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Infinite Series and Logarithmic Integrals Associated to Differentiation with Respect to Parameters of the Whittaker $W_{\kappa,\mu}(x)$ Function II

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Abstract: In the first part of this investigation, we considered the parameter differentiation of the Whittaker function $M_{\kappa,\mu}(x)$. In this second part, first derivatives with respect to the parameters of the Whittaker function $W_{\kappa,\mu}(x)$ are calculated. Using the confluent hypergeometric function, these derivatives can be expressed as infinite sums of quotients of the digamma and gamma functions. Furthermore, it is possible to obtain these parameter derivatives in terms of infinite integrals, with integrands containing elementary functions (products of algebraic, exponential, and logarithmic functions), from the integral representation of $W_{\kappa,\mu}(x)$. These infinite sums and integrals can be expressed in closed form for particular values of the parameters. Finally, an integral representation of the integral Whittaker function $\mathrm{wi}_{\kappa,\mu}(x)$ and its derivative with respect to κ, as well as some reduction formulas for the integral Whittaker functions $\mathrm{Wi}_{\kappa,\mu}(x)$ and $\mathrm{wi}_{\kappa,\mu}(x)$, are calculated.

Keywords: derivatives with respect to parameters; Whittaker functions; integral Whittaker functions; incomplete gamma functions; sums of infinite series of psi and gamma; infinite integrals involving Bessel functions

MSC: 33B15; 33B20; 33C10; 33C15; 33C20; 33C50; 33E20

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1. Introduction

Two functions, $M_{\kappa,\mu}(x)$ and $W_{\kappa,\mu}(x)$, were introduced to the mathematical literature by Whittaker [1] in 1903, and they are linearly independent solutions of the following second-order differential equation:

$$\frac{d^2y}{dx^2} + \left(\frac{\frac{1}{4}-\mu}{x^2} + \frac{\kappa}{x} - \frac{1}{4}\right)y = 0,$$
$$y(x) = C_1\, M_{\kappa,\mu}(x) + C_2\, W_{\kappa,\mu}(x),$$
$$2\mu \neq -1, -2, \ldots$$

where κ and μ are parameters. For particular values of these parameters, the Whittaker functions $M_{\kappa,\mu}(x)$ and $W_{\kappa,\mu}(x)$ can be reduced to a variety of elementary and special functions (such as modified Bessel functions, incomplete gamma functions, parabolic cylinder functions, error functions, logarithmic and cosine integrals, as well as the generalized Hermite and Laguerre polynomials). Recently, Mainardi et al. [2] investigated the special case whereby the Wright function can be expressed in terms of Whittaker functions.

The Whittaker functions can be expressed as [3] (Eqn. 13.14.2):

$$\begin{aligned} \mathrm{M}_{\kappa,\mu}(z) &= z^{\mu+1/2}e^{-z/2}\,{}_1F_1\left(\left.\begin{array}{c}\frac{1}{2}+\mu-\kappa\\ 1+2\mu\end{array}\right|z\right) \\ 2\mu &\neq -1,-2,\ldots \end{aligned} \tag{1}$$

and [3] (Eqn. 13.14.33):

$$\begin{aligned} \mathrm{W}_{\kappa,\mu}(z) &= \frac{\Gamma(-2\mu)}{\Gamma\left(\frac{1}{2}-\mu-\kappa\right)}\mathrm{M}_{\kappa,\mu}(z)+\frac{\Gamma(2\mu)}{\Gamma\left(\frac{1}{2}+\mu-\kappa\right)}\mathrm{M}_{\kappa,-\mu}(z), \\ 2\mu &\notin \mathbb{Z}, \end{aligned} \tag{2}$$

where $\Gamma(x)$ denotes the gamma function, and the Kummer function is defined as [4] (Eqn. 47:3:1):

$$ {}_1F_1\left(\left.\begin{array}{c}a\\ b\end{array}\right|z\right)=\sum_{n=0}^{\infty}\frac{(a)_n}{(b)_n}\frac{z^n}{n!}, \tag{3}$$

where $(\alpha)_n=\frac{\Gamma(\alpha+n)}{\Gamma(\alpha)}$ denotes the Pochhammer polynomial and

$$ {}_pF_q\left(\left.\begin{array}{c}a_1,\ldots,a_p\\ b_1,\ldots,b_q\end{array}\right|x\right)=\sum_{n=0}^{\infty}\frac{(a_1)_n\cdots(a_p)_n}{(b_1)_n\cdots(b_q)_n}\frac{x^n}{n!}, \tag{4}$$

is the generalized hypergeometric function.

Also, the Whittaker function $\mathrm{W}_{\kappa,\mu}(x)$ can be expressed as [3] (Eqn. 13.14.3):

$$\mathrm{W}_{\kappa,\mu}(z)=e^{-z/2}z^{\mu+1/2}\,\mathrm{U}\left(\frac{1}{2}+\mu-\kappa,1+2\mu,z\right), \tag{5}$$

where $\mathrm{U}(a,b,z)$ denotes the Tricomi function.

The analytical properties of the Whittaker functions (see [3–11]) are of great interest in mathematical physics, because these functions are involved in many applications, such as the solutions of the wave equation in paraboloidal coordinates, the behavior of charged particles in fields with Coulomb potentials, stationary Green's functions in atomic and molecular calculations in quantum mechanics (i.e., the solution of the Schrödinger equation for a harmonic oscillator), probability density functions, and in many other physical and engineering problems [10,12–14].

Mostly, the Whittaker functions are regarded as a function of variable x with fixed values of parameters κ and μ, although there are a few investigations where mathematical operations associated with both parameters are considered, especially for the κ parameter [13,15–17]. In this context, it is worthwhile mentioning Laurenzi's paper [13], where the calculation of the derivative of $\mathrm{W}_{\kappa,1/2}(x)$ with respect to κ, when this parameter is an integer, is derived. In [17], Buschman showed that the derivative of $\mathrm{W}_{\kappa,\mu}(x)$ with respect to the parameters can be expressed in terms of finite sums of these $\mathrm{W}_{\kappa,\mu}(x)$ functions. Higher derivatives of the Whittaker functions with respect to parameter κ were discussed by Abad and Sesma [15], and integrals with respect to parameter μ by Becker [16]. Since the Whittaker functions are related to the confluent hypergeometric function, it is worth mentioning the investigation of the derivatives of generalized hypergeometric functions presented by Ancarini and Gasaneo [18] and Sofostasios and Brychkov [19].

The integral Whittaker functions were introduced by us [20] as follows:

$$\mathrm{Wi}_{\kappa,\mu}(x) = \int_0^x \frac{\mathrm{W}_{\kappa,\mu}(t)}{t}dt, \tag{6}$$

$$\mathrm{wi}_{\kappa,\mu}(x) = \int_x^{\infty} \frac{\mathrm{W}_{\kappa,\mu}(t)}{t}dt. \tag{7}$$

In the first part of this investigation, we calculated some reduction formulas for the first derivatives, with respect to the parameters of the Whittaker function $\mathrm{M}_{\kappa,\mu}(x)$. In the current paper, the main attention will be devoted to the calculation of reduction formulas for the first parameter derivatives of the Whittaker function $W_{\kappa,\mu}(x)$. For this purpose, we analyze the first derivative of this function with respect to the parameters from the corresponding series and integral representations. Direct differentiation of the Whittaker functions, leads to infinite sums of quotients of the digamma and gamma functions. It is possible to calculate these sums in closed form in some cases, with the aid of the MATHEMATICA program. When the integral representations of the Whittaker function $W_{\kappa,\mu}(x)$ are taken into account, the results of differentiation can be expressed in terms of Laplace transforms of elementary functions. Integrands of the these Laplace-type integrals include products of algebraic, exponential, and logarithmic functions. New groups of infinite integrals, comparable to those investigated by Kölbig [21], Geddes et al. [22], and Apelblat and Kravitzky [23], are calculated in this paper.

Also, we will focus our attention on the integral Whittaker functions $\mathrm{Wi}_{\kappa,\mu}(x)$ and $\mathrm{wi}_{\kappa,\mu}(x)$, in order to derive some new reduction formulas, as well as an integral representation of $\mathrm{wi}_{\kappa,\mu}(x)$ and its first derivative with respect to parameter κ.

2. Parameter Differentiation of $W_{\kappa,\mu}$ via Kummer Function ${}_1F_1$

Notation 1. *Unless indicated otherwise, it is assumed throughout the paper that x is a real variable and z is a complex variable.*

Definition 1. *According to the notation introduced by Ancarini and Gasaneo [18,24], define*

$$G^{(1)}\left(\begin{array}{c} a \\ b \end{array}\middle| x\right) = \frac{\partial}{\partial a}\left[{}_1F_1\left(\begin{array}{c} a \\ b \end{array}\middle| x\right)\right], \tag{8}$$

and

$$H^{(1)}\left(\begin{array}{c} a \\ b \end{array}\middle| x\right) = \frac{\partial}{\partial b}\left[{}_1F_1\left(\begin{array}{c} a \\ b \end{array}\middle| x\right)\right]. \tag{9}$$

2.1. Derivative with Respect to the First Parameter $\partial W_{\kappa,\mu}(x)/\partial\kappa$

Taking into account (1) and (8), direct differentiation of (2) yields:

$$\begin{aligned} &\frac{\partial W_{\kappa,\mu}(x)}{\partial\kappa} \\ = &\frac{\Gamma(-2\mu)}{\Gamma\left(\frac{1}{2}-\mu-\kappa\right)}\left[\psi\left(\frac{1}{2}-\mu-\kappa\right)M_{\kappa,\mu}(x) - x^{1/2+\mu}e^{-x/2}\,G^{(1)}\left(\begin{array}{c} \frac{1}{2}+\mu-\kappa \\ 1+2\mu \end{array}\middle| x\right)\right] \\ &+\frac{\Gamma(2\mu)}{\Gamma\left(\frac{1}{2}+\mu-\kappa\right)}\left[\psi\left(\frac{1}{2}+\mu-\kappa\right)M_{\kappa,-\mu}(x) - x^{1/2-\mu}e^{-x/2}\,G^{(1)}\left(\begin{array}{c} \frac{1}{2}-\mu-\kappa \\ 1-2\mu \end{array}\middle| x\right)\right]. \end{aligned} \tag{10}$$

If we first apply Kummer's transformation formula [3] (Eqn. 13.2.39):

$${}_1F_1\left(\begin{array}{c} a \\ b \end{array}\middle| x\right) = e^x\,{}_1F_1\left(\begin{array}{c} b-a \\ b \end{array}\middle| -x\right), \tag{11}$$

we can rewrite (10) as

$$\begin{aligned} &\frac{\partial W_{\kappa,\mu}(x)}{\partial\kappa} \\ = &\frac{\Gamma(-2\mu)}{\Gamma\left(\frac{1}{2}-\mu-\kappa\right)}\left[\psi\left(\frac{1}{2}-\mu-\kappa\right)M_{\kappa,\mu}(x) + x^{1/2+\mu}e^{x/2}\,G^{(1)}\left(\begin{array}{c} \frac{1}{2}+\mu+\kappa \\ 1+2\mu \end{array}\middle| -x\right)\right] \\ &+\frac{\Gamma(2\mu)}{\Gamma\left(\frac{1}{2}+\mu-\kappa\right)}\left[\psi\left(\frac{1}{2}+\mu-\kappa\right)M_{\kappa,-\mu}(x) - x^{1/2-\mu}e^{-x/2}\,G^{(1)}\left(\begin{array}{c} \frac{1}{2}-\mu-\kappa \\ 1-2\mu \end{array}\middle| x\right)\right]. \end{aligned} \tag{12}$$

Theorem 1. *For $2\mu \notin \mathbb{Z}$, the following parameter derivative formula of $\mathrm{W}_{\kappa,\mu}(x)$ holds true:*

$$\left.\frac{\partial \mathrm{W}_{\kappa,\pm\mu}(x)}{\partial\kappa}\right|_{\kappa=\mu+1/2} = \sqrt{x}e^{-x/2} \tag{13}$$
$$\left\{x^{\mu}\left[\psi(-2\mu) - \frac{x}{2\mu+1}\,{}_2F_2\left(\left.\begin{array}{c}1,1\\2\mu+2,2\end{array}\right|x\right)\right] + \Gamma(2\mu+1)x^{-\mu}(-x)^{2\mu}\gamma(-2\mu,-x)\right\},$$

where $\gamma(\nu,z)$ denotes the lower incomplete gamma function (117).

Proof. First, note that
$$\frac{\partial \mathrm{W}_{\kappa,\mu}(x)}{\partial\kappa} = \frac{\partial \mathrm{W}_{\kappa,-\mu}(x)}{\partial\kappa}, \tag{14}$$
since [3] (Eqn. 13.14.31):
$$\mathrm{W}_{\kappa,\mu}(x) = \mathrm{W}_{\kappa,-\mu}(x). \tag{15}$$

Now, let us calculate $\partial \mathrm{W}_{\kappa,\mu}(x)/\partial\kappa\big|_{\kappa=\mu+1/2}$. For this purpose, take $\kappa = \mu + 1/2 - \epsilon$ in (12), to obtain

$$\left.\frac{\partial \mathrm{W}_{\kappa,\mu}(x)}{\partial\kappa}\right|_{\kappa=\mu+1/2-\epsilon} \tag{16}$$
$$= \frac{\Gamma(-2\mu)}{\Gamma(-2\mu+\epsilon)}\left[\psi(-2\mu+\epsilon)\mathrm{M}_{\mu+1/2-\epsilon,\mu}(x) + x^{1/2+\mu}e^{x/2}\,G^{(1)}\left(\left.\begin{array}{c}1+2\mu-\epsilon\\1+2\mu\end{array}\right|-x\right)\right]$$
$$+\frac{\Gamma(2\mu)}{\Gamma(\epsilon)}\left[\psi(\epsilon)\mathrm{M}_{\mu+1/2-\epsilon,-\mu}(x) - x^{1/2-\mu}e^{-x/2}\,G^{(1)}\left(\left.\begin{array}{c}-2\mu+\epsilon\\1-2\mu\end{array}\right|x\right)\right].$$

Note that, according to [3] (Eqn. 13.18.2)
$$\mathrm{M}_{\mu+1/2,\mu}(x) = e^{-x/2}x^{1/2+\mu}. \tag{17}$$

Further, from (1) and (11), we have
$$\begin{aligned}\mathrm{M}_{\mu+1/2,-\mu}(x) &= e^{x/2}x^{1/2-\mu}\,{}_1F_1\left(\left.\begin{array}{c}1\\1+2\mu\end{array}\right|-x\right)\\ &= e^{x/2}x^{1/2-\mu}\sum_{n=0}^{\infty}\frac{(-x)^n}{(1-2\mu)_n}.\end{aligned} \tag{18}$$

Taking into account [4] (Eqn. 45:6:2):
$$e^{x}\gamma(\nu,x) = \frac{x^{\nu}}{\nu}\sum_{n=0}^{\infty}\frac{x^n}{(1+\nu)_n},$$
rewrite (18) as
$$\mathrm{M}_{\mu+1/2,-\mu}(x) = -2\mu\,e^{-x/2}x^{1/2-\mu}(-x)^{2\mu}\gamma(-2\mu,-x). \tag{19}$$

Consider as well the reduction formula given in Equation (A1) in Appendix A:
$$G^{(1)}\left(\left.\begin{array}{c}a\\a\end{array}\right|x\right) = \frac{x\,e^{x}}{a}\,{}_2F_2\left(\left.\begin{array}{c}1,1\\a+1,2\end{array}\right|-x\right). \tag{20}$$

Finally, according to the property [4] (Eqn. 44:5:3):
$$\psi(z+1) = \frac{1}{z} + \psi(z),$$

see that

$$\lim_{\epsilon\to 0}\frac{\psi(\epsilon)}{\Gamma(\epsilon)}=\lim_{\epsilon\to 0}\frac{1}{\Gamma(\epsilon)}\left[\psi(\epsilon+1)-\frac{1}{\epsilon}\right]=-1. \quad (21)$$

Now, take the limit $\epsilon\to 0$ in (16), considering the results given in (14), (17), (19)–(21), to obtain (13), as we wanted to prove. □

Table 1 presents some explicit expressions for particular values of (13), obtained with the help of the MATHEMATICA program.

