial_\mu \phi(x). \tag{2.72}$$

We can introduce a gauge field to redefine the derivative so it transforms in a covariant way. Let $A_\mu(x)$ denote a gauge field that transform as

$$A_\mu(x) \to U(x)\left[A_\mu(x) + \frac{i}{g}\partial_\mu\right]U^{-1}(x). \quad (2.73)$$

Now we can define a derivative that transforms in a covariant way. This is the covariant derivative and it is defined as

$$D_\mu\phi(x) = \partial_\mu\phi(x) \pm igA_\mu(x)\phi(x), \quad (2.74)$$

where the plus sign correspond to fields that transform in the anti-fundamental representation, that is, they transform as

$$\phi(x) \to \phi(x)U^{-1}(x), \quad (2.75)$$

and the minus sign correspond to fields that transform in the fundamental representation as in (2.71). If there are no anomalies, we can convert a field theory with global symmetries into a theory with local symmetry if we substitute all the derivatives that appear in the action by covariant derivatives.

Until this point, gauge fields $A_\mu(x)$ do not have any dynamics. If we want them to have dynamics, we have to add a kinetic term to the action by hand. To do so, we define the field strength $F_{\mu\nu}$ of the gauge field $A_\mu(X)$ as

$$[D_\mu, D_\nu]\,\phi = -ig(\partial_\mu A_\nu - \partial_\nu A_\mu - ig\,[A_\mu, A_\nu])\phi \equiv -igF_{\mu\nu}\phi. \quad (2.76)$$

The field strength transforms in the adjoint representation, that is,

$$F_{\mu\nu} \to UF_{\mu\nu}U^{-1}. \quad (2.77)$$

In components it can be written as

$$F^a_{\mu\nu} = \partial_\mu A^a_\nu - \partial_\nu A^a_\mu + gf_{bca}A^b_\mu A^c_\nu. \quad (2.78)$$

The field strength satisfies Bianchi identities,

$$D_{[\lambda}F_{\mu\nu]} = 0. \quad (2.79)$$

One could think that $F_{\mu\nu}F^{\mu\nu}$ would be a suitable kinetic term to add to the action, but it is not gauge invariant since it would transform as $F_{\mu\nu}F^{\mu\nu} \to UF_{\mu\nu}F^{\mu\nu}U^{-1}$. This problem can be solved by taking the trace, that is,

$$S_{YM}\,[A] = -\frac{1}{C}\int d^4x\mathrm{Tr}(F_{\mu\nu}F^{\mu\nu})\,. \quad (2.80)$$

This action receives the name of Yang-Mills action. Now making use of the property $\mathrm{Tr}(T_a, T_b) = C\delta_{ab}$ where T_a, T_b are the generators of the symmetry group and the definition of the structure constants $\left[T^a, T^b\right] = f_{abc}T^c$, we can write the action in components as

$$
\begin{aligned}
S_{YM}[A] &= -\frac{1}{4}\int d^4x \left[\partial^\mu A^\nu_a \partial_\mu A^a_\nu - \partial^\nu A^\mu_a \partial_\mu A^a_\nu + 2g f_{bca} A^b_\mu A^c_\nu \partial^\mu A^\nu_a \right. \\
& \left. \frac{1}{2} g^2 f_{efa} f_{bca} A^\mu_e A^\nu_f A^b_\mu A^c_\nu \right], \qquad (2.81)
\end{aligned}
$$

We can see in (2.81) that, apart from the kinetic term that we wanted to add, we have obtained two interaction terms. One of them is a 3 point self-interaction and the other is a 4 point self-interaction. The equations of motion in this case are

$$
D_\mu F^{\mu\nu} = 0, \qquad (2.82)
$$

where we can see that the Yang-Mills action can be used to describe non-Abelian gauge fields in vacuum. Recall that a group is Abelian if all its elements commute between them.

As a particular case of Yang-Mills action we can consider Maxwell's action. This is a Yang-Mills theory whose symmetry group is U(1). Since this group is Abelian, all the structure constants vanish and we can write is action as

$$
S_M[A] = -\frac{1}{4}\int d^4x F_{\mu\nu} F^{\mu\nu}. \qquad (2.83)
$$

The main difference between Maxwell's action and the Yang-Mills action as presented in (2.81) is that Maxwell's action does not contain self-interaction terms. Since the gauge field in this theory is related to the photon, it would mean that the photon does not suffer self-interactions.

3 Symmetry breaking

We have already stated that symmetries are a very relevant resource to study a theory, for instance, through conserved quantities given by Noether's theorem, but there are more aspects about symmetries that are important to such purposes. One of these aspects is the breaking of the symmetries. We will distinguish between spontaneously and explicitly broken symmetries. We can know some features of the spectrum of theories with spontaneous symmetry breaking a priori via Goldstone's theorem, a theorem that states that the theory will contain at least one massless mode if the theory possesses spontaneous symmetry breaking. The dynamics and interactions of these massless modes can be explained in terms of fluctuations around the vacua of the theory and the effective theory.

A Lagrangian is symmetric under a given transformation if it remains unchanged or if the only term appearing after the transformation is a total derivative. If the Lagrangian of a theory is symmetric but the vacua of the theory are not, the symmetry is said to be spontaneously broken. The study of the theory at low energy will show no symmetries, but the original theory will still be symmetric. Spontaneous symmetry breaking (SSB) has a great relevance because it explains, for instance, Higgs mechanism.

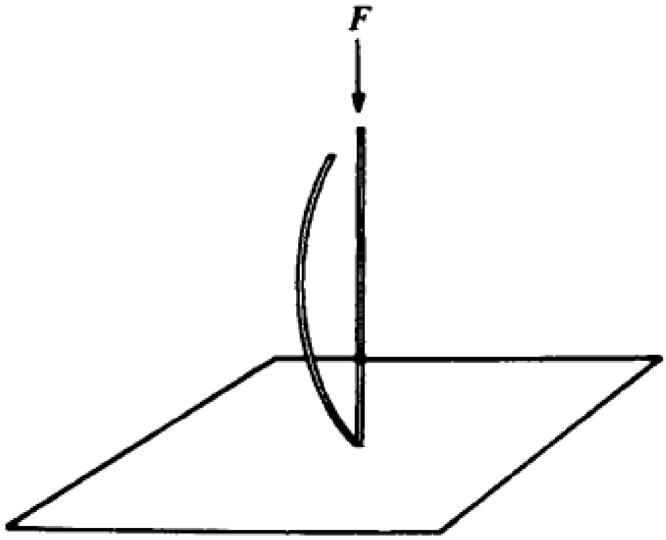

Figure 3.1: Rod placed vertically on a table and its configuration when it is pushed down along its axis, giving a graphical visualization of SSB [5]

As a simple example we can consider a rod with cylindrical cross section placed vertically on a plane surface as in Figure 3.1. If we push down this rod along its axis with a force F strong enough, the rod will bend in one direction randomly chosen. Note that the symmetric configuration becomes unstable when the applied force is bigger than some critical force and that the new ground state is not symmetric. However, there is a collection of ground states connected by rotational symmetry. From this example we deduce that there is a parameter that can take a critical value. Beyond this critical value, the symmetric configuration of the system becomes unstable, so the system evolves to a new ground state that is degenerate.

We say that a theory has explicit symmetry breaking if its action S can be interpreted as a term that is invariant under a given symmetry plus another term controlled by a parameter that can be infinitesimal that is not invariant under the same symmetry group. Explicit symmetry breaking can be regarded as an expansion of a theory that has a given symmetry. This theories are of the form $S = S_{\text{sym}} + S_{\text{no sym}}$, where S_{sym} is the part of the action that is symmetric under the given transformation and $S_{\text{no sym}}$ is the part that is not symmetric and it is multiplied by an infinitesimal parameter.

3.1 Ising model

The next sections will be devoted to the study of spontaneous symmetry breaking. For completeness sake we present an example where explicit symmetry breaking is present. This model first introduced by E. Ising and W. Lenz is used to study ferromagnetism in Statistical Mechanics. However, we will not explore its implications for magnetism and we will just investigate the symmetries of the theory.

The Hamiltonian of the model when there is not an external field is

$$H = -J \sum_{\text{n.n.}} \sigma_i \sigma_j, \tag{3.1}$$

where n.n. stand for near neighbours, that is, we consider interactions only between consecutive spins. The constant J indicates the type of material, being a ferromagnetic material when $J > 0$ and anti-ferromagnetic when $J < 0$. The symbol σ can take the values ± 1, indicating spin "up" or "down" respectively.

The Hamiltonian (3.1) is invariant under the transformation

$$\sigma \to -\sigma, \tag{3.2}$$

that can be regarded as a rotation. If we introduce an external field into the model, the Hamiltonian (3.1) will be modified. Since the field would act on individual spins, after the modification the Hamiltonian is

$$H = -J \sum_{\text{n.n.}} \sigma_i \sigma_j - \mu \sum_j h \sigma_j, \tag{3.3}$$

where μ is the magnetic moment and h is the external field. The new term is linear in σ, thus (3.3) is no longer symmetric under (3.2). Note that the explicit symmetry breaking is due to the term containing the external field.

4 Goldstone's theorem

This theorem states a relation between masses and symmetries. It was postulated by Y. Nambu and improved by J. Goldstone by defining the concept of spontaneous symmetry breaking (SSB) and by giving importance to the fact that the broken symmetry must be continuous.

First of all, we will consider the restricted form of the theorem for relativistic classical fields and we will give an example and the proof of this restricted theorem. After that we will present the most general form of the theorem applied to quantum fields and we will also prove it.

Theorem 2. *Given a classical relativistic field theory with a group of continuous internal symmetries, this theory must contain a massless particle for every spontaneously broken symmetry.*

As we already mentioned, we can predict the number of massless modes that will appear (called Goldstone's bosons) by counting how many linearly independent broken generators are contained in the given symmetry group. In order to illustrate this statement, we consider a rotation in N dimensions. This rotation can be performed in any of the $\frac{N(N-1)}{2}$ planes, meaning that the group $O(N)$ has $\frac{N(N-1)}{2}$ linearly independent continuous symmetries. After the symmetry breaking, there will be $\frac{(N-1)(N-2)}{2}$ symmetries, so $N-1$ symmetries have been broken, which means that there will be $N-1$ Goldstone's bosons in this theory.

