Review

Some Applications of Affine in Velocities Lagrangians in Two-Dimensional Systems

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Abstract: The two-dimensional inverse problem for first-order systems is analysed and a method to construct an affine Lagrangian for such systems is developed. The determination of such Lagrangians is based on the theory of the Jacobi multiplier for the system of differential equations. We illustrate our analysis with several examples of families of forces that are relevant in mechanics, on one side, and of some relevant biological systems, on the other.

1. Introduction

The time evolution of a system is usually described in classical mechanics by means of systems of second-order ordinary differential equations. Sometimes, these equations can be derived from a variational principle, and the solution of the inverse problem in classical mechanics [1–5], i.e., finding the corresponding variational (or Lagrangian) description, requires then the use of a (regular) Lagrangian function L depending on the first-order velocities, that is, $L = L(t, q, \dot{q})$, the $\dot{q}$-dependence normally being at most quadratic (included in a kinetic energy term), although there are Lagrangians of other forms, known as non-standard Lagrangians [6–10]. In any case, the corresponding Euler–Lagrange equations are of the desired second-order type except when L is linear or, more generally, affine in the velocities [11,12], for which actually the equations are of the first-order type.

Although it may seem that the class of Lagrangians that are affine in the velocities is of scarcely any interest in mechanics, this is not the case, see, e.g., [13], and were considered a long time ago in the framework of classical field theory [14]. Moreover, every system of second-order differential equations can be converted into a related one of first-order equations by doubling the number of independent variables. Consequently, one can think about the possibility of having such a description in terms of affine in the velocities Lagrangians for a given mechanical system. The problem is now that we have no basic principles from which to derive the affine in velocities Lagrangian (or the corresponding action integral). The only way is to consider it as an 'inverse variational problem' of the system of first-order differential equations, and ask whether given a mechanical system with a system of second-order differential equations, one can find an affine in velocities Lagrangian whose Euler–Lagrange equations are equivalent in the above sense to the given system of second-order differential equations. In fact, this motivated a geometric analysis of this kind of Lagrangians in the framework of autonomous systems [12], with the aim of studying the inverse problem of Lagrangian mechanics [2–5] and the theory of non-point symmetries. Almost simultaneously, Faddeev and Jackiw developed a method for the

quantization of such singular systems which soon became very popular and received much attention from many theoretical physicists. This procedure of approaching such systems is usually referred to as the Faddeev–Jackiw (FJ) quantization method [15–17].

Obviously, our interest can be extended to include arbitrary systems of first-order differential equations, not only those coming from systems of second-order differential equations, because systems of first-order differential equations play a relevant rôle in many cases, not only in physics, where many equations are first-order, as in the Dirac equation, but also in other fields such as biology dynamics [18,19], economy and chemistry. In this article, we only consider the simplest case, the two-dimensional one, that is, the inverse problem of the first-order system on $\mathbb{R}^2$ for such Lagrangians.

The search for Lagrangians for a single second-order differential equation has received much attention during the last years, see [6–8,20–23]. The main reason for that interest is that the knowledge of first integrals is very useful to integrate the system and the analysis of the infinitesimal point symmetries of the Lagrangian can be used, via the well-known Noether theorem, to find first integrals. The study of the integrability, as well as linearisability, of a given system is, therefore, usually based on the existence of such constants of motion (see, e.g., the series of papers [24–28]). Another reason is the search for a Hamiltonian formulation in order to proceed to its possible quantization. We, therefore, believe that a first-order approach to this problem may also be of interest. The general treatment of the variational inverse problem for systems of first-order equations began with the work of Havas [29]. Santilli also approached in [30] this subject and developed what he called Birkhoffian mechanics [31,32], which he considered as a generalisation of Hamiltonian formalism; see also [5] and references therein. More recent works on a geometric approach to the problem are [11,33].

The plan of the paper is the following. In Section 2, the main features of a two-dimensional affine in velocities Lagrangian are analysed. In order to make the paper more self-contained and to fix notation, we summarise some concepts used in the modern geometrical formulation of mechanics (operations with vector fields and differential forms), which can be found in classical textbooks (see, e.g., [20,34,35]). In Section 3, although we start with a time-dependent dynamical system in $\mathbb{R} \times T\mathbb{R}^2$ of the Lagrangian type, it is shown that we finally have to deal with a time-dependent dynamical system $\Gamma \in \mathfrak{X}(\mathbb{R} \times \mathbb{R}^2)$, which actually is Hamiltonian with respect to an appropriate Hamiltonian structure. Section 4 is devoted to analysing the two-dimensional inverse problem for systems of first-order differential equations. As the main result is to be expressed in terms of a Jacobi multiplier of the given system, we first review in Section 4.1 the theory of Jacobi multipliers in geometrical terms [21,36–53]. The main result asserts that in order to have a Lagrangian description for a given system of two first-order differential equations, it is necessary and sufficient to establish the existence of a Jacobi multiplier for the system. Moreover, these Jacobi multipliers can be used to find constants of motion via Hojman symmetries [44]. The equations determining the multipliers have always a local solution, so this inverse problem has a positive answer; in fact, there are infinitely many (not necessarily gauge-equivalent) Lagrangians for the given system, opening in this way the possibility of finding constants of motion (see, e.g., [38,43,51]) and alternative Lagrangians [54]. For more information on the rôle of Jacobi multipliers in integrability and with other approaches to integrability see, e.g., [55–58]. In Section 5, we present explicit Lagrangians for some important examples of differential equations, as mechanical systems, and interesting results on biological examples [59], as a generalisation of the Lotka–Volterra model [18,60,61] and a host–parasite model [19], are derived from this new perspective and their Hamiltonian functions and a set of canonical variables are also given. Finally, in Section 6, we summarise the previous results and the generalisation to systems involving more variables is proposed.

2. Affine Lagrangians on $\mathbb{R}^2$

The higher order in the velocities terms of the functions usually appearing as Lagrangians of classical mechanical systems are quadratic, and the corresponding systems

of Euler–Lagrange equations are systems of second-order differential equations as it happens for more general regular Lagrangians. In physics, it is also frequent to use singular Lagrangians, and constraints are present. An instance of singular Lagrangians are those depending linearly (or, more generally, affinely) on the velocities which give rise to systems of first-order differential equations (see, for instance, [11,12] and references therein).

In order to analyse the structure of this kind of Lagrangians, let us consider a general time-dependent Lagrangian system on a configuration space $\mathbb{R}^2$ described by a Lagrangian function $L \in C^\infty(\mathbb{R} \times T\mathbb{R}^2)$ which is affine in the velocities. In the local coordinates (t, x, y) in $\mathbb{R} \times \mathbb{R}^2$ and the corresponding fibred ones (t, x, y, v_x, v_y) in $\mathbb{R} \times T\mathbb{R}^2$, such a Lagrangian is of the form

$$L(t, x, y, v_x, v_y) = m_x(t, x, y)\, v_x + m_y(t, x, y)\, v_y + H(t, x, y), \tag{1}$$

where $m_x, m_y, H \in C^\infty(\mathbb{R} \times \mathbb{R}^2)$. Such a Lagrangian (1) can be intrinsically defined by the 1-form on $\mathbb{R} \times \mathbb{R}^2$

$$\lambda = m_x(t, x, y)\, dx + m_y(t, x, y)\, dy + H(t, x, y)\, dt \tag{2}$$

by contraction with the total-time derivative operator

$$\mathbf{T} = \frac{\partial}{\partial t} + v_x \frac{\partial}{\partial x} + v_y \frac{\partial}{\partial y},$$

that is, $L = i_{\mathbf{T}}\lambda$, where $dt \in \bigwedge^1(\mathbb{R} \times \mathbb{R}^2)$ is the pull-back of the volume form $dt \in \bigwedge^1(\mathbb{R})$ such that $i(d/dt)dt = 1$. Note that $\mathbf{T}$ is a vector field along the natural projection $\pi : \mathbb{R} \times T\mathbb{R}^2 \to \mathbb{R} \times \mathbb{R}^2$, given by $\pi(t, x, y, v_x, v_y) = (t, x, y)$, while λ is the above-mentioned 1-form (2) on $\mathbb{R} \times \mathbb{R}^2$; the contraction $i_{\mathbf{T}}\lambda$ makes sense and it is a real function on the manifold $\mathbb{R} \times T\mathbb{R}^2$. See [11,33] and references therein for more details.