Table 1. Derivative of $\mathrm{W}_{\kappa,\mu}$ with respect to κ, by using (13).

κ	μ	$\frac{\partial \mathrm{W}_{\kappa,\mu}(x)}{\partial\kappa}$
$-\frac{3}{4}$	$\pm\frac{5}{4}$	$\frac{1}{3}x^{-3/4}e^{-x/2}\left[2x\,{}_2F_2\left(1,1;-\frac{1}{2},2;x\right)+3\pi\,\mathrm{erfi}\left(\sqrt{x}\right)+2\sqrt{\pi\,x}\,e^{x}(2x-3)-3\gamma+8-3\ln 4\right]$
$-\frac{1}{4}$	$\pm\frac{3}{4}$	$x^{-1/4}e^{-x/2}\left[2x\,{}_2F_2\left(1,1;\frac{1}{2},2;x\right)+\pi\,\mathrm{erfi}\left(\sqrt{x}\right)-2\sqrt{\pi\,x}\,e^{x}-\gamma+2-\ln 4\right]$
$-\frac{1}{6}$	$\pm\frac{2}{3}$	$\frac{1}{6}x^{-5/6}e^{-x/2}\Big\{3x^{2/3}\left[6x\,{}_2F_2\left(1,1;\frac{2}{3},2;x\right)-2\gamma+6-3\ln 3\right]$ $-6x^{2}\Gamma\left(-\frac{1}{3}\right)\mathrm{E}_{-1/3}(-x)-\sqrt{3}\pi\left[x^{2/3}+4(-x)^{2/3}\right]\Big\}$
$\frac{1}{6}$	$\pm\frac{1}{3}$	$\frac{1}{6}x^{-1/6}e^{-x/2}\Big\{-3x^{1/3}\left[6x\,{}_2F_2\left(1,1;\frac{4}{3},2;x\right)+2\gamma+3\ln 3\right]$ $-6x\,\Gamma\left(\frac{1}{3}\right)\mathrm{E}_{1/3}(-x)+\sqrt{3}\pi\left[x^{1/3}-4(-x)^{1/3}\right]\Big\}$
$\frac{1}{4}$	$\pm\frac{1}{4}$	$-x^{1/4}e^{-x/2}\left[2x\,{}_2F_2\left(1,1;\frac{3}{2},2;x\right)-\pi\,\mathrm{erfi}\left(\sqrt{x}\right)+\gamma+\ln 4\right]$
$\frac{3}{4}$	$\pm\frac{1}{4}$	$\frac{1}{3}e^{-x/2}\Big\{x^{3/4}\left[-2x\,{}_2F_2\left(1,1;\frac{5}{2},2;x\right)+3\left(\pi\,\mathrm{erfi}\left(\sqrt{x}\right)-\gamma+2-\ln 4\right)\right]-3\sqrt{\pi}\,x^{1/4}e^{x}\Big\}$
$\frac{5}{6}$	$\pm\frac{1}{3}$	$\frac{1}{30}x^{1/6}e^{-x/2}\Big\{-18\,x^{5/3}\,{}_2F_2\left(1,1;\frac{8}{3},2;x\right)+15\,x^{2/3}(3-2\gamma-3\ln 3)$ $-30\,\Gamma\left(\frac{5}{3}\right)\mathrm{E}_{5/3}(-x)-5\sqrt{3}\pi\left[x^{2/3}+4(-x)^{1/3}\right]\Big\}$
$\frac{5}{4}$	$\pm\frac{3}{4}$	$\frac{1}{30}x^{-1/4}e^{-x/2}\Big\{-2x^{3/2}\left[6x\,{}_2F_2\left(1,1;\frac{7}{2},2;x\right)-5\left(\pi\,\mathrm{erfi}\left(\sqrt{x}\right)-3\gamma+8-3\ln 4\right)\right]-15\sqrt{\pi}\,e^{x}(2x+1)\Big\}$

Next, we present another reduction formula of $\partial\mathrm{W}_{\kappa,\mu}(x)/\partial\kappa$, from the result found in [13].

Theorem 2. *The following reduction formula holds true for $n=1,2,\ldots$*

$$\left.\frac{\partial \mathrm{W}_{\kappa,\pm 1/2}(x)}{\partial\kappa}\right|_{\kappa=n} = (-1)^{n}(n-1)!\,e^{-x/2}\left[\sum_{\ell=0}^{n-1}\frac{n-\ell}{n+\ell}\,L_{\ell}^{(-1)}(x)+n\,L_{\ell}^{(-1)}(x)\ln x\right], \quad (22)$$

where $L_n^{(\alpha)}(x)$ denotes the Laguerre polynomial.

Proof. First, note that according to (14), we have

$$\frac{\partial \mathrm{W}_{\kappa,1/2}(x)}{\partial\kappa}=\frac{\partial \mathrm{W}_{\kappa,-1/2}(x)}{\partial\kappa}. \quad (23)$$

Therefore, let us calculate $\partial \mathrm{W}_{\kappa,1/2}(x)/\partial\kappa$. For this purpose, consider the formula [13]:

$$\left.\frac{\partial \mathrm{W}_{\kappa,1/2}(x)}{\partial\kappa}\right|_{\kappa=n} = (-1)^n (n-1)! \sum_{\ell=0}^{n-1} \frac{(-1)^\ell (n-\ell)}{\ell!(n+\ell)} \mathrm{W}_{\ell,1/2}(x) + \mathrm{W}_{n,1/2}(x) \ln x \quad (24)$$

Furthermore, from [3] (Eqn. 13.18.17), we have for $n = 0, 1, 2, \ldots$

$$\mathrm{W}_{\kappa+n,\kappa-1/2}(x) = (-1)^n n! e^{-x/2} x^\kappa L_n^{(2\kappa-1)}(x), \quad (25)$$

thus, applying (15) and taking $\kappa = 0$ in (25), we have

$$\mathrm{W}_{n,1/2}(x) = \mathrm{W}_{n,-1/2}(x) = (-1)^n n!\, e^{-x/2} L_n^{(-1)}(x). \quad (26)$$

Finally, insert (26) into (22) and consider (23), to obtain (22), as we wanted to prove. □

In Table 2 we collect some particular cases of (22), obtained with the help of the MATHEMATICA program.

Table 2. Derivative of $\mathrm{W}_{\kappa,\mu}$ with respect to κ, by using (22).

κ	μ	$\frac{\partial \mathrm{W}_{\kappa,\mu}(x)}{\partial\kappa}$
1	$\pm\frac{1}{2}$	$e^{-x/2}(x\ln x - 1)$
2	$\pm\frac{1}{2}$	$e^{-x/2}[x(x-2)\ln x - 3x + 1]$
3	$\pm\frac{1}{2}$	$e^{-x/2}\left[x\left(x^2-6x+6\right)\ln x - 5x^2 + 14x - 2\right]$

Note that for $n = 0$, we obtain an indeterminate expression in (22). We calculate this particular case with a result in the next section.

Theorem 3. *The following reduction formula holds true:*

$$\left.\frac{\partial \mathrm{W}_{\kappa,\pm 1/2}(x)}{\partial\kappa}\right|_{\kappa=0} = e^{-x/2} \left\{\ln x + \frac{1}{4\sqrt{\pi}}\left[G_{2,4}^{3,1}\left(\frac{x^2}{4}\middle|\begin{array}{c}\frac{1}{2},1\\ 0,0,\frac{1}{2},-\frac{1}{2}\end{array}\right) - (e^x-1)\, G_{1,3}^{3,0}\left(\frac{x^2}{4}\middle|\begin{array}{c}1\\ -\frac{1}{2},0,0\end{array}\right)\right]\right\}, \quad (27)$$

where $G_{p,q}^{m,n}\left(z\middle|\begin{array}{c}a_1,\ldots,a_p\\ b_1,\ldots,b_q\end{array}\right)$ *denotes the Meijer G-function.*

Proof. According to [3] (Eqn. 13.18.2), we have

$$\mathrm{W}_{\kappa,\kappa-1/2}(x) = e^{-x/2} x^\kappa, \quad (28)$$

thus, performing the derivative with respect to κ,

$$\left.\frac{\partial \mathrm{W}_{\kappa,\mu}(x)}{\partial\kappa}\right|_{\mu=\kappa-1/2} + \left.\frac{\partial \mathrm{W}_{\kappa,\mu}(x)}{\partial\mu}\right|_{\mu=\kappa-1/2} = e^{-x/2} x^\kappa \ln x.$$

Taking $\kappa = 0$ and considering (23), we have

$$\left.\frac{\partial \mathrm{W}_{\kappa,\pm 1/2}(x)}{\partial\kappa}\right|_{\kappa=0} = -\left.\frac{\partial \mathrm{W}_{0,\mu}(x)}{\partial\mu}\right|_{\mu=-1/2} + e^{-x/2}\ln x.$$

Finally, apply (31) and (33), to arrive at (27), as we wanted to prove. □

2.2. Derivative with Respect to the Second Parameter $\partial W_{\kappa,\mu}(x)/\partial\mu$

Theorem 4. *For $2\mu \notin \mathbb{Z}$, the following parameter derivative formula of $W_{\kappa,\mu}(x)$ holds true:*

$$\left.\frac{\partial W_{\kappa,\pm\mu}(x)}{\partial\mu}\right|_{\kappa=\mu+1/2} = \pm\sqrt{x}e^{-x/2} \tag{29}$$
$$\left\{x^{\mu}\left[\frac{x}{2\mu+1}\,{}_2F_2\left(\left.\begin{array}{c}1,1\\2\mu+2,2\end{array}\right|x\right) - \psi(-2\mu) + \ln x\right] - \Gamma(2\mu+1)x^{-\mu}(-x)^{2\mu}\gamma(-2\mu,-x)\right\}.$$

Proof. Differentiate the following reduction formula with respect to parameter μ [3] (Eqn. 13.18.2):

$$W_{\mu+1/2,\pm\mu}(x) = e^{-x/2}x^{1/2+\mu},$$

to obtain

$$\left.\frac{\partial W_{\kappa,\pm\mu}(x)}{\partial\kappa}\right|_{\kappa=\mu+1/2} \pm \left.\frac{\partial W_{\kappa,\pm\mu}(x)}{\partial\mu}\right|_{\kappa=\mu+1/2} = e^{-x/2}x^{1/2+\mu}\ln x. \tag{30}$$

Insert (13) into (30) to arrive at (29), as we wanted to prove. □

Table 3 shows the derivative of $W_{\kappa,\mu}(x)$ with respect to μ for particular values of κ and μ, using (29) and the help of the MATHEMATICA program.

Table 3. Derivative of $W_{\kappa,\mu}$ with respect to μ, by using (29).

κ	μ	$\frac{\partial W_{\kappa,\mu}(x)}{\partial\mu}$
$-\frac{3}{4}$	$\pm\frac{5}{4}$	$\pm\frac{1}{3}x^{-3/4}e^{-x/2}\left[2x\,{}_2F_2\left(1,1;-\frac{1}{2},2;x\right) + 3\pi\,\mathrm{erfi}\left(\sqrt{x}\right) + 2\sqrt{\pi x}\,e^{x}(2x-3) - 3\gamma + 8 - 3\ln(4x)\right]$
$-\frac{1}{4}$	$\pm\frac{3}{4}$	$\pm x^{-1/4}e^{-x/2}\left[2x\,{}_2F_2\left(1,1;\frac{1}{2},2;x\right) + \pi\,\mathrm{erfi}\left(\sqrt{x}\right) - 2\sqrt{\pi x}\,e^{x} - \gamma + 2 - \ln(4x)\right]$
$-\frac{1}{6}$	$\pm\frac{2}{3}$	$\pm\frac{1}{6}x^{-5/6}e^{-x/2}\left\{3x^{2/3}\left[6x\,{}_2F_2(1,1;\frac{2}{3},2;x) - 2\gamma + 6 - 3\ln 3 - 2\ln x\right]\right.$ $\left.-6x^2\Gamma\left(-\frac{1}{3}\right)E_{-1/3}(-x) - \sqrt{3}\pi\left[x^{2/3} + 4(-x)^{2/3}\right]\right\}$
$\frac{1}{6}$	$\pm\frac{1}{3}$	$\pm\frac{1}{6}x^{-1/6}e^{-x/2}\left\{-3x^{1/3}\left[6x\,{}_2F_2\left(1,1;\frac{4}{3},2;x\right) + 2\gamma + 3\ln 3 + 2\ln x\right]\right.$ $\left.-6x\,\Gamma\left(\frac{1}{3}\right)E_{1/3}(-x) + \sqrt{3}\pi\left[x^{1/3} - 4(-x)^{1/3}\right]\right\}$
$\frac{1}{4}$	$\pm\frac{1}{4}$	$\pm x^{1/4}e^{-x/2}\left[-2x\,{}_2F_2(1,1;\frac{3}{2},2;x) + \pi\,\mathrm{erfi}\left(\sqrt{x}\right) - \gamma - \ln(4x)\right]$
$\frac{3}{4}$	$\pm\frac{1}{4}$	$\pm\frac{1}{3}x^{1/4}e^{-x/2}\left\{\sqrt{x}\left[2x\,{}_2F_2(1,1;\frac{5}{2},2;x) - 3\left(\pi\,\mathrm{erfi}\left(\sqrt{x}\right) - \gamma + 2 - \ln(4x)\right)\right] + 3\sqrt{\pi}\,e^{x}\right\}$
$\frac{5}{6}$	$\pm\frac{1}{3}$	$\pm\frac{1}{30}x^{1/6}e^{-x/2}\left\{18\,x^{5/3}\,{}_2F_2(1,1;\frac{8}{3},2;x) + 15\,x^{2/3}(2\gamma + 3\ln 3 + 2\ln x - 3)\right.$ $\left.+30\,\Gamma\left(\frac{5}{3}\right)E_{5/3}(-x) + 5\sqrt{3}\pi\left[x^{2/3} + 4(-x)^{1/3}\right]\right\}$
$\frac{5}{4}$	$\pm\frac{3}{4}$	$\pm\frac{1}{30}x^{-1/4}e^{-x/2}\left\{2x^{3/2}\left[6x\,{}_2F_2(1,1;\frac{7}{2},2;x) - 5\left(\pi\,\mathrm{erfi}\left(\sqrt{x}\right) - 3\gamma + 8 - 3\ln(4x)\right)\right] + 15\sqrt{\pi}\,e^{x}(2x+1)\right\}$

Theorem 5. *The following parameter derivative formula of $W_{\kappa,\mu}(x)$ holds true:*

$$\frac{\partial W_{0,\mu}(x)}{\partial\mu} = \mathrm{sgn}(\mu)\sqrt{\frac{x}{\pi}}\left.\frac{\partial K_{\mu}(x/2)}{\partial\mu}\right|_{|\mu|}, \tag{31}$$

where $K_{\nu}(x)$ denotes the modified Bessel of the second kind (Macdonald function).

