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Principal Blocks for Different Primes, II

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Abstract

If G is a finite group, we have proposed three new conjectures on the interaction between different primes and their corresponding Brauer principal blocks. In this paper,we give strong support to the validity of Conjectures B and C.

Keywords Characters · Principal blocks · Commuting Sylow subgroups · Height zero

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1 Introduction

Meaningful interaction between the representation theory of finite groups from the perspective of different primes is extremely rare. However, in [25], we proposed the following three plausible conjectures, which extended work of several authors (see [3, 16, 19, 28]).

If p is a prime and G is a finite group, we denote by $B_p(G)$ the principal p-block of G. The main subject of our work is the set $\mathrm{Irr}_{p'}(B_p(G))$ of the irreducible complex characters in the principal p-block of G whose degree is not divisible by p. This set seems to possess remarkable properties.

Conjecture A *Let G be a finite group and let p and q be different primes. If*

$$\mathrm{Irr}_{p'}(B_p(G)) \cap \mathrm{Irr}_{q'}(B_q(G)) = \{1_G\},$$

To Pham Huu Tiep, on his 60th birthday.

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then there are a Sylow p-subgroup P of G and a Sylow q-subgroup Q of G such that $xy = yx$ for all $x \in P$ and $y \in Q$.

Conjecture B *Let G be a finite group and let p and q be primes dividing the order of G. If $\mathrm{Irr}_{p'}(B_p(G)) = \mathrm{Irr}_{q'}(B_q(G))$, then $p = q$.*

Conjecture C *Let G be a finite group, and let p and q be different primes. Then q does not divide $\chi(1)$ for all $\chi \in \mathrm{Irr}_{p'}(B_p(G))$ and p does not divide $\chi(1)$ for all $\chi \in \mathrm{Irr}_{q'}(B_q(G))$ if and only if there are a Sylow p-subgroup P of G and a Sylow q-subgroup Q of G such that $xy = yx$ for all $x \in P$ and $y \in Q$.*

The main result of [25] was to reduce Conjecture A to a problem on almost simple groups and to prove it in the case that one of the primes is 2. To prove Conjecture A for almost simple groups in the case where p and q are both odd remains quite a challenge.

In the present paper, we focus on Conjectures B and C. Using the Classification of Finite Simple Groups, in our first main theorem we prove the following.

Theorem D *Conjecture C implies Conjecture B.*

After proving Theorem D, therefore, we concentrate our efforts in the remainder of the paper towards Conjecture C. Since the "if" direction of Conjecture C follows from "if" direction of the main result [19], we shall only focus on the "only if" direction.

Theorem E *Conjecture C holds for finite simple groups.*

Besides Conjecture C being true for simple groups, in Theorem 3.3 below, we shall also prove that Conjecture C is true for p-solvable groups, assuming the inductive Alperin–McKay condition. This gives strong support to the validity of this conjecture.

Unfortunately, at the time of this writing, we still do not know how to reduce Conjecture C to a question on almost simple groups.

2 Theorem D

In this Section we prove that Conjecture C implies Conjecture B. This will require the following result on simple groups.

Theorem 2.1 *Let p, q be different primes and let S be a non-abelian simple group with $pq \mid |S|$. Assume that $[P, Q] = 1$ for some Sylow p-subgroup P of S and a Sylow q-subgroup Q of S. Then one of the following holds:*

(a) *There exists $\alpha \in \mathrm{Irr}_{p'}(B_p(S)) - \mathrm{Irr}(B_q(S))$ which is $\mathrm{Aut}(S)_p$-invariant, where $\mathrm{Aut}(S)_p$ is some Sylow p-subgroup of $\mathrm{Aut}(S)$.*
(b) *There exists $\alpha \in \mathrm{Irr}_{q'}(B_q(S)) - \mathrm{Irr}(B_p(S))$ which is $\mathrm{Aut}(S)_q$-invariant, where $\mathrm{Aut}(S)_q$ is some Sylow q-subgroup of $\mathrm{Aut}(S)$.*

Proof First, note that by [19, Lemma 3.1], the condition $[P, Q] = 1$ implies that $[\bar{P}, \bar{Q}] = 1$ for some Sylow p- and q- subgroups $\bar{P}$, $\bar{Q}$ of any covering group of S. If S is one of the sporadic groups J_1 or J_4, then we may use GAP [8] and its Character Table Library to see that the statement holds. Then [19, Propositions 3.2–3.4] further imply that we may assume S to be a simple group of Lie type that is not isomorphic to a sporadic or alternating group and is defined in characteristic $r_0 \notin \{p, q\}$. (We remark that this information was first found in [2].)

So, let S be of the form $S = G/\mathbf{Z}(G)$ for G a group of Lie type of simply connected type defined in characteristic $r_0 \neq p, q$. In this case, we may further assume that p and q are both odd and that the Sylow p- and q-subgroups of G are abelian, using [19, Proposition 3.5] (see also [2, Theorem 2.1]).

Since p and q are odd, note that we may therefore assume without loss that $p \geq 5$. Furthermore, using [18, Lemma 2.1 and Proposition 2.2], we have p is good for G and $p \nmid |\mathbf{Z}(G)|$. This implies that p does not divide the order of any diagonal or graph outer automorphism, so that $\mathrm{Aut}(S)_p$ may be taken as a subgroup of $S \rtimes \langle F\rangle$, where F is a generating field automorphism.

Now, let t be a p-element in G^* whose G^*-class is $\langle F\rangle_P$-invariant. Then the semisimple character χ_t of G lies in $B_p(G)$ by [12, Corollary 3.4] and has degree prime to p since $\chi_t(1) = [G^* : \mathbf{C}_{G^*}(t)]_{r_0'}$ (see e.g. [5, Theorem 8.4.8]) and Sylow p-subgroups of G^* are abelian. But since χ_t lies in the Lusztig series $\mathcal{E}(G, t)$ and $B_q(G)$ contains only characters lying in Lusztig series $\mathcal{E}(G, s)$ with $|s|$ a power of q (see [4, Theorem 9.12]), we see that χ_t does not lie in $\mathrm{Irr}(B_q(S))$. Furthermore, as $p \nmid |\mathbf{Z}(G)|$, we have $t \in O^{p'}(G^*)$, and hence χ_t is trivial on $\mathbf{Z}(G)$. Finally, since $p \nmid |\mathbf{Z}(G)|$, we also have $\mathbf{C}_{\mathbf{G}^*}(t)$ is connected, where $\mathbf{G}^*$ is the ambient reductive group whose fixed points under an appropriate Frobenius endomorphism yields G^* (see [21, Exercise 20.16]). Then for $\varphi \in \langle F\rangle$, we have $\chi_t^{\varphi} = \chi_{t^{\varphi^*}}$, where φ^* is an appropriate field automorphism of G^* (see [27, Corollary 2.5]). Hence χ_t is $\mathrm{Aut}(G)_P$-invariant by our choice of t. ☐

Lemma 2.2 *Suppose that N is a minimal normal subgroup of G, which is a direct product of the different G-conjugates of a non-abelian simple group S. Let $P \in \mathrm{Syl}_p(G)$. Suppose that $\alpha \in \mathrm{Irr}(S)$ is $\mathrm{Aut}(S)_p$-invariant, where $\mathrm{Aut}(S)_p$ is some Sylow p-subgroup of $\mathrm{Aut}(S)$. Then there are $g_i \in G$, $h_i \in P$ and $\sigma_i \in \mathrm{Aut}(S^{g_i})$ such that $(\alpha^{g_1})^{\sigma_1 h_1} \times \cdots \times (\alpha^{g_m})^{\sigma_m h_m} \in \mathrm{Irr}(N)$ is P-invariant.*

Proof Suppose that N is the direct product of $\Omega = \{S^g \mid g \in G\}$. Now, write

$$\Omega = \mathcal{O}(S_1) \cup \cdots \cup \mathcal{O}(S_t),$$

where $\mathcal{O}(S_i)$ is the P-orbit of some $S_i = S^{x_i}$, for some $x_i \in G$. Then $N = N_1 \times \cdots \times N_t$, where N_i is the product of the elements in $\mathcal{O}(S_i)$. Let us fix an i until the end of the proof. Of course, N_i is P-invariant. Let $\alpha_i = \alpha^{x_i} \in \mathrm{Irr}(S_i)$. Suppose that $\{S_i^{y_1}, \ldots, S_i^{y_r}\}$ are the different P-conjugates of S_i, where $y_j \in P$. Hence $N_i = S_i^{y_1} \times \cdots \times S_i^{y_r}$, and

$$P = \bigcup_{j=1}^{r} \mathbf{N}_P(S_i) y_j$$

is a disjoint union.