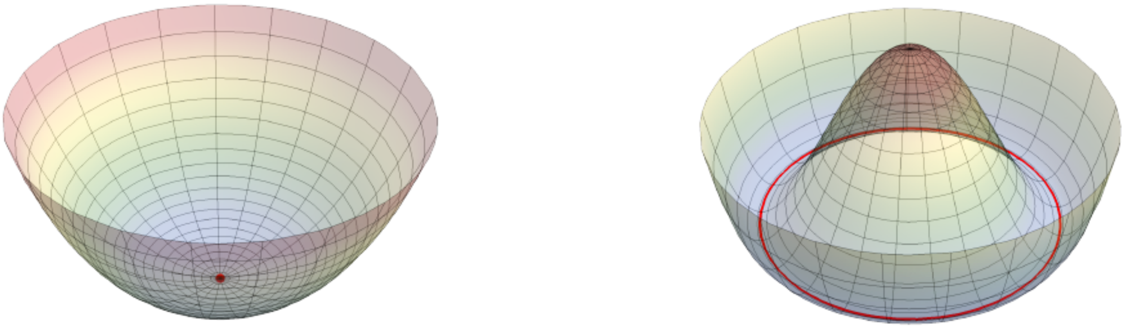

Figure 4.1: Left. Potential if $\mu^2 > 0$ and $\lambda < 0$ in (4.1). There is only one configuration of minimum energy (red dot), hence disallowing spontaneous symmetry breaking to take place. Right. Potential with $\mu^2 > 0$ and $\lambda > 0$ in (4.1). Now there is an infinite number of configurations that minimize the energy (red circumference), allowing then spontaneous symmetry breaking. Figure taken from [7].

As a concrete example of the mechanism of spontaneous symmetry breaking we consider a complex scalar field ruled by

$$S = \int d^d x \left[(\partial_\mu \varphi^*)(\partial^\mu \varphi) + \mu^2 \varphi^* \varphi - \lambda (\varphi^* \varphi)^2 \right], \tag{4.1}$$

where $\mu^2 > 0$, $\lambda > 0$. The need for the parameters to have these signs can be seen in Figure 4.1. In order to spontaneous symmetry breaking to take place, the theory need to have a set of degenerate vacua and this is precisely what we are achieving by choosing these signs for the parameters.

The transformation

$$\begin{aligned} \varphi(x) &\to e^{i\alpha}\varphi(x) \\ \varphi^*(x) &\to e^{-i\alpha}\varphi^*(x), \end{aligned} \tag{4.2}$$

is a symmetry transformation because it leaves the action invariant (it is easy to check this because all the terms contain a product of the field and the conjugated field, so the exponential appearing in the transformed fields cancels out). Moreover, we can see that it is an internal symmetry since it acts only on the space of fields (and not on the coordinates) and a continuous one since the parameter $\alpha \in \mathbb{R}$.

We can parametrize the complex field using two real fields. Identifying the complex field as

$$\varphi = \frac{1}{\sqrt{2}}(\varphi_1 + i\varphi_2), \tag{4.3}$$

the action transforms into

$$S = \int d^d x \left[\frac{1}{2}(\partial_\mu \varphi_1)(\partial^\mu \varphi_1) + \frac{1}{2}(\partial_\mu \varphi_2)(\partial^\mu \varphi_2) + \frac{\mu^2}{2}(\varphi_1^2 + \varphi_2^2) - \frac{\lambda}{4}(\varphi_1^2 + \varphi_2^2)^2 \right]. \quad (4.4)$$

With this we can see that the minimum of the potential is a circumference whose equation is

$$\varphi_1^2 + \varphi_2^2 = \frac{\mu}{\sqrt{\lambda}}, \quad (4.5)$$

where we observe that the minimum is degenerated, that is, there are infinite states with this same energy. We will denote by $\bar{\varphi}$ any field configuration that fulfils the condition (4.5), that is, any configuration that minimizes the energy (vacua). Now we perturb this vacua with fluctuations around the background (4.3). Note that we have required vacua to be constant. We have imposed this condition because it cancels all the terms containing derivatives in the action and then the states of minimum energy are those that minimize the potential. We can write the perturbations around the vacua as

$$\varphi(x) = \bar{\varphi} + \frac{1}{\sqrt{2}} \left[\eta(x) + i\xi(x) \right], \quad (4.6)$$

and the complex conjugated expression for $\varphi^*(x)$. The quadratic action for the fluctuations is now

$$S = \int d^d x \left[\frac{1}{2}(\partial_\mu \eta)(\partial^\mu \eta) + \frac{1}{2}(\partial_\mu \xi)(\partial^\mu \xi) - \mu^2 \eta^2 \right], \quad (4.7)$$

where there is only one massive field. The action does not contain a mass term for the field $\xi(x)$, so it is a massless mode that we identify with the Goldstone boson. This boson has appeared because the $U(1)$ symmetry has been broken. We can see that (4.1) is invariant under (4.2), but (4.7) is not. The only symmetry present in this action is a shift in the field $\xi(x)$, that is $\xi(x) \rightarrow \xi(x) + \alpha$, where α is a constant.

Note that this symmetry is now realized in a non-linear way, unlike in (4.2) where it was realized through a product of the field and a unitary complex number, that is a linear operation on the field. In this sense, spontaneous symmetry breaking could be called non-linear realization of a symmetry. In fact, in the literature both are understood as synonyms.

Moreover, the symmetry under shifts has deep implications in the theory at low energy. According to Goldstone's theorem, this theory can not contain a mass term for the field $\xi(x)$. In fact, the Nambu-Goldstone field can only appear derivated (one or more times). In Fourier space this would mean that this field can only appear multiplied by some power of the momentum. If we consider the regime of low momentum, every term involving a Nambu-Goldstone will be suppressed by a power of the momentum. The low-energy theorem states that in the effective theory at low momentum, Nambu-Goldstone modes are massless and can be well approximated by free modes.

The mass associated to the field $\eta(x)$ can be read from its mass term by interpreting it as

$$\frac{1}{2}m^2\eta^2 = \mu^2\eta^2. \tag{4.8}$$

The mass associated to this field is then $m_\eta = \sqrt{2\mu^2}$.

4.1 Example: spontaneous breaking of SO(3) symmetry

We will consider a theory described by the Lagrangian

$$\mathcal{L} = \frac{1}{2}(\partial_\mu\phi_1)(\partial^\mu\phi_1) + \frac{1}{2}(\partial_\mu\phi_2)(\partial^\mu\phi_2) + \frac{1}{2}(\partial_\mu\phi_3)(\partial^\mu\phi_3) - V(\phi_1,\phi_2,\phi_3), \tag{4.9}$$

where there are three real scalar fields $\phi_i,\ i = 1, 2, 3$ and the potential has the form

$$V(\phi_1,\phi_2,\phi_3) = -\mu^2(\phi_1^2+\phi_2^2+\phi_3^2) + \lambda(\phi_1^2+\phi_2^2+\phi_3^2)^2 \tag{4.10}$$

We can interpret these fields as the components of a vector field $\vec{\phi}$, that is

$$\vec{\phi} = \begin{pmatrix}\phi_1\\ \phi_2\\ \phi_3\end{pmatrix}. \tag{4.11}$$

We are considering rotations in 3 dimensions. The matrices that perform this rotations are

$$R_1(\theta) = \begin{pmatrix}1 & 0 & 0\\ 0 & \cos\theta & -\sin\theta\\ 0 & \sin\theta & \cos\theta\end{pmatrix},\ R_2(\theta) = \begin{pmatrix}\cos\theta & 0 & \sin\theta\\ 0 & 1 & 0\\ -\sin\theta & 0 & \cos\theta\end{pmatrix},\ R_3(\theta) = \begin{pmatrix}\cos\theta & -\sin\theta & 0\\ \sin\theta & \cos\theta & 0\\ 0 & 0 & 1\end{pmatrix}, \tag{4.12}$$

where the subindex written for each matrix indicates the axis around which the system is rotated. It can be proven that the Lagrangian in (4.9) is invariant under the action of any matrix to the field as written in (4.11). Note that the most general form of the rotation matrices in SO(3) would be given in terms of Euler's angles, but for our purposes is enough with considering the presented matrices.

Now, we look for the states of minimum energy, which coincide with the ones that minimize the potential since we are considering vacua with constant value. This is because the terms containing derivatives are defined positive, so they will not contribute to lower the energy. We find the minima of potential by imposing the conditions

$$\left.\frac{\partial V}{\partial\phi_i}\right|_{\phi=\bar{\phi}} = 0, \qquad \left.\frac{\partial^2 V}{\partial\phi_i^2}\right|_{\phi=\bar{\phi}} > 0, \tag{4.13}$$

where $\bar{\phi}$ denotes the value of the fields in the minimum. With this we find that the states with minimum energy are given by the condition

$$\bar{\phi}_1^2 + \bar{\phi}_2^2 + \bar{\phi}_3^2 = \frac{\mu^2}{2\lambda}, \tag{4.14}$$

that is, they form a spherical surface of radius given by

$$R^2 = \frac{\mu^2}{2\lambda}. \tag{4.15}$$

We can write a generic vacuum of this theory in terms of the fields $R(x^\mu), \theta(x^\mu), \phi(x^\mu)$ as

$$\begin{pmatrix} \phi_1 \\ \phi_2 \\ \phi_3 \end{pmatrix} = R \begin{pmatrix} \sin\theta\cos\varphi \\ \sin\theta\sin\varphi \\ \cos\theta \end{pmatrix}. \tag{4.16}$$

Now we consider fluctuations around this general vacuum, that is, we consider the field

$$\vec{\phi} = (R + \delta r) \begin{pmatrix} \sin(\theta+\delta\theta)\cos(\varphi+\delta\varphi) \\ \sin(\theta+\delta\theta)\sin(\varphi+\delta\varphi) \\ \cos(\theta+\delta\theta) \end{pmatrix}, \tag{4.17}$$

where the perturbations are $\delta r(x), \delta\theta(x), \delta\varphi(x) << 1$. Substituting this parametrization in the Lagrangian we find that it can be rewritten as

$$\mathcal{L}(\delta r, \delta\theta, \delta\varphi) = \frac{1}{2}(\partial_\mu \delta r)(\partial^\mu \delta r) + \frac{1}{2}R^2(\partial_\mu \delta\theta)(\partial^\mu \delta\theta) + \frac{1}{2}R^2\sin^2\theta(\partial_\mu \delta\varphi)(\partial^\mu \delta\varphi) - V(\delta r, \delta\theta, \delta\varphi), \tag{4.18}$$

where now the potential $V(r, \theta, \varphi)$ takes the form

$$V(\delta r, \delta\theta, \delta\varphi) = 2\mu^2 \delta r^2. \tag{4.19}$$

We have only kept terms up to quadratic order in the perturbations. We see that there are not quadratic terms in the fields $\theta(x), \phi(x)$, so they are massless (Goldstone bosons). The only symmetry that preserves the vacuum is given by a shift in $\delta\varphi$, which means that two symmetries have been broken.