The system of Euler–Lagrange equations corresponding to the function $L \in C^\infty(\mathbb{R} \times T\mathbb{R}^2)$ given by (1) is a system of two first-order differential equations that can be written in matrix form as

$$M\dot{X} = W, \tag{3}$$

with the matrices

$$M = \begin{pmatrix} 0 & \mu \\ -\mu & 0 \end{pmatrix}, \qquad X = \begin{pmatrix} x \\ y \end{pmatrix} \qquad \text{and} \qquad W = \begin{pmatrix} \mathrm{w}_x \\ \mathrm{w}_y \end{pmatrix}$$

being given by

$$\mu = \frac{\partial m_y}{\partial x} - \frac{\partial m_x}{\partial y}, \quad \mathrm{w}_x = \frac{\partial m_x}{\partial t} - \frac{\partial H}{\partial x}, \quad \mathrm{w}_y = \frac{\partial m_y}{\partial t} - \frac{\partial H}{\partial y}. \tag{4}$$

Observe that the functions μ, w_x, and w_y are not completely independent because we have, as a direct consequence of definition (4),

$$\frac{\partial \mu}{\partial t} + \frac{\partial(-\mathrm{w}_y)}{\partial x} + \frac{\partial \mathrm{w}_x}{\partial y} = 0. \tag{5}$$

Moreover, taking into account that the 2-form $d\lambda$ is given by

$$d\lambda = \mathrm{w}_x\, dt \wedge dx + \mathrm{w}_y\, dt \wedge dy + \mu\, dx \wedge dy, \tag{6}$$

the relation (5) expresses nothing but that $d^2\lambda = 0$.

Obviously, the only case of interest is the *regular* one, namely, the case $\mu \neq 0$ everywhere; then, M is invertible and consequently, the system (3) reads $\dot{X} = M^{-1}W$, that is

$$\begin{cases} \dot{x} = -\dfrac{w_y}{\mu} \\[2mm] \dot{y} = \dfrac{w_x}{\mu} \end{cases}. \tag{7}$$

In summary, the system of Euler–Lagrange equations of a Lagrangian L affine in the velocities turns out to be a system of first-order differential equations in normal form.

Let us analyse the above result in geometric terms. The system of differential Equation (7) determines the integral curves of the vector field

$$\Gamma = \frac{\partial}{\partial t} - \frac{w_y}{\mu} \frac{\partial}{\partial x} + \frac{w_x}{\mu} \frac{\partial}{\partial y} \in \mathfrak{X}(\mathbb{R} \times \mathbb{R}^2). \tag{8}$$

This vector field and the 2-form $d\lambda$ on $\mathbb{R} \times \mathbb{R}^2$ enjoy the following important properties:

1. The closed 2-form $d\lambda$ on $\mathbb{R} \times \mathbb{R}^2$ has rank 2, because if $X = C\,\partial_t + A\,\partial_x + B\,\partial_y$, where $A, B, C \in C^\infty(\mathbb{R} \times \mathbb{R}^2)$, is in the kernel of $d\lambda$, $i(X)\,d\lambda = 0$, as we assumed that $\mu \neq 0$ in each point, then,

$$B = C\,\frac{w_x}{\mu}, \quad A = -C\,\frac{w_y}{\mu},$$

and therefore, $X = C\,\Gamma$. Thus, the vector field Γ is the only vector field such that $i(\Gamma)dt = 1$ and $i(\Gamma)d\lambda = 0$; the first one of these two conditions, $i(\Gamma)dt = 1$, means that the time coordinate t is the parameter for the integral curves of Γ, while the second equation is the intrinsic expression of the Euler–Lagrange Equation (3). Note also that as the dynamics Γ is only determined, up to reparametrization, by $d\lambda$ and not by λ, one can add to λ any closed 1-form $\alpha \in \bigwedge^1(\mathbb{R} \times \mathbb{R}^2)$, and, therefore, we can change λ by $\lambda' = \lambda + df$, with $f \in C^\infty(\mathbb{R} \times \mathbb{R}^2)$; that is to say, the dynamics is invariant under the gauge transformation

$$m_x \mapsto m_x + \partial_x f,\ m_y \mapsto m_y + \partial_y f,\ H \mapsto H + \partial_t f. \tag{9}$$

2. Let $\omega_0 = dx \wedge dy \in \bigwedge^2(\mathbb{R}^2)$ be the 'natural' symplectic structure on $\mathbb{R}^2$ and $\Omega = dt \wedge dx \wedge dy$ be the induced volume form on $\mathbb{R} \times \mathbb{R}^2$ (in the Cartesian coordinates (t, x, y)). The 3-form $dt \wedge d\lambda$ is a volume form too and, therefore, it is proportional to Ω; actually, from the expression (6) we see that $dt \wedge d\lambda = \mu\,\Omega$, and by contracting with the vector field Γ, both sides of this last relation we immediately get $\mu\,i(\Gamma)\Omega = d\lambda$. That is to say, although the differential 2-form $\alpha_\Gamma = i(\Gamma)\Omega$ is not closed, the 2-form $\beta_\Gamma = \mu\,\alpha_\Gamma$ is exact, $\beta_\Gamma = d\lambda$. Note also that the Lie derivative $\pounds_{\mu\Gamma}\Omega$ vanishes; as we will see later on (Section 4), such a property of μ, which is equivalent to $\pounds_\Gamma(\mu\Omega) = 0$, defines the so-called 'Jacobi multipliers' for the vector field Γ with respect to the volume form Ω.

Remark 1. *It is worth noting that although we have started with a dynamical system on $\mathbb{R} \times T\mathbb{R}^2$ of the Lagrangian type, we arrive at an equivalent reduced dynamics on $\mathbb{R} \times \mathbb{R}^2$ given by the vector field (8). The Lagrangian dynamics on $\mathbb{R} \times T\mathbb{R}^2$ is given by the integral curves of the first prolongation Γ^1 of Γ, i.e., the first prolongation of the integral curves of Γ. In this approach, the affine Lagrangian (1) is singular, and, consequently, it is subjected to constraints. In fact, the constraints are exactly the equations of motion (3). We do not insist on this aspect because more details can be found in [11].*

3. Hamiltonian Formulation

Now, we can analyse whether there exists an, in general, time-dependent, Hamiltonian formulation for the dynamical system $\Gamma \in \mathfrak{X}(\mathbb{R} \times \mathbb{R}^2)$ describing the dynamics of our Lagrangian (1) in the regular case $\mu(t, x, y) \neq 0$. The problem is to look for a pair $(\omega, \widetilde{H})$, where $\omega \in \bigwedge^2(\mathbb{R} \times \mathbb{R}^2)$ is a rank two closed 2-form and the function $\widetilde{H} \in C^\infty(\mathbb{R} \times \mathbb{R}^2)$ is such that $i(\Gamma)\omega_{\widetilde{H}} = 0$, where $\omega_{\widetilde{H}} = \omega + d\widetilde{H} \wedge dt$ [34,62]. For a given Γ, the pair $(\omega, \widetilde{H})$ may be not unique. When ω is written in Darboux coordinates, i.e., $\omega = dq \wedge dp$, we

have a Hamiltonian description with canonical conjugate variables q, p and Hamiltonian function $\widetilde{H}$.