By hypothesis, α is invariant under $X \in \mathrm{Syl}_p(\mathrm{Aut}(S))$. Therefore α_i is invariant under $X_i = X^{x_i} \in \mathrm{Syl}_p(\mathrm{Aut}(S_i))$. Let $M_i = \mathbf{N}_G(S_i)$ and $C_i = \mathbf{C}_G(S_i)$. We have that M_i/C_i embeds into $\mathrm{Aut}(S_i)$. Then $\mathbf{N}_P(S_i)C_i/C_i$ is a p-subgroup of $\mathrm{Aut}(S_i)$, and therefore, there is $\sigma_i \in \mathrm{Aut}(S_i)$ such that $(\mathbf{N}_P(S_i)C_i/C_i) \subseteq X_i^{\sigma_i}$. Since α_i is X_i-invariant, it follows that $\beta_i = (\alpha_i)^{\sigma_i} \in \mathrm{Irr}(S_i)$ is $X_i^{\sigma_i}$-invariant, and therefore $\mathbf{N}_P(S_i)$-invariant. We claim that $\gamma_i =$

$\beta_i^{y_1} \times \cdots \times \beta_i^{y_r} \in \mathrm{Irr}(N_i)$ is P-invariant. Indeed, if $x \in P$, then $y_k x = w_k y_{\sigma(k)}$ for some $w_k \in \mathbf{N}_P(S_i)$, $1 \leq k \leq r$, and σ a permutation of S_r. Now, if $u \in S_i$, then

$$\gamma_i^{x^{-1}}(u^{y_k}) = \gamma_i(u^{y_k x}) = \alpha(1)^{r-1}\beta_i^{y_{\sigma(k)}}(u^{w_k y_{\sigma(k)}}) = \alpha(1)^{r-1}\beta_i(u^{w_k}) \\ = \alpha(1)^{r-1}\beta_i(u) = \gamma_i(u^{y_k}).$$

This proves that γ_i is P-invariant. Hence $\gamma = \gamma_1 \times \cdots \times \gamma_t \in \mathrm{Irr}(N)$ is P-invariant. □

We will need the following well-known result of J. Alperin and E. C. Dade.

Theorem 2.3 *Suppose that N is a normal subgroup of G, with G/N a p'-group. Let $P \in \mathrm{Syl}_p(G)$ and assume that $G = N\mathbf{C}_G(P)$. Then restriction of characters defines a natural bijection between the irreducible characters of the principals p-blocks of G and N.*

Proof The case where G/N is solvable was proved in [1] and the general case in [7]. □

Theorem 2.4 *Assume that the "only if" direction of Conjecture C is true for all finite groups. Let p and q be primes. Assume that G is a finite group of order divisible by p and q. If $\mathrm{Irr}_{p'}(B_p(G)) = \mathrm{Irr}_{q'}(B_q(G))$, then $p = q$.*

Proof We argue by induction on $|G|$. Assume that $p \neq q$. By Conjecture C, we know that $[P, Q] = 1$ for some $P \in \mathrm{Syl}_p(G)$ and $Q \in \mathrm{Syl}_q(G)$.

Step 0. If $1 \neq N \lhd G$, then p divides $|N|$ or q divides $|N|$.

Otherwise by [23, Theorem 9.9(c)]

$$\mathrm{Irr}_{p'}(B_p(G/N)) = \mathrm{Irr}_{p'}(B_p(G)) = \mathrm{Irr}_{q'}(B_q(G)) = \mathrm{Irr}_{q'}(B_q(G/N))$$

and by induction we are done.

Step 1. Let L be a proper normal subgroup of G. Then G/L has order divisible by p or q.

Suppose that G/L has p' and q'-order. We claim that $\mathrm{Irr}_{p'}(B_p(L)) = \mathrm{Irr}_{q'}(B_q(L))$. Indeed, let $\theta \in \mathrm{Irr}_{q'}(B_q(L))$. Then there exists $\chi \in \mathrm{Irr}(B_q(G))$ over θ. Then χ has q'-degree by [15, Theorem 11.29] and therefore $\chi \in \mathrm{Irr}_{p'}(B_p(G))$, by hypothesis. Therefore $\theta \in \mathrm{Irr}_{p'}(B_p(L))$. By symmetry, the claim is proved. Therefore, Step 1 follows by using the inductive hypothesis.

Step 2. Let N be a minimal normal subgroup of G and suppose that N is an elementary abelian p-group. Then $G = \mathbf{C}_G(N)\mathbf{C}_G(Q)$.

Write $M = \mathbf{C}_G(N)$ and $L = M\mathbf{C}_G(Q)$. We have that $Q \subseteq \mathbf{C}_G(P) \subseteq \mathbf{C}_G(N) = M$ and G/M is a q'-group. Also by the Frattini argument $G = M\mathbf{N}_G(Q)$ and $L \lhd G$. Notice that G/L is a q'-group and a p'-group (because $P \subseteq \mathbf{C}_G(Q) \subseteq L$). Then we use Step 1.

Step 3. Let N be a minimal normal subgroup of G and suppose that N is abelian. Then $G = \mathbf{C}_G(N)$.

By Step 0, we may assume that N is a p-group or a q-group. By symmetry, assume that N is a p-group. Write $M = \mathbf{C}_G(N)$. We prove first that $B_p(G)$ is the only p-block of G covering $B_p(M)$. Let B be a p-block of G covering $B_p(M)$ and let D be a defect group of B. Then $N \subseteq D$ by [23, Theorem 4.8] and $\mathbf{C}_G(D) \subseteq M = \mathbf{C}_G(N)$. Then B is regular with respect to M ([23, Lemma 9.20]) and hence by [23, Theorem 9.19] we have that $B = B_p(M)^G = B_p(G)$ by the third main theorem (see [23, Theorem 6.7]). Hence $B_p(G)$ is the only p-block of G covering $B_p(M)$. In particular, we have that $\mathrm{Irr}(G/M) \subseteq \mathrm{Irr}(B_p(G))$.

Next, we prove that $G = M$. Recall that G/M is a q'-group, because $[Q, N] = 1$. By Step 2, we have $G = M\mathbf{C}_G(Q)$. Hence by Theorem 2.3, we have that the restriction map

$$\mathrm{res}: \ \mathrm{Irr}(B_q(G)) \ \rightarrow \ \mathrm{Irr}(B_q(M))$$

is a bijection. We claim that $\mathrm{Irr}_{p'}(G/M) = \{1_G\}$. Indeed, if $\chi \in \mathrm{Irr}_{p'}(G/M)$, then $\chi \in \mathrm{Irr}_{p'}(B_p(G)) = \mathrm{Irr}_{q'}(B_q(G))$. Therefore, we have that χ_M is irreducible and since χ lies over 1_M we have $\chi_M = 1_M$. Hence $\chi = 1_G$ by the injectivity of the restriction map. Thus $\mathrm{Irr}_{p'}(G/M) = \{1_{G/M}\}$ and $G = M$ by [25, Lemma 2.2].

Step 4. Let N be a minimal normal subgroup of G, then N is not abelian.