The number of Goldstone bosons that appear when a $O(N)$ symmetry is broken has been discussed after stating the theorem. In this case we have $O(3)$ symmetry breaking, meaning $N = 3$. The number of generators is then $\frac{N(N-1)}{2} = 3$ and $N - 1 = 2$ symmetries have been broken, which coincides with the symmetries of (4.18).

4.2 Proof for classical scalar fields

Consider a relativistic theory that contains fields $\phi^a(x)$ and that has a Lagrangian of the form

$$\mathcal{L} = (\text{terms with derivatives}) - V(\phi). \tag{4.20}$$

Let ϕ_0^a be a constant field that minimizes the potential $V(\phi)$,

$$\frac{\partial V}{\partial \phi^a}\Big|_{\phi_0} = 0, \qquad \frac{\partial^2 V}{\partial \phi^{a2}}\Big|_{\phi_0} > 0. \tag{4.21}$$

The expansion of the potential around this minimum gives

$$V(\phi) = V(\phi_0) + \frac{1}{2}(\phi - \phi_0)^a(\phi - \phi_0)^b \left(\frac{\partial^2 V}{\partial \phi^a \partial \phi^b}\right)\Big|_{\phi_0} + O(\phi^3), \tag{4.22}$$

where we can identify a symmetric matrix whose eigenvalues are the masses of the fields. This matrix is the coefficient of the term of second order, that is,

$$\frac{\partial^2 V}{\partial \phi^a \partial \phi^b} \equiv m_{ab}^2. \tag{4.23}$$

In order to prove the theorem we have to show that every continuous symmetry of the Lagrangian in Eq. 4.20 that is not a symmetry of the ground state ϕ_0 produces a null eigenvalue in the matrix defined in 4.23. To do so we consider a general continuous symmetry,

$$\phi^a \to \phi^a + \alpha \Delta^a(\phi), \tag{4.24}$$

where α is an infinitesimal parameter and $\Delta^a(\phi)$ is a function which depends on every field. Since we are considering minimums of energy whose value is constant, we can get rid of the terms with derivatives, thus we will always deal with the potential from this point. Taking this into account, we need to check that the potential $V(\phi)$ is invariant under the transformation on Eq. 4.24. This conditions means

$$V(\phi^a) = V[\phi^a + \alpha \Delta^a(\phi)] \iff \Delta^a(\phi)\frac{\partial V(\phi)}{\partial \phi^a} = 0. \tag{4.25}$$

Differentiating with respect to ϕ^b and setting $\phi = \phi_0$ in the second condition on Eq. 4.25 we get

$$\frac{\partial \Delta^a}{\partial \phi^b}\Big|_{\phi_0} \cancelto{0}{\frac{\partial V}{\partial \phi^a}}\Big|_{\phi_0} + \Delta^a(\phi_0)\left(\frac{\partial^2 V}{\partial \phi^a \partial \phi^b}\right)\Big|_{\phi_0} = 0, \tag{4.26}$$

where the first term vanishes because ϕ_0 is a minimum. The second term would vanish if $\Delta^a(\phi_0) = 0$, but spontaneously broken symmetries are those that verify $\Delta^a(\phi_0) \neq 0$. In this case, $\Delta^a(\phi_0)$ is the vector with null eigenvalue that we were looking for, concluding the proof.

4.3 General form of Goldstone's theorem

Theorem 3. *We consider a physical field theory at a quantum level. This theory has a global continuous symmetry group G such that it is spontaneously broken to a subgroup H which is not equal to G and where the notion of gap is well-defined. Under this conditions, the spectrum of the theory will contain at least one gapless particle.*

Since G is a generic group, the only condition imposed on the symmetry is that it must be continuous, so it can be of any kind. Moreover, it also works for thermal theories, that is, theories such that the chemical potential and the temperature are finite. The theory must be physical, that means that the concept of locality has to be respected.

4.3.1 Proof of the general form

In this section we provide the proof of Goldstone theorem in its most general form. We present the so-called spectral decomposition proof.

We consider a symmetry group G that is global, continuous and uniform so that we can define Noether's current. Note that imposing G to be continuous rules out gauge symmetries, whose conserved currents are just equivalence relations. This is a key point because the proof of the theorem starts with a non-zero conserved charge. G being uniform implies that we are not considering spacetime symmetries.

Since we demand the group G to be global, conserved currents are well-defined. Moreover, we will demand the interactions to be local enough such that these currents vanish at the boundaries of the space,

$$\int_{\partial V} dS_i j^i(x) = 0, \tag{4.27}$$

where V is the volume of the system. With this,

$$\frac{\mathrm{d}Q}{\mathrm{d}t} = \int_V d^{d-1}x\partial_0 j^o(x) = -\int_V d^{d-1}x\partial_i j^i(x) = -\int_{\partial V} dS_i j^i(x) = 0, \tag{4.28}$$

where we have used the continuity equation on the first step and Gauss' theorem on the last step. Since we are not considering the spontaneous breaking of spacetime symmetries, the vacuum will be an eigenstate of the momentum operator P_μ. Moreover, the vacuum will be the zero of energy because we are not considering gravity, so $P_\mu |0\rangle = 0$.

Let Q be a generator of G such that Q is spontaneously broken. In this situation, there must exist a field ϕ such that

$$\langle 0| [Q, \phi(x)] |0\rangle \neq 0. \tag{4.29}$$

The spectral decomposition of this expression is

$$\begin{aligned}\langle 0|\left[Q,\phi(x)\right]|0\rangle &= \int d^{d-1}x' \langle 0|\left[j^0(x'),\phi(x)\right]|0\rangle \\ &= \int d^{d-1}x' \sum_n \int \frac{d^{d-1}k}{(2\pi)^{d-1}} \left[\langle 0| j^0(x') \left|n_{-\vec{k}}\right\rangle \left\langle n_{-\vec{k}}\right| \phi(x)|0\rangle\right. \\ &\quad \left. - \langle 0|\phi(x)\left|n_{\vec{k}}\right\rangle \left\langle n_{\vec{k}}\right| j^0(x')|0\rangle\right], \end{aligned} \tag{4.30}$$

where we have made use of the closure relation. Taking into account that P_μ is the generator of translations,

$$\begin{aligned}\langle 0|\left[Q,\phi(x)\right]|0\rangle &= \int d^{d-1}x' \sum_n \int \frac{d^{d-1}k}{(2\pi)^{d-1}} e^{ik_\mu x'^\mu} \left[\langle 0| j^0(x') \left|n_{-\vec{k}}\right\rangle \left\langle n_{-\vec{k}}\right| \phi(x)|0\rangle\right. \\ &\quad \left. - \langle 0|\phi(x)\left|n_{\vec{k}}\right\rangle \left\langle n_{\vec{k}}\right| j^0(x')|0\rangle\right] \\ &= \sum_n \int d^{d-1}k e^{iE_n(\vec{k})t} \phi(\vec{k}) \left[\langle 0| j^0(0) \left|n_{-\vec{k}}\right\rangle \left\langle n_{-\vec{k}}\right| \phi(x)|0\rangle\right. \\ &\quad \left. \langle 0|\phi(x)\left|n_{\vec{k}}\right\rangle \left\langle \vec{k}\right| j^0(0)|0\rangle\right], \end{aligned} \tag{4.31}$$

where

$$\int \frac{d^{d-1}k}{(2\pi)^{d-1}} e^{i\vec{k}\vec{x}} = \varphi(\vec{k}), \tag{4.32}$$

and in the limit $V \to \infty$ it transforms into $\delta^{d-1}(\vec{k})$. This means that the only modes that contribute to (4.31) are the ones in the zero-momentum limit. Moreover, given that the charge is conserved over time the only time dependence in (4.31) comes from the field $\phi(x)$, so the exponential does not intervene. This implies that only the modes with $E_n(\vec{k}) \to 0$ when $\vec{k} \to 0$ (that is, massless modes) contribute to the sum over n. Due to (4.29) we know that the result is not zero, meaning that there is at least one massless mode that contributes to the integral, hence proving the theorem.

4.4 New directions of research

In this part we present some possible aspects that can be studied within the context of Goldstone's theorem.

The theorem states that there exists at least one massless mode always a global continuous symmetry is spontaneously broken. However, the theorem does not predict the number of massless modes that will appear (it only states a minimum number of massless modes). A possible direction of research would be how to make predictions of how many massless modes will appear after a symmetry breaking.

Once we know the number of Goldstone bosons that appear after a certain spontaneous symmetry breaking, one can compute their dispersion relations and then compute some thermodynamic observables, for instance. In this sense, classifying and studying the different possible dispersion relation would be of interest.

The existence of Goldstone bosons is guaranteed at low energies because they are massless. However, they are not the only light modes there exist. Another possible direction of research would be the understanding of the interactions between Goldstonce bosons.

More directions of research involve extensions of Goldstone's theorem, for instance on curved spacetimes or the study of Goldstone bosons in systems out of thermal equilibrium.

5 Superfluidity

The formalism of spontaneous symmetry breaking was first defined and used in Condensed Matter Physics to study magnets, superfluids, superconductors and phase transitions. Afterwards, this formalism was extended to high energy physics until it reached the relevance to become one of the cornerstones of the Standard Model of Particle Physics. In this section we will abandon (momentarily) High Energy Physics to make a little incursion into an application of this formalism in Condensed Matter Physics: the superfluid.