As we already know, $i(\Gamma)dt = 1$ and $i(\Gamma)d\lambda = 0$, so the problem may be reduced to see whether $d\lambda$ can be put in the needed form $\omega + d\widetilde{H} \wedge dt$. Recall that $d\lambda$ given by (6) is also a rank-two closed 2-form, as indicated above, and that by making use of the definition (2), it can be expressed as

$$d\lambda = \beta_0 + dH \wedge dt, \tag{10}$$

where β_0 is the 2-form $\beta_0 = dm_x \wedge dx + dm_y \wedge dy$, which is also closed and is of rank two, as it is straightforward to check. Therefore, the solution of the Hamiltonisation problem is achieved by choosing $\omega = \beta_0$ and $\widetilde{H} = H$, so that $\omega_{\widetilde{H}} = \omega_H = d\lambda$.

In practical cases, the simplest solution is obtained when considering that we can impose restrictive conditions to the components of the 1-form λ without altering the dynamics: as we know, the 1-form $\lambda' = \lambda + df$, $f \in C^\infty(\mathbb{R} \times \mathbb{R}^2)$, is gauge-equivalent to λ; when choosing f in such a way that $m_y + \partial_y f = 0$, we have $\lambda' = (m_x + \partial_x f)dx + (H + \partial_t f)dt$. For this reason, the reduced form $\lambda = m_x dx + H dt$ can always be used and then $\omega_H = d\lambda = dx \wedge d(-m_x) + dH \wedge dt$. That means that a possible set of conjugate canonical variables is $q = x$, $p = -m_x$, with Hamiltonian function H.

Remark 2. *The formulation we have obtained is not the Hamiltonian counterpart (via the Legendre transformation) on $\mathbb{R} \times T^*\mathbb{R}^2$ of the singular Lagrangian (1) we have started with; the dynamical system (8) is a time-dependent regular Hamiltonian system $(d\lambda, H)$ on $\mathbb{R} \times \mathbb{R}^2$.*

4. The Inverse Problem for First-Order Systems

We have shown that an affine Lagrangian (1) gives rise to the system of first-order differential Equation (7) and we immediately ask for the inverse problem; namely, given a system of ordinary first-order differential equations, is there a Lagrangian of type (1) whose associated Euler–Lagrange equations are equivalent to that system? In this section, we analyse this problem and conclude that, in general, it has always a positive answer; moreover, there are infinitely many affine Lagrangians for a given first-order system. The main result is expressed in terms of the so-called Jacobi multipliers [21,36,38,40,51], a notion which we present in geometrical terms [37,43].

4.1. Theory of the Jacobi Multipliers

Although we present the notion and the basic properties of Jacobi multipliers in the case of $\mathbb{R}^3$, the constructions can be easily extended to a multidimensional oriented manifold.

Definition 1. *We say that the nonvanishing function $\mu \in C^\infty(\mathbb{R} \times \mathbb{R}^2)$ on the oriented manifold $(\mathbb{R} \times \mathbb{R}^2, \Omega)$ is a Jacobi multiplier for the vector field $V \in \mathfrak{X}(\mathbb{R} \times \mathbb{R}^2)$ with respect to the volume form Ω if the 2-form $\beta_V = \mu\, i(V)\Omega$ is closed. That is, β_V is locally exact, which means that there exists a locally defined 1-form $\lambda \in \bigwedge^1(\mathbb{R} \times \mathbb{R}^2)$ such that $\beta_V = d\lambda$.*

Note that as Ω and $\mu\,\Omega$ are volume forms, and then $d\Omega = d(\mu\,\Omega) = 0$, the condition defining a Jacobi multiplier can be expressed in terms of Lie derivatives either as $\pounds_V(\mu\,\Omega) = 0$ or as $\pounds_{\mu V}\Omega = 0$, because $\pounds_V(\mu\,\Omega) = d(\mu\, i(V)\Omega) = \pounds_{\mu V}\Omega$, that is,

$$\operatorname{div}(\mu\, V) = 0, \tag{11}$$

where "div" stands for the divergence operator on vector fields associated to the volume form Ω, which is defined by $\pounds_V\Omega = (\operatorname{div} V)\Omega$, with $V \in \mathfrak{X}(\mathbb{R} \times \mathbb{R}^2)$ (see for instance [35]).

In the Cartesian coordinates (t, x, y) of $\mathbb{R} \times \mathbb{R}^2$, if Ω is such that $\Omega = dt \wedge dx \wedge dy$ and

$$V = V_t \frac{\partial}{\partial t} + V_x \frac{\partial}{\partial x} + V_y \frac{\partial}{\partial y} \tag{12}$$

we recover the familiar expression for the divergence operator of a vector field

$$\operatorname{div} V = \frac{\partial V_t}{\partial t} + \frac{\partial V_x}{\partial x} + \frac{\partial V_y}{\partial y}. \tag{13}$$

Thus, if the vector field $V \in \mathfrak{X}(\mathbb{R} \times \mathbb{R}^2)$ is such that $V_t = i(V)dt = 1$, the Equation (11) for its Jacobi multipliers μ is the partial differential equation

$$\operatorname{div}(\mu V) = \frac{\partial \mu}{\partial t} + \frac{\partial(\mu V_x)}{\partial x} + \frac{\partial(\mu V_y)}{\partial y} = 0. \tag{14}$$

Taking into account the properties of the div-operator, in particular, $\operatorname{div}(fV) = f \operatorname{div} V + V(f)$, because

$$\pounds_{fV}\Omega = \pounds_V(f\Omega) = V(f)\Omega + f\pounds_V\Omega = \big(V(f) + f \operatorname{div} V\big)\Omega,$$

and the fact that $V(\log f) = V(f)/f$ (with f a positive function), we see that the relation (11) between μ and V can be written as

$$V(\mu) + \mu \operatorname{div} V = 0, \tag{15}$$

or equivalently as

$$V(\log \mu) + \operatorname{div} V = 0. \tag{16}$$

From here, one can immediately derive several properties of the multipliers:

(1) The Jacobi multipliers for divergence-free vector fields are its first-integrals;

(2) The multiplier for a vector field V is not unique, every two multipliers μ and μ' being related by $\mu' = f \mu$, where f is a first integral of the vector field V (in the cases of interest f is non-trivial); the corresponding exact 2-forms $d\lambda$ and $d\lambda'$ are related in the same way, $d\lambda' = f \, d\lambda$;

(3) Given a function R such that $V(R) = \operatorname{div} V$, then, having in mind that $V(e^{-R}) = -e^{-R}V(R)$, one sees that $\mu = e^{-R}$ is a Jacobi multiplier for V. For instance, in the particular case of $\operatorname{div} V = v(t)$, the function $\mu(t) = \exp\left(-\int^t v(t')dt'\right)$ is a Jacobi multiplier for V, because $V\left(\int^t v(t')dt'\right) = v(t) = \operatorname{div} V$.