Suppose the contrary and assume without loss of generality that N is an elementary abelian p-group, so by Step 3 we have $G = \mathbf{C}_G(N)$ and $N \subseteq \mathbf{Z}(G)$. By [23, Theorem 9.10] we have that $B_p(G/N)$ is the unique p-block of G/N contained in $B_p(G)$. We claim that $\mathrm{Irr}_{p'}(B_p(G/N)) = \mathrm{Irr}_{q'}(B_q(G/N))$. Indeed, we have that

$$\mathrm{Irr}_{p'}(B_p(G/N)) \subseteq \mathrm{Irr}_{p'}(B_p(G)) = \mathrm{Irr}_{q'}(B_q(G)) = \mathrm{Irr}_{q'}(B_q(G/N)),$$

where we have used [23, Theorem 9.9] in the last equality. On the other hand, let $\chi \in \mathrm{Irr}_{q'}(B_q(G/N))$, so $\chi \in \mathrm{Irr}_{q'}(B_q(G)) = \mathrm{Irr}_{p'}(B_p(G))$ and $N \subseteq \mathrm{Ker}(\chi)$. Since $B_p(G/N)$ is the only p-block of G/N contained in $B_p(G)$, we have that $\chi \in \mathrm{Irr}_{p'}(B_p(G/N))$ and the claim is proven. By using the inductive hypothesis, we have that p does not divide $|G/N|$. Therefore, $\{1_G\} = \mathrm{Irr}(B_p(G/N)) = \mathrm{Irr}_{p'}(B_p(G/N)) = \mathrm{Irr}_{q'}(B_q(G/N))$ and $q \nmid |G/N|$ by [25, Lemma 2.1]). Hence $q \nmid |G|$ and this is a contradiction.

Step 5. Let N be a minimal normal subgroup of G, then pq divides $|N|$.

Suppose that N is a p'-group. We claim first that NQ does not have a normal q-complement. Indeed, suppose the contrary and let $X \lhd NQ$ be a normal q-complement. Then $N \cap X$ is a normal q-complement of N and by the minimality of N we have that either $N \cap X = 1$ or $N \cap X = N$. If $N \cap X = N$, N is q' and p', contradiction with Step 0. If $N \cap X = 1$ then $N \cong XN/X$ is a q-group, which is a contradiction with Step 4. Therefore NQ does not have a normal q-complement. By [13, Corollary 3] there is $\tau \in \mathrm{Irr}(B_q(QN))$ non-linear of q'-degree. Therefore $1 \neq \tau_N \in \mathrm{Irr}_{q'}(B_q(N))$. By [22, Lemma 4.3] we have that there is some $\gamma \in \mathrm{Irr}_{q'}(B_q(G))$ lying over τ_N. By hypothesis, we have that γ is in the principal p-block of G, and therefore τ_N is in the principal p-block of N, which is a contradiction since N is a p'-group and $\tau_N \neq 1$.

Final Step. If N is a minimal normal subgroup of G, then N is semisimple by Step 4. Suppose that N is a direct product of all the different G-conjugates of a simple group S of order divisible by pq. Suppose that (a) of Theorem 2.1 holds and let α be the character in $\mathrm{Irr}_{p'}(B_p(S))$ (not in $\mathrm{Irr}(B_q(S))$) which is $\mathrm{Aut}(S)_p$-invariant. Notice that any G-conjugate or $\mathrm{Aut}(G)$-conjugate of α is in the principal p-block and not in the principal q-block of S. By Lemma 2.2, there exists $\tau \in \mathrm{Irr}_{p'}(B_p(N))$ which is P-invariant, and such that each of its factors does not belong to the principal q-block. In particular, τ does not belong to the principal q-block of N. Now τ extends to PN by [14, Corollary 8.16]. By [22, Lemma 4.3] there is $\chi \in \mathrm{Irr}_{p'}(B_p(G))$ lying over τ. Then $\chi \in \mathrm{Irr}_{q'}(B_q(G))$ and thus $\tau \in \mathrm{Irr}_{q'}(B_q(N))$, which is a contradiction. Assuming (b) in Theorem 2.1 and reasoning analogously we get again a contradiction. □

3 Conjecture C and *p*-Solvable Groups

As of the writing of this article, we can only prove Conjecture C for p-solvable groups by assuming the so called Inductive Alperin–McKay condition (for the prime q). We shall need the following.

Theorem 3.1 *Suppose that N is normal in G, and let $P \in \mathrm{Syl}_p(N)$. Assume that all the non-abelian simple groups involved in N satisfy the inductive Alperin–McKay condition. Then there is a bijection*

$$^* : \mathrm{Irr}_{p'}(B_p(N)) \rightarrow \mathrm{Irr}_{p'}(B_p(\mathbf{N}_N(P)))$$

such that for each $\theta \in \mathrm{Irr}_{p'}(B_p(N))$, there is a bijection $f_\theta : \mathrm{Irr}(B_p(G)|\theta) \rightarrow \mathrm{Irr}(B_p(\mathbf{N}_G(P))|\theta^*)$ such that $\chi(1)/\theta(1) = f_\theta(\chi)/\theta^*(1)$ for all $\chi \in \mathrm{Irr}(B_p(G)|\theta)$.

Proof This is Theorem B and Theorem 7.1 of [26]. □

Let us remark that we shall only need Theorem 3.1 in the case where G/N is a p'-group.

In our proof, we shall also need a McKay divisibility theorem, which was made possible after M. Geck proved a remarkable conjecture on Glauberman correspondents [9].

In the following, we follow the proof of [29, Theorem A], and then use Geck's result.

Theorem 3.2 *Let G be a p-solvable group and let $P \in \mathrm{Syl}_p(G)$. Then there is a bijection*

$$^* : \mathrm{Irr}_{p'}(G) \rightarrow \mathrm{Irr}_{p'}(\mathbf{N}_G(P))$$

such that $\chi^*(1)$ divides $\chi(1)$ and $\chi(1)/\chi^*(1)$ divides $|G : \mathbf{N}_G(P)|$.

Proof We argue by induction on $|G|$. As in the proof of [29, Theorem A] we may assume that $\mathbf{O}_p(G) = 1$, and hence $K = \mathbf{O}_{p'}(G) > 1$.

Let $S/K = \mathbf{O}_p(G/K)$, and notice that $P_0 = P \cap S$ is a Sylow p-subgroup of S. By the Frattini argument we have that $G = K\mathbf{N}_G(P_0)$ and $\mathbf{N}_G(P_0) < G$ since $\mathbf{O}_p(G) = 1$. Let $\theta_1, \ldots, \theta_s$ be a complete set of representatives of the orbits of the action of $\mathbf{N}_G(P)$ on the P-invariant irreducible characters of K. By [24, Lemma 9.3], we have that

$$\mathrm{Irr}_{p'}(G) = \mathrm{Irr}_{p'}(G \,|\, \theta_1) \cup \cdots \cup \mathrm{Irr}_{p'}(G \,|\, \theta_s)$$

is a disjoint union. Fix $\theta_i \in \mathrm{Irr}(K)$ and observe that θ_i is also P_0-invariant. Let $\theta_i^* \in \mathrm{Irr}(\mathbf{C}_K(P_0))$ be the Glauberman correspondent of θ_i (see [24, Theorem 2.9], for instance) and let T_i be the stabilizer of θ_i in G. Since the Glauberman correspondence and the action of $\mathbf{N}_G(P_0)$ commute (see [24, Lemma 2.10]), it follows that $\mathbf{N}_{T_i}(P_0) = T_i \cap \mathbf{N}_G(P_0)$ is the stabilizer of θ_i^* in $\mathbf{N}_G(P_0)$.