This section has two clearly differentiated parts. The first part is devoted to explain some concepts of Statistical Mechanics that will be useful in the second part of this section. In the first section we will not use the field theory formalism since it is not necessary to introduce the topics that we treat. Moreover, from the point of view of field theory, the computations would be more difficult and we would obtain the same results as we present. In particular, the introduced topics are the Landau-Ginzburg theory of phase transitions and second order phase transitions. In the second part of the section we recover the field theory formalism to compute the critical temperature at which He-4 turns into a superfluid.

5.1 Phase transitions and Landau-Ginzburg theory

When a phase transition takes place, the system changes from a less ordered state to a more ordered state. The quantity that is used to measure this order is the order parameter. As an example, we consider the transition from paramagnetic to ferromagnetic. When the system is in paramagnetic state, the atomic magnetic moment is randomly oriented, but in the ferromagnetic phase all the magnetic moments are aligned. Taking this into account, we can use the average magnetic moment[2] as the order parameter. The order parameter is usually a field that we will call $\theta(x)$ in general. We have already faced some order parameters, but since they were not relevant to the point that we were making, we did not mention it. For instance, $\bar{\varphi}$ in the $U(1)$ symmetry breaking was an order parameter.

The microscopic description of a phase transition would involve many other degrees of freedom such as the electronic states or interatomic interaction in the case of the ferromagnetic transition. However, there is another way of studying the microscopic description that does not show explicitly all these degrees of freedom and expresses all

[2] This average moment is computed by taking the average magnetic moment within a region of the system that contains a large number of atoms

the quantities in terms of the order parameter alone. This approach is known as Landau-Ginzburg phenomenological theory.

To simplify the presentation of the theory we will consider that the order parameter is a scalar field $\theta(x)$ where x is an arbitrary number of dimensions. The state of the system will then be given by this scalar field and the statistical ensemble that describes the system will be a set of these fields. This theory is based in the functional $E[\theta]$

$$E[\theta] = \int dx \left[|\nabla\theta(x)|^2 + W[\theta(x)] - h(x)\theta(x) \right], \tag{5.1}$$

which receives the name of Landau free energy. The first term is the kinetic energy. The function $W(\theta)$ is similar to a potential and it can be expanded in terms of the order parameter as

$$W[\theta(x)] = g_2\theta^2(x) + g_3\theta^3(x) + g_4\theta^4(x) + \ldots, \tag{5.2}$$

where g_n are the phenomenological constants and we have assumed that the order parameter is small enough to expand in a power series. The physical meaning of the smallness of the order parameter is that we are studying the system in the region close to the transition phase. Note that the subindex is equal to the power of $\theta(x)$ in each term. The linear term is written separately in (5.1) with a coupling $h(x)$ that is an external field whose purpose will be seen later.

Given this free energy, we define the partition function as

$$Z[h] = \int \mathcal{D}\theta e^{-\beta E[\theta]}, \tag{5.3}$$

where $\beta^{-1} = k_B T$. From this partition function we define the average of any quantity O as

$$\langle O \rangle = \frac{1}{Z} \int \mathcal{D}\theta O e^{-\beta E[\theta]}. \tag{5.4}$$

5.2 Second order phase transitions

As an example we will consider spontaneous magnetization. This phenomenon takes place suddenly when the temperature of the system drops below a given temperature T_c when there is no external field. In Figure 5.1 there is a plot representing the magnetization of a material in terms of the temperature.

Recall that the magnetization is a vector field that indicates the density of magnetic dipoles in a magnetic material. This means that if the magnetization is zero, the dipoles will be randomly oriented as in a paramagnetic material. If the magnetization is non-zero, there will be a preferred orientation for the dipoles due to, for instance, an external field. When there is no external field applied, a ferromagnetic material undergoes an abrupt phase transition into paramagnetic state as soon as the temperature reaches the critical value (Curie temperature), but when we apply an external field, there will be some residual magnetization even if the temperature raises above Curie temperature because the field will orient the magnetic dipoles in some direction, hence there will be magnetization.

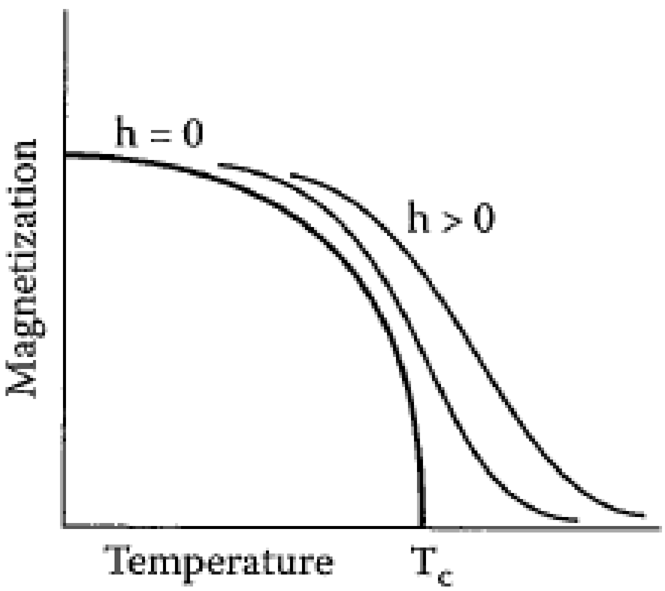

Figure 5.1: Difference in the drop of the magnetization with the temperature in the presence of an external field. If there is an external field, it decreases very rapidly. Recall that T_C is the temperature at which the phase transition takes place (it is also known as Curie temperature). Picture taken from [6]

This behaviour and other similar behaviours can be explained with a potential $W(\theta)$ of the form

$$W(\theta) = r_0\theta^2 + u_0\theta^4, \tag{5.5}$$

and it is represented in Figure 5.2. In order to ensure the stability of the theory, u_0 must be positive. In this case the potential has only one minimum if $r_0 > 0$ but it has two

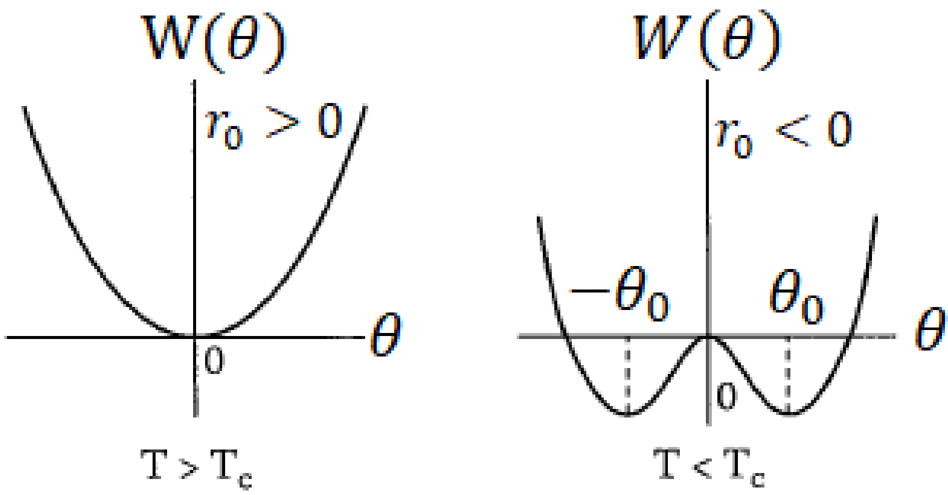

Figure 5.2: Graphical representation of $W(\theta)$ for second-order phase transitions. When the temperature drops below the critical temperature, the potential presents two minimum because $r_0 < 0$.

if $r_0 < 0$. The sign of r_0 can change because, since it is a phenomenological parameter, it depends on the temperature as we will see. Given this change of sign, we can use r_0 as the indicator of whether there has been a phase transition or not. To accomplish this, we can write r_0 as

$$r_0 = b\frac{T - T_c}{T_c}, \tag{5.6}$$

where b is a real positive constant. Note that u_0 has to be positive to ensure stability.

5.3 Computation of the critical temperature for He-4

We have already studied the $U(1)$ symmetry breaking in the model (4.1). Now we will apply this results to the study of superfluidity with a similar model. We will focus on the potential part of the Lagrangian since the kinetical part has no relevance in the aspect we will develop. The potential has the form

$$V(\varphi^*\varphi) = -\mu\varphi^*\varphi + \frac{\lambda}{2}(\varphi^*\varphi)^2, \tag{5.7}$$

where φ is again a scalar complex field. We will use this potential to derive an equation for the critical temperature at which the transition from normal state to superfluid state begins in the case of He-4. This transition temperature has been experimentally measured to be $T_\lambda = 2,2$ K for He-4. The first attemp to explain this value for the critical temperature is due to F. London in the 20th century. He assumed that the process was equivalent to a Bose-Einstein condensation and he assumed that the temperature at which the phase transition took place was the temperature at which an ideal Bose gas transforms into a

Bose-Einstein condensate. His formula depended on two physical parameters, the mass of the helium and the number density and gave a value of $T = 3,13$ K. Even if it is of the same order, it still does not explain the measured temperature.

In order to give an analytical formula to this temperature, we will interpret λ as the coupling of the interactions between helium molecules. This coupling can be expressed as

$$\lambda = \frac{2\pi\hbar^2\sigma}{m}, \tag{5.8}$$

where σ is the hard-sphere diameter of helium molecules and m is the mass of the helium, while μ can be interpreted as the chemical potential. A physical system undergoes a phase transition when the $\mu > 0$, that can be translated into $T < T_c$ for some critical temperature T_c according to (5.6). The ground states of (5.7) are those that satisfy

$$|\varphi| = \sqrt{\frac{\mu}{\lambda}}, \tag{5.9}$$

so we require $\mu > 0$, thus allowing the phase transition. The relation between $|\varphi|$ and the temperature can be derived by interpreting $|\varphi|^2$ as the number density n. This gives the relation

$$|\varphi| = \sqrt{\frac{\mu}{\lambda}} = \sqrt{n} = \sqrt{\frac{\rho_s}{m}}, \tag{5.10}$$

where ρ_s is the superfluid density. From Eq. 5.10 we can deduce a linear relation between μ and ρ_s. In Figure 5.3 there is a comparative between experimental data taken from [8] and several models that provide the relation $T - \rho_s$ as a power law. The best-fitting model is the one given by a power of 6, giving a relation $T - \mu$ given by

$$\mu = \mu_0 \left[1 - \left(\frac{T}{T_\lambda}\right)^6\right], \tag{5.11}$$

where $\mu_0 = \lambda\rho_0/m$ and ρ_0 is the total density at absolute zero.