Finally, we note some additional properties. As $\Omega = dt \wedge dx \wedge dy$ is a volume form on a three-dimensional space, we can replace the vector field V (12) by the 2-form $\alpha_V = i(V)\Omega = V_t \, dx \wedge dy + V_x \, dy \wedge dt + V_y \, dt \wedge dx$. If μ is a Jacobi multiplier, then the contractions with V of the (locally) exact 2-forms α_V and $\beta_V = \mu \, \alpha_V$ vanish trivially, i.e., $i(V)\beta_V = 0$, which means that β_V is an absolute integral invariant [34,43] of V because also $d\beta_V = 0$.

For a vector field such that $i(V)dt = 1$, the volume form Ω is such that $\Omega = dt \wedge \alpha_V$. To prove this identity, it is enough to observe that Ω and $dt \wedge \alpha_V$ are both 3-forms on a three-dimensional manifold, so they are proportional, and, as $i(V)\alpha_V = 0$, the condition $V_t = 1$ immediately yields the identity. It can also be checked by using the corresponding local expressions. Moreover, the 2-form α_V also satisfies the identities $\pounds_V\Omega = d\alpha_V$ and $\pounds_V\alpha_V = (\operatorname{div} V)\,\alpha_V$, because $\pounds_V\alpha_V = \pounds_V(i(V)\Omega) = i(V)\pounds_V\Omega = \operatorname{div} V\,\alpha_V$.

4.2. The Inverse Problem on $\mathbb{R}^2$

Let us now consider the inverse problem on $\mathbb{R}^2$. Given a non-autonomous system of ordinary first-order differential equations on $\mathbb{R}^2$

$$\begin{cases} \dot{x} = X(t,x,y) \\ \dot{y} = Y(t,x,y) \end{cases}, \tag{17}$$

we can consider an associated system

$$
\begin{cases}
\dfrac{dt}{ds} = 1 \\[2mm]
\dfrac{dx}{ds} = X(t, x, y) \\[2mm]
\dfrac{dy}{ds} = Y(t, x, y)
\end{cases}
\tag{18}
$$

whose solutions are the integral curves of the vector field on $\mathbb{R} \times \mathbb{R}^2$

$$
\Gamma = \frac{\partial}{\partial t} + X(t, x, y) \frac{\partial}{\partial x} + Y(t, x, y) \frac{\partial}{\partial y} \in \mathfrak{X}(\mathbb{R} \times \mathbb{R}^2).
\tag{19}
$$

The existence of a Hamiltonian description for the system was achieved in [63,64] and it was related to the determination of a Jacobi multiplier for the vector field (19). We want to study here the inverse problem of Lagrangian mechanics for this system. In other words, is there a Lagrangian $L(t, x, y, v_x, v_y)$ affine in the velocities whose system of Euler–Lagrange equations is equivalent to the system (17)? Once an affine Lagrangian has been found, we have also a Hamiltonian formulation (see the previous section). The answer to this question is given by the following theorem.

Theorem 1. *Given a vector field $\Gamma \in \mathfrak{X}(\mathbb{R} \times \mathbb{R}^2)$ as in (19), there exists a 1-form $\lambda \in \bigwedge^1(\mathbb{R} \times \mathbb{R}^2)$ defining an affine in velocities Lagrangian $L = i_{\mathbf{T}}\lambda$ giving rise to the first-order system (17) if, and only if, there exists a Jacobi multiplier μ for Γ with respect to the volume form $\Omega = dt \wedge dx \wedge dy$ in $\mathbb{R} \times \mathbb{R}^2$. In this case, the 1-form λ is such that $d\lambda = \mu\, i(\Gamma)\Omega$.*

Proof. If Γ derives from the affine Lagrangian $L = i_{\mathbf{T}}\lambda$, where $\lambda \in \bigwedge^1(\mathbb{R} \times \mathbb{R}^2)$, then $i(\Gamma)d\lambda = 0$. As we point out in Section 2, $dt \wedge d\lambda = \mu\, \Omega$, where the proportionality factor μ is given by the first expression in (4). Then, by contraction with the vector field Γ, we immediately obtain $d\lambda = \mu\, i(\Gamma)\Omega$. Conversely, if μ is a Jacobi multiplier for the vector field Γ, then there exists a 1-form λ such that $d\lambda = \mu\, i(\Gamma)\Omega$ and hence $i(\Gamma)d\lambda = 0$, which means that Γ is of the Lagrangian type, with associated 1-form λ, i.e., $L = i_{\mathbf{T}}\lambda$. $\quad\square$

It is clear that every Jacobi multiplier for Γ gives rise to various (gauge-equivalent) Lagrangians. In fact, a given multiplier μ determines the cohomology class of 1-forms $\lambda + df$, with $f \in C^{\infty}(\mathbb{R} \times \mathbb{R}^2)$, and the Lagrangians determined by λ and $\lambda + df$ are gauge-equivalent, $L' = L + df/dt$. The gauge-equivalence imposes some restrictive conditions (2) on the coefficients m_x, m_y and H.

On the other hand, given two different multipliers μ and μ' the function f such that $\mu' = f\,\mu$ is a first-integral of Γ, as noted previously. Then, $d\lambda' = f\,d\lambda$ and if f is not a constant function the respective Lagrangians L and L' are (solution-)equivalent, but not gauge-equivalent Lagrangians (see, e.g., [65]).

In conclusion, there are infinitely many affine Lagrangians for the system (17) or the vector field (19), a result also reflected in the system of PDE which must satisfy the coefficients m_x, m_y and H (see (20) below).

Now, in order to find an affine Lagrangian for a vector field Γ given by (19), we can devise a specific procedure by using the result of the theorem. We proceed along the following steps:

(1) We find a particular solution of the PDE corresponding to (14) for the Jacobi multiplier μ, i.e., (14) with $V_x = X$ and $V_y = Y$;

(2) Once such a particular solution μ has been found, we have to look for the coefficients m_x, m_y and H to build the 1-form $\lambda = m_x dx + m_y dy + H dt$ such that $d\lambda = \mu\, i(\Gamma)\Omega$. At this stage, we first construct the 2-form $\beta_{\Gamma} = \mu\, i(\Gamma)\Omega = \mu(dx \wedge dy + Y\, dt \wedge dx - X\, dt \wedge dy)$ and then the condition $\beta_{\Gamma} = d\lambda$ yields the system of linear PDE for these coefficients:

$$\begin{cases} \dfrac{\partial m_y}{\partial x} - \dfrac{\partial m_x}{\partial y} = \mu \\[2ex] \dfrac{\partial m_x}{\partial t} - \dfrac{\partial H}{\partial x} = \mu\, Y \\[2ex] \dfrac{\partial m_y}{\partial t} - \dfrac{\partial H}{\partial y} = -\mu\, X \end{cases} . \tag{20}$$

Assuming that the needed regularity conditions are fulfilled, this system has always a solution (Cauchy–Kovaleski theorem), but no general method to solve it does exist, of course.

However, we only need one particular solution of (20) and the gauge-invariance property of the Lagrangian allows us to impose additional restrictive conditions to the m-coefficients in order to simplify the final expression of L. In several of the following examples, we look for a solution such that $m_x \neq 0$, $m_y = 0$, and $H \neq 0$, as indicated at the end of Section 3. If such a solution does exist, every Lagrangian related to this Jacobi multiplier μ is gauge-equivalent to the Lagrangian $L_1 = m_x v_x + H$, i.e., $L = L_1 + df/dt$ for some function $f = f(t, x, y)$.