Proof. Differentiate with respect to μ the expression [3] (Eqn. 13.18.9):

$$\mathrm{W}_{0,\mu}(x) = \sqrt{\frac{x}{\pi}} K_\mu\left(\frac{x}{2}\right), \tag{32}$$

to obtain

$$\left.\frac{\partial \mathrm{W}_{0,\pm\mu}(x)}{\partial \mu}\right|_{\mu\geq 0} = \pm \left.\frac{\partial \mathrm{W}_{0,\mu}(x)}{\partial \mu}\right|_{\mu\geq 0} = \pm\sqrt{\frac{x}{\pi}} \left.\frac{\partial K_\mu(x/2)}{\partial \mu}\right|_{\mu\geq 0},$$

as we wanted to prove. □

The order derivative of $K_\mu(x)$ is given in terms of Meijer G-functions for $\mathrm{Re}\, x > 0$, and $\mu \geq 0$ [25]:

$$\begin{aligned} &\frac{\partial K_\mu(x)}{\partial \mu} \\ &= \frac{\mu}{2}\left[\frac{K_\mu(x)}{\sqrt{\pi}} G^{3,1}_{2,4}\left(x^2 \left| \begin{array}{c} \frac{1}{2}, 1 \\ 0, 0, \mu, -\mu \end{array}\right.\right) - \sqrt{\pi} I_\mu(x)\, G^{4,0}_{2,4}\left(x^2 \left| \begin{array}{c} \frac{1}{2}, 1 \\ 0, 0, \mu, -\mu \end{array}\right.\right)\right], \end{aligned} \tag{33}$$

where $I_\nu(x)$ is the *modified Bessel function*; or in terms of generalized hypergeometric functions for $\mathrm{Re}\, x > 0$, $\mu > 0$, and $2\mu \notin \mathbb{Z}$ [26]:

$$\begin{aligned} &\frac{\partial K_\mu(x)}{\partial \mu} \\ &= \frac{\pi}{2}\csc(\pi\mu)\Bigg\{\pi\cot(\pi\mu)\, I_\mu(x) - \left[I_\mu(x) + I_{-\mu}(x)\right] \\ &\quad \left[\frac{x^2}{4(1-\mu^2)}\, {}_3F_4\left(\begin{array}{c} 1, 1, \frac{3}{2} \\ 2, 2, 2-\mu, 2+\mu \end{array}\middle| x^2\right) + \ln\left(\frac{x}{2}\right) - \psi(\mu) - \frac{1}{2\mu}\right]\Bigg\} \\ &\quad + \frac{1}{4}\Bigg\{I_{-\mu}(x)\Gamma^2(-\mu)\left(\frac{x}{2}\right)^{2\mu} {}_2F_3\left(\begin{array}{c} \mu, \frac{1}{2}+\mu \\ 1+\mu, 1+\mu, 1+2\mu \end{array}\middle| x^2\right) \\ &\qquad - I_\mu(x)\Gamma^2(\mu)\left(\frac{x}{2}\right)^{-2\mu} {}_2F_3\left(\begin{array}{c} -\mu, \frac{1}{2}-\mu \\ 1-\mu, 1-\mu, 1-2\mu \end{array}\middle| x^2\right)\Bigg\}. \end{aligned} \tag{34}$$

There are different expressions for the order derivatives of the Bessel functions [23,27]. This subject is summarized in [28], where general results are presented in terms of convolution integrals, and order derivatives of Bessel functions are found for particular values of the order.

Using (31), (33) and (34), some derivatives of $\mathrm{W}_{\kappa,\mu}(x)$ with respect to μ have been calculated with the help of the MATHEMATICA program, and they are presented in Table 4.

Table 4. Derivative of $\mathrm{W}_{\kappa,\mu}$ with respect to μ, by using (31).

κ	μ	$\frac{\partial \mathrm{W}_{\kappa,\mu}(x)}{\partial \mu}$
0	0	0
0	$\pm\frac{1}{4}$	$\pm\frac{1}{8\sqrt{\pi}}\Bigg\{4\pi\sqrt{2x}\left(\pi I_{1/4}\left(\frac{x}{2}\right) - \left[I_{1/4}\left(\frac{x}{2}\right) + I_{-1/4}\left(\frac{x}{2}\right)\right]\left[\frac{x^2}{15}\, {}_3F_4\left(\begin{array}{c} 1, 1, \frac{3}{2} \\ 2, 2, \frac{7}{4}, \frac{9}{4} \end{array}\middle| \frac{x^2}{4}\right) + \ln\left(\frac{x}{4}\right) - \psi\left(\frac{1}{4}\right) - 2\right]\right)$ $- 4\,\Gamma^2\left(\frac{1}{4}\right) I_{1/4}\left(\frac{x}{2}\right) {}_2F_3\left(\begin{array}{c} -\frac{1}{4}, \frac{1}{4} \\ \frac{3}{4}, \frac{3}{4}, \frac{1}{2} \end{array}\middle| \frac{x^2}{4}\right) + x\,\Gamma^2\left(-\frac{1}{4}\right) I_{-1/4}\left(\frac{x}{2}\right) {}_2F_3\left(\begin{array}{c} \frac{1}{4}, \frac{3}{4} \\ \frac{5}{4}, \frac{5}{4}, \frac{3}{2} \end{array}\middle| \frac{x^2}{4}\right)\Bigg\}$

Table 4. *Cont.*

κ	μ	$\frac{\partial \mathrm{W}_{\kappa,\mu}(x)}{\partial \mu}$
0	$\pm\frac{1}{3}$	$\pm\frac{x^{-1/6}}{384\sqrt{\pi}}\Bigg\{\pi\, x^{2/3}\Bigg(128\pi\, I_{1/3}\left(\frac{x}{2}\right)-\sqrt{3}\left[I_{1/4}\left(\frac{x}{2}\right)+I_{-1/4}\left(\frac{x}{2}\right)\right]\Bigg[9\,x^2\,{}_3F_4\left(\left.\begin{array}{c}1,1,\frac{3}{2}\\2,2,\frac{5}{3},\frac{7}{3}\end{array}\right\vert\frac{x^2}{4}\right)+64\left(2\ln\left(\frac{x}{4}\right)-2\psi\left(\frac{1}{3}\right)-3\right)\Bigg]\Bigg)$ $-48\sqrt[3]{2}\Bigg[3x\,\Gamma\left(-\frac{1}{3}\right){}_0F_1\left(\left.\begin{array}{c}-\\\frac{2}{3}\end{array}\right\vert\frac{x^2}{16}\right){}_2F_3\left(\left.\begin{array}{c}\frac{1}{3},\frac{5}{6}\\\frac{4}{3},\frac{4}{3},\frac{5}{3}\end{array}\right\vert\frac{x^2}{4}\right)+\Gamma^2\left(\frac{1}{3}\right)I_{1/3}\left(\frac{x}{2}\right){}_2F_3\left(\left.\begin{array}{c}-\frac{1}{3},\frac{1}{6}\\\frac{1}{3},\frac{2}{3},\frac{2}{3}\end{array}\right\vert\frac{x^2}{4}\right)\Bigg]\Bigg\}$
0	$\pm\frac{1}{2}$	$\pm\frac{1}{4\sqrt{\pi}}e^{-x/2}\left[G^{3,1}_{2,4}\left(\frac{x^2}{4}\left\vert\begin{array}{c}\frac{1}{2},1\\0,0,\frac{1}{2},-\frac{1}{2}\end{array}\right.\right)-(e^x-1)\,G^{3,0}_{1,3}\left(\frac{x^2}{4}\left\vert\begin{array}{c}1\\-\frac{1}{2},0,0\end{array}\right.\right)\right]$
0	$\pm\frac{2}{3}$	$\pm\frac{1}{\sqrt{\pi}}\sqrt{x}\Bigg\{-\frac{1}{3}\pi^2\,I_{2/3}\left(\frac{x}{2}\right)-\frac{\pi}{\sqrt{3}}\left[I_{-2/3}\left(\frac{x}{2}\right)+I_{2/3}\left(\frac{x}{2}\right)\right]\Bigg[\frac{9}{80}x^2\,{}_3F_4\left(\left.\begin{array}{c}1,1,\frac{3}{2}\\2,2,\frac{4}{3},\frac{8}{3}\end{array}\right\vert\frac{x^2}{4}\right)+\ln\left(\frac{x}{4}\right)-\psi\left(\frac{2}{3}\right)-\frac{3}{4}\Bigg]$ $+2^{-14/3}x^{4/3}\,\Gamma^2\left(-\frac{2}{3}\right)I_{-2/3}\left(\frac{x}{2}\right){}_2F_3\left(\left.\begin{array}{c}\frac{2}{3},\frac{7}{6}\\\frac{5}{3},\frac{5}{3},\frac{7}{3}\end{array}\right\vert\frac{x^2}{4}\right)-2^{2/3}x^{-4/3}\,\Gamma^2\left(\frac{2}{3}\right)I_{2/3}\left(\frac{x}{2}\right){}_2F_3\left(\left.\begin{array}{c}-\frac{2}{3},-\frac{1}{6}\\-\frac{1}{3},\frac{1}{3},\frac{1}{3}\end{array}\right\vert\frac{x^2}{4}\right)\Bigg\}$
0	$\pm\frac{3}{4}$	$\pm\frac{1}{672\sqrt{\pi x}}\Bigg\{x^{3/2}\Bigg(-8\sqrt{2}\pi\left[I_{-3/4}\left(\frac{x}{2}\right)+I_{3/4}\left(\frac{x}{2}\right)\right]\Bigg[6\,x^2\,{}_3F_4\left(\left.\begin{array}{c}1,1,\frac{3}{2}\\2,2,\frac{5}{4},\frac{11}{4}\end{array}\right\vert\frac{x^2}{4}\right)+42[\ln(2x)+\gamma]-28\Bigg]$ $+\,21\,x^{3/2}\,\Gamma^2\left(-\frac{3}{4}\right)I_{-3/4}\left(\frac{x}{2}\right){}_2F_3\left(\left.\begin{array}{c}\frac{3}{4},\frac{5}{4}\\\frac{7}{4},\frac{7}{4},\frac{5}{2}\end{array}\right\vert\frac{x^2}{4}\right)+336\pi\,K_{3/4}\left(\frac{x}{2}\right)\Bigg)-1344\,\Gamma^2\left(\frac{3}{4}\right)I_{3/4}\left(\frac{x}{2}\right){}_2F_3\left(\left.\begin{array}{c}-\frac{3}{4},-\frac{1}{4}\\-\frac{1}{2},\frac{1}{4},\frac{1}{4}\end{array}\right\vert\frac{x^2}{4}\right)\Bigg\}$
0	± 1	$\pm\frac{1}{2\pi}\sqrt{x}\left[K_1\left(\frac{x}{2}\right)G^{2,1}_{1,3}\left(\frac{x^2}{4}\left\vert\begin{array}{c}\frac{1}{2}\\0,0,-1\end{array}\right.\right)-\pi\,I_1\left(\frac{x}{2}\right)G^{3,0}_{1,3}\left(\frac{x^2}{4}\left\vert\begin{array}{c}\frac{1}{2}\\-1,0,0\end{array}\right.\right)\right]$
0	$\pm\frac{3}{2}$	$\pm\frac{1}{4\sqrt{\pi x}}e^{-x/2}\left[3(x+2)G^{3,1}_{2,4}\left(\frac{x^2}{4}\left\vert\begin{array}{c}\frac{1}{2},1\\0,0,\frac{3}{2},-\frac{3}{2}\end{array}\right.\right)-3[e^x(x-2)+x+2]\,G^{4,0}_{2,4}\left(\frac{x^2}{4}\left\vert\begin{array}{c}\frac{1}{2},1\\-\frac{3}{2},0,0,\frac{3}{2}\end{array}\right.\right)\right]$
0	± 2	$\pm\frac{1}{\pi}\sqrt{x}\left[K_2\left(\frac{x}{2}\right)G^{3,1}_{2,4}\left(\frac{x^2}{4}\left\vert\begin{array}{c}\frac{1}{2},1\\0,0,2,-2\end{array}\right.\right)-\pi\,I_2\left(\frac{x}{2}\right)G^{4,0}_{2,4}\left(\frac{x^2}{4}\left\vert\begin{array}{c}\frac{1}{2},1\\-2,0,0,2\end{array}\right.\right)\right]$

3. Parameter Differentiation of $\mathrm{W}_{\kappa,\mu}$ via Integral Representations

3.1. Derivative with Respect to the First Parameter $\partial \mathrm{W}_{\kappa,\mu}(x)/\partial\kappa$

Integral representations of the Whittaker function $\mathrm{W}_{\kappa,\mu}(z)$ are given in the form of a Laplace transform for $\mathrm{Re}(\mu-\kappa) > -\frac{1}{2}$ and $|\arg z| < \frac{\pi}{2}$ [8] (Section 7.4.2):

$$\begin{aligned}&\mathrm{W}_{\kappa,\mu}(z)\\ =\;& \frac{z^{\mu+1/2}e^{-z/2}}{\Gamma\left(\mu-\kappa+\frac{1}{2}\right)}\int_0^\infty e^{-z\,t}t^{\mu-\kappa-1/2}(1+t)^{\mu+\kappa-1/2}dt,\end{aligned} \tag{35}$$

and as the infinite integral:

$$\begin{aligned}&\mathrm{W}_{\kappa,\mu}(z)\\ =\;& \frac{z^{\mu+1/2}e^{z/2}}{\Gamma\left(\mu-\kappa+\frac{1}{2}\right)}\int_1^\infty e^{-z\,t}t^{\mu+\kappa-1/2}(t-1)^{\mu-\kappa-1/2}dt.\end{aligned} \tag{36}$$

In order to calculate the first derivative of $\mathrm{W}_{\kappa,\mu}(x)$ with respect to parameter κ, let us introduce the following finite logarithmic integrals.