Another relevant aspect of this theory is the dependence of the potential on the temperature. This situation is represented in Figure 5.4. In the case $T > T_\lambda$ there is only one minimum at $\varphi = 0$ and the curvature is positive at that point. If $T = T_\lambda$, there is only one minimum and the curvature at this point vanishes. The case $T < T_\lambda$ presents negative curvature at $\varphi = 0$, giving an unstable local maximum. The system will evolve to a stable minima given by $|\varphi| = \sqrt{\mu/\lambda}$.

At $T = 0$, the system will be in some non-zero minimum and when the temperature raises over the critical temperature, the system will be located in the minimum $\varphi = 0$, recovering the $U(1)$ that is broken if $T < T_\lambda$. The critical temperature is then determined by the depth of the potential well at zero temperature. This depth is given by

$$V(|\varphi| = \sqrt{\mu_0/\lambda}) = \frac{\mu_0 n_0}{2}, \tag{5.12}$$

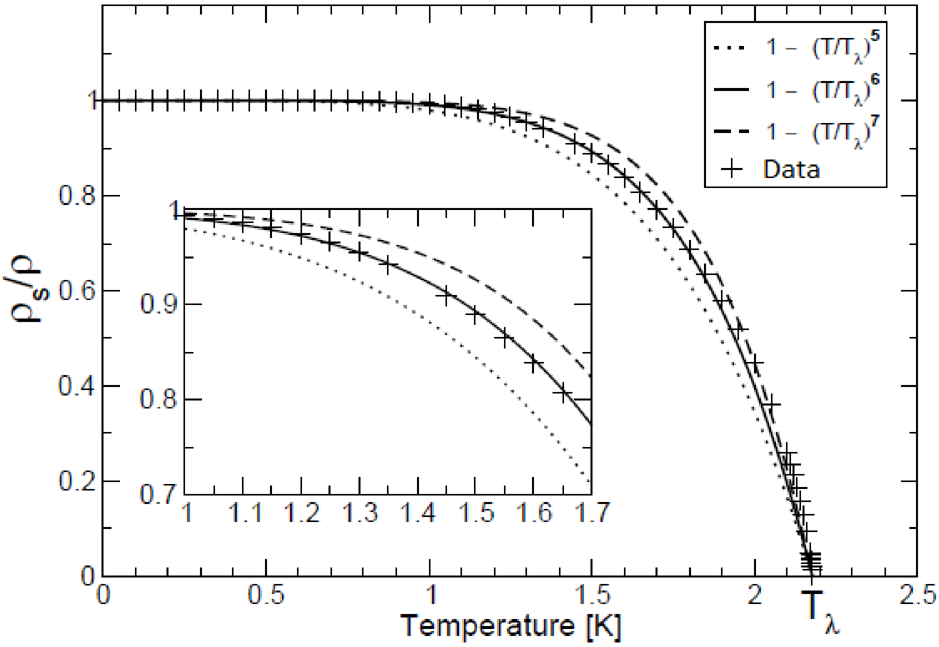

Figure 5.3: Relation between temperature and superfluid density compared with three power-law models. Experimental data taken from [8]

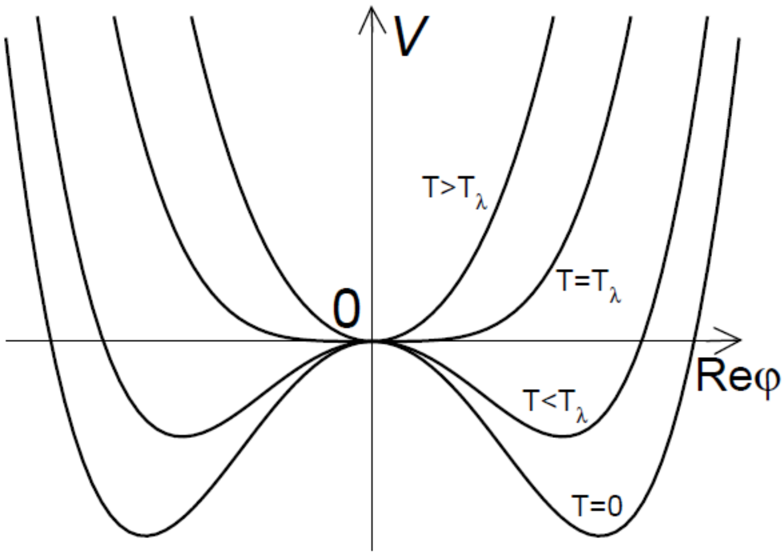

Figure 5.4: Dependence of the potential on the temperature

where n_0 is the number density at zero temperature. This depth is the energy that is needed to recover the broken symmetry and it will be equal to $k_B T_c n_0$ since it is a thermal energy. From this we obtain

$$k_B T_c = \frac{\mu_0}{2}. \tag{5.13}$$

Taking into account the relation in Eq. 5.10 and the identification of λ in terms of the hard-sphere diameter and the mass, we get the final expression,

$$T_c = \frac{\rho_0 \pi \hbar^2 \sigma}{m^2 k_B}. \tag{5.14}$$

Substituting the measured values of $\rho_0 = 0,1451$ g/cm^3, $m = 6,6465 \times 10^{-24}$ g and $\sigma = 2,639$ Å we get a critical temperature of $T_c = 2,194$ K, which is more precise than the one given by F. London. It differs with the measured one by 0,8%.

6 Spontaneous breaking of translations

After our little incursion into Condensed Matter Physics we return to High Energy Physics at zero temperature and zero chemical potential. In this section we consider the action of a complex scalar field ϕ

$$\begin{aligned} S &= \int d^3x \{ -(\partial^t \phi^*)(\partial_t \phi) + A(\partial^i \phi^*)(\partial_i \phi) - B\left[(\partial^i \phi^*)(\partial_i \phi)\right]^2 \\ & -F\phi^*\phi(\partial^i\partial^j\phi^*)(\partial_i\partial_j\phi) + G(\partial_i\phi^*)(\partial^i\phi^*)(\partial^j\phi)(\partial_j\phi)\}, \end{aligned} \tag{6.1}$$

where $i, j = \{1, 2\}$ are the spatial indices and A, B, F, G are positive constants. We will parametrize the fields as

$$\phi(t, x, y) = \rho(t, x, y) e^{i\varphi(t,x,y)}, \tag{6.2}$$

which means that the symmetry of the action (6.1) under $U(1)$ is translated into a symmetry under shifts of the phase φ. The Euler-Lagrange equations of motion of a general Lagrangian

$$\mathcal{L}(\phi, \partial_{\mu(1)}\phi, \partial_{\mu(1)\mu(2)}\phi \ldots, \partial_{\mu(1)}\partial_{\mu(2)} \ldots \partial_{\mu(N)}\phi), \tag{6.3}$$

can be computed using variational calculus (recall that the equations of motion satisfy $\delta S = 0$)

$$\delta S = \int_{\mathcal{D}} d^D x \sum_{n=0}^{N} \frac{\partial \mathcal{L}}{\partial(\partial_{\mu(n)}\phi)} \delta(\partial_{\mu(n)}\phi) = 0, \tag{6.4}$$

where we will assume

$$\delta(\partial_{\mu(n)}\phi)\Big|_{\partial\mathcal{D}} = 0, \quad n = 0, \ldots, N-1. \tag{6.5}$$

Integrating (6.4) by parts and taking into account the conditions that must satisfy the field and its derivatives in the boundary (6.5) we get that the equations of motion can be computed as

$$\sum_{n=0}^{N}(-1)^n\left[\partial_{\mu(n)}\frac{\partial\mathcal{L}}{\partial(\partial_{\mu(n)}\phi}\right] = 0, \tag{6.6}$$

so the equations of motion of our theory (6.1) are

$$\begin{aligned}\text{EOM}[\phi] &= \partial^t\partial_t\phi - A\partial^i\partial_i\phi + 2B\partial^i\left[\partial^j\phi^*\partial_j\phi\partial_i\phi\right] - F\partial^i\partial^j\left[\phi^*\phi\partial_i\partial_j\phi\right] \\ &\quad -F\phi(\partial^i\partial^j\Phi^*)(\partial_i\partial_j\phi) - 2G\partial^i\left[\partial_i\phi^*\partial^j\phi\partial_j\phi\right] = 0,\end{aligned} \tag{6.7}$$

and its complex conjugate for $\text{EOM}[\phi^*]$. In terms of the parametrization (6.2), the equations of motion are

$$\rho\text{EOM}[\rho] = \phi^*\text{EOM}[\phi^*] + \phi\text{EOM}[\phi], \tag{6.8}$$

$$i\text{EOM}[\varphi] = \phi^*\text{EOM}[\phi^*] - \phi\text{EOM}[\phi]. \tag{6.9}$$

Now we will use an ansatz with the form of a plane wave, that is,

$$\rho(t,x,y) = \bar{\rho}, \qquad \varphi(t,x,y) = kx, \tag{6.10}$$

where both $\bar{\rho}$ and k are constants. With this, one can see that (6.9) is satisfied and that if we plug it into (6.8) we get

$$\rho\text{EOM}[\rho] = 2k^2\bar{\rho}^2\left[A - 2k^2\bar{\rho}^2(B+F-G)\right] = 0. \tag{6.11}$$

Solving for k we find that the solutions are

$$k^{(1,2)} = 0, \tag{6.12}$$

$$k^{(3,4)} = \pm\frac{1}{\bar{\rho}}\left[\frac{A}{2(B+F-G)}\right]^{1/2}, \tag{6.13}$$

where we impose $B+F-G>0$ in order to have real solutions. The energy density of a general Lagrangian (6.3) is the component T^{tt} of the canonical energy-momentum tensor

$T^{\mu\nu}$, which is defined as the conserved current generated from space-time translation invariance. In a similar way as we did deriving the generalized Euler-Lagrange equations of motion (6.4), we can compute the energy-momentum tensor as

$$T^{\mu\nu} = -\eta^{\mu\nu}\mathcal{L} + \sum_{n=0}^{N-1}\sum_{m=0}^{N-n-1}(-1)^m \partial_{\beta(m)}\left[\frac{\partial\mathcal{L}}{\partial(\partial_\mu\partial_{\beta(m)}\partial_{\alpha(n)}\phi)}\right]\partial_{\alpha(n)}\partial^\nu\phi. \tag{6.14}$$

In the case of the configuration in (6.10) we have

$$\epsilon = T^{tt} = \eta^{tt}L_0 = -L_0, \tag{6.15}$$

where we have defined L_0 as the Lagrangian in (6.1) written in terms of (6.10). With this, the energy configuration of the configuration (6.10) is given by

$$\epsilon(k,\bar{\rho}) = k^2\bar{\rho}^2\left[-A + k^2\bar{\rho}^2(B+F-G)\right], \tag{6.16}$$

and we see that it does not depend on any spatial coordinate, thus it is spatially homogeneous. We will investigate more about the concept of homogeneity in the next section. The energy density evaluated on the solutions $k^{(1,2)}, k^{(3,4)}$ is

$$\epsilon(k^{(1,2)}) = 0, \tag{6.17}$$

$$\epsilon(k^{(3,4)}) = -\frac{1}{2}A(k^{(3,4)})^2\bar{\rho}^2, \tag{6.18}$$

where $\epsilon(k^{(3,4)})$ gives the degenerate global minima since $A > 0$.