5. Applications in Mechanical and Biological Systems

To solve an inverse problem for a given t-dependent vector field Γ as in (19), one must first look for a Jacobi multiplier μ, that is, a particular solution of the equation corresponding to (14) with $V_x = X$ and $V_y = Y$ for $V = \Gamma$. There are methods for the determination of Jacobi multipliers, in particular, when the vector field is of a polynomial type (see e.g., [66,67] for two recent examples). Once a multiplier μ has been found, we have to look for a 1-form λ such that $d\lambda = \beta_\Gamma = \mu(dx \wedge dy + Y\, dt \wedge dx - X\, dt \wedge dy)$, which can be achieved by direct inspection or by solving (20). Then, we can construct the Lagrangian and Hamiltonian descriptions according to Sections 2 and 3. In the following applications, the volume form $\Omega = dt \wedge dx \wedge dy$ is always understood.

5.1. Mechanical Systems

Usual mechanical systems are described by a second-order differential equation (a SODE) $\ddot{x} = F(t, x, \dot{x})$, where F is the total force. Denoting the basic coordinates by (t, x, v), this differential equation has associated a system of first-order differential equations

$$\begin{cases} \dot{x} = v \\ \dot{v} = F(t, x, v) \end{cases}' \tag{21}$$

and according to (19), the equivalent vector field $\Gamma \in \mathfrak{X}(\mathbb{R} \times \mathbb{R}^2)$ is

$$\Gamma = \frac{\partial}{\partial t} + v\frac{\partial}{\partial x} + F(t, x, v)\frac{\partial}{\partial v}. \tag{22}$$

If $\Omega = dt \wedge dx \wedge dv$, $\operatorname{div}\Gamma = \partial F/\partial v$ and the Equation (16) for the Jacobi last multiplier μ is

$$\Gamma(\log \mu) + \frac{\partial F}{\partial v} = 0. \tag{23}$$

It is evident in this expression that for v-independent forces, the Jacobi multipliers are but the first integrals. Consider, therefore, as a first example the particular case of a velocity-independent force, $F = F(t, x)$. Thus, Equation (23) is $\Gamma(\log \mu) = 0$, and then any real constant is a Jacobi multiplier. The simplest choice, $\mu = 1$, yields

$$\beta_\Gamma = \alpha_\Gamma = dx \wedge dv - v\, dt \wedge dv + F(t, x)\, dt \wedge dx,$$

and then, we see that

$$\beta_\Gamma = d\left(-v\,dx - \left(\int^x F(t,z)\,dz - \frac{v^2}{2}\right)dt\right).$$

Consequently, $\beta_\Gamma = d\lambda$, with

$$\lambda = -v\,dx + \left(\frac{v^2}{2} - \int^x F(t,z)\,dz\right)dt,$$

from where we see that m_x, m_v and H in (2) are given by

$$m_x = -v, \quad m_v = 0, \quad H = \frac{v^2}{2} - \int^x F(t,z)\,dz, \tag{24}$$

and the Lagrangian $L = i_\mathbf{T}\lambda$ is then

$$L(t,x,v,\dot{x},\dot{v}) = -v\dot{x} + \frac{v^2}{2} - \int^x F(t,z)\,dz. \tag{25}$$

Here, and in all the examples of mechanical type, the 'velocities' of x and v are directly denoted, as usual, by $\dot{x}$ and $\dot{v}$, respectively.

It is an easy task to check that (24) is a solution of the system (20), where y is v, $\mu = 1$, $X = v$, and $Y = F(t,x)$. It is also easy to derive from the affine Lagrangian (25) that we have found the system of first-order equations of motion (21) equivalent to the second-order differential equation $\ddot{x} = F(t,x)$. The Lagrangian energy is $E_L = \frac{v^2}{2} - \int^x F(t,z)\,dz$, which in general is not a conserved quantity.

As far as the Hamiltonian formulation is concerned, from the expression of λ we have $\omega_H = d\lambda = dx \wedge dv + dH \wedge dt$, so that the set of canonical conjugate variables is, up to a canonical transformation, $q = x$, $p = v$. The Hamiltonian function is written

$$H = \frac{p^2}{2} - \int^q F(t,z)\,dz.$$

The inverse problem for forces that may depend on the velocities clearly plays an interesting rôle, as it is next illustrated. Thus, as a second example, generalising the first one, consider the family of mechanical systems with force function of the form

$$F(t,x,v) = k(t)\,v + \varphi(t,x),$$

with $k(t)$ and $\varphi(t,x)$ being arbitrary functions. In this case, for Γ given by (22), div $\Gamma = k(t)$, and then forces of this type admit as solution of Equation (23) the multiplier

$$\mu(t) = \exp\left(-\int^t k(z)\,dz\right), \tag{26}$$

as pointed out above (Section 4.1). A solution of the system (20) is

$$m_x = -\mu(t)v, \quad m_v = 0, \quad H = \mu(t)\left(\frac{v^2}{2} - \int^x \varphi(t,z)\,dz\right).$$

Then, the affine Lagrangian is

$$L(t,x,v,\dot{x},\dot{v}) = \exp\left(-\int^t k(z)\,dz\right)\left(-v\dot{x} + \frac{v^2}{2} - \int^x \varphi(t,z)\,dz\right). \tag{27}$$

Choosing $q = x$ and $p = v\exp\left(-\int^t k(z)\,dz\right)$ as canonical conjugate coordinates, the Hamiltonian function is

$$H(t,q,p) = \frac{p^2}{2} \exp\left(\int^t k(z)\, dz \right) - \exp\left(-\int^t k(z)\, dz \right) \int^q \varphi(t,z)\, dz.$$

As a particular instance, we can consider a damped harmonic oscillator $\ddot{x} = -\omega^2 x - 2b\,\dot{x}$, with $b > 0$, $\omega > 0$, for which the affine Lagrangian

$$L(t, x, v, \dot{x}, \dot{v}) = e^{2bt}\left(-v\,\dot{x} + \frac{1}{2}(v^2 + \omega^2 x^2) \right) \tag{28}$$

is obtained. The Lagrangian energy $E_L = -\frac{1}{2} e^{2bt}(v^2 + \omega^2 x^2)$ is not conserved. Passing to the Hamiltonian description, $\omega_H = d\lambda = dx \wedge d(ve^{2bt}) + dH \wedge dt$, so that a set of canonical variables is provided by $q = x$ and $p = ve^{2bt}$, with Hamiltonian function $H(t,q,p) = \frac{1}{2}(p^2 e^{-2bt} + \omega^2 e^{2bt} q^2)$, which corresponds to Bateman [68] and Caldirola [69] formulations of dissipative harmonic oscillator (see also [62]).

Incidentally, this result can be generalised for the three-dimensional isotropic harmonic oscillator with a linear damping, described by the equation

$$\ddot{\mathbf{r}} + 2b\,\dot{\mathbf{r}} + \omega^2 \mathbf{r} = \mathbf{0}, \tag{29}$$

and which is equivalent to three independent equally damped harmonic oscillators (the normal modes). The Lagrangian is now

$$L = e^{2bt}\left(-\mathbf{v} \cdot \dot{\mathbf{r}} + \frac{\mathbf{v}^2}{2} + \frac{1}{2}\omega^2 \mathbf{r}^2 \right). \tag{30}$$

This very simple derivation is to be compared with others previous arguments as, for instance, the one given by Havas [29].