Definition 2. *For* $\mathrm{Re}(\mu-\kappa) > -\frac{1}{2}$ *and* $x>0$, *define:*

$$I_1^*(\kappa,\mu;x) = \int_0^\infty e^{-xt}t^{\mu-\kappa-1/2}(1+t)^{\mu+\kappa-1/2}\ln\left(\frac{1+t}{t}\right)dt, \tag{37}$$

$$I_2^*(\kappa,\mu;x) = \int_1^\infty e^{-xt}t^{\mu+\kappa-1/2}(t-1)^{\mu-\kappa-1/2}\ln\left(\frac{t}{t-1}\right)dt. \tag{38}$$

For $x>0$, differentiation of (35) and (36) with respect to parameter κ yields, respectively,

$$\begin{aligned}&\frac{\partial \mathrm{W}_{\kappa,\mu}(x)}{\partial\kappa}\\ =\;& \psi\left(\mu-\kappa+\frac{1}{2}\right)\mathrm{W}_{\kappa,\mu}(x)+\frac{x^{\mu+1/2}e^{-x/2}}{\Gamma\left(\mu-\kappa+\frac{1}{2}\right)}I_1^*(\kappa,\mu;x) &(39)\\ =\;& \psi\left(\mu-\kappa+\frac{1}{2}\right)\mathrm{W}_{\kappa,\mu}(x)+\frac{x^{\mu+1/2}e^{x/2}}{\Gamma\left(\mu-\kappa+\frac{1}{2}\right)}I_2^*(\kappa,\mu;x). &(40)\end{aligned}$$

Note that, from (39) and (40) we have

$$I_2^*(\kappa,\mu;x) = e^{-x}I_1^*(\kappa,\mu;x). \tag{41}$$

Theorem 6. *The following integral holds true for* $\frac{1}{2}+\mu-\kappa>0$ *and* $x>0$:

$$\begin{aligned}&I_1^*(\kappa,\mu;x) &(42)\\ =\;& \mathrm{B}\left(\frac{1}{2}+\mu-\kappa,-2\mu\right)\\ &\left\{\left[\psi\left(\frac{1}{2}-\mu-\kappa\right)-\psi\left(\frac{1}{2}+\mu-\kappa\right)\right]{}_1F_1\left(\left.\begin{array}{c}\frac{1}{2}+\mu-\kappa\\ 1+2\mu\end{array}\right|x\right)\right.\\ &\left.-G^{(1)}\left(\left.\begin{array}{c}\frac{1}{2}+\mu-\kappa\\ 1+2\mu\end{array}\right|x\right)\right\}-\Gamma(2\mu)\,x^{-2\mu}\,G^{(1)}\left(\left.\begin{array}{c}\frac{1}{2}-\mu-\kappa\\ 1-2\mu\end{array}\right|x\right),\end{aligned}$$

where $\mathrm{B}(x,y)=\frac{\Gamma(x)\Gamma(y)}{\Gamma(x+y)}$ *denotes the beta function.*

Proof. Compare (10) to (39) and take into account (1) to arrive at (42), as we wanted to prove. □

Now, we derive a Lemma that will be applied throughout this section and the next one.

Lemma 1. *For* $\nu\geq 0$ *and* $x>0$, *the following Laplace transform holds true:*

$$\begin{aligned}&\mathcal{I}_\pm(\nu,x) &(43)\\ =\;& \int_0^\infty e^{-xt}t^\nu\ln\left(t^{\pm1}(1+t)\right)dt\\ =\;& \frac{\Gamma(\nu+1)}{x^{\nu+1}}\left\{\frac{x}{\nu+1}\,{}_2F_2\left(\left.\begin{array}{c}1,1\\ 2,2+\nu\end{array}\right|-x\right)\right.\\ &\left.-e^{-i\pi\nu}\,\Gamma(-\nu,x)\,\gamma(\nu+1,-x)+(1\pm1)[\psi(\nu+1)-\ln x]\right\},\end{aligned}$$

where $\Gamma(\nu,z)$ *and* $\gamma(\nu,z)$ *denote, respectively, the upper and lower incomplete gamma functions, (117) and (119).*

Proof. Split the integral into two terms as follows:

$$\mathcal{I}_{\pm}(\nu,x) = \underbrace{\int_0^\infty e^{-xt} t^\nu \ln(1+t) dt}_{\mathcal{I}_a(\nu,x)} \pm \underbrace{\int_0^\infty e^{-xt} t^\nu \ln t\, dt}_{\mathcal{I}_b(\nu,x)},$$

and apply the Laplace transform for $x > 0$ [9] (Eqn. 2.5.2(4)) (it is worth noting that there is an incorrect sign in the reference cited):

$$\begin{aligned}
&\int_0^\infty e^{-xt} t^\nu \ln(at+b) dt \\
= &-\frac{\pi}{(\nu+1)\sin\pi\nu}\left(\frac{b}{a}\right)^{\nu+1} {}_1F_1\left(\begin{array}{c}\nu+1\\ \nu+2\end{array}\middle| \frac{b\,x}{a}\right) \\
&+\frac{\Gamma(\nu+1)}{x^{\nu+1}}\left[\psi(\nu+1) - \ln\left(\frac{x}{a}\right) + \frac{b\,x}{a\,\nu}\, {}_2F_2\left(\begin{array}{c}1,1\\ 2,1-\nu\end{array}\middle| \frac{b\,x}{a}\right)\right],
\end{aligned}$$

to obtain

$$\begin{aligned}
&\mathcal{I}_a(\nu,x) \\
= &-\frac{\pi}{(\nu+1)\sin\pi\nu}\, {}_1F_1\left(\begin{array}{c}\nu+1\\ \nu+2\end{array}\middle| x\right) \\
&+\frac{\Gamma(\nu+1)}{x^{\nu+1}}\left[\psi(\nu+1) - \ln x + \frac{x}{\nu}\, {}_2F_2\left(\begin{array}{c}1,1\\ 2,1-\nu\end{array}\middle| x\right)\right],
\end{aligned} \tag{44}$$

and

$$\mathcal{I}_b(\nu,x) = \frac{\Gamma(\nu+1)}{x^{\nu+1}}[\psi(\nu+1) - \ln x]. \tag{45}$$

Note that, according to Kummer's transformation (11), and to the reduction formula [9] (Eqn. 7.11.1(14)):

$${}_1F_1\left(\begin{array}{c}1\\ b\end{array}\middle| z\right) = (b-1) z^{1-b} e^z \gamma(1-b,z),$$

we have for $x > 0$

$$\begin{aligned}
{}_1F_1\left(\begin{array}{c}a\\ a+1\end{array}\middle| x\right) &= e^x\, {}_1F_1\left(\begin{array}{c}1\\ a+1\end{array}\middle| -x\right) \\
&= a(-x)^{-a}\gamma(a,-x) \\
&= a\, e^{-i\pi a} x^{-a}\gamma(a,-x),
\end{aligned} \tag{46}$$

thus (44) becomes

$$\begin{aligned}
&\mathcal{I}_a(\nu,x) \\
= &\frac{1}{x^{\nu+1}}\left\{\frac{\pi}{\sin\pi\nu} e^{-i\pi\nu}\gamma(\nu+1,-x)\right. \\
&\left. +\Gamma(\nu+1)\left[\psi(\nu+1) - \ln x + \frac{x}{\nu}\, {}_2F_2\left(\begin{array}{c}1,1\\ 2,1-\nu\end{array}\middle| x\right)\right]\right\}.
\end{aligned} \tag{47}$$

Now, insert (45) and (47) in (78), to arrive at

$$\begin{aligned}
&\mathcal{I}_{\pm}(\nu,x) = \frac{1}{x^{\nu+1}} \\
&\left\{\frac{\pi}{\sin\pi\nu} e^{-i\pi\nu}\gamma(\nu+1,-x) + x\,\Gamma(\nu)\, {}_2F_2\left(\begin{array}{c}1,1\\ 2,1-\nu\end{array}\middle| x\right)\right\} \\
&+(1\pm 1)\frac{\Gamma(\nu+1)}{x^{\nu+1}}[\psi(\nu+1) - \ln x].
\end{aligned} \tag{48}$$

Next, apply the transformation formula [9] (Eqn. 7.12.1(7)):

$$\begin{aligned} &{}_2F_2\left(\begin{array}{c} 1,a \\ a+1,b \end{array}\middle| z\right) + \frac{b-1}{a-b+1}\,{}_2F_2\left(\begin{array}{c} 1,a \\ a+1,2+a-b \end{array}\middle| -z\right) \\ = &\frac{a}{a-b+1}\,{}_1F_1\left(\begin{array}{c} a-b+1 \\ a-b+2 \end{array}\middle| z\right){}_1F_1\left(\begin{array}{c} b-1 \\ b \end{array}\middle| -z\right), \end{aligned}$$

taking $a = 1$ and $b = 1 - \nu$, and applying again (46), to arrive at

$$\begin{aligned} &{}_2F_2\left(\begin{array}{c} 1,1 \\ 2,1-\nu \end{array}\middle| x\right) \\ = &\nu\left\{\frac{1}{\nu+1}\,{}_2F_2\left(\begin{array}{c} 1,1 \\ 2,2+\nu \end{array}\middle| -x\right) + \frac{e^{-i\pi\nu}}{x}\gamma(-\nu,x)\,\gamma(\nu+1,-x)\right\}. \end{aligned} \tag{49}$$

Insert (49) into (48) to get

$$\begin{aligned} &\mathcal{I}_{\pm}(\nu,x) \\ = &\frac{1}{x^{\nu+1}}\left\{e^{-i\pi\nu}\gamma(\nu+1,-x)\left[\frac{\pi}{\sin\pi\nu} + \Gamma(\nu+1)\gamma(-\nu,x)\right]\right. \\ &\left.+\Gamma(\nu+1)\left[\frac{x}{\nu+1}\,{}_2F_2\left(\begin{array}{c} 1,1 \\ 2,1-\nu \end{array}\middle| x\right) + (1\pm 1)[\psi(\nu+1) - \ln x]\right]\right\}. \end{aligned} \tag{50}$$

Applying the properties [4] (Eqn. 45:0:1)

$$\Gamma(\nu) = \gamma(\nu,z) + \Gamma(\nu,z), \tag{51}$$

and [29] (Eqn. 1.2.2)

$$\Gamma(z)\Gamma(1-z) = \pi\csc\pi z,$$

rewrite (50) as (43), as we wanted to prove. □

Theorem 7. *The following integral holds true for $\mu > 0$ and $x > 0$:*

$$I_1^*\left(\frac{1}{2}-\mu,\mu;x\right) \tag{52}$$

$$= \mathcal{I}_-(2\mu-1,x) \tag{53}$$

$$= \frac{\Gamma(2\mu)}{x^{2\mu}}\left\{\frac{x}{2\mu}\,{}_2F_2\left(\begin{array}{c} 1,1 \\ 2,1+2\mu \end{array}\middle| -x\right) + e^{-2\pi i\mu}\,\Gamma(1-2\mu,x)\,\gamma(2\mu,-x)\right\}.$$

Proof. From (37) and (43), we obtain the desired result. □

Remark 1. *If we insert (48) in (53), we obtain the following alternative form:*

$$\begin{aligned} &I_1^*\left(\frac{1}{2}-\mu,\mu;x\right) \\ = &\frac{1}{x^{2\mu}}\left\{\pi[\cot(2\pi\mu) - i]\,\gamma(2\mu,-x) + x\,\Gamma(2\mu-1)\,{}_2F_2\left(\begin{array}{c} 1,1 \\ 2,2-2\mu \end{array}\middle| x\right)\right\}. \end{aligned} \tag{54}$$

Theorem 8. *The following reduction formula holds true for $-2\mu \neq 0,1,\ldots$ and $x > 0$:*

$$\begin{aligned} &\left.\frac{\partial W_{\kappa,\mu}(x)}{\partial\kappa}\right|_{\kappa=-\mu+1/2} = e^{-x/2}x^{1/2-\mu} \\ &\left\{\psi(2\mu) + \frac{x}{2\mu}\,{}_2F_2\left(\begin{array}{c} 1,1 \\ 2,1+2\mu \end{array}\middle| -x\right) + e^{-2\pi i\mu}\,\Gamma(1-2\mu,x)\,\gamma(2\mu,-x)\right\}. \end{aligned} \tag{55}$$

Proof. Insert into (39) the reduction formula [3] (Eqn. 13.18.2), with $\kappa = -\mu + 1/2$, i.e.,

$$\mathrm{W}_{1/2-\mu,\mu}(x) = e^{-x/2} x^{1/2-\mu}, \tag{56}$$

and the result given in (52), to arrive at (55). □

Remark 2. *If we consider (54), we obtain the following alternative form:*

$$\left.\frac{\partial \mathrm{W}_{\kappa,\mu}(x)}{\partial \kappa}\right|_{\kappa=-\mu+1/2} = e^{-x/2} x^{1/2-\mu} \left\{ \psi(2\mu) + \frac{\pi[\cot(2\pi\mu) - i]}{\Gamma(2\mu)} \gamma(2\mu, -x) + \frac{x}{2\mu - 1} \, {}_2F_2\left(\begin{array}{c} 1,1 \\ 2, 2-2\mu \end{array} \middle| x \right) \right\}. \tag{57}$$

Table 5 shows the first derivative of $\mathrm{W}_{\kappa,\mu}(x)$ with respect to parameter κ for some particular values of κ and μ, and $x > 0$, calculated with the aid of the MATHEMATICA program from (57). Note that the $\mathrm{erfi}(x)$ function that appears in Table 5 denotes the imaginary error function [29] (Eqn. (2.3.1)).

Table 5. Derivative of $W_{\kappa,\mu}$ with respect to κ, by using (57).

κ	μ	$\frac{\partial \mathrm{W}_{\kappa,\mu}(x)}{\partial \kappa} \quad (x > 0)$
$-\frac{1}{4}$	$\pm\frac{3}{4}$	$x^{-1/4} e^{-x/2} \left[2 - \gamma - \ln 4 - 2\, e^x \sqrt{\pi\, x} + \pi\, \mathrm{erfi}\left(\sqrt{x}\right) + 2x\, {}_2F_2\left(1, 1; \frac{1}{2}, 2; x\right) \right]$
$\frac{1}{4}$	$\pm\frac{1}{4}$	$x^{1/4} e^{-x/2} \left[\pi\, \mathrm{erfi}\left(\sqrt{x}\right) - 2x\, {}_2F_2\left(1, 1; \frac{3}{2}, 2; x\right) - \gamma - \ln 4 \right]$
$\frac{3}{4}$	$\pm\frac{1}{4}$	$e^{-x/2} \left\{ x^{3/4} \left[2 - \gamma - \ln 4 + \pi\, \mathrm{erfi}\left(\sqrt{x}\right) - \frac{2}{3} x\, {}_2F_2\left(1, 1; \frac{5}{2}, 2; x\right) \right] - \sqrt{\pi}\, x^{1/4}\, e^x \right\}$
$\frac{5}{4}$	$\pm\frac{3}{4}$	$\frac{1}{30} x^{-1/4} e^{-x/2} \left\{ 2x^{3/2} \left[40 - 15\gamma - 30 \ln 2 + 15\pi\, \mathrm{erfi}\left(\sqrt{x}\right) - 12x\, {}_2F_2\left(1, 1; \frac{7}{2}, 2; x\right) \right] - 15\sqrt{\pi} e^x (2x + 1) \right\}$

Notice that for $-2\mu = 0, 1, \ldots$, we obtain an indeterminate expression in (55) and (57). For these cases, we present the following result.