Again as in the proof of [29, Theorem A] we obtain a bijection

$$^* : \mathrm{Irr}_{p'}(T_i \,|\, \theta_i) \rightarrow \mathrm{Irr}_{p'}(\mathbf{N}_{T_i}(P_0) \,|\, \theta_i^*)$$

satisfying $\psi(1)/\psi^*(1) = \theta_i(1)/\theta_i^*(1)$. Hence we have that $\psi^*(1)$ divides $\psi(1)$ by the main result of [9]. Moreover, since $G = \mathbf{N}_G(P_0)K$ we have that $|K : \mathbf{C}_K(P_0)| = |T_i : \mathbf{N}_{T_i}(P_0)|$ and since $\theta_i(1)/\theta_i^*(1)$ divides $|K : \mathbf{C}_K(P_0)|$ we conclude that $\psi(1)/\psi^*(1)$ divides $|T_i : \mathbf{N}_{T_i}(P_0)|$.

The remaining part of the proof proceeds exactly as in the proof of [29, Theorem A]. □

Theorem 3.3 *Let G be a finite p-solvable group. Assume that the inductive Alperin–McKay condition (for q) holds for every non-abelian simple group involved in G. Then Conjecture C is true for G. In particular, Conjecture C holds for $\{p, q\}$-solvable groups.*

Proof By [19], we only need to prove that if all $\mathrm{Irr}_{p'}(B_p(G))$ have q'-degree and all $\mathrm{Irr}_{q'}(B_q(G))$ have p'-degree, then $[P, Q] = 1$ for some $P \in \mathrm{Syl}_p(G)$ and $Q \in \mathrm{Syl}_q(G)$. Let N be a normal subgroup of G. Since the hypothesis is satisfied by G/N, by induction, we know that $[P, Q] \subseteq N$ for some $P \in \mathrm{Syl}_p(G)$ and $Q \in \mathrm{Syl}_q(G)$.

Suppose that $\mathbf{O}_{p'}(G) = 1$. Then we know that $\mathrm{Irr}(B_p(G)) = \mathrm{Irr}(G)$, by Theorem 10.20 of [23]. Hence, all the irreducible characters in $\mathrm{Irr}_{p'}(G)$ have q'-degree. Let $L = \mathbf{O}_p(G)$ and let $P \in \mathrm{Syl}_p(G)$ and $Q \in \mathrm{Syl}_q(G)$ such that $[P, Q] \subseteq L$. Therefore Q normalizes P and $|G : \mathbf{N}_G(P)|$ is not divisible by q. By Theorem 3.2, we have that all characters in $\mathrm{Irr}(\mathbf{N}_G(P)/P')$ have q'-degree. By the Itô–Michler theorem we have that QP' is normal in $\mathbf{N}_G(P)$, and hence $[QP', P] \subseteq P'$. Then Q acts trivially on P/P', and therefore Q acts trivially on P by coprime action (see [15, Corollary 3.29]). Thus $[Q, P] = 1$ and we are done in this case.

Suppose that $L = \mathbf{O}_{p'}(G) > 1$ and let $P \in \mathrm{Syl}_p(G)$ and $Q \in \mathrm{Syl}_q(G)$ with $[Q, P] \subseteq L$. By Hall–Higman 1.2.3 Lemma (see [15, Theorem 3.21]) we have that $\mathbf{C}_{G/L}(\mathbf{O}_p(G/L)) \subseteq \mathbf{O}_p(G/L)$. Since $QL/L \subseteq \mathbf{C}_{G/L}(PL/L)$ we conclude that $QL/L \subseteq \mathbf{O}_p(G/L)$ and hence $Q \subseteq L$. Thus G/L is q' and $G = L\mathbf{N}_G(Q)$, in particular $|G : \mathbf{N}_G(Q)|$ is not divisible by p. We claim that all the irreducible characters in $\mathrm{Irr}_{q'}(B_q(\mathbf{N}_G(Q)))$ have p'-degree. Indeed, let $\chi^* \in \mathrm{Irr}_{q'}(B_q(\mathbf{N}_G(Q)))$ and let $\theta^* \in \mathrm{Irr}_{q'}(B_q(\mathbf{N}_L(Q)))$ under χ^*. Let $\theta \in \mathrm{Irr}_{q'}(B_q(L))$ be the pre-image of θ^* given by the bijection in Theorem 3.1. Again using Theorem 3.1, let $\chi \in \mathrm{Irr}(B_q(G)|\theta)$ be such that $f_\theta(\chi) = \chi^*$, so we know that

$$\frac{\chi(1)}{\theta(1)} = \frac{\chi^*(1)}{\theta^*(1)}.$$

Then $\chi(1)/\theta(1)$ is not divisible by q and thus $\chi \in \mathrm{Irr}_{q'}(B_q(G))$. By hypothesis, $\chi(1)$ is not divisible by p. Hence $\chi^*(1)/\theta^*(1)$ is not divisible by p, and therefore, since θ^* is of p'-degree we have that $\chi^*(1)$ is not divisible by p, and the claim follows.

Let $X = \mathbf{O}_{q'}(\mathbf{N}_G(Q))$. Then all the elements in $\mathrm{Irr}(\mathbf{N}_G(Q)/Q'X) = \mathrm{Irr}_{q'}(B_q(\mathbf{N}_G(Q)))$ have degree not divisible by p. By the Itô–Michler theorem, we have that this group has a normal Sylow p-subgroup (which is a Sylow p-subgroup of G, since $|G : \mathbf{N}_G(Q)|$ is p'), and therefore P centralizes Q/Q'. By coprime action, $[P, Q] = 1$. □

We thank the referee for pointing out the “in particular” in Theorem 3.3 above. In fact, this observation by the referee gives a different proof of Theorem E in [25], something which we have not noticed before.

4 Conjecture C and Simple Groups

As we have mentioned in the Introduction, we note that the “if” direction of Conjecture C follows from the work of Malle–Navarro in [19], namely [19, Theorem 4.1].

Hence we focus on the "only if" direction. First, we consider the cases easily dealt with in GAP:

Proposition 4.1 *Conjecture C holds for sporadic simple groups, alternating and symmetric groups $\mathfrak{A}_n$ and $\mathfrak{S}_n$ with $n \leq 8$, the Tits group ${}^2\mathrm{F}_4(2)'$, $\mathrm{G}_2(2)'$, and groups of Lie type with exceptional Schur multipliers.*

Proof This can be seen using [8] and its Character Table Library. □

4.1 Conjecture C for Alternating and Symmetric Groups

Here we prove Conjecture C in the case of alternating groups $\mathfrak{A}_n$ and symmetric groups $\mathfrak{S}_n$. Note that it follows from [19, Proposition 3.3] that $[P, Q] \neq 1$ for every Sylow p-subgroup P and Sylow q-subgroup Q of $\mathfrak{S}_n$ or $\mathfrak{A}_n$.

Proposition 4.2 *Let G be an alternating or symmetric group $\mathfrak{A}_n$ or $\mathfrak{S}_n$ with $n \geq 9$ and let p, q be primes dividing $|G|$. Then either there exists $\chi \in \mathrm{Irr}_{p'}(B_p(S))$ with degree divisible by q or there exists $\chi \in \mathrm{Irr}_{q'}(B_q(S))$ with degree divisible by p.*

The Strategy We first recall some facts and give the basic idea of the proof. The set $\mathrm{Irr}(\mathfrak{S}_n)$ is indexed by partitions of n, and two characters χ_λ, χ_μ corresponding to partitions λ, μ lie in the same p-block if λ, μ have the same p-core (and similar for q). In particular, writing $n = pm + b$ with $0 \leq b < p$, the set $\mathrm{Irr}(B_p(\mathfrak{S}_n))$ consists of the characters χ_λ such that λ has p-core (b). Furthermore, recall that the degree of the character χ_λ is given by the hooklength formula $\chi_\lambda(1) = \frac{n!}{\prod h_\lambda}$, where the denominator is the product of all hooklengths in the tableau corresponding to the partition λ. Furthermore, if λ is not self-conjugate, then the corresponding character restricts irreducibly to $\mathfrak{A}_n$.

So, our strategy will be (up to switching p and q) to illustrate a non-self-conjugate partition λ of n with p-core (b) such that the numerator in the hooklength formula has the same p-part as the denominator and larger q-part than the denominator. Our proof will require several technical cases and analysis of the degrees given by the hooklength formula.