The Lane–Emden equation is also an interesting example of this type of forces, appearing, among other sources, from the Poisson equation for the gravitational field of fluids with a spherical symmetry in a hydrostatic equilibrium and that appears in many other different contexts. For instance, it is used in astrophysics [70] for modelling a star as a symmetric star of gases in equilibrium under its own weight. In the case of a polytropic state equilibrium, the Lane–Emden equation is

$$\ddot{x} + \frac{2}{t}\dot{x} + x^n = 0, \tag{31}$$

and, therefore, it admits the Lagrangian

$$L(t, x, v, \dot{x}, \dot{v}) = t^2\left(-v\,\dot{x} + \frac{v^2}{2} + \frac{x^{n+1}}{n+1} \right). \tag{32}$$

This result can be extended to its obvious generalisation, the Emden equations $\ddot{x} + a(t)\,\dot{x} + b(t)\,x^n = 0$, as, for instance, the Bessel equation.

The Hamiltonian formulation is achieved by choosing $q = x$, $p = v\,t^2$ as canonical variables, and the Hamiltonian function is

$$H(t, q, p) = \frac{p^2}{2t^2} + t^2 \frac{q^{n+1}}{n+1},$$

the same result as the one given in [63].

Further generalisation of the previous cases leads us to consider forces F of the form

$$F(t, x, v) = A(t, x) + B(t, x)v + C(t, x)v^2, \tag{33}$$

which admit a v-independent multiplier $\mu(t, x)$ when there exists a function $\varphi(t, x)$ such that $d\varphi = B\,dt + 2C\,dx$, and then the multiplier and the force are just $\mu = e^{-\varphi}$ and $F = A(t, x) + (\partial\varphi/\partial t)v + (1/2)(\partial\varphi/\partial x)v^2$, a result already obtained by Jacobi, as indicated in [21].

With the general prescription, we find that an affine Lagrangian for this family of forces is

$$L(t, x, v, \dot{x}, \dot{v}) = e^{-\varphi(t,x)}\left(\frac{v^2}{2} - v\,\dot{x}\right) - \int^x e^{-\varphi(t,z)}A(t, z)\,dz. \tag{34}$$

As an application of this result, let us consider the second-order differential equation

$$\ddot{x} = \frac{3\dot{x}^2}{x} + \frac{\dot{x}}{t}, \quad x \neq 0, \tag{35}$$

derived by Buchdahl in General Relativity [71], for which $A = 0$, $\varphi(t, x) = \log(tx^6)$ and $\mu = 1/(tx^6)$. Consequently,

$$L(t, x, v, \dot{x}, \dot{v}) = \frac{1}{t\,x^6}\left(\frac{v^2}{2} - v\,\dot{x}\right). \tag{36}$$

Additionally, we can reverse the meaning of the Equation (23) by fixing the form of the Jacobi multipliers and asking for the family of forces whose associated second-order differential equations admit such functions as Jacobi multipliers.

The trivial case of the multiplier being a real constant, for instance, $\mu = 1$, demands a v-independent force $F = \varphi(t, x)$ and it has been studied as a first example. The case in which the Jacobi multiplier μ is a function only of t was studied in (3) of Section 4.1, and then the Equation (23) reduces to

$$\frac{d\log\mu}{dt} + \frac{\partial F}{\partial v} = 0,$$

which shows that

$$F(t, x, v) = k(t)v + \varphi(t, x), \tag{37}$$

with φ being an arbitrary function and $k(t) = -d(\log\mu)/dt$. Conversely, a Jacobi multiplier for the force (37) is given by $\mu(t) = \exp\left(-\int^t k(\zeta)d\zeta\right)$. This is the second case we have studied above.

When we consider the particular case for which the function μ only depends on x and look for the forces F admitting such a multiplier, as the Equation (23) reduces to

$$v\frac{d\log\mu}{dx} + \frac{\partial F}{\partial v} = 0,$$

we see from here that F must be of the form

$$F(t, x, v) = k(x)\,v^2 + \varphi(t, x), \tag{38}$$

where φ is an arbitrary function and $2k(x) = -d(\log\mu)/dx$. The multiplier for a force such as (38) is of the form

$$\mu(x) = \exp\left(-2\int^x k(\zeta)\,d\zeta\right). \tag{39}$$

This is a particular case of those of (33) that we have studied before.

The case of a multiplier that depends only on the velocity v can be analysed along the same lines. In this case, the Equation (23) is

$$F\frac{d\log\mu}{dv} + \frac{\partial F}{\partial v} = 0,$$

or written in a different way,

$$\frac{\partial}{\partial v}(\log \mu + \log F) = 0,$$

and then the force F which admits such a multiplier $\mu(v)$ must be of the form

$$F(t, x, v) = \varphi(t, x)\, \Phi(v), \tag{40}$$

with $\Phi(v) = 1/\mu(v)$ and $\varphi(t, x)$ an arbitrary function.

For a v-independent multiplier, $\mu = \mu(t, x)$, the Equation (23) is

$$\frac{\partial \log \mu}{\partial t} + v \frac{\partial \log \mu}{\partial x} + \frac{\partial F}{\partial v} = 0,$$

and one obtains that the forces F admitting such a multiplier are those of the form

$$F(t, x, v) = A(t, x) + B(t, x)v + C(t, x)v^2, \tag{41}$$

where $A(t, x)$ is an arbitrary function, and

$$B(t, x) = -\frac{\partial \log \mu}{\partial t}, \qquad C(t, x) = -\frac{1}{2}\frac{\partial \log \mu}{\partial x}.$$

That is, B and C are functions satisfying the relation $\partial_x B = 2\partial_t C$, which is equivalent to say that the 1-form $B\, dt + 2C\, dx$ is closed (i.e., locally exact), and then there exists a locally defined function $\varphi(t, x)$ such that $d\varphi = B\, dt + 2C\, dx$. Therefore, this is the case studied in (33), and then a Jacobi multiplier is $\mu = e^{-\varphi}$ and the force,

$$F = A(t, x) + \frac{\partial \varphi}{\partial t} v + \frac{1}{2}\frac{\partial \varphi}{\partial x} v^2,$$

a result already obtained by Jacobi, as indicated in [21].

The same procedure can be applied to other cases, as, for instance, a multiplier of the form $\mu(t, v) = a(t)b(v)$, for which the Equation (23) is

$$\frac{\dot{a}}{a} + F\frac{b'}{b} + \frac{\partial F}{\partial v} = 0,$$

a non-homogeneous linear differential equation which can be integrated, and we find the family of forces

$$F(t, x, v) = B(v)\left(\varphi(t, x) + A(t)\int^v \frac{dz}{B(z)}\right), \tag{42}$$

with $A(t) = -\dot{a}/a$, $B(v) = 1/b(v)$, and φ being an arbitrary function of t and x.

5.2. Biological Systems

We can give a mathematical formulation in order to the study the evolution of certain biological populations of two competing species as a system of first-order differential equations in the plane, so that they admit Lagrangian and Hamiltonian descriptions. This can now be illustrated by means of two examples: a generalisation of the Lotka–Volterra model [61] and a host–parasite model [19].