Theorem 9. *The following reduction formula holds true for* $m = 0, 1, 2, \ldots$*:*

$$\begin{aligned} &\left.\frac{\partial \mathrm{W}_{\kappa,\mu}(x)}{\partial \kappa}\right|_{\kappa=(1+m)/2, \mu=\pm m/2} \\ &= e^{-x/2} x^{(1+m)/2} \left\{ \ln x - \sum_{k=1}^{m} x^{-k} \left[e^x\, \Gamma(k) + \binom{m}{k} \gamma(k, -x) \right] \right\}. \end{aligned} \tag{58}$$

Proof. Take $\nu = 2\mu$ in (57) and perform the limit $\nu \to -m = 0, -1, -2, \ldots$

$$\left.\frac{\partial \mathrm{W}_{\kappa,\mu}(x)}{\partial \kappa}\right|_{\kappa=(m+1)/2, \mu=-m/2} = e^{-x/2} x^{(1+m)/2} \left\{ \lim_{\nu \to -m} \left[\psi(\nu) + \frac{\pi[\cot(\pi\nu) - i]}{\Gamma(\nu)} \gamma(\nu, -x) \right] - \frac{x}{m+1} \, {}_2F_2\left(\begin{array}{c} 1,1 \\ 2, 2+m \end{array} \middle| x \right) \right\}. \tag{59}$$

On the one hand, let us prove the following asymptotic formulas for $\nu \to -m = 0, -1, -2, \ldots$

$$\psi(\nu) \approx -\gamma + H_m - \frac{1}{\nu+m}, \tag{60}$$

$$\pi \cot(\pi\nu) \approx \frac{1}{\nu+m}, \tag{61}$$

$$\Gamma(\nu) \approx \frac{(-1)^m}{m!}\frac{1}{\nu+m}. \tag{62}$$

In order to prove (60), consider [4] (Eqn. 44:5:4)

$$\begin{aligned} \psi(\nu+m+1) &= \psi(\nu) + \sum_{j=0}^{m}\frac{1}{\nu+j} \\ &= \psi(\nu) + \sum_{j=1}^{m}\frac{1}{\nu+j-1} + \frac{1}{\nu+m}, \end{aligned}$$

thus, knowing that [29] (Eqn. 1.3.6)

$$\psi(1) = -\gamma, \tag{63}$$

and performing the substitution $k = j - m - 1$, we have

$$\begin{aligned} \lim_{\nu\to -m}\psi(\nu) &= \lim_{\nu\to -m}\left[-\gamma - \frac{1}{\nu+m} - \sum_{j=1}^{m}\frac{1}{j-m-1}\right] \\ &= \lim_{\nu\to -m}\left[-\gamma - \frac{1}{\nu+m} + H_m\right], \end{aligned}$$

where $H_n = \sum_{k=1}^{n}\frac{1}{k}$ denotes the n-th harmonic number. In order to prove (61), note that $\cot x = \cot(x+\pi)$ and for $x \in (-\pi, \pi)$ we have the expansion [4] (Eqn. 44:6:2)

$$\cot x = \frac{1}{x} - \frac{x}{3} - \frac{x^3}{45} - \cdots$$

Finally, notice that (62) follows directly from [29] (Eqn. 1.1.5). Therefore, taking into account (60)–(62), and taking into account (51), we conclude

$$\begin{aligned} &\lim_{\nu\to -m}\left[\psi(\nu) + \frac{\pi[\cot(\pi\nu) - i]}{\Gamma(\nu)}\gamma(\nu, -x)\right] \\ =\ & H_m - \gamma - i\pi + (-1)^{m+1} m!\,\Gamma(-m, -x). \end{aligned} \tag{64}$$

Insert (64) into (59) to arrive at

$$\begin{aligned} &\left.\frac{\partial W_{\kappa,\mu}(x)}{\partial\kappa}\right|_{\kappa=(m+1)/2,\mu=-m/2} = e^{-x/2}x^{(1+m)/2} \\ &\left\{H_m - \gamma - i\pi + (-1)^{m+1} m!\,\Gamma(-m,-x) - \frac{x}{m+1}\,{}_2F_2\left(\left.\begin{array}{c}1,1\\2,2+m\end{array}\right| x\right)\right\}. \end{aligned} \tag{65}$$

On the other hand, consider the reduction Formula (A6), derived in the Appendix B,

$${}_2F_2\left(\left.\begin{array}{c}1,1\\2,2+m\end{array}\right| x\right) = \frac{m+1}{x}\left\{H_m - \mathrm{Ein}(-x) + \sum_{k=1}^{m}\binom{m}{k}x^{-k}\gamma(k,-x)\right\}, \tag{66}$$

and the formula [3] (Eqn. 8.4.15)

$$\Gamma(-m,z) = \frac{(-1)^m}{m!}\left[\mathrm{E}_1(z) - e^{-z}\sum_{k=0}^{m-1}\frac{(-1)^k k!}{z^{k+1}}\right], \tag{67}$$

where $\mathrm{E}_1(z)$ denotes the exponential integral [3] (Eqn. 6.2.1), which is defined as

$$\mathrm{E}_1(z) = \int_z^\infty \frac{e^{-t}}{t}dt, \quad z \neq 0,$$

where the path does not cross the negative real axis or pass through the origin. Furthermore, consider the property [3] (Eqn. 6.2.4)

$$\mathrm{E}_1(z) = \mathrm{Ein}(z) - \ln z - \gamma. \tag{68}$$

Therefore, substituting (66) and (67) into (65), and taking into account (68), we arrive at (58), as we wanted to prove. □

Remark 3. *It is worth noting that from [17],*

$$\left.\frac{\partial \mathrm{W}_{\kappa,\mu}(x)}{\partial\kappa}\right|_{\kappa=(N+1)/2,\mu=M/2} = \mathrm{W}_{\frac{N+1}{2},\frac{N}{2}}(x)\ln x \tag{69}$$
$$+\sum_{k=1}^{(N+M)/2}\frac{(-1)^k\left(\frac{N+M}{2}\right)!}{k\left(\frac{N+M}{2}-k\right)!}\mathrm{W}_{\frac{N+1}{2}-k,\frac{M}{2}}(x) + \sum_{k=1}^{(N-M)/2}\frac{(-1)^k\left(\frac{N-M}{2}\right)!}{k\left(\frac{N-M}{2}-k\right)!}\mathrm{W}_{\frac{N+1}{2}-k,\frac{M}{2}}(x),$$

where $-N \le M \le N$ and M, N are integers of like parity, we can derive an equivalent reduction formula to (58). Indeed, taking $N = M = m$, (69) is reduced to

$$\left.\frac{\partial \mathrm{W}_{\kappa,\mu}(x)}{\partial\kappa}\right|_{\kappa=(m+1)/2,\mu=m2} = \mathrm{W}_{\frac{m+1}{2},\frac{m}{2}}(x)\ln x + \sum_{k=1}^{m}\frac{(-1)^k\, m!}{k(m-k)!}\mathrm{W}_{\frac{m}{2}-k,\frac{m}{2}}(x). \tag{70}$$

Note that from (28), we have

$$\mathrm{W}_{\frac{m+1}{2},\frac{m}{2}}(x) = e^{-x/2}x^{(1+m)/2}. \tag{71}$$

Further, from (5) and the reduction formula for $n = 0, 1, \ldots$ given in [3] (Eqn. 13.2.8)

$$\mathrm{U}(a, a+n+1, z) = z^{-a}\sum_{s=0}^{n}\binom{n}{s}(a)_s z^{-s},$$

we obtain

$$\mathrm{W}_{\frac{m}{2}-k,\frac{m}{2}}(x) = \frac{e^{-x/2}x^{(1+m)/2}x^{-k}}{\Gamma(k)}\sum_{s=0}^{m-k}\binom{m-k}{s}\Gamma(k+s)x^{-s}. \tag{72}$$

Therefore, substituting (71) and (72) into (70), and simplifying, we arrive at

$$\left.\frac{\partial \mathrm{W}_{\kappa,\mu}(x)}{\partial\kappa}\right|_{\kappa=(m+1)/2,\mu=m/2}$$
$$= e^{-x/2}x^{(1+m)/2}\left[\ln x + m!\sum_{k=1}^{m}\frac{(-1)^k}{k!}x^{-k}\sum_{s=0}^{m-k}\frac{(k+s-1)!}{s!(m-k-s)!}x^{-s}\right]. \tag{73}$$

Perform the index substitution $s \to s + k$ and exchange the sum order in (73), to arrive at

$$\begin{aligned} &\left.\frac{\partial \mathrm{W}_{\kappa,\mu}(x)}{\partial \kappa}\right|_{\kappa=(m+1)/2,\mu=m/2} \\ = \;& e^{-x/2}x^{(1+m)/2}\left[\ln x + m!\sum_{s=1}^{m}\frac{x^{-s}}{s(m-s)!}\sum_{k=1}^{s}\binom{s}{k}(-1)^k\right]. \end{aligned} \tag{74}$$

By virtue of the binomial theorem, the inner sum in (74) is just -1, thus we finally obtain:

$$\left.\frac{\partial \mathrm{W}_{\kappa,\mu}(x)}{\partial \kappa}\right|_{\kappa=(1+m)/2,\mu=\pm m/2} = e^{-x/2}x^{(1+m)/2}\left[\ln x - m!\sum_{k=1}^{m}\frac{x^{-k}}{k(m-k)!}\right]. \tag{75}$$

Theorem 10. *For $n = 0, 1, 2, \ldots$, and $x > 0$, the following integral holds true:*

$$\begin{aligned} I_1^*\left(\frac{n}{2},\frac{n+1}{2};x\right) &= \frac{e^x\,\mathrm{Ein}(x)}{x^{n+1}}\Gamma(n+1,x) + n!\sum_{k=0}^{n}\frac{x^{-k-1}}{(n-k)!} \\ &\left\{(-1)^{k+1}\Gamma(-k,x)\,\gamma(k+1,-x) - H_k - \sum_{\ell=1}^{k}\binom{k}{\ell}(-x)^{-\ell}\gamma(\ell,x)\right\}. \end{aligned} \tag{76}$$

Proof. From (37), we have

$$\begin{aligned} &I_1^*\left(\mu-\frac{1}{2},\mu;x\right) \\ = &\int_0^\infty e^{-xt}(1+t)^{2\mu-1}\ln\left(\frac{1+t}{t}\right)dt, \end{aligned}$$

thus, taking $\mu = \frac{n+1}{2}$ with $n = 0, 1, 2, \ldots$ and applying the binomial theorem, we get

$$\begin{aligned} I_1^*\left(\frac{n}{2},\frac{n+1}{2};x\right) &= \sum_{k=0}^{n}\binom{n}{k}\int_0^\infty e^{-xt}t^k\ln\left(\frac{1+t}{t}\right)dt \\ &= \sum_{k=0}^{n}\binom{n}{k}\mathcal{I}_-(k,x). \end{aligned} \tag{77}$$

Insert the result obtained in (43) for $\nu = k$ into (77) to arrive at

$$\begin{aligned} I_1^*\left(\frac{n}{2},\frac{n+1}{2};x\right) &= n!\sum_{k=0}^{n}\frac{x^{-k-1}}{(n-k)!} \\ &\left\{(-1)^{k+1}\Gamma(-k,x)\,\gamma(k+1,-x) + \frac{x}{k+1}\,{}_2F_2\left(\begin{array}{c}1,1\\2,2+k\end{array}\middle|-x\right)\right\}. \end{aligned}$$

Now, take into account (66), to get

$$\begin{aligned} I_1^*\left(\frac{n}{2},\frac{n+1}{2};x\right) &= n!\sum_{k=0}^{n}\frac{x^{-k-1}}{(n-k)!} \\ &\left\{(-1)^{k+1}\Gamma(-k,x)\,\gamma(k+1,-x) - H_k + \mathrm{Ein}(x) - \sum_{\ell=1}^{k}\binom{k}{\ell}(-x)^{-\ell}\gamma(\ell,x)\right\}. \end{aligned} \tag{78}$$

Finally, note that using the *exponential polynomial*, defined as

$$\mathrm{e}_n(x) = \sum_{k=0}^{n}\frac{x^k}{k!},$$

and the property for $n = 0, 1, 2, \ldots$ [4] (Eqn. 45:4:2):

$$\Gamma(1+n,x) = n!\,\mathrm{e}_n(x)\,e^{-x},$$

we calculate the following finite sum as:

$$\sum_{k=0}^{n} \frac{x^{-k}}{(n-k)!} = x^{-n} \sum_{s=0}^{n} \frac{x^s}{s!} = \frac{x^{-n} e^x}{n!} \Gamma(1+n,x). \tag{79}$$

Apply (79) to (78) in order to obtain (76), as we wanted to prove. □

Theorem 11. *For $n = 0, 1, 2, \ldots$, the following reduction formula holds true:*

$$\begin{aligned} &\left.\frac{\partial \mathrm{W}_{\kappa,\mu}(x)}{\partial \kappa}\right|_{\kappa=n/2,\mu=\pm(n+1)/2} \qquad (80)\\ = \;& x^{-n/2} e^{x/2} \Gamma(1+n,x)[\mathrm{E}_1(x) + \ln x] + n!\, x^{n/2} e^{-x/2} \\ & \sum_{k=0}^{n} \frac{x^{-k}}{(n-k)!} \left[(-1)^{k+1} \Gamma(-k,x)\,\gamma(k+1,-x) - H_k - \sum_{\ell=1}^{k} \binom{k}{\ell} (-x)^{-\ell} \gamma(\ell,x) \right]. \end{aligned}$$

Proof. Applying (5) and [3] (Eqn. 13.6.6)

$$\mathrm{U}(1, 2-a, z) = z^{a-1} e^z\, \Gamma(1-a, z),$$

see that for $n = 0, 1, 2, \ldots$

$$\mathrm{W}_{n/2,(n+1)/2}(z) = z^{-n/2} e^{z/2} \Gamma(1+n,z). \tag{81}$$

Taking into account (63) and (68), insert (76) and (81) into (39) for $\kappa = \frac{n}{2}$ and $\mu = \frac{n+1}{2}$, to arrive at (80), as we wanted to prove. □

Theorem 12. *For $n = 0, 1, 2, \ldots$, and $x > 0$, the following integral holds true:*

$$\begin{aligned} I_1^*\left(0, n+\frac{1}{2}; x\right) = \frac{n!\, e^{x/2}}{\sqrt{\pi}\, x^{n+1/2}} K_{n+1/2}\left(\frac{x}{2}\right) \mathrm{Ein}(x) + \sum_{k=0}^{n} \binom{n}{k} \frac{(n+k)!}{x^{n+k+1}} \qquad (82)\\ \left\{ (-1)^{n+k+1} \Gamma(-n-k,x)\,\gamma(n+k+1,-x) - H_{n+k} - \sum_{\ell=1}^{n+k} \binom{n+k}{\ell} (-x)^{-\ell} \gamma(\ell,x) \right\}. \end{aligned}$$