Setting Notation Throughout our proof, we will assume without loss that $q < p$. We will write $mp = wq + r$ with $0 \leq r < q$, and let $n = mp + b$ with $0 \leq b < p$. Note that we may assume that $m > 1$, since otherwise a Sylow p-subgroup is abelian, and the result follows from [10, Theorem 3.5], together with the principal block version of Brauer's height zero conjecture [20]. In studying the degrees of the characters that we construct, several expressions will appear repeatedly. Hence we define once and for all:

$$Y := \frac{(mp+b)\cdots(mp+1)}{b!}, \qquad Y' := \frac{(mp+b-1)\cdots(mp+1)}{(b-1)!},$$
$$Z := \frac{(mp+r)\cdots(mp+1)}{r!}, \quad \text{and} \quad Z' := \frac{(mp-1)\cdots(mp-r)}{r!}$$

which we see are each relatively prime to p.

It will also be useful to set the p-adic and q-adic expansions of mp: Let

$$mp = a_1 q^{t_1} + \cdots + a_k q^{t_k} = b_1 p^{s_1} + \cdots + b_{k'} p^{s_{k'}}$$

with $t_i < t_{i+1}$, $s_j < s_{j+1}$, $0 < a_i < q$, and $0 < b_j < p$ for each appropriate value of i, j. With these established, we further define

$$X := \frac{\prod_{i=1}^{a_1 q^{t_1}} (mp - i)}{\prod_{i=1}^{a_1 q^{t_1}} i} \quad \text{and} \quad X' := \frac{\prod_{i=1}^{b_1 p^{s_1}} (mp - i)}{\prod_{i=1}^{b_1 p^{s_1}} i}.$$

Note that $mp \neq a_1 q^{t_1}$, since $p \nmid a_1$. In the situation that the expression X' becomes relevant, we will see that also $mp \neq b_1 p^{s_1}$. Finally, for an integer x, we will write $(x)_p$ (or just x_p if it is clear) for the p-part of x.

Proof of Proposition 4.2 Keep the notation above.

(I) First, suppose that $r > 0$. If $b = 0$, so that $n = mp$, then consider $\lambda = (1^{mp-r-1}, 1 + r)$. Then $\chi_\lambda \in \mathrm{Irr}(B_p(\mathfrak{S}_n))$ and $\chi_\lambda(1) = Z'$, which is p' but divisible by q since $mp - r = wq$ and $r < q$. If $b \neq 0$ and $0 < r < b$, consider $\lambda = (1^{mp-r-1}, 1 + r, b)$. If $0 < b < r$, consider $\lambda = (1^{mp-r-1}, 1 + b, r)$. In these cases, $\chi_\lambda \in \mathrm{Irr}(B_p(\mathfrak{S}_n))$ and $\chi_\lambda(1) = Y \cdot Z' \cdot \frac{|b-r|}{mp-r+b}$. Note that the q-part of the numerator of Y must be at least as large as the q-part of the denominator. (Each remainder modulo b appears once as a factor in the numerator.) Hence, this character still has degree that is p' and divisible by q, with the possible exception of if $q \mid b$ and $b!(wq + b)$ has larger q-part than $(mp + 1) \cdots (mp + b)$. In the latter case, $(1^{mp}, b)$, giving degree Y', works instead.

If $b = r > 0$, note that $r + 1 < wq$, as otherwise we would have $r = q - 1$ and $w = 1 = m$, contradicting our assumption that $m > 1$. Then let $\lambda = (1^r, r + 1, wq - 1)$, so that $\chi_\lambda(1) = Z \cdot Z' \cdot (wq - r - 1)/(2r + 1)$, which is p' and divisible by q unless $2r + 1 = p$ and $p \mid (m - 1)$ or if $2r + 1 = q$ and $q \nmid w$. In the latter cases, the partition $(1^{wq-2}, 1 + r, 1 + r)$ works, unless we were in the case $2r + 1 = p$ with $p \mid (m - 1)$ and $r + 1 = q$ with $q \nmid w$. In this case, if $q \neq 2$ (and hence $r \neq 1$), take $\lambda = (1^{mp}, r)$, which corresponds to a character χ_λ that lies in $B_p(\mathfrak{S}_n)$ and has degree $Y' = (mp + 1) \cdots (mp + r - 1)/(q - 2)!$. This is relatively prime to p, and is divisible by q since there must be a number between mp and $mp + q - 1 = mp + r$ divisible by q, but neither $mp = (w + 1)q - 1$ nor $n = mp + r = wq + 2q - 2$ can be divisible by q. If $q = 2$, we have $r = 1, q = 2, p = 3$, and the character corresponding to $(1^{n-2}, 2)$ lies in $B_2(\mathfrak{S}_n)$ and has degree $n - 1 = 3m = 2w + 1$, which is odd and divisible by 3.

From now on, we may therefore assume that $r = 0$, so that $mp = wq$.

(II) First, assume that $a_1 q^{t_1} < b_1 p^{s_1}$. If $b = 0$, so $n = mp = wq$, consider $\lambda = (1^{mp-a_1q^{t_1}-1}, 1 + a_1 q^{t_1})$. Then the corresponding degree is X, which we see is equal to

$$X = \frac{\prod_{i=0}^{a_1 q^{t_1}-1} (i + a_2 q^{t_2} + \cdots + a_k q^{t_k})}{\prod_{i=1}^{a_1 q^{t_1}} i} = \frac{\prod_{i=1}^{a_1 q^{t_1}} (b_1 p^{s_1} - i + b_2 p^{s_2} + \cdots + b_{k'} p^{s_{k'}})}{\prod_{i=1}^{a_1 q^{t_1}} i}.$$

Note that the q-part of this is q^{t_2}/q^{t_1}, which is divisible by q. Furthermore, the p-part is 1, since $a_1 q^{t_1} < b_1 p^{s_1}$ implies that the p-part of $-i + b_1 p^{s_1} + \cdots + b_{k'} p^{s_{k'}}$ is the same as that of i for $1 \leq i \leq a_1 q^{t_1}$.

Now suppose that $b > 0$, so $n = mp + b = wq + b$. If $a_1 q^{t_1} \neq b$, consider either $(1^{mp-a_1q^{t_1}-1}, 1 + a_1 q^{t_1}, b)$ or $(1^{mp-a_1q^{t_1}-1}, 1 + b, a_1 q^{t_1})$, depending on whether b is larger or smaller than $a_1 q^{t_1}$. Then the corresponding character lies in $B_p(\mathfrak{S}_n)$ and has degree $X \cdot Y \cdot \frac{|b - a_1 q^{t_1}|}{n - a_1 q^{t_1}}$. Note that since $a_1 q^{t_1} < b_1 p^{s_1}$, the p-part of $|b - a_1 q^{t_1}|_p \leq p^{s_1}$, and from this we see $|b - a_1 q^{t_1}|$ and $n - a_1 q^{t_1} = mp + b - a_1 q^{t_1}$ have the same p-part. Hence this character is a member of $\mathrm{Irr}_{p'}(B_p(\mathfrak{S}_n))$. Furthermore, its degree is still divisible by q, except possibly if $|b-a_1q^{t_1}|_q < (n-a_1q^{t_1})_q$. This can only happen if $(b)_q \geq q^{t_2}$. In the latter case, consider again the partition $(1^{mp}, b)$. The degree is Y' and hence we have removed $\frac{mp+b}{b}$ from the expression Y, which is p' and, from before, has q-part of the numerator at least as large as that of the denominator. Since in our situation $(mp+b)_q = q^{t_1} < q^{t_2} \leq (b)_q$, this degree Y' is also divisible by q.