(1) Consider first the two-dimensional system of nonlinear differential equations

$$\begin{cases} \dot{x} = x(A + Bx + Cy) \\ \dot{y} = y(K + Mx + Ny) \end{cases}, \tag{43}$$

where $A, B, C, K, \ldots$, are constant parameters and the domain of interest is the region of $\mathbb{R}^2$ where $x > 0$ and $y > 0$. The model described by (43) is frequently found in analysing the behaviour of biological interacting species, the parameters $A, B, \ldots$, being the growth

and interaction (encounters) coefficients; for instance, in the Lotka–Volterra model for the evolution of a predator–prey system assuming logistic growth for preys (x) when there are no predators (y), the coefficients A and M are positive, B, C and K negative, and $N = 0$ [72]. The Equation (43) is of the first order already [72] and the associated vector field is

$$\Gamma = \frac{\partial}{\partial t} + x(A + Bx + Cy)\frac{\partial}{\partial x} + y(K + Mx + Ny)\frac{\partial}{\partial y} \,. \tag{44}$$

Therefore, as div $\Gamma = A + 2B\,x + C\,y + K + 2N\,y + M\,x$, the equation corresponding to (14) for the Jacobi multiplier is

$$\frac{\partial \mu}{\partial t} + x(A + Bx + Cy)\frac{\partial \mu}{\partial x} + y(K + Mx + Ny)\frac{\partial \mu}{\partial y} + \mu\big(A + K + (2B + M)x + (C + 2N)y\big) = 0.$$

A simple solution μ, valid regardless the value of the constants $A, B, \ldots$, can be easily achieved by assuming that the multiplier is of the form

$$\mu(t, x, y) = e^{rt}x^p y^q, \tag{45}$$

that is, such that $\partial_t \mu = r\,\mu$, $x\partial_x \mu = p\,\mu$ and $y\partial_y \mu = q\,\mu$. The needed exponents p, q and r are solution of the linear system

$$\begin{cases} Bp + Mq + 2B + M = 0 \\ Cp + Nq + 2N + C = 0 \\ r + A(p + 1) + K(q + 1) = 0 \end{cases} \tag{46}$$

When $\Delta = BN - CM \neq 0$, the solution is unique and with the resulting multiplier we obtain, by solving the system (20), the particular solution

$$m_x = -\frac{e^{rt}x^p y^{q+1}}{q + 1}, \quad m_y = 0, \quad H = \frac{e^{rt}x^{p+1}y^{q+1}}{q + 1}\left(A + Bx + C\frac{q + 1}{q + 2}y\right), \tag{47}$$

a solution that obviously is only valid when $(q + 1)(q + 2) \neq 0$.

In this way, we obtain an affine Lagrangian for the two-dimensional system (43):

$$L = \frac{e^{rt}x^{p+1}y^{q+1}}{q + 1}\left(-\frac{v_x}{x} + A + Bx + C\frac{q + 1}{q + 2}y\right). \tag{48}$$

The connection between the parameters A, B, C, K, M and N and the exponents r, p, q is given by (46).

Note that the classical Lotka–Volterra system, that is, the predator–prey system with exponential growth [72], corresponds to putting $B = N = 0$ in (43), then $p + 1 = q + 1 = 0$ and $r = 0$. Equation (45) gives then the following multiplier

$$\mu = \frac{1}{xy}, \tag{49}$$

but now (47) and (48) are not valid and it is necessary to analyse this case separately. The right expressions for m_x and H are now

$$m_x = -\frac{\log y}{x}, \quad H = -K\log x - Mx + A\log y + Cy.$$

In this way, we obtain a Lagrangian for the general two-dimensional Lotka–Volterra system, by taking $m_y = 0$ once again:

$$L(t, x, y, v_x, v_y) = -\frac{\log y}{x}v_x - K\log x - Mx + A\log y + Cy. \tag{50}$$

The Lagrangian energy, $E_L = -H = K \log x + Mx - A \log y - Cy$, is conserved. There are other solutions with the same multiplier, with every one differing from each other by a total time derivative [18].

As for the other previous examples, we can give a Hamiltonian formulation of the Lotka–Volterra system: the canonical conjugate variables are $\tilde{q} = x$ and $\tilde{p} = (\log y)/x$, with Hamiltonian function $H(\tilde{q}, \tilde{p}) = -K \log \tilde{q} - M\tilde{q} + A\tilde{p}\,\tilde{q} + Ce^{\tilde{p}\,\tilde{q}}$.

(2) Another interesting example is the one given by the nonlinear system of differential equations

$$\begin{cases} \dot{x} = x(A - B\,y) \\ \dot{y} = y\left(C - D\dfrac{y}{x}\right) \end{cases} \,,$$

known as the "host–parasite" model [19]. The region of interest is again $x > 0$, $y > 0$, and A, B, C and D are positive real constants. The relevant vector field is

$$\Gamma = \frac{\partial}{\partial t} + x(A - B\,y)\frac{\partial}{\partial x} + y\left(C - D\frac{y}{x}\right)\frac{\partial}{\partial y},$$

and as $\operatorname{div}\Gamma = A - B\,y + C - 2D\dfrac{y}{x}$, the Equation (14) for multiplier results in

$$\frac{\partial \mu}{\partial t} + x(A - B\,y)\frac{\partial \mu}{\partial x} + y\left(C - D\frac{y}{x}\right)\frac{\partial \mu}{\partial y} + \mu\left(A - B\,y + C - 2D\frac{y}{x}\right) = 0.$$

As in the preceding example, we can look for a solution such as $\mu = e^{rt}x^p y^q$, and the resulting multiplier is, up to a constant factor,

$$\mu = \frac{e^{Ct}}{xy^2}.$$

Writing Equation (20), we find for the coefficients m and the function H

$$m_x = \frac{e^{Ct}}{xy}, \quad m_y = 0, \quad H = -e^{Ct}\left(\frac{D}{x} + \frac{A}{y} + B \log y\right),$$

so that an affine Lagrangian is

$$L(t, x, y, v_x, v_y) = e^{Ct}\left(\frac{v_x}{xy} - \frac{D}{x} - \frac{A}{y} - B \log y\right).$$

Other Lagrangians with the same multiplier can be easily found, all of them being gauge-equivalent. For example, the affine Lagrangian we have obtained differs from the one given in [19], but they are gauge-equivalent.

With canonical variables $q = x$ and $p = -m_x = -e^{Ct}/(xy)$ and Hamiltonian

$$H = e^{Ct}\left(-\frac{D}{q} + Ae^{-Ct}pq - BCt + B \log|pq|\right),$$

the dynamical equations are recovered as $\dot{q} = \partial_p H$, $\dot{p} = -\partial_q H$.

6. Summary and Outlook

We have analysed the two-dimensional inverse problem for first-order systems and devised a method to construct an affine Lagrangian for every system of two first-order ordinary differential equations. A time-dependent Hamiltonian formulation for the dynamical system $\Gamma \in \mathfrak{X}(\mathbb{R} \times \mathbb{R}^2)$ describing the dynamics of our Lagrangian (1) is also studied in the regular case $\mu(t, x, y) \neq 0$. The construction of the affine Lagrangian is based on the knowledge of a Jacobi multiplier for the given system of differential equations, with

respect to the usual volume form. Of course, the Lagrangian so obtained is singular. The method is particularly suitable for the case of the first-order systems equivalent to a single second-order differential equation, typically, those describing the evolution of mechanical systems, and the theory has been illustrated with several examples, depending on the explicit form of the forces. We have also reversed the method and found, in the case of mechanical systems, the family of forces admitting a given function as a multiplier. This point of view is fruitful and produces new examples of affine Lagrangians.