Proof. Applying the binomial theorem to (37) for $\kappa = 0$ and $\mu = n + \frac{1}{2}$, we have

$$\begin{aligned} I_1^*\left(0, n+\frac{1}{2}; x\right) &= \sum_{k=0}^{n} \binom{n}{k} \int_0^\infty e^{-xt} t^{n+k} \ln\left(\frac{1+t}{t}\right) dt \\ &= \sum_{k=0}^{n} \binom{n}{k} \mathcal{I}_-(n+k,x) \qquad (83) \end{aligned}$$

Insert the result obtained in (43) for $\nu = n + k$ into (83), to get

$$\begin{aligned} & I_1^*\left(0, n+\frac{1}{2}; x\right) = \sum_{k=0}^{n} \binom{n}{k} \frac{(n+k)!}{x^{n+k+1}} \\ & \left\{ (-1)^{n+k+1} \Gamma(-n-k,x)\,\gamma(n+k+1,-x) + \frac{x}{n+k+1}\, {}_2F_2\left(\left. \begin{matrix} 1,1 \\ 2, 2+n+k \end{matrix} \right| -x \right) \right\}. \end{aligned}$$

Now, take into account (66), to obtain

$$I_1^*\left(0,n+\frac{1}{2};x\right) = \sum_{k=0}^{n}\binom{n}{k}\frac{(n+k)!}{x^{n+k+1}}$$
$$\left\{(-1)^{n+k+1}\Gamma(-n-k,x)\,\gamma(n+k+1,-x) + \mathrm{Ein}(x) - H_{n+k} - \sum_{\ell=1}^{n+k}\binom{n+k}{\ell}(-x)^{-\ell}\gamma(\ell,x)\right\}.$$

Finally, consider [3] (Eqns. 10.47.9,12)

$$\sqrt{\frac{z}{\pi}}K_{n+1/2}\left(\frac{z}{2}\right) = \frac{z}{\pi}\,\mathrm{k}_n\left(\frac{z}{2}\right) = e^{-z/2}\sum_{k=0}^{n}\frac{(n+k)!\,z^{-k}}{k!(n-k)!}, \tag{84}$$

where $\mathrm{k}_n(z)$ is the *modified spherical Bessel function of the second kind*, to arrive at the desired result. □

Theorem 13. *For $n = 0,1,2,\ldots$, the following reduction formula holds true:*

$$\left.\frac{\partial \mathrm{W}_{\kappa,\mu}(x)}{\partial\kappa}\right|_{\kappa=0,\mu=\pm(n+1/2)} \tag{85}$$
$$= \sqrt{\frac{x}{\pi}}K_{n+1/2}\left(\frac{x}{2}\right)[H_n + \mathrm{E}_1(x) + \ln x] + e^{-x/2}\sum_{k=0}^{n}\frac{(n+k)!\,x^{-k}}{k!(n-k)!}$$
$$\left[(-1)^{n+k+1}\,\Gamma(-n-k,x)\,\gamma(n+k+1,-x) - H_{n+k} - \sum_{\ell=1}^{n+k}\binom{n+k}{\ell}(-x)^{-\ell}\gamma(\ell,x)\right].$$

Proof. Take $\kappa = 0$ and $\mu = n+\frac{1}{2}$ in (39), to obtain

$$\left.\frac{\partial \mathrm{W}_{\kappa,\mu}(x)}{\partial\kappa}\right|_{\kappa=0,\mu=n+1/2} = \psi(n+1)\mathrm{W}_{0,n+1/2}(x) + \frac{x^{n+1}e^{-x/2}}{n!}\,I_1^*\left(0,n+\frac{1}{2};x\right). \tag{86}$$

Consider [29] (Eqn. 1.3.7)

$$\psi(n+1) = -\gamma + H_n, \tag{87}$$

and [3] (Eqn. 13.18.9)

$$\mathrm{W}_{0,n+1/2}(z) = \sqrt{\frac{z}{\pi}}K_{n+1/2}\left(\frac{z}{2}\right). \tag{88}$$

Substitute (82), (87) and (88) into (86), and take into account (14) and (68), to arrive at (85), as we wanted to prove. □

Table 6 shows the first derivative of $\mathrm{W}_{\kappa,\mu}(x)$ with respect to parameter κ for some particular values of κ and μ, calculated with the aid of the MATHEMATICA program, from (58), (80), and (85). Note that, in Table 6 the $\mathrm{Shi}(x)$ and the $\mathrm{Chi}(x)$ functions appear, which denote the sine and cosine integrals [3] (Eqns. 6.2.15–16).

Table 6. First derivative of $W_{\kappa,\mu}(x)$ with respect to parameter κ for particular values of κ and μ.

κ	μ	$\frac{\partial W_{\kappa,\mu}(x)}{\partial \kappa}$
0	$\pm\frac{1}{2}$	$e^{-x/2}[\ln x + e^x\,\Gamma(0,x)]$
0	$\pm\frac{3}{2}$	$x^{-1}e^{-x/2}\{(x-2)e^x[\mathrm{Chi}(x)-\mathrm{Shi}(x)]+(x+2)\ln x+2\}$
0	$\pm\frac{5}{2}$	$x^{-2}e^{-x/2}\{(x^2+6x+12)\ln x+18-(x^2-6x+12)e^x[\mathrm{Chi}(x)-\mathrm{Shi}(x)]\}$
$\frac{1}{2}$	0	$\sqrt{x}e^{-x/2}\ln x$
$\frac{1}{2}$	± 1	$x^{-1/2}e^{-x/2}[(x+1)\ln x + e^x\,\Gamma(0,x)]$
1	$\pm\frac{1}{2}$	$e^{-x/2}(x\ln x-1)$
1	$\pm\frac{3}{2}$	$x^{-1}e^{-x/2}\{(x^2+2x+2)\ln x-2\,e^x[\mathrm{Chi}(x)-\mathrm{Shi}(x)]-x\}$
$\frac{3}{2}$	± 1	$x^{-1/2}e^{-x/2}(x^2\ln x-2x-1)$
$\frac{3}{2}$	± 2	$x^{-3/2}e^{-x/2}\{(x^3+3x^2+6x+6)\ln x-2x^2-4-6\,e^x[\mathrm{Chi}(x)-\mathrm{Shi}(x)]\}$
2	$\pm\frac{3}{2}$	$e^{-x/2}(x^2\ln x-3x-3-\frac{2}{x})$

3.2. Application to the Calculation of Infinite Integrals

Additional integral representations of the Whittaker function $W_{\kappa,\mu}(x)$ in terms of Bessel functions [8] (Section 7.4.2) are known:

$$\begin{aligned} &W_{\kappa,\mu}(x) \\ =\; &\frac{2\sqrt{x}e^{-x/2}}{\Gamma\left(\frac{1}{2}+\mu-\kappa\right)\Gamma\left(\frac{1}{2}-\mu-\kappa\right)}\int_0^\infty e^{-t}t^{-\kappa-1/2}K_{2\mu}\left(2\sqrt{xt}\right)dt \\ &\mathrm{Re}\left(\frac{1}{2}\pm\mu-\kappa\right)>0. \end{aligned} \tag{89}$$

Let us introduce the following infinite logarithmic integral.

Definition 3.

$$\mathcal{H}(\kappa,\mu;x)=\int_0^\infty e^{-t}t^{-\kappa-1/2}K_{2\mu}\left(2\sqrt{xt}\right)\ln t\,dt. \tag{90}$$

Theorem 14. *For $\kappa,\mu\in\mathbb{R}$ with $|\mu|<\frac{1}{2}-\kappa$, the following integral holds true:*

$$\begin{aligned} \mathcal{H}(\kappa,\mu;x) &= \frac{1}{2}\Gamma\left(\frac{1}{2}-\mu-\kappa\right) \\ &\left\{\frac{\Gamma\left(\frac{1}{2}+\mu-\kappa\right)\psi\left(\frac{1}{2}-\mu-\kappa\right)}{\sqrt{x}\,e^{-x/2}}W_{\kappa,\mu}(x)+x^\mu\,I_1^*(\kappa,\mu;x)\right\}, \end{aligned} \tag{91}$$

where $I_1^*(\kappa,\mu;x)$ is given by (42).

Proof. Differentiation of (89) with respect to parameter κ yields:

$$\begin{aligned} \frac{\partial W_{\kappa,\mu}(x)}{\partial\kappa} &= \left[\psi\left(\frac{1}{2}-\mu-\kappa\right)+\psi\left(\frac{1}{2}+\mu-\kappa\right)\right]W_{\kappa,\mu}(x) \\ &-\frac{2\sqrt{x}e^{-x/2}}{\Gamma\left(\frac{1}{2}+\mu-\kappa\right)\Gamma\left(\frac{1}{2}-\mu-\kappa\right)}\mathcal{H}(\kappa,\mu;x) \end{aligned} \tag{92}$$

Equate (39) to (92) to arrive at (91), as we wanted to prove. □

3.3. Derivative with Respect to the Second Parameter $\partial \mathrm{W}_{\kappa,\mu}(x)/\partial\mu$

First, note that

$$\frac{\partial \mathrm{W}_{\kappa,\pm\mu}(x)}{\partial\mu} = \pm\frac{\partial \mathrm{W}_{\kappa,\mu}(x)}{\partial\mu}, \tag{93}$$

since (15) is satisfied. Next, let us introduce the following definitions in order to calculate the first derivative of $\mathrm{W}_{\kappa,\mu}(x)$ with respect to parameter μ.

Definition 4. *Following the notation introduced in (8) and (9), define*

$$\tilde{G}^{(1)}(a,b,z) = \frac{\partial}{\partial a}[\mathrm{U}(a,b,z)], \tag{94}$$

and

$$\tilde{H}^{(1)}(a,b,z) = \frac{\partial}{\partial b}[\mathrm{U}(a,b,z)]. \tag{95}$$

Direct differentiation of (5) yields:

$$\begin{aligned} &\frac{\partial \mathrm{W}_{\kappa,\mu}(x)}{\partial\mu} \\ = \; &\ln x\, \mathrm{W}_{\kappa,\mu}(x) + x^{\mu+1/2}e^{-x/2} \\ &\left[\tilde{G}^{(1)}\left(\frac{1}{2}-\kappa+\mu, 1+2\mu, x\right) + 2\,\tilde{H}^{(1)}\left(\frac{1}{2}-\kappa+\mu, 1+2\mu, x\right)\right] \end{aligned} \tag{96}$$

Definition 5. *For* $\mathrm{Re}(\mu-\kappa) > -\frac{1}{2}$ *and* $x > 0$, *define:*

$$I_3^*(\kappa,\mu;x) = \int_0^\infty e^{-xt} t^{\mu-\kappa-1/2}(1+t)^{\mu+\kappa-1/2} \ln[t(1+t)]dt, \tag{97}$$

$$I_4^*(\kappa,\mu;x) = \int_1^\infty e^{-xt} t^{\mu+\kappa-1/2}(t-1)^{\mu-\kappa-1/2} \ln[t(t-1)]dt. \tag{98}$$

These integrals are interrelated by

$$I_4^*(\kappa,\mu;x) = e^{-x} I_3^*(\kappa,\mu;x).$$

Differentiation of (35) with respect to parameter μ gives

$$\begin{aligned} &\frac{\partial \mathrm{W}_{\kappa,\mu}(x)}{\partial\mu} \\ = \; &\left[\ln x - \psi\left(\mu-\kappa+\frac{1}{2}\right)\right]\mathrm{W}_{\kappa,\mu}(x) + \frac{x^{\mu+1/2}e^{-x/2}}{\Gamma\left(\mu-\kappa+\frac{1}{2}\right)} I_3^*(\kappa,\mu;x). \end{aligned} \tag{99}$$

Theorem 15. *According to the notation introduced in (94) and (95), the following integral holds true for* $x > 0$*:*

$$\begin{aligned} &I_3^*(\kappa,\mu;x) \\ = \; &\Gamma\left(\frac{1}{2}-\kappa+\mu\right)\left\{\mathrm{U}\left(\frac{1}{2}-\kappa+\mu, 1+2\mu, x\right)\psi\left(\frac{1}{2}-\kappa+\mu\right)\right. \\ &\left. + \tilde{G}^{(1)}\left(\frac{1}{2}-\kappa+\mu, 1+2\mu, x\right) + 2\,\tilde{H}^{(1)}\left(\frac{1}{2}-\kappa+\mu, 1+2\mu, x\right)\right\}. \end{aligned} \tag{100}$$

Proof. Comparing (96) to (99), taking into account (5), we arrive at (100), as we wanted to prove. □

Theorem 16. *For $-2\mu \neq 0, 1, 2, \ldots$ and $x > 0$, the following reduction formula holds true:*

$$\left.\frac{\partial W_{\kappa,\mu}(x)}{\partial \mu}\right|_{\kappa=1/2-\mu} = x^{1/2-\mu} e^{-x/2} \tag{101}$$

$$\left\{ \frac{x}{2\mu} \, {}_2F_2\left(\begin{array}{c} 1,1 \\ 2, 1+2\mu \end{array} \middle| -x \right) + e^{-2\pi i \mu} \, \Gamma(1-2\mu, x) \, \gamma(2\mu, -x) + \psi(2\mu) - \ln x \right\}.$$

Proof. According to (43) and (97), note that

$$\begin{aligned} I_3^*\left(\frac{1}{2} - \mu, \mu; x\right) &= \mathcal{I}_+(2\mu - 1, x) \\ &= \frac{\Gamma(2\mu)}{x^{2\mu}} \left\{ \frac{x}{2\mu} \, {}_2F_2\left(\begin{array}{c} 1,1 \\ 2, 1+2\mu \end{array} \middle| -x \right) \right. \\ &\left. + e^{-2\pi i \mu} \, \Gamma(1-2\mu, x) \, \gamma(2\mu, -x) + 2[\psi(2\mu) - \ln x] \right\}. \end{aligned} \tag{102}$$

Taking $\kappa = 1/2 - \mu$ in (99), substitute (102) and (56) to arrive at the desired result, given in (101). □

Remark 4. *If we take into account (48) in (102), we obtain the alternative form:*

$$\begin{aligned} & I_3^*\left(\frac{1}{2} - \mu, \mu; x\right) \\ = & \frac{1}{x^{2\mu}} \left\{ \pi[\cot(2\pi\mu) - i] \gamma(2\mu, -x) + x \, \Gamma(2\mu - 1) \, {}_2F_2\left(\begin{array}{c} 1,1 \\ 2, 2-2\mu \end{array} \middle| x \right) \right. \\ + & \left. 2 \, \Gamma(2\mu)[\psi(2\mu) - \ln x] \right\}, \end{aligned}$$

thus for $-2\mu \neq 0, 1, 2, \ldots$ and $x > 0$, we have

$$\left.\frac{\partial W_{\kappa,\mu}(x)}{\partial \mu}\right|_{\kappa=1/2-\mu} = x^{1/2-\mu} e^{-x/2} \tag{103}$$

$$\left\{ \frac{\pi[\cot(2\pi\mu) - i]}{\Gamma(2\mu)} \gamma(2\mu, -x) + \frac{x}{2\mu - 1} \, {}_2F_2\left(\begin{array}{c} 1,1 \\ 2, 2-2\mu \end{array} \middle| x \right) + \psi(2\mu) - \ln x \right\}.$$

Table 7 shows the first derivative of $W_{\kappa,\mu}(x)$ with respect to parameter μ for some particular values of κ and μ, with $x > 0$, calculated from (103) with the aid of the MATHEMATICA program.