Now we consider fluctuations around (6.10), that is

$$\phi(t,x,y) = \left[\bar{\rho} + \delta\eta\,(t,x,y)\right]e^{ikx}, \tag{6.19}$$

where the complex fluctuation field can be parametrized as $\delta\eta = \sigma(t,x,y) + i\tau(t,x,y)$. In order to obtain the dispersion relation, we perform the Fourier transform of the action to quadratic order in the fluctuations, obtaining

$$S_{(2)} = \int \frac{d^3q}{(2\pi)^3}\tilde{v}(-q)^T M\tilde{v}(q), \tag{6.20}$$

where $q = (\omega, q_x, q_y)$ and

$$\tilde{v}(q) = \begin{pmatrix}\tilde{\sigma}(q)\\ \tilde{\tau}(q)\end{pmatrix}. \tag{6.21}$$

The entries of the matrix M are given by

$$M_{\sigma\sigma} = \omega^2 - 2Ak^2 - 4k^2\bar{\rho}^2(2Fq_x^2 + Gq_y^2), \tag{6.22}$$

$$M_{\tau\tau} = \omega^2 - 2Aq_x^2 - F\bar{\rho}^2(q_x^2 + q_y^2)^2, \tag{6.23}$$

$$M_{\sigma\tau} = M_{\tau\sigma}^* = -2ikq_x\left[A + 2F\bar{\rho}^2(q_x^2 + q_y^2)\right]. \tag{6.24}$$

We can now obtain the dispersion relation by equating the eigenvalues of M $e_{1,2}$ to zero. Since they have a complicated analytic expression, we expand the relation for low momenta. With this, we obtain

$$\omega^2 = 2Ak^2 + 4k^2\bar{\rho}^2\left[(B + 3F - G)q_x^2 + Gq_y^2\right] + \dots \tag{6.25}$$

$$\omega^2 = \bar{\rho}^2\left[2(2G - 3F)q_x^2q_y^2 + F(q_x^4 + q_y^4)\right] + \dots \tag{6.26}$$

In both of these dispersion relations k is given by (6.13). Since we require the quartic term in momenta to be positive, $G \geq F$. We also require $F \neq 0$ in order to have either pure longitudinal ($q_y = 0$) or pure transverse ($q_x = 0$) propagation in (6.26). Combining both requirements, we have $G \geq F > 0$.

We can see that (6.26) describes a gapless mode because if we set $q_x = q_y = 0$ we obtain $\omega^2 = 0$, indicating that the mode does not have mass. This gapless mode receives the name of phonon and it is the Goldstone boson resulting from the spontaneous breaking of the product of phase shifts and translations into its diagonal subgroup. In coming sections there is a more explicit computation of these dispersion relations in the case where $F = 0$.

6.1 Homogeneous translation symmetry and diagonal subgroup

Recall that the vacua (6.10) are of the type plane wave when plugged into the parametrization (6.2). This plane wave is not invariant under translations as we can see from the following transformation

$$\phi(x) = \rho e^{ikx} \to \phi(x+\delta) = \rho e^{ik(x+\delta)} = \rho e^{ikx}e^{ik\delta}, \tag{6.27}$$

which is clearly not equal to the original field. However, energy density (6.16) does not depend on any spatial coordinate. This situation where the vacua of a theory are not invariant under translations but the energy density of these states does not depend on spatial coordinates is called homogeneous translation symmetry.

Now if we consider again the action (6.1), we can see that it is invariant under $U(1)$, that is, a global transformation of the fields

$$\phi(x) \to e^{i\alpha}\phi(x) \tag{6.28}$$

$$\phi^*(x) \to e^{-i\alpha}\phi^*(x), \tag{6.29}$$

where $\partial_\mu \alpha = 0$, leaves the action invariant. The action of this transformation on the parametrization of the plane wave (6.10) is manifested as a phase shift,

$$\phi(x) = \rho e^{ikx} \to e^{i\alpha}\phi(x) = \rho e^{ikx} e^{i\alpha}. \tag{6.30}$$

Combining (6.27) and (6.30) we can see that the vacua are invariant simultaneously under translations and $U(1)$ if the parameters of both transformations satisfy the relation

$$\alpha = -k\delta. \tag{6.31}$$

The symmetry group of the action (6.1) is $T \times U(1)$, where T stands for the group of translations, but the vacua are not invariant under the action of these groups. However, we have shown that if the parameters of the transformation are related by (6.31) the vacua are symmetric. This is what we mean by saying that the product of phase shifts and translations has spontaneously broken into its diagonal subgroup.

6.2 Study of a simplified model

In this section we consider a model

$$S = \int d^3x \{ -(\partial^t \phi^*)(\partial_t \phi) + A(\partial^i \phi^*)(\partial_i \phi) - B \left[(\partial^i \phi^*)(\partial_i \phi) \right]^2 \}, \tag{6.32}$$

which is just (6.1) with $F = G = 0$. We can see that this theory is invariant under $U(1)$. The field will be parametrized as

$$\phi(t,x,y) = \alpha \left[x + u_x(t,x,y) \right] + i\beta \left[(y + u(t,x,y) \right], \tag{6.33}$$

where the perturbations are u_x, u_y. For this configuration, the ground states are those that satisfy the relation

$$\alpha^2 + \beta^2 = \frac{A}{2B}. \tag{6.34}$$

The action at second order in perturbations for this background is

$$\begin{aligned} S &= \int d^3x \left\{ -4\beta^4 B(\partial_y u_y)(\partial^y u_y) + 4\beta^2(-A + 2\beta^2 B)(\partial_y u_y)(\partial^x u_x) \right. \\ &\left. + \frac{1}{2B} \left[-2(A - 2\beta^2 B)^2 (\partial_x u_x)(\partial^x u_x) + (A^2 - 2\beta^2 B)(\partial_t u_x)(\partial^t u_x) + 2\beta^2 B(\partial_t u_y)(\partial^t u_y) \right] \right\}, \end{aligned} \tag{6.35}$$

and the equations of motion obtained from it are

$$\frac{A - 2\beta^2 B}{B} \left[4\beta^2 B \partial_x \partial_y u_y + 2(A - 2\beta^2 B)\partial_x \partial_x u_x - \partial_t \partial_t u_x \right] = 0, \tag{6.36}$$

$$2\beta^2 \left[4\beta^2 B \partial_y \partial_y u_y + 2(A - 2\beta^2 B)\partial_x \partial_y u_x - \partial_t \partial_t u_y \right] = 0. \tag{6.37}$$

In order to compute the dispersion relations we take the Fourier transform of the perturbations,

$$u_x(\omega,\vec{q}) = u_x e^{i(\omega t-\vec{q}\cdot\vec{x})}, \tag{6.38}$$

$$u_y(\omega,\vec{q}) = u_y e^{i(\omega t-\vec{q}\cdot\vec{x})}. \tag{6.39}$$

Given this, the equations of motion (6.36), (6.37) transform into

$$-\frac{A-2\beta^2 B}{B}\left[2Aq_x^2 u_x + 4\beta^2 B q_x(-q_x u_x + q_y u_y) - \omega^2 u_x\right] = 0, \tag{6.40}$$

$$2\beta^2\left[-2Aq_x q_y u_x + 4\beta^2 B q_y(q_x u_x - q_y u_y) + \omega^2 u_y\right] = 0. \tag{6.41}$$

Taking derivatives of these equation we can build the matrix

$$M = \begin{pmatrix} -(-A+2\beta^2 B)(-2Aq_x^2 + 4\beta^2 B q_x^2 + \omega^2)/B & -4\beta^2(A-2\beta^2 B)q_x q_y \\ -4\beta^2(A-2\beta^2 B)q_x q_y & 2\beta^2(-4\beta^2 B q_y^2 + \omega^2) \end{pmatrix}, \tag{6.42}$$

whose eigenvalues give the dispersion relations. These relations are

$$\omega^2 = 0, \tag{6.43}$$

$$\omega^2 = 2Aq_x^2 + 4\beta^2 B(-q_x^2 - q_y^2). \tag{6.44}$$

The relation (6.43) indicates the appearance of a non-propagating mode which receives the name of fracton in the literature and its implications will be discussed in the next section. In (6.44) we see the emergence of another massless mode, but propagating in this case. It can propagate in either direction.