It is also remarkable that the method is equally applicable to systems of first-order differential equations used in the mathematical modelling of interacting biological species [19] and several examples, as a generalisation of the Lotka–Volterra model and a host–parasite model, have explicitly been developed in Section 5.2.

We believe that the generalisation to higher dimensional cases would be of interest. We give only a summary of the theoretical results generalising the two-dimensional case we have treated, and mention a pair of four-dimensional examples.

An affine Lagrangian on a $2n$-dimensional manifold M (for instance, $\mathbb{R}^{2n}$) is a differentiable function L locally given by

$$L = \sum_{i=1}^{n} m_i(t,x)\, v^i + H(t,x), \quad x = (x^1, \ldots, x^n), \tag{51}$$

where (t, x^i), $i = 1, \ldots, 2n$, are a system of local coordinates on $\mathbb{R} \times M$ and (t, x^i, v^i) the corresponding fibred ones on $\mathbb{R} \times TM$, and $m_i, H \in C^\infty(\mathbb{R} \times M)$.

As in the two-dimensional case, the function L is obtained by contraction of the 1-form $\lambda = \sum_{i=1}^{n} m_i\, dx^i + H\, dt$ with the time-derivative operator $\mathbf{T} = \partial_t + \sum_{i=1}^{n} v^i \partial_{x^i}$. The corresponding system of Euler–Lagrange equations is the system of first-order differential equations

$$\sum_{j=1}^{n} \mu_{ij} v^j = \mathrm{w}_i, \quad i = 1, \ldots, n, \tag{52}$$

where $\mu_{ij} = \partial_{x^i} m_j - \partial_{x^j} m_i$ and $\mathrm{w}_i = \partial_t m_i - \partial_{x^i} H$; they are the obvious generalisations of (7) and (4) and we are interested only in the 'regular' case, namely, when the matrix μ is regular. The equations in (52) are sometimes known as Birkhoff's equations, see e.g., [30,73].

The inverse problem is formulated in the same terms as before: given the vector field $\Gamma = \partial_t + X^i \partial_{x^i} \in \mathfrak{X}(\mathbb{R} \times M)$, is there an affine Lagrangian (51) whose system of Euler–Lagrange equations is equivalent to the first-order system $\dot{x}^i = X^i(t,x)$ for $i = 1, \ldots, n$? The necessary and sufficient condition is the existence of a non-degenerate exact invariant integral 2-form α for Γ, that is, a non-degenerate 2-form $\alpha \in \bigwedge^2(\mathbb{R} \times M)$ such that $d\alpha = 0$ and $i(\Gamma)\alpha = 0$.

If in the local coordinates (t, x) the 2-form α is

$$\alpha = \frac{1}{2} \sum_{i,j=1}^{n} A_{ij}\, dx^i \wedge dx^j + \sum_{i=1}^{n} B_i\, dt \wedge dx^i,$$

with the matrix (A_{ij}) being skew-symmetric, these conditions are expressed by means of $\det A \neq 0$, $B_i = \sum_{j=1}^{n} A_{ij} X^j$, for $i = 1, \ldots, n$, and the following system of linear PDE that the A-coefficients must satisfy:

$$\frac{\partial A_{ij}}{\partial t} + \sum_{k=1}^{n} \left(X^k \frac{\partial A_{ij}}{\partial x^k} + A_{kj} \frac{\partial X^k}{\partial x^i} + A_{ik} \frac{\partial X^k}{\partial x^j} \right) = 0, \quad i,j = 1, \ldots, 2n. \tag{53}$$

The main problem is now to find a particular solution of (53) for the $n(2n-1)$ skew-symmetric coefficients A_{ij}, a problem that might be a tough task in practical cases. After

that, it is necessary to find the 1-form λ that integrates α and construct the affine Lagrangian $L = i_{\mathbf{T}}\lambda$. For instance, the system of two second-order differential equations $\ddot{x} = -\dot{y}$, $\ddot{y} = -y$, a non-Lagrangian example in Douglas' classification [4], is equivalently represented in the variables $x_1 = x$, $x_2 = y$, $x_3 = v_x$, $x_4 = v_y$ by the linear first-order system on $\mathbb{R}^4$

$$\begin{cases} \dot{x}_1 = x_3 \\ \dot{x}_2 = x_4 \\ \dot{x}_3 = -x_4 \\ \dot{x}_4 = -x_2 \end{cases}. \tag{54}$$

Solving (53) for (54) requires searching for six A-coefficients. A simple solution, all of them being constant, does exist and yields various affine Lagrangians, such as, for instance,

$$L = (x_2 + x_3)\dot{x}_1 - x_3\dot{x}_4 + \frac{1}{2}(-x_3^2 + x_4^2 - 2x_2x_3), \tag{55}$$

which is gauge-equivalent to the Hojman–Urrutia Lagrangian [5,74], obtained by previous integration of the equation. There are other non-gauge-equivalent possibilities, such as

$$L_1 = (x_2 + x_3)\dot{x}_1 + (x_2 - x_3)\dot{x}_4 + \frac{1}{2}(x_2^2 - x_3^2 + 2x_4^2 - 2x_2x_3). \tag{56}$$

Clearly, there is no function f such that $L_1 - L = df/dt$.

Another example we can consider is $\ddot{x} = y^2$, $\ddot{y} = x^2$. We can check that the equivalent first-order system $\dot{x}_1 = x_3$, $\dot{x}_2 = x_4$, $\dot{x}_3 = x_2^2$, $\dot{x}_4 = x_1^2$ admits an affine Lagrangian description with $L = x_4\dot{x}_1 + x_3\dot{x}_2 + (x_1^3 + x_2^3)/3 - x_3x_4$.

As a final example, let us consider briefly a more involved example: the four-dimensional Lotka–Volterra system given by the system of differential equations

$$\begin{cases} \dot{x}_1 = x_1(-1 + x_2) \\ \dot{x}_2 = x_2(1 - x_1 + ax_3) \\ \dot{x}_3 = x_3(-1 - ax_2 + x_4) \\ \dot{x}_4 = x_4(1 - x_3) \end{cases}, \tag{57}$$

with a being a constant parameter, proposed in [75]. A particular solution of Equation (53) for the six A-coefficients appearing in the expression of the 2-form α is $A_{12} = (x_1x_2)^{-1}$, $A_{14} = a(x_1x_4)^{-1}$, $A_{34} = (x_3x_4)^{-1}$, $A_{13} = A_{23} = A_{24} = 0$, while

$$B_1 = \sum_{j=1}^{3} A_{1j}X^j = -1 + \frac{1+a}{x_1}, \qquad B_2 = \sum_{j=1}^{3} A_{2j}X^j = -1 + \frac{1}{x_2},$$

$$B_3 = \sum_{j=1}^{3} A_{3j}X^j = -1 + \frac{1}{x_3}, \quad B_4 = \sum_{j=1}^{3} A_{4j}X^j = -1 + \frac{1+a}{x_4},$$

to which corresponds the affine Lagrangian

$$L = \frac{\log x_1}{x_2}v_2 - \frac{\log x_4}{x_3}v_3 + \frac{\log x_1}{x_4}v_4 + x_1 + x_2 + x_3 + x_4 - \log(x_1^{1+a} x_2 x_3 x_4^{1+a}).$$

It is easy to check that the corresponding Euler–Lagrange equations are equivalent to (57).

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