Table 7. Derivative of $W_{\kappa,\mu}$ with respect to μ, by using (103).

κ	μ	$\frac{\partial W_{\kappa,\mu}(x)}{\partial \mu} \quad (x > 0)$
$-\frac{1}{4}$	$\pm\frac{3}{4}$	$\pm x^{-1/4} e^{-x/2} \left[2 - \gamma - \ln(4x) - 2 e^x \sqrt{\pi x} + \pi \, \mathrm{erfi}\left(\sqrt{x}\right) + 2x \, {}_2F_2\left(1, 1; \frac{1}{2}, 2; x\right) \right]$
$\frac{1}{4}$	$\pm\frac{1}{4}$	$\pm x^{1/4} e^{-x/2} \left[\pi \, \mathrm{erfi}\left(\sqrt{x}\right) - 2x \, {}_2F_2\left(1, 1; \frac{3}{2}, 2; x\right) - \gamma - \ln(4x) \right]$
$\frac{3}{4}$	$\pm\frac{1}{4}$	$\pm e^{-x/2} \left\{ x^{3/4} \left[\frac{2}{3} x \, {}_2F_2\left(1, 1; \frac{5}{2}, 2; x\right) - 2 + \gamma + \ln(4x) - \pi \, \mathrm{erfi}\left(\sqrt{x}\right) \right] + \sqrt{\pi} \, x^{1/4} \, e^x \right\}$
$\frac{5}{4}$	$\pm\frac{3}{4}$	$\pm \frac{1}{30} x^{-1/4} e^{-x/2} \left\{ 15 \sqrt{\pi} e^x (2x + 1) - 2 \, x^{3/2} \right.$ $\left. \left[40 - 15\gamma - 30 \ln(2x) + 15\pi \, \mathrm{erfi}\left(\sqrt{x}\right) - 12x \, {}_2F_2\left(1, 1; \frac{7}{2}, 2; x\right) \right] \right\}$

Notice that for $-2\mu = 0, 1, \ldots$, we obtain an indeterminate expression in (101) or (103). For these cases, we present the following result.

Theorem 17. *The following reduction formula holds true for* $m = 0, 1, 2, \ldots$*:*

$$\left.\frac{\partial W_{\kappa,\mu}(x)}{\partial \mu}\right|_{\kappa=(1+m)/2,\mu=\pm m/2} = \pm e^{-x/2} x^{(1+m)/2} \sum_{k=1}^{m} x^{-k} \left[e^{x}\, \Gamma(k) + \binom{m}{k} \gamma(k,-x) \right]. \tag{104}$$

Proof. Take $\nu = 2\mu$ in (103) and perform the limit $\nu \to -m = 0, -1, -2, \ldots$

$$\left.\frac{\partial W_{\kappa,\mu}(x)}{\partial \mu}\right|_{\kappa=(m+1)/2,\mu=-m/2} = e^{-x/2} x^{(1+m)/2}$$
$$\left\{ \lim_{\nu \to -m} \left[\psi(\nu) + \frac{\pi[\cot(\pi\nu) - i]}{\Gamma(\nu)} \gamma(\nu,-x) \right] - \frac{x}{m+1}\, {}_2F_2\left(\left. \begin{array}{c} 1,1 \\ 2, 2+m \end{array} \right| x \right) - \ln x \right\}.$$

Applying the result given in (64), we get

$$\left.\frac{\partial W_{\kappa,\mu}(x)}{\partial \mu}\right|_{\kappa=(m+1)/2,\mu=-m/2} = e^{-x/2} x^{(1+m)/2} \tag{105}$$
$$\left\{ H_m - \gamma - i\,\pi + (-1)^{m+1} m!\, \Gamma(-m,-x) - \frac{x}{m+1}\, {}_2F_2\left(\left. \begin{array}{c} 1,1 \\ 2, 2+m \end{array} \right| x \right) - \ln x \right\}.$$

Now, compare (58) to (65) to see that

$$\begin{aligned} & H_m - \gamma - i\,\pi + (-1)^{m+1} m!\, \Gamma(-m,-x) - \frac{x}{m+1}\, {}_2F_2\left(\left. \begin{array}{c} 1,1 \\ 2, 2+m \end{array} \right| x \right) \\ = \; & \ln x - \sum_{k=1}^{m} x^{-k} \left[e^{x}\, \Gamma(k) + \binom{m}{k} \gamma(k,-x) \right] \end{aligned} \tag{106}$$

Therefore, inserting (106) into (105), and taking into account (93), we arrive at (104), as we wanted to prove. □

Remark 5. *It is worth noting that from [17],*

$$\begin{aligned} & \left.\frac{\partial W_{\kappa,\mu}(x)}{\partial \mu}\right|_{\kappa=(N+1)/2,\mu=M/2} \\ = \; & \sum_{k=1}^{(N+M)/2} \frac{(-1)^k \left(\frac{N+M}{2}\right)!}{k \left(\frac{N+M}{2} - k\right)!} W_{\frac{N+1}{2}-k,\frac{M}{2}}(x) + \sum_{k=1}^{(N-M)/2} \frac{(-1)^k \left(\frac{N-M}{2}\right)!}{k \left(\frac{N-M}{2} - k\right)!} W_{\frac{N+1}{2}-k,\frac{M}{2}}(x), \end{aligned} \tag{107}$$

where $-N \leq M \leq N$ *and* M, N *are integers of like parity, we can derive an equivalent reduction formula to (104). Indeed, following similar steps as in Remark 3, we arrive at:*

$$\left.\frac{\partial W_{\kappa,\mu}(x)}{\partial \mu}\right|_{\kappa=(1+m)/2,\mu=\pm m/2} = \pm m!\, e^{-x/2} x^{(1+m)/2} \sum_{k=1}^{m} \frac{x^{-k}}{k(m-k)!}. \tag{108}$$

Theorem 18. *For* $n = 0, 1, 2, \ldots$*, the following reduction formula holds true:*

$$\begin{aligned} & \left.\frac{\partial W_{\kappa,\mu}(x)}{\partial \mu}\right|_{\kappa=n/2,\mu=\pm(n+1)/2} \\ = \; & \pm x^{-n/2} e^{x/2}\, E_1(x)\, \Gamma(1+n,x) \pm n!\, x^{n/2} e^{-x/2} \\ & \sum_{k=0}^{n} \frac{x^{-k}}{(n-k)!} \left\{ H_k + (-1)^{k+1}\, \Gamma(-k,x)\, \gamma(k+1,-x) - \sum_{\ell=1}^{k} \binom{k}{\ell} (-x)^{-\ell} \gamma(\ell,x) \right\}. \end{aligned} \tag{109}$$

Proof. According to (97) and (43), using the binomial theorem and taking into account (87), we have

$$I_3^*\left(\frac{n}{2},\frac{n+1}{2};x\right)=\int_0^\infty e^{-xt}(1+t)^n\ln[t(1+t)]dt \tag{110}$$

$$=\sum_{k=0}^{n}\binom{n}{k}\mathcal{I}_+(k,x)=n!\sum_{k=0}^{n}\frac{x^{-k-1}}{(n-k)!}$$

$$\left\{\frac{x}{k+1}\,{}_2F_2\left(\begin{array}{c}1,1\\2,2+k\end{array}\middle|-x\right)+(-1)^{k+1}\,\Gamma(-k,x)\,\gamma(k+1,-x)+2[H_k-\gamma-\ln x]\right\}.$$

Consider (66), (68) and (79) in order to rewrite (110) as

$$I_3^*\left(\frac{n}{2},\frac{n+1}{2};x\right)=\frac{\mathrm{E}_1(x)-\ln x-\gamma}{x^{n+1}}\,e^x\,\Gamma(n+1,x) \tag{111}$$

$$+n!\sum_{k=0}^{n}\frac{x^{-k-1}}{(n-k)!}\left\{H_k+(-1)^{k+1}\,\Gamma(-k,x)\,\gamma(k+1,-x)-\sum_{\ell=1}^{k}\binom{k}{\ell}(-x)^{-\ell}\gamma(\ell,x)\right\}.$$

Therefore, substituting (81), (63), and (111) into (99), we obtain (109), as we wanted to prove. □

Theorem 19. *For $n=0,1,2,\ldots$, the following reduction formula holds true:*

$$\left.\frac{\partial \mathrm{W}_{\kappa,\mu}(x)}{\partial\mu}\right|_{\kappa=0,\mu=\pm(n+1/2)} \tag{112}$$

$$=\pm\sqrt{\frac{x}{\pi}}K_{n+1/2}\left(\frac{x}{2}\right)[\mathrm{E}_1(x)-H_n]\pm e^{-x/2}\sum_{k=0}^{n}\frac{(n+k)!\,x^{-k}}{k!(n-k)!}$$

$$\left\{H_{n+k}+(-1)^{n+k+1}\,\Gamma(-n-k,x)\,\gamma(n+k+1,-x)-\sum_{\ell=1}^{k}\binom{k}{\ell}(-x)^{-\ell}\gamma(\ell,x)\right\}.$$

Proof. Applying the binomial theorem to (97) for $\kappa=0$ and $\mu=n+\frac{1}{2}$, and taking into account (43), (66), (68), and (84) for $x>0$, we arrive at

$$I_3^*\left(0,n+\frac{1}{2};x\right)=\int_0^\infty e^{-xt}[t(1+t)]^n\ln[t(1+t)]dt \tag{113}$$

$$=\sum_{k=0}^{n}\binom{n}{k}\int_0^\infty e^{-xt}t^{n+k}\ln[t(1+t)]dt=\sum_{k=0}^{n}\binom{n}{k}\mathcal{I}_+(n+k,x)$$

$$=\frac{n!e^{x/2}K_{n+1/2}\left(\frac{x}{2}\right)}{\sqrt{\pi}\,x^{n+1/2}}[\mathrm{E}_1(x)-\gamma-\ln x]+\frac{n!}{x^{n+1}}\sum_{k=0}^{n}\frac{(n+k)!\,x^{-k}}{k!(n-k)!}$$

$$\left\{H_{n+k}+(-1)^{n+k+1}\,\Gamma(-n-k,x)\,\gamma(n+k+1,-x)-\sum_{\ell=1}^{k}\binom{k}{\ell}(-x)^{-\ell}\gamma(\ell,x)\right\}.$$

Take $\kappa=0$ and $\mu=n+\frac{1}{2}$ in (99), and substitute (113) and (88) in order to arrive at (112), as we wanted to prove. □

Table 8 shows $\mathrm{W}_{\kappa,\mu}(x)$ with respect to parameter μ for some particular values of κ and μ, which has been calculated from (104), (109), and (112) with the aid of the MATHEMATICA program.

Table 8. Derivative of $W_{\kappa,\mu}$ with respect to μ, by using (109) and (112).

κ	μ	$\frac{\partial W_{\kappa,\mu}(x)}{\partial \mu}$
0	$\pm\frac{1}{2}$	$\pm e^{x/2}[\mathrm{Shi}(x) - \mathrm{Chi}(x)]$
0	$\pm\frac{3}{2}$	$\pm x^{-1}e^{-x/2}\{e^x(x-2)[\mathrm{Shi}(x) - \mathrm{Chi}(x)] + 4\}$
0	$\pm\frac{5}{2}$	$\pm x^{-2}e^{-x/2}\{4(x+8) - e^x(x^2-6x+12)[\mathrm{Shi}(x) - \mathrm{Chi}(x)]\}$
$\frac{1}{2}$	± 1	$\pm x^{-1/2}e^{-x/2}\{e^x[\mathrm{Shi}(x) - \mathrm{Chi}(x)] + 2\}$
$\frac{1}{2}$	0	0
1	$\pm\frac{1}{2}$	$\pm e^{-x/2}$
1	$\pm\frac{3}{2}$	$\pm x^{-1}e^{-x/2}\{2\,e^x[\mathrm{Shi}(x) - \mathrm{Chi}(x)] + 3(x+2)\}$
$\frac{3}{2}$	± 1	$\pm x^{-1/2}e^{-x/2}(2x+1)$
$\frac{3}{2}$	± 2	$\pm x^{-3/2}e^{-x/2}\{2(2x^2+7x+11) - 6\,e^x[\mathrm{Shi}(x) - \mathrm{Chi}(x)]\}$
2	$\pm\frac{3}{2}$	$\pm e^{-x/2}(3x+3+\frac{2}{x})$
2	$\pm\frac{5}{2}$	$\pm x^{-2}e^{-x/2}\{5(x^3+5x^2+14x+20) - 24\,e^x[\mathrm{Shi}(x) - \mathrm{Chi}(x)]\}$

4. Integral Whittaker Functions $\mathrm{Wi}_{\kappa,\mu}$ and $\mathrm{wi}_{\kappa,\mu}$

In [20], we found some reduction formulas for the integral Whittaker function $\mathrm{Wi}_{\kappa,\mu}(x)$. Next, we derive some new reduction formulas for $\mathrm{Wi}_{\kappa,\mu}(x)$ and $\mathrm{wi}_{\kappa,\mu}(x)$ from reduction formulas of the Whittaker function $\mathrm{W}_{\kappa,\mu}(x)$.

Theorem 20. *The following reduction formula holds true for $n = 0, 1, 2, \ldots$ and $\kappa > 0$:*

$$\mathrm{Wi}_{\kappa+n,\kappa-1/2}(x) = (-1)^n (2\kappa)_n\, 2^\kappa \sum_{m=0}^{n} \binom{n}{m} \frac{(-2)^m}{(2\kappa)_m} \gamma(\kappa+m, x/2). \tag{114}$$

Proof. According to [3] (Eqn. 13.18.17)

$$\mathrm{W}_{\kappa+n,\kappa-1/2}(x) = (-1)^n n!\, e^{-x/2} x^\kappa L_n^{(2\kappa-1)}(x), \tag{115}$$

where [29] (Eqn. 4.17.2)

$$L_n^{(\alpha)}(x) = \sum_{m=0}^{n} \frac{\Gamma(n+\alpha+1)}{\Gamma(m+\alpha+1)} \frac{(-x)^m}{m!(n-m)!}, \tag{116}$$

denotes the Laguerre polynomials. Insert (116) into (115) and integrate term by term according to the definition of the integral Whittaker function (6), to get

$$\begin{aligned} & \mathrm{Wi}_{\kappa+n,\kappa-1/2}(x) \\ = \; & (-1)^n (2\kappa)_n \sum_{m=0}^{n} \binom{n}{m} \frac{(-1)^m}{(2\kappa)_m} \int_0^x e^{-t/2} t^{\kappa+m-1} dt. \end{aligned}$$

Finally, take into account the definition of the lower incomplete gamma function [3] (Eqn. 8.2.1):

$$\gamma(\nu, z) = \int_0^z t^{\nu-1} e^{-t} dt, \qquad \mathrm{Re}\,\nu > 0, \tag{117}$$

and simplify the result, to arrive at (114), as we wanted to prove. □

Remark 6. *Taking $n = 0$ in (114), we recover the formula given in [20].*

Theorem 21. *The following reduction formula holds true for $x > 0$, $n = 0, 1, 2, \ldots$ and $\kappa \in \mathbb{R}$:*

$$\mathrm{wi}_{\kappa+n,\kappa-1/2}(x) = (-1)^n (2\kappa)_n \, 2^\kappa \sum_{m=0}^{n} \binom{n}{m} \frac{(-2)^m}{(2\kappa)_m} \Gamma(\kappa + m, x/2), \tag{118}$$

where $\Gamma(\nu, z)$ denotes the upper incomplete gamma function (119).