6.3 Dispersion relations and exotic modes

In this section we will work out a simplified version of (6.1) taking $F = 0$,

$$S = \int d^3x\{-(\partial^t\phi^*)(\partial_t\phi) + A(\partial^i\phi^*)(\partial_i\phi) - B\left[(\partial^i\phi^*)(\partial_i\phi)\right]^2$$
$$+G(\partial_i\phi^*)(\partial^i\phi^*)(\partial^j\phi)(\partial_j\phi)\}. \tag{6.45}$$

The first line of the equation is the simplest action one can write to study the spontaneous breaking of translations with a model similar to the Mexican hat. The term proportional to G is the one that will allow us to study some exotic features. We will parametrize the field as

$$\phi(x) = \rho e^{ikx}, \tag{6.46}$$

where $\phi(x) = \phi(t, \vec{x})$ and the condition to be in a minimum of energy is then

$$\rho^2 k^2 = \frac{1}{2} \frac{A}{B - G}. \tag{6.47}$$

Let $\bar{\phi}$ be a vacuum configuration of the field. We can write the perturbations around the vacuum as

$$\phi(x) = \bar{\phi}(x) + \delta\phi(x) = [\rho + \alpha(x) + i\beta(x)]\, e^{ikx}, \tag{6.48}$$

where $\alpha(x), \beta(x) \ll 1$ are the perturbations. The equations of motion obtained from (6.45) are

$$\partial_0(\partial^0\phi) + A\partial_i(\partial^i\phi) - 2B\partial_i\left[(\partial_j\phi)(\partial^j\phi^*)\right](\partial^i\phi) - 2B(\partial_j\phi)(\partial^j\phi^*)\partial_i(\partial^i\phi^*)$$
$$+2G\partial_i(\partial^i\phi^*)(\partial_j\phi^*)(\partial^j\phi) + 2G(\partial^i\phi^*)\partial_i\left[(\partial_j\phi)(\partial^j\phi)\right] = 0, \tag{6.49}$$

and the field as written in (6.46) satisfies the equation as long as the condition (6.47) is satisfied. The equations of motion at first order in the perturbations are

$$\partial_0(\partial^0\alpha) + \left[A - 2\rho^2k^2(B+G)\right]\partial_i(\partial^i\alpha) + 4G\rho^2(k\partial_\mu)^2\alpha + 2Ak(\partial_i\beta) + 2Ak^2\alpha = 0, \tag{6.50}$$

$$\partial_0(\partial^0\beta) - 4\rho^2(B-G)(k\partial_\mu)^2\beta - 2Ak(\partial_i\alpha) = 0. \tag{6.51}$$

Note that these are coupled equations. This will become relevant afterwards. In order to obtain the dispersion relations we have to switch to momentum space. We will take the Fourier ansatz of the fields to be

$$\begin{aligned} \alpha(t, \vec{x}) &= \tilde{\alpha} e^{-i(\omega t - \vec{q}\cdot\vec{x})} \\ \beta(t, \vec{x}) &= \tilde{\beta} e^{-i(\omega t - \vec{q}\cdot\vec{x})}. \end{aligned} \tag{6.52}$$

With this, we can express (6.50) and (6.51) in momentum space as

$$\left[-\omega^2 + 4G\rho^2 k^2 q_t^2 + 2Ak^2\right]\alpha = -2iAkqb \tag{6.53}$$

$$\left[-\omega^2 + 2Aq_l^2\right]\beta = 2iAkqa, \tag{6.54}$$

where we have defined the longitudinal momentum q_l

$$q_l \equiv \frac{\vec{k}\cdot\vec{q}}{|\vec{k}|}, \tag{6.55}$$

and the transversal momentum q_t,

$$q_t^2 \equiv q^2 - q_l^2. \tag{6.56}$$

Assuming that the term $2Ak^2$ in (6.53) is much bigger than any other term appearing in this expression, we can get that the field at second order in ω and q_t is

$$\alpha(\omega, q) = -2iAkqb\left\{\frac{1}{2Ak^2} - \frac{1}{4A^2k^2}\left[-\omega^2 + 4G\rho^2k^2q_t^2 + \dots\right]\right\}. \tag{6.57}$$

Substituting (6.57) into (6.54) we obtain the first dispersion relation

$$\omega^2 = 4G\rho^2 q_l^2 q_t^2, \tag{6.58}$$

which corresponds to a gapless mode because if we set $q_t = q_l = 0$, then $\omega = 0$. Combining (6.53) and (6.54) we get

$$\left[-\omega^2 + 2Aq_l^2\right]\left[-\omega^2 + 4G\rho^2k^2q_t^2 + 2Ak^2\right]\alpha = 4A^2k^2q_l^2\alpha. \tag{6.59}$$

Assuming a perturbative solution of ω in terms of q we obtain

$$\omega^2 = 2Ak^2 + 2Aq_l^2 + 4G\rho^2k^2q_t^2 + \dots, \tag{6.60}$$

that now represents the dispersion relation of a gapped mode. The quadratic action for the fluctuations in position space is

$$\mathcal{L} = (\partial_0\alpha)(\partial^0\alpha) + (\partial_0\beta)(\partial^0\beta) - 4Gk^2\rho^2\left[(\partial_i\alpha)(\partial^i\alpha) - \frac{(k\partial_\mu\alpha)(k\partial^\mu\alpha)}{k^2}\right] - 2Ak^2\left(\alpha + i\frac{k\partial_\mu\beta}{k^2}\right)^2. \tag{6.61}$$

Now we can obtain the effective theory for β by putting the field α on-shell at leading order in derivatives, that is

$$\alpha = -i\frac{k\partial_\mu\beta}{k^2} + \dots, \tag{6.62}$$

obtaining that the effective theory is

$$\mathcal{L} = \frac{1}{4}(\partial_0\beta)(\partial^0\beta) + G\rho^2\left[\frac{(k\partial_\mu\partial_i\beta)(k\partial^\mu\partial^i\beta)}{k^2} - \frac{(k\partial_\mu k\partial_\mu\beta)(k\partial^\mu k\partial^\mu\beta)}{k^4}\right] + \dots. \tag{6.63}$$

This Lagrangian is valid only if the energy of the system is lower than the energy required to create particles of the α type. Effective theories must give consistent dispersion relations. To check that, we can see that in the limit $G \to 0$, (6.58) would transform into a trivial relation ($\omega^2 = 0$) and we would obtain the same relation from (6.63). More details on why a theory with only kinetic terms yields a trivial dispersion relation will be found later in this section.

Another relevant comment about exotic modes that can be made at this point is that (6.58) decouples energy and momentum. Non-exotic dispersion relations are such that the energy is proportional to the momentum of the particle. In the case of this dispersion relation we see that the energy depends on the product of the longitudinal and the transverse moment. This means that the energy can be zero even if the momentum is very large in one direction. Typically, low energy implies low momentum and low momentum and this can be translated into the non-sensitivity to the aspects of the theory at short distances. This observation plays a key role in quantum field theory, especially when renormalizing a theory. This exotic model (and other models where the energy and the momentum decoupled in the mentioned sense) could potentially break the mechanism of renormalization.

In (6.63), in the limit $G \to 0$, we can also see that a new symmetry has appeared. This new symmetry is

$$\beta(t, x, y) \to \beta(t, x, y) + f(x, y), \tag{6.64}$$

where $f(x, y)$ is a completely arbitrary function that depends only on the spatial coordinates. This symmetry can not be anticipated from (6.45). In fact, this symmetry is valid only in the approximation of low energies. Symmetries that appear in effective field theories receive the name of emergent symmetries.

6.3.1 Comments on trivial dispersion relations

Now we return to our discussion of the consequences of trivial dispersion relations such as (6.58) in the limit $G \to 0$ or (6.43). Recall that dispersion relations are a mathematical relation between the energy and the momentum of a particle. Trivial dispersion relations mean that the energy of the particle is zero irrespectively of its momentum.

To gain an insight on the physical meaning of a trivial dispersion relation we consider the simple but interesting elastic string. We consider a set of N masses, each one of mass m, connected by ideal springs. When the elongation of the springs is zero, the distance between consecutive masses is ϵ. We will study the dynamics of the collection of balls, considered by simplicity that they can only move in one direction, using the displacement of the ith ball with respect to its position of equilibrium, $\phi_i(t)$.

The kinetic energy of the system can be written as

$$T = \sum_{i=1}^{N} \frac{m}{2} \dot{\phi}_i^2(t) = \rho \sum_{i=1}^{N} \epsilon \dot{\phi}_i^2(t), \tag{6.65}$$

where we have defined the linear mass density as

$$\rho = \frac{M}{L} = \frac{Nm}{(N-1)\epsilon} \simeq \frac{m}{\epsilon}, \tag{6.66}$$

being M the total mass and L the total length of this chain. The potential can be written as

$$V_i = \frac{k_\epsilon}{2}(l_i - \epsilon)^2, \tag{6.67}$$

where k_ϵ is the elastic constant, which depends on the rest length and l_i is the elongation of the ith spring. With this, $l_i - \epsilon = \phi_{i+1}(t) - \phi_i(t)$, and the potential can be rewritten as

$$V = \sum_{i=1}^{N-1} \frac{k_\epsilon}{2}(\phi_{i+1}(t) - \phi_i(t))^2 = \rho \sum_{i=1}^{N-1} \frac{c^2}{2\epsilon}(\phi_{i+1}(t) - \phi_i(t))^2, \tag{6.68}$$

where we have defined $\rho c^2 \equiv k_\epsilon \epsilon$. With this, the Lagrangian of the system is

$$L = T - V = \rho \sum_{i=1}^{N} \epsilon \dot{\phi}_i^2(t) - \rho \sum_{i=1}^{N-1} \frac{c^2}{2\epsilon}(\phi_{i+1}(t) - \phi_i(t))^2. \tag{6.69}$$

Now we take the so-called continuum limit, which consists on taking the limits

$$\epsilon \to 0, \tag{6.70}$$
$$N \to \infty, \tag{6.71}$$
$$m \to 0, \tag{6.72}$$

simultaneously but with $M = Nm$ and $L = (N-1)\epsilon$ fixed. We introduced the quantities $\rho\epsilon$ and ρc^2 because they remain constant when taking this limit. In the continuum limit, (6.69) takes the form

$$L = \frac{\rho}{2} \int_0^L dx \left[\left(\frac{\partial \phi(t,x)}{\partial t} \right)^2 - c^2 \left(\frac{\partial \phi(t,x)}{\partial x} \right)^2 \right]. \tag{6.73}$$

The equations of motion obtained from this Lagrangian are

$$\Box \phi = \partial_t^2 \phi - \nabla^2 \phi = 0. \tag{6.74}$$

Transforming this equation into Fourier space, that is, substituting

$$\phi = \phi e^{-i(\omega t - qx)}, \tag{6.75}$$

we obtain a dispersion relation of the type $\omega^2 = q^2$. The fact of having a non-trivial dispersion relation comes from the masses being connected via springs with a given restoring force. This makes possible the transmission of waves through the chain. In the case there was no restoring force, the potential term in (6.69) would not have appeared and neither the term involving spatial derivatives in (6.74), meaning that the dispersion relation would be $\omega^2 = 0$ and no wave could be transmitted through the chain. Since there is no restoring force, the chain can be deformed without turning to its original position. In this sense, we can regard $\omega^2 = 0$ as a plastic medium that does not restore its form when deformed.