Now assume $b = a_1q^{t_1}$. If $b+1 \neq p$ or $p \nmid (m-1)$, we take $\lambda = (1^{mp-b-2}, b+1, b+1)$, with $\chi_\lambda(1) = Y \cdot \frac{(mp-2)\cdots(mp-b-1)}{(b+1)!}$. Then $\chi_\lambda \in \mathrm{Irr}(B_p(\mathfrak{S}_n))$ and $p \nmid \chi_\lambda(1)$ due to the assumption $b+1 \neq p$ or $m-1$ is not divisible by p. Furthermore, $\chi_\lambda(1)$ is divisible by q since the q-part of the numerators of each of the two fractions is at least that of the denominators, as before, and in this case, the factor $(mp-b)$ is divisible by q^{t_2}, but no factor in the denominator is. Now, if $p = b+1 = 1+a_1q^{t_1}$ and $p \mid (m-1)$, this forces also $b_1p^{s_1} = p$, as $mp-p = \sum b_ip^{s_i} - p$ must be divisible by p^2. Here consider the partition $(1^{mp-p}, b+p)$, which gives a character with degree $(mp+b-1)\cdots(mp-p+1)/(p+b-1)!$ in $B_p(\mathfrak{S}_n)$. Note that the only factor in the numerator divisible by p is mp, which is divisible by p exactly once. Then since the denominator is divisible by p, we see this character lies in $\mathrm{Irr}_{p'}(B_p(\mathfrak{S}_n))$. Furthermore, since $mp-p+1 = \sum_{i\geq 2} a_iq^{t_i}$ in this case, we see $(mp-p+1+j)_q \geq (j)_q$ for $1 \leq j \leq p+b-2$, and that $\frac{mp-p+1}{p+b-1} = \frac{\sum_{i\geq 2} a_iq^{t_i}}{2a_1q^{t_1}}$ is divisible by q except possibly if $q=2$ and $t_2 = t_1+1$. In the latter case, the character corresponding to $(1^{mp-1}, b+1)$, which has degree $\frac{mp\cdot\prod_{i=1}^{2^{t_1}-1}(mp+i)}{2^{t_1}!}$ lies in $\mathrm{Irr}_{2'}(B_2(\mathfrak{S}_n))$ and has degree divisible by p.

(III) Finally, suppose that $a_1q^{t_1} > b_1p^{s_1}$. Note here that $mp \neq b_1p^{s_1}$, as $mp \geq a_1q^{t_1}$. If $b = 0$, then reversing the roles of p and q in the corresponding case in (II) above yields a character in $B_q(\mathfrak{S}_n)$ with degree X', which is relatively prime to q but divisible by p.

Hence we assume $b > 0$, so $n = mp+b = wq+b$. Note here that $b = w'q+b'$ for some integers w', b' with $0 \leq b' < q$, and $B_q(\mathfrak{S}_n)$ consists of those characters whose corresponding partitions have q-core (b').

Now, the partition $(1^{mp-b_1p^{s_1}-1}, b+1, b_1p^{s_1})$ gives a character in $B_q(\mathfrak{S}_n)$ with degree $Y \cdot X' \cdot \frac{b_1p^{s_1}-b}{n-b_1p^{s_1}}$. Note that the third factor is not divisible by p nor q, since $p \nmid b$ and $b_1p^{s_1} - b < a_1q^{t_1}$ so $b_1p^{s_1}-b$ and $n - b_1p^{s_1} = mp - (b_1p^{s_1}-b)$ have the same q-part. Furthermore, $p \nmid Y$, and also $q \nmid Y$ as long as $b < (q-a_1)q^{t_1}$. So if $b < (q-a_1)q^{t_1}$, this character lies in $\mathrm{Irr}_{q'}(B_q(\mathfrak{S}_n))$ with degree divisible by p.

So, we now assume that $q \mid Y$, so $b \geq (q-a_1)q^{t_1}$. Then the partition $(b+1, mp-1)$ corresponds to a character in $B_p(\mathfrak{S}_n)$ with degree $Y \cdot \frac{mp-(b+1)}{b+1}$, which is divisible by q and is relatively prime to p if $b+1 \neq p$ or $p \nmid (m-1)$. Hence we may now assume that further $p = b+1$ and $p \mid (m-1)$. Then setting $\lambda = (1^{mp}, b)$ yields $\chi_\lambda \in \mathrm{Irr}(B_p(\mathfrak{S}_n))$ and $\chi_\lambda(1) = Y'$, which is prime to p. Since $q \mid Y$, we have $q \mid Y'$ unless $q \mid b$ and $(mp+b)_q > (b)_q$, which forces $b = (q-a_1)q^{t_1}$.

So we are reduced to the case $q \nmid Y'$, $b = (q-a_1)q^{t_1} = p-1$, and $p \mid (m-1)$. Then the partition $(1^{mp-1}, 1+b)$ gives a character in $B_q(\mathfrak{S}_n)$ with degree $Y' \cdot mp/b$, which is divisible by p but is not divisible by q since the q-part of both b and mp are q^{t_1}.

Finally, note that in all cases, the λ described is not self-conjugate, and hence the characters restrict irreducibly to $\mathfrak{A}_n$, completing the proof. $\square$

4.2 Conjecture C for Simple Groups of Lie Type

Let r_0 be a prime and $r := r_0^a$ be some power of r_0. For p another prime, we denote by $d_p(r)$ the order of r modulo p, respectively modulo 4, if p is odd, respectively $p = 2$. Here we will prove the remaining direction of Conjecture C for simple groups of Lie type S defined in characteristic r_0. (For the case of Suzuki and Ree groups, we let r be 2^{2n+1} or 3^{2n+1}, as appropriate.)

Several cases here also follow quickly from [19]:

Proposition 4.3 *Let S be a simple group of Lie type defined over $\mathbb{F}_r$, and let p, q be primes such that $[P, Q] \neq 1$ for every Sylow p-subgroup P and Sylow q-subgroup Q of S but that at least one of the following conditions holds:*

1. *$r_0 \in \{p, q\}$;*
2. *a Sylow p-subgroup or a Sylow q-subgroup of S is abelian; or*
3. *$d_p(r) \neq d_q(r)$;*

Then either there exists $\chi \in \mathrm{Irr}_{p'}(B_p(S))$ with degree divisible by q or there exists $\chi \in \mathrm{Irr}_{q'}(B_q(S))$ with degree divisible by p.

Proof First suppose that $r_0 \in \{p, q\}$ and without loss, say $r_0 = p$. Then $\mathrm{Irr}(B_p(S)) = \mathrm{Irr}(S) \setminus \{\mathrm{St}_S\}$ contains all characters of degree divisible by q. Let G be a quasisimple group of Lie type of simply connected type such that $G/\mathbf{Z}(G) = S$. Suppose first that there is a semisimple element s of the dual group G^* that does not centralize a Sylow q-subgroup of G^* and which lies in $O^{p'}(G^*)$. (See e.g. [5, Chapter 4] for a discussion of the dual group G^*.) Considering a semisimple character χ_s corresponding to s (in particular, we may fix χ_s to correspond to the trivial character of $\mathbf{C}_{G^*}(s)$ under Jordan decomposition), we see that $\chi_s(1) = [G^* : \mathbf{C}_{G^*}(s)]_{p'}$ (see e.g. [17, (2.1)]) and χ_s is trivial on $\mathbf{Z}(G)$ (see e.g. [31, Proposition 2.7]), so χ_s has degree divisible by q but lies in $\mathrm{Irr}_{p'}(B_p(S))$ as a character of S.