Proof. Follow similar steps as in the previous theorem, but consider the definition of the upper incomplete gamma function [3] (Eqn. 8.2.2):

$$\Gamma(\nu, z) = \int_z^\infty t^{\nu-1} e^{-t} dt. \tag{119}$$

□

Theorem 22. *The following reduction formula holds true for $x > 0$, and $n = 0, 1, 2, \ldots$:*

$$\mathrm{wi}_{0,n+1/2}(x) = \sum_{m=0}^{n} \frac{(n+k)! 2^{-k}}{k!(n-k)!} \Gamma(-k, x/2). \tag{120}$$

Proof. From (84) and (88), we have

$$\mathrm{W}_{0,n+1/2}(z) = e^{-z/2} \sum_{k=0}^{n} \frac{(n+k)! \, z^{-k}}{k!(n-k)!},$$

thus, integrating term by term, we obtain

$$\mathrm{wi}_{0,n+1/2}(x) = \sum_{k=0}^{n} \frac{(n+k)!}{k!(n-k)!} \int_x^\infty e^{-t/2} \, t^{-k-1} dt.$$

Finally, taking into account (119), we arrive at (120), as we wanted to prove. □

Theorem 23. *For $x > 0$ and $\mathrm{Re}\left(\frac{1}{2} + \mu - \kappa\right) > 0$, the following integral representation holds true:*

$$\mathrm{wi}_{\kappa,\mu}(x) = \frac{1}{\Gamma\left(\frac{1}{2} + \mu - \kappa\right)} \int_0^\infty \frac{t^{\mu-\kappa-1/2}(1+t)^{\mu+\kappa-1/2}}{\left(\frac{1}{2} + t\right)^{\mu+1/2}} \Gamma\left(\frac{1}{2} + \mu, x\left(t + \frac{1}{2}\right)\right) dt. \tag{121}$$

Proof. According to (7) and (35), we have

$$\begin{aligned} &\mathrm{wi}_{\kappa,\mu}(x) \\ = &\frac{1}{\Gamma\left(\mu - \kappa + \frac{1}{2}\right)} \int_x^\infty dt \; t^{\mu-1/2} e^{-t/2} \int_0^\infty e^{-x\xi} \xi^{\mu-\kappa-1/2} (1+\xi)^{\mu+\kappa-1/2} d\xi. \end{aligned}$$

Exchange the integration order and calculate the inner integral using (119), to arrive at (121), as we wanted to prove. □

Remark 7. *It is worth noting that we cannot follow the above steps to derive the integral representation of $\mathrm{Wi}_{\kappa,\mu}(x)$, because the corresponding integral does not converge, except for some special cases such as the ones given in (114).*

Theorem 24. *For $x > 0$ and $\mathrm{Re}\left(\frac{1}{2}+\mu-\kappa\right) > 0$, the following integral representation holds true:*

$$\frac{\partial \mathrm{wi}_{\kappa,\mu}(x)}{\partial \kappa} = \frac{1}{\Gamma\left(\frac{1}{2}+\mu-\kappa\right)} \tag{122}$$

$$\int_0^{\infty}\left[\psi\left(\frac{1}{2}+\mu-\kappa\right)+\ln\left(\frac{1+t}{t}\right)\right]\frac{t^{\mu-\kappa-1/2}(1+t)^{\mu+\kappa-1/2}}{\left(\frac{1}{2}+t\right)^{\mu+1/2}}\Gamma\left(\frac{1}{2}+\mu, x\left(t+\frac{1}{2}\right)\right)dt.$$

Proof. Direct differentiation of (121) with respect to κ yields (122), as we wanted to prove. □

5. Conclusions

The Whittaker function $\mathrm{W}_{\kappa,\mu}(x)$ is defined in terms of the Tricomi function, hence its derivative with respect to the parameters κ and μ can be expressed as infinite sums of quotients of the digamma and gamma functions. In addition, the parameter differentiation of some integral representations of $\mathrm{W}_{\kappa,\mu}(x)$ leads to infinite integrals of elementary functions. These sums and integrals have been calculated for some particular cases of the parameters κ and μ, in closed form. As an application of these results, we have calculated an infinite integral containing the Macdonald function. It is worth noting, that all the results presented in this paper have been both numerically and symbolically checked with the MATHEMATICA program.

In Appendix A, we calculate a reduction formula for the first derivative of the Kummer function, i.e., $G^{(1)}(a;a;z)$, which is necessary for the derivation of Theorem 1.

In Appendix B, we calculate a reduction formula of the hypergeometric function ${}_2F_2(1,1;2,2+m;x)$ for a non-negative integer m, since it is not found in most common literature, such as [9]. This reduction formula is used throughout Section 3 in order to simplify the results obtained.

Finally, we collect some reduction formulas for the Whittaker function $\mathrm{W}_{\kappa,\mu}(x)$ in Appendix C.

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Appendix A. Calculation of $G^{(1)}(a;a;z)$

Theorem A1. *The following reduction formula holds true:*

$$G^{(1)}\left(\begin{array}{c} a \\ a \end{array}\middle| x\right) = \frac{x\,e^{x}}{a}\,{}_2F_2\left(\begin{array}{c} 1,1 \\ a+1,2 \end{array}\middle| -x\right). \tag{A1}$$

Proof. According to the definition of the Kummer function (3), we have

$${}_1F_1\left(\begin{array}{c} b \\ a \end{array}\middle| x\right) = 1+\sum_{n=0}^{\infty}\frac{(b)_{n+1}}{(a)_{n+1}}\frac{x^{n+1}}{(n+1)!}. \tag{A2}$$

Taking into account [4] (Eqn. 18:5:7)

$$(\alpha)_{n+1} = \alpha(\alpha+1)_n,$$

and the definition of the generalized hypergeometric function (4), we may recast (A2) as

$${}_1F_1\left(\begin{array}{c} b \\ a \end{array}\middle| x\right) = 1 + \frac{b}{a} x\, {}_2F_2\left(\begin{array}{c} 1, b+1 \\ 2, a+1 \end{array}\middle| x\right),$$

thus, for $b \neq 0$, we obtain (it is worth noting that there is an error in (Eqn. 7.12.1(5)) in [9])

$${}_2F_2\left(\begin{array}{c} 1, b+1 \\ 2, a+1 \end{array}\middle| x\right) = \frac{a}{b\,x}\left[{}_1F_1\left(\begin{array}{c} b \\ a \end{array}\middle| x\right) - 1\right]. \tag{A3}$$

Applying L'Hôpital's rule, calculate the limit $b \to 0$ in (A3), considering the notation given in (8),

$${}_2F_2\left(\begin{array}{c} 1, 1 \\ 2, a+1 \end{array}\middle| x\right) = \frac{a}{x} G^{(1)}\left(\begin{array}{c} 0 \\ a \end{array}\middle| x\right). \tag{A4}$$

Finally, differentiate Kummer's transformation formula (11) with respect to the first parameter, to obtain:

$$G^{(1)}\left(\begin{array}{c} b \\ a \end{array}\middle| x\right) = -e^x\, G^{(1)}\left(\begin{array}{c} b-a \\ b \end{array}\middle| -x\right). \tag{A5}$$

Apply (A5) in order to rewrite (A4) as (A1), as we wanted to prove. □

Appendix B. Calculation of ${}_2F_2(1,1;2,2+m;x)$

Theorem A2. *For* $m = 0, 1, 2, \ldots$, *the following reduction formula holds true:*

$${}_2F_2\left(\begin{array}{c} 1, 1 \\ 2, 2+m \end{array}\middle| x\right) = \frac{m+1}{x}\left\{H_m - \mathrm{Ein}(-x) + \sum_{k=1}^{m}\binom{m}{k} x^{-k}\gamma(k,-x)\right\}, \tag{A6}$$

where $\mathrm{Ein}(z)$ *denotes the complementary exponential integral.*

Proof. Consider the function

$$R_m(x) = \frac{1}{m!}\, {}_2F_2\left(\begin{array}{c} 1, 1 \\ 2, 1+m \end{array}\middle| x\right) = \sum_{k=0}^{\infty} \frac{x^k}{(m+k)!(k+1)},$$

thus

$$\frac{d}{dx}[x^m R_m(x)] = x^{m-1} R_{m-1}(x),$$

and by induction

$$\frac{d^m}{dx^m}[x^m R_m(x)] = R_0(x) = \frac{1}{x}\sum_{k=0}^{\infty}\frac{x^{k+1}}{(k+1)!} = \frac{e^x - 1}{x}.$$

Now, apply the repeated integral formula [3] (Eqn. 1.4.31)

$$f^{(-n)}(x) = \frac{1}{(n-1)!}\int_0^x (x-t)^{n-1} f(t)\,dt,$$

to obtain

$$\begin{aligned} R_{m+1}(x) &= \frac{1}{(m+1)!}\,{}_2F_2\left(\begin{array}{c} 1,1 \\ 2,2+m \end{array}\middle| x\right) \\ &= \frac{x^{-m-1}}{m!}\int_0^x (x-t)^m \left(\frac{e^t-1}{t}\right) dt. \end{aligned} \tag{A7}$$

Use the binomial theorem to expand (A7) as

$$\begin{aligned} &{}_2F_2\left(\begin{array}{c} 1,1 \\ 2,2+m \end{array}\middle| x\right) \\ &= \frac{m+1}{x}\left\{\int_0^x \frac{e^t-1}{t}\,dt + \sum_{k=1}^{m}\binom{m}{k} x^{-k}(-1)^k \int_0^x t^{k-1}\left(e^t-1\right)dt\right\}. \end{aligned} \tag{A8}$$

According to [3] (Eqn. 6.2.3), we have

$$\int_0^x \frac{e^t-1}{t}\,dt = -\mathrm{Ein}(-x). \tag{A9}$$

Further, taking into account the definition of the lower incomplete gamma function [4] (Eqn. 45:3:1), we calculate for $k = 1, 2, \ldots$

$$\int_0^x t^{k-1}\left(e^t-1\right)dt = (-1)^k \gamma(k,-x) - \frac{x^k}{k}. \tag{A10}$$

Therefore, substituting (A9) and (A10) into (A8), we have

$${}_2F_2\left(\begin{array}{c} 1,1 \\ 2,2+m \end{array}\middle| x\right) = \frac{m+1}{x}\left\{-\mathrm{Ein}(-x) + \sum_{k=1}^{m}\binom{m}{k}\left[x^{-k}\gamma(k,-x) + \frac{(-1)^{k+1}}{k}\right]\right\}.$$

Finally, consider the formula [7] (Eqn. 0.155.4)

$$\sum_{k=1}^{m}\binom{m}{k}\frac{(-1)^{k+1}}{k} = H_m,$$

to arrive at (A6), as we wanted to prove □

Appendix C. Reduction Formulas for the Whittaker Function $\mathrm{W}_{\kappa,\mu}(x)$

For the convenience of the reader, reduction formulas for the Whittaker function $\mathrm{W}_{\kappa,\mu}(x)$ are presented in their explicit form in Table A1.

Table A1. Whittaker function $W_{\kappa,\mu}(x)$ for particular values of κ and μ.

κ	μ	$\mathrm{W}_{\kappa,\mu}(x)$
$-\frac{1}{4}$	$\pm\frac{1}{4}$	$\sqrt{\pi}e^{x/2}x^{1/4}\mathrm{erfc}\left(\sqrt{x}\right)$
$-\frac{1}{2}$	$\pm\frac{1}{2}$	$\frac{x}{\sqrt{\pi}}\left[K_1\left(\frac{x}{2}\right) - K_0\left(\frac{x}{2}\right)\right]$
$-\frac{1}{2}$	$\pm\frac{1}{6}$	$3\frac{x}{\sqrt{\pi}}\left[K_{2/3}\left(\frac{x}{2}\right) - K_{1/3}\left(\frac{x}{2}\right)\right]$
$-\frac{1}{2}$	± 1	$x^{-1/2}e^{-x/2}$
0	0	$\sqrt{\frac{x}{\pi}}\,K_0\left(\frac{x}{2}\right)$
0	$\pm\frac{1}{2}$	$e^{-x/2}$
0	± 1	$\sqrt{\frac{x}{\pi}}\,K_1\left(\frac{x}{2}\right)$

Table A1. *Cont.*

κ	μ	$\mathbf{W}_{\kappa,\mu}(x)$
0	$\pm\frac{3}{2}$	$x^{-1}e^{-x/2}(x+2)$
0	$\pm\frac{5}{2}$	$x^{-2}e^{-x/2}(x^2+6\,x+12)$
$\frac{1}{4}$	$\pm\frac{1}{4}$	$x^{1/4}e^{-x/2}$
$\frac{1}{2}$	$\pm\frac{1}{6}$	$\frac{x}{2\sqrt{\pi}}\left[K_{1/3}\left(\frac{x}{2}\right)+K_{2/3}\left(\frac{x}{2}\right)\right]$
$\frac{1}{2}$	$\pm\frac{1}{4}$	$\frac{x}{2\sqrt{\pi}}\left[K_{1/4}\left(\frac{x}{2}\right)+K_{3/4}\left(\frac{x}{2}\right)\right]$
$\frac{1}{2}$	$\pm\frac{1}{2}$	$\frac{x}{2\sqrt{\pi}}\left[K_0\left(\frac{x}{2}\right)+K_1\left(\frac{x}{2}\right)\right]$
$\frac{1}{2}$	± 1	$x^{-1/2}e^{-x/2}\,(x+1)$
$\frac{1}{2}$	± 2	$x^{-3/2}e^{-x/2}\,(x^2+4\,x+6)$
1	$\pm\frac{3}{2}$	$x^{-1}e^{-x/2}(x^2+2\,x+2)$
1	± 1	$\frac{1}{2}\sqrt{\frac{x}{\pi}}\left[x\,K_0\left(\frac{x}{2}\right)+(x+1)K_1\left(\frac{x}{2}\right)\right]$
1	± 2	$\frac{1}{2\sqrt{\pi\,x}}\left[x(x+3)K_0\left(\frac{x}{2}\right)+\left(x^2+4\,x+12\right)K_1\left(\frac{x}{2}\right)\right]$
2	± 2	$\frac{1}{4\sqrt{\pi\,x}}\left[x\left(2\,x^2+2x+3\right)K_0\left(\frac{x}{2}\right)+2\left(x^3+2\,x^2+4\,x+6\right)K_1\left(\frac{x}{2}\right)\right]$

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