7 Conclusions

During this thesis we have studied the effective theory and dispersion relations resulting from the spontaneous breaking of translational symmetry in two theories. The massless mode of the full theory (the one with $F \neq 0$) received the name of phonon since it arose from the spontaneous breaking of translational symmetry. When considering this same theory in the case $F = G = 0$, we obtained a trivial dispersion relation corresponding to a fractonic mode. In the case where only $F = 0$, we obtained another fractonic mode but its dispersion relation (6.58) was trivial only in the limit $G \to 0$ as we expected. However, this dispersion relation was of big interest because it decouples energy and momentum in the sense that we can obtain null energy even if the momentum is very large in one component. This implies that there is not a clear limit between the low energy regime and the high energy limit and it can have deep implications while renormalizing the theory, thus providing a new line of research.

We have also made clear that this formalism can be applied to other fields. In particular, we have seen that, when applied to superfluids, spontaneous symmetry breaking can improve the theoretical predictions of some values, in particular, we provided the example of the critical temperature of the phase transition of superfluidity for He-4. The shown model improved the prediction that was made previously with other models.

The study of the spontaneous breaking of translational symmetry we performed was at zero temperature and chemical potential, thus imposing severe restrictions on possible applications of the models we presented. Further analysis at finite temperature and/or chemical potential could be of applicative interest, hence providing a new line of research. Other lines of research include the study in curved geometries, the classification of Goldstone bosons or the study of its interactions with other modes present in effective theories.

A Proof of Wigner's theorem

In this appendix we proof Wigner's theorem following the steps given in [11]. However, we will see that the proof is not complete. The full proof can be found, for instance, in [12]. We start by recalling that a ray is a vector in Hilbert space up to a phase. That means that two unit vectors φ, $\bar{\varphi}$ related by

$$\varphi = e^{i\alpha}\bar{\varphi}, \tag{A.1}$$

belong to the same equivalence class, a ray $\tilde{\varphi}$ in the Hilbert space. Note that the modulus of the scalar product of two rays $\tilde{\varphi}$, $\tilde{\chi}$ is well-defined if we choose two arbitrary representatives of the equivalence class,

$$|(\varphi, \chi)| = |(\bar{\varphi}, \bar{\chi})|, \tag{A.2}$$

where $(\cdot, \cdot)$ indicates the scalar product. Now suppose there is a correspondence between rays in the Hilbert space $\mathcal{H}$

$$\tilde{\varphi} \to T\tilde{\varphi}, \tag{A.3}$$

that leaves the modulus of the scalar product invariant,

$$|(\tilde{\varphi}, \tilde{\chi})| = |(T\tilde{\varphi}, T\tilde{\chi})|. \tag{A.4}$$

According to Wigner's theorem, we will be able to choose the phases of the vectors in such a way that the correspondence between rays becomes a correspondence between vectors,

$$\varphi \to U\varphi, \quad |(U\varphi, U\chi)| = |(\varphi, \chi)|, \tag{A.5}$$

where U is either linear and unitary,

$$(U\varphi, U\chi) = (\varphi, \chi), \tag{A.6}$$

or antiunitary (that is, antilinear and unitary)

$$(U\varphi, U\chi) = (\chi, \varphi) = (\varphi, \chi)^*. \tag{A.7}$$

Let $\{\chi_i\}$ be an N-dimensional orthonormal basis of $\mathcal{H}$, that is $(\chi_i, \chi_j) = \delta_{ij}$. From this point, the sub-indices i, k will run from 1 to N and j, l from 2 to N.

Now we choose a representative $\chi_1'' \equiv \chi_1$ of the equivalence class $T\tilde{\chi}_1$ and another representative $\tilde{\chi}_j''$ of the equivalence class $T\tilde{\chi}_j'$. According to (A.4), the set formed by (χ_1'', χ_j'') is another orthonormal basis of $\mathcal{H}$.

We consider a set of vectors $\varphi_j = \chi_1 + \chi_j$ and the transformation of the ray $\tilde{\varphi}$ to be $T\tilde{\varphi}$. If φ_j'' is a representative of $T\tilde{\varphi}_j$,

$$|(\chi_1', \varphi_j'')| = |(\chi_1, \varphi_j)| = 1$$

$$|(\chi_j'', \varphi_l'')| = |(\chi_j, \varphi_l)| = \delta_{jl}, \tag{A.8}$$

so its only components will be in the directions of χ_1' and χ_j'', that is

$$\varphi_j'' = c_j\chi_1' + d_j\chi_j'', \tag{A.9}$$

where $|c_j| = |d_j| = 1$. Recall that even if the indices are repeated, we are not adopting Einstein's summation convention and the index j is not summed over. Now we can choose new representatives

$$\varphi_j' = \frac{1}{c_j}\varphi_j'', \quad \chi_j' = \frac{d_j}{c_j}\chi_j''. \tag{A.10}$$

With this,

$$\varphi_j' = \frac{1}{c_j}(c_j\chi_1' + d_j\chi_j'') = \chi_1' + \chi_j'. \tag{A.11}$$

Note that we have defined an operation on vectors of $\mathcal{H}$,

$$\chi_1 + \chi_j \to (\chi_1 + \chi_j)' = \chi_1' + \chi_j', \tag{A.12}$$

where $\chi_1' \in T\tilde{\chi}_1$, $\chi_j' \in T\tilde{\chi}_j$ and $\chi_1' + \chi_j' \in T(\widetilde{\chi_1 + \chi_j})$. Now we have to check if an arbitrary vector ψ transforms as

$$\psi = \sum_{k=1}^{N} c_k\chi_k \to \psi' = \sum_{k=1}^{N} c_k'\chi_k'. \tag{A.13}$$

If the vector transforms in this way, we have

$$|c_k'| = |(\chi_k', \psi')| = |(\chi_k, \psi)| = |c_k|, \tag{A.14}$$

and

$$(\chi_1 + \chi_j, \psi) = c_1 + c_j, \quad (\chi_1' + \chi_j', \psi') = c_1' + c_j' \Rightarrow |c_1 + c_j| = |c_1' + c_j|. \tag{A.15}$$

Given this last relation and the conditions $c_1 = c_1', c_j = c_j'$ we can define

$$c_1 = |c_1|e^{i\theta_1} \quad , \quad c_j = |c_j|e^{i\theta_j},$$

$$c_1' = |c_1'|e^{i\theta_1'} \quad , \quad c_j' = |c_j'|e^{i\theta_j'}, \tag{A.16}$$

where the angles verify the relation

$$\cos(\theta_1 - \theta_j) = \cos\big(\theta'_1 - \theta'_j\big), \tag{A.17}$$

whose solutions are

$$\theta_1 - \theta_j = \theta'_1 - \theta'_j, \tag{A.18}$$

$$\theta_1 - \theta_j = -(\theta'_1 - \theta'_j). \tag{A.19}$$

In the first case we can define the phase of ψ' in such a way hat $c'_1 = c_1$, which implies $\theta_1 = \theta'_1$. Moreover, $\theta'_j = \theta_j$ as a consequence of (A.18) and $c'_j = c_j$, so

$$\psi' = \sum_{k=1}^{N} c_k \chi'_k. \tag{A.20}$$

If we consider now a vector

$$\eta = \sum_{k=1}^{N} d_k \chi_k, \tag{A.21}$$

where $d'_1 = d_1$ (repeating the computation that lead us to $c'_1 = c_1$), then

$$(\lambda\psi + \mu\eta)' = \sum_{k=1}^{N} (\lambda c_k + \mu d_k)\chi'_k = \lambda\psi' + \mu\eta', \tag{A.22}$$

where λ and μ are complex numbers. Now given that the modulus of the scalar product is conserved and that we can choose a phase, we can choose the transformation T to be linear and unitary. In the second case we can define the phase of ψ' such that $c'_1 = c^*_1$, so that $c'_j = c^*_j$ and

$$\psi' = \sum_{k=1}^{N} c^*_k \chi'_k. \tag{A.23}$$

Now the transformation of $\lambda\psi + \mu\eta$ is

$$(\lambda\psi + \mu\eta)' = \sum_{k=1}^{N} \left[(\lambda c_k + \mu d_k)\chi_k\right]' = \lambda^*\psi' + \mu^*\eta', \tag{A.24}$$

and then the scalar product transforms as

$$(\psi', \eta') = (\psi, \eta)^* = (\eta, \psi), \tag{A.25}$$

so the transformation is antiunitary (it is antilinear and unitary), concluding the proof of the theorem.

In order to obtain the full proof, one has to show that (A.18) can not hold for c_j and that (A.19) can not hold for c_l with $l \neq j$. To prove this, one has to examine the properties of a vector $\psi = \chi_1 + \chi_j + \chi_l$ under a transformation.

References

[1] A. J. Beekman, L. Rademaker, J. van Wezel, "An introduction to Spontaneous Symmetry Breaking"

[2] P. Di Francesco, P. Mathieu, D. Sénéchal, "Conformal Field Theory", Springer, 1997

[3] D. Naegels, "Perturbative and holographic study of symmetry breaking in non-realtivistic theories", Université Libre de Bruxelles, 2022

[4] M. E. Peskin, D. V. Schroeder, "An introduction to Quantum Field Theory", Westview Press, 1995

[5] L. H. Ryder, "Quantum Field Theory", Cambridge University Press, 1996

[6] K. Huang, "Introduction to Statistical Physics", CRC Press, 2010

[7] A. Jain, "Notes on symmetries in particle physics", arXiv:2109.12087v1

[8] R. J. Donnelly and C. F. Barenghi, "The Observed Properties of Liquid Helium at the Saturated Vapor Pressure," J. Phys. Chem. Ref. Data. Vol. 27, No. 6 (1998).

[9] D. Musso, "Simplest phonons and pseudo-phonons in field theory", arXiv:1810.01799v3

[10] R. Thibes, "Natural Higher-Derivatives Generalization for the Klein-Gordon Equation", arXiv:2011.02567v1

[11] M. le Bellac, "Quantum Physics", Cambridge University Press, 2006

[12] S. Weinberg, "The Quantum Theory of Fields", Cambridge University Press, 1995