Now suppose that every semisimple element of G^* that lies in $O^{p'}(G^*)$ centralizes a Sylow q-subgroup of G^*. In particular, this means that $O^{p'}(G^*)$ has abelian Sylow q-subgroups, so S also has abelian Sylow q-subgroups. (Indeed, in many cases, $S \cong O^{p'}(G^*)$, and otherwise the claim can be seen from the observations in [18, Section 2.1].) In this case, $\mathrm{Irr}_{q'}(B_q(S)) = \mathrm{Irr}(B_q(S))$ by Brauer's height zero conjecture for principal blocks [20]. Then let χ be any nontrivial unipotent character in $\mathrm{Irr}(B_q(S))$. This character will have degree divisible by p by [17, Theorem 6.8], unless $p = r \in \{2, 3\}$ and S is one of the exceptions given in loc. cit. Using Proposition 4.1, this leaves only the case $S = \mathrm{B}_n(2) = \mathrm{C}_n(2) = \mathrm{Sp}_{2n}(2)$ for $n \geq 4$. Let e be the order of $r^2 = 4$ modulo q and write $n = me + b$ with $b < e$. Now, following the proof in [17, Theorem 6.8], we see the nontrivial unipotent characters with degree not divisible by 2 are those whose corresponding symbols contain only the numbers 0, 1, n. Using the theory of e-core and e-cocore partitions (the relevant details of which we have summarized in the section on B_n and C_n in the proof of [25, Proposition 3.7]), we have one of the characters indexed by symbols $\binom{0,b+1}{me}$, $\binom{0,me}{b+1}$, or $\binom{b+1,me}{0}$ lies in $B_q(S)$. These characters are not the odd-degree characters, unless $b = 0$, i.e. $e \mid n$. In the latter case, the Steinberg character has degree a power of 2 and lies in $B_q(S)$.

Next, suppose that $r_0 \notin \{p, q\}$ and that a Sylow p-subgroup of S is abelian. Then $\mathrm{Irr}_{p'}(B_p(S)) = \mathrm{Irr}(B_p(S))$ by Brauer's height zero conjecture for principal blocks [20], and hence the proof of [19, Theorem 5.1] yields the desired character.

Finally, assume that $r_0 \notin \{p, q\}$, that no Sylow p- or q- subgroup of S is abelian, and that $d_p(r) \neq d_q(r)$. Note that the assumption $[P, Q] \neq 1$ for every choice of Sylow p- and Sylow q-subgroups of S (and hence analogously for G) implies that also $[P^*, Q^*] \neq 1$ for any Sylow p- and q- subgroups of the dual group G^*. (Again, this is pointed out already in [19, Theorem 5.1].) Now, let $d_p(r) \le d_q(r)$ and suppose that there exists $1 \ne s \in \mathbf{Z}(Q^*)$ for some $Q^* \in \mathrm{Syl}_q(G^*)$ such that s centralizes a Sylow p-subgroup of G^*. Then the same argument as in the first two paragraphs of [19, Proposition 3.5], but now applied to G^*, yields that $d_p(r) = d_q(r)$, a contradiction. Hence, given $1 \ne s \in \mathbf{Z}(Q^*)$, we havestop $\mathbf{C}_{G^*}(s)$ contains Q^* but does not contain a Sylow p-subgroup P^* of G^*. Then the corresponding semisimple character χ_s of G has degree divisible by p but lies in $\mathrm{Irr}_{q'}(B_q(G))$, since $\chi_s(1) = [G^* : \mathbf{C}_{G^*}(s)]_{r_0'}$ and using [12, Corollary 3.4]. Furthermore, arguing as in [11, Theorem 3.5] shows that such s can be chosen so that χ_s is trivial on $\mathbf{Z}(G)$, completing the proof. □

Our task is now to prove Conjecture C in the case that neither p nor q is the defining characteristic, $d_p(r) = d_q(r)$, and no Sylow p- or q-subgroup of S is abelian. We begin with the case of exceptional groups of Lie type, by which we mean the groups $S = \mathrm{G}_2(r)$, $\mathrm{F}_4(r)$, $\mathrm{E}_6^\epsilon(r)$, $\mathrm{E}_7(r)$, $\mathrm{E}_8(r)$, ${}^3\mathrm{D}_4(r)$, ${}^2\mathrm{G}_2(r)$, ${}^2\mathrm{F}_4(r)$, and ${}^2\mathrm{B}_2(r)$.

Proposition 4.4 *Let S be an exceptional simple group of Lie type defined over $\mathbb{F}_r$, and let p, q be primes such that $[P, Q] \ne 1$ for every Sylow p-subgroup P and Sylow q-subgroup Q of S. Then either there exists $\chi \in \mathrm{Irr}_{p'}(B_p(S))$ with degree divisible by q or there exists $\chi \in \mathrm{Irr}_{q'}(B_q(S))$ with degree divisible by p.*

Proof By Proposition 4.3, we may assume that r is not a power of p nor q, that no Sylow p- or q- subgroup of S is abelian, and that $d_p(r) = d_q(r)$. Let $d := d_p(r) = d_q(r)$. With these constraints, we see that S is not of Suzuki or Ree type, that p and q are at most 7, and that d is a regular number in the sense of Springer [33] (see also [32, Definition 2.5]). Hence we see that the principal blocks $B_p(S)$ and $B_q(S)$ are the unique blocks of S containing p'-, respectively, q'-degree unipotent characters (see, e.g., [30, Lemma 3.6]). Under these conditions, we see by observing the explicit list of unipotent character degrees in [5, Section 13.9] that there exists a unipotent character χ satisfying either $\chi \in \mathrm{Irr}_{p'}(B_p(S))$ and $q \mid \chi(1)$ or $\chi \in \mathrm{Irr}_{q'}(B_q(S))$ and $p \mid \chi(1)$. □

We next consider the case of linear and unitary groups.

Proposition 4.5 *Let $S = \mathrm{PSL}_n^\epsilon(r)$ with $n \ge 2$, and let $p \ne q$ be primes such that $[P, Q] \ne 1$ for every Sylow p-subgroup P and Sylow q-subgroup Q of S. Then either there exists $\chi \in \mathrm{Irr}_{p'}(B_p(S))$ with degree divisible by q or there exists $\chi \in \mathrm{Irr}_{q'}(B_q(S))$ with degree divisible by p.*

Proof By Proposition 4.1, we may assume S is not isomorphic to a sporadic or alternating group. Furthermore, by Proposition 4.3, we may assume that r is not a power of p nor q, that no Sylow p- or q- subgroup of S is abelian, and that $d_p(r) = d = d_q(r)$.

Write $G := \mathrm{SL}_n^\epsilon(r)$, $\widetilde{G} := \mathrm{GL}_n^\epsilon(r)$, and $e := d_p(\epsilon r) = d_q(\epsilon r)$. Note that $\widetilde{G}^* \cong \widetilde{G}$, so we will make this identification. Let $\widetilde{P} \in \mathrm{Syl}_p(\widetilde{G})$ and $\widetilde{Q} \in \mathrm{Syl}_q(\widetilde{G})$. It suffices to show that (up to switching p and q) there is some $s \in \mathbf{Z}(\widetilde{P}) \cap G$ such that $|\mathbf{C}_{\widetilde{G}}(s)|$ is not divisible by $|\widetilde{Q}|$ and sz is not $\widetilde{G}$-conjugate to s for any $1 \ne z \in \mathbf{Z}(\widetilde{G})$. (Indeed, that $s \in G = \mathrm{O}^{r_0'}(\widetilde{G})$ implies that the corresponding semisimple character χ_s is trivial on $\mathbf{Z}(\widetilde{G})$ and that sz is not conjugate to s for nontrivial $z \in \mathbf{Z}(\widetilde{G})$ implies that χ_s is irreducible on restriction to G using e.g. [31, Proposition 2.7] and [30, Lemma 1.4]; the remaining conditions imply $\chi_s \in \mathrm{Irr}(B_p(\widetilde{G}))$ using [12, Corollary 3.4] and $\chi_s(1)$ is p' but divisible by q since $\chi_s(1) = [\widetilde{G} : \mathbf{C}_{\widetilde{G}}(s)]_{r_0'}$.)

Now, if $2 \notin \{p, q\}$, we see using the results of Weir [34] that $\widetilde{P}$ and $\widetilde{Q}$ are naturally isomorphic to the corresponding Sylow subgroups of $\mathrm{GL}^{\epsilon}_{we}(r)$, embedded naturally into $\mathrm{GL}^{\epsilon}_{we}(r) \times \mathrm{GL}^{\epsilon}_{b}(r) \leq \mathrm{GL}^{\epsilon}_{n}(r)$ where $n = we + b$ with $0 \leq b < e$. The results of Carter–Fong [6] yield the same when p or q is 2, except that if $(e, b) = (2, 1)$, then $|\mathrm{GL}^{\epsilon}_{b}(r)|$ is divisible by 2 exactly once, and the Sylow 2-subgroup of $\mathrm{GL}^{\epsilon}_{n}(r)$ in this case is that of $\mathrm{GL}^{\epsilon}_{we}(r) \times \mathrm{GL}_{b}(r)$.

Let $w = a_1 p^{t_1} + a_2 p^{t_2} + \cdots + a_k p^{t_k} = b_1 q^{m_1} + \cdots + b_{k'} q^{m_{k'}}$ be the p-adic and q-adic expansions of w, with $t_1 < \cdots < t_k$; $m_1 < \cdots < m_{k'}$; $1 \leq a_i < p$ for each $1 \leq i \leq k$; and $1 \leq b_j < q$ for each $1 \leq j \leq k'$. By [6, 34], we have $\widetilde{P} \cong P_1^{a_1} \times \cdots \times P_k^{a_k} \times X$, where P_i is a Sylow p-subgroup of $\mathrm{GL}^{\epsilon}_{ep^{t_i}}(r)$ (which, if p is odd, is a Sylow p-subgroup of $\mathrm{GL}^{\epsilon}_{p^{t_i}}(r^e)$ embedded naturally) for each $1 \leq i \leq k$ and $X \in \mathrm{Syl}_p(\mathrm{GL}^{\epsilon}_{b}(r))$ is isomorphic to C_2 if $(p, e, b) = (2, 2, 1)$ and is trivial otherwise. Here we view $\prod \mathrm{GL}^{\epsilon}_{ep^{t_i}}(r)$ as the natural diagonally-embedded subgroup. Similarly, $\widetilde{Q} \cong Q_1^{b_1} \times \cdots \times Q_{k'}^{b_{k'}} \times Y$ where $Q_j \in \mathrm{Syl}_q(\mathrm{GL}^{\epsilon}_{eq^{m_j}}(r))$ for $1 \leq j \leq k'$ and $Y \in \mathrm{Syl}_q(\mathrm{GL}^{\epsilon}_{b}(r))$.

Without loss of generality, assume $a_1 p^{t_1} < b_1 q^{m_1}$. (Note that we cannot have $a_1 p^{t_1} = b_1 q^{m_1}$, as this would contradict that either $a_1 < p < q$ or $b_1 < q < p$.) Let $x \in \mathbf{Z}(P_1)$ have no eigenvalues equal to 1 (indeed, this can be done by taking x as an element of a Sylow p-subgroup of $\mathbf{Z}(\mathrm{GL}^{\epsilon}_{p^{t_1}}(r^e))$ embedded naturally into $\mathrm{GL}^{\epsilon}_{ep^{t_1}}(r)$) and consider the element $s = \mathrm{diag}(x, \ldots, x, I_{n-ea_1p^{t_1}}) \in \mathbf{Z}(\widetilde{P})$, with a_1 copies of x. In fact, taking x (and hence its eigenvalues) to have order p, we obtain $\det(x) = 1$ and hence $s \in G$. Here $\mathbf{C}_{\widetilde{G}}(s) = \mathbf{C}_{\mathrm{GL}^{\epsilon}_{a_1ep^{t_1}}(r)}(\mathrm{diag}(x, \ldots, x)) \times \mathrm{GL}^{\epsilon}_{n-ea_1p^{t_1}}(r)$. Now, by considering the structure, and hence size, of $\widetilde{Q}$ (namely, each Q_j is a wreath product $Q' \wr C_q$ where Q' is a Sylow q-subgroup of $\mathrm{GL}^{\epsilon}_{eq^{m_j-1}}(r)$), we see $|\mathrm{GL}^{\epsilon}_{a_1ep^{t_1}}(r) \times \mathrm{GL}^{\epsilon}_{n-ea_1p^{t_1}}(r)|$ is not divisible by $|\widetilde{Q}|$. Furthermore, by considering the block sizes, we see that sz and z cannot have the same eigenvalues (and hence they cannot be $\widetilde{G}$-conjugate) for any nontrivial scalar matrix $z \in \mathbf{Z}(\mathrm{GL}^{\epsilon}_{n}(r))$. This completes the proof. □

We next consider the remaining classical types, for which the proof is very similar to the linear and unitary case.

Proposition 4.6 *Let $S = \mathrm{PSp}_{2n}(r)$ with $n \geq 2$, $\mathrm{P\Omega}_{2n+1}(r)$ with $n \geq 3$, or $\mathrm{P\Omega}^{\epsilon}_{2n}(r)$ with $n \geq 4$. Let $p \neq q$ be primes such that $[P, Q] \neq 1$ for every Sylow p-subgroup P and Sylow q-subgroup Q of S. Then either there exists $\chi \in \mathrm{Irr}_{p'} B_p(S)$ with degree divisible by q or there exists $\chi \in \mathrm{Irr}_{q'} B_q(S)$ with degree divisible by p.*

Proof As before, we may assume S is not isomorphic to a sporadic or alternating group, r is not a power of p nor q, no Sylow p- or q- subgroup of S is abelian, and that $d_p(r) = d = d_q(r)$.

First we set some notation. We define $H_n := \mathrm{Sp}_{2n}(r)$, $\mathrm{SO}_{2n+1}(r)$, and $\mathrm{SO}^{\epsilon}_{2n}(r)$ in the cases $S = \mathrm{PSp}_{2n}(r)$, $\mathrm{P\Omega}_{2n+1}(r)$, and $\mathrm{P\Omega}^{\epsilon}_{2n}(r)$, respectively. Let $H := H_n$ and let $\Omega := \mathrm{O}^{r_0'}(H)$ so that Ω is perfect and $S = \Omega/\mathbf{Z}(\Omega)$. Note that the dual groups are $H_n^* = \mathrm{SO}_{2n+1}(r)$, $\mathrm{Sp}_{2n}(r)$, and $\mathrm{SO}^{\epsilon}_{2n}(r)$, respectively, and we will write $H^* := H_n^*$. Note that $\mathbf{Z}(\Omega) \leq \mathbf{Z}(H)$ and that H/Ω and $\mathbf{Z}(H)$ are 2-groups. Let $\widetilde{P}$ and $\widetilde{Q}$ be Sylow p- and q-subgroups of H^*.

In this situation, it suffices to show that (up to switching p and q) there is some $s \in \mathbf{Z}(\widetilde{P})$ such that $\mathbf{C}_{\widetilde{G}}(s)$ is not divisible by $|\widetilde{Q}|$, using similar reasoning to the above case. Indeed, if p is odd, then $\widetilde{P}$ may be considered as a Sylow p-subgroup of $\mathrm{O}^{r'_0}(H^*)$ and sz cannot be H^*-conjugate to s for any $1 \neq z \in \mathbf{Z}(H^*)$ since $\mathbf{Z}(H^*)$ is a 2-group, so a corresponding semisimple character χ_s of H is trivial on $\mathbf{Z}(H)$ and restricts irreducibly to Ω. If instead $p = 2$, then such a character χ_s would have odd degree, and therefore restrict irreducibly to Ω since H/Ω is a 2-group. Then since Ω is perfect, $\mathbf{Z}(\Omega)$ is a 2-group, and χ_s has odd degree, this forces χ_s to be trivial on $\mathbf{Z}(\Omega)$. Furthermore, as before, $\chi_s \in \mathrm{Irr}(B_p(H))$ in either case.

Assume first that p and q are odd. In these cases, the work of Weir [34] again describes the structure of $\widetilde{P}$ and $\widetilde{Q}$, building off of the case of linear groups. If $H^* = \mathrm{SO}_{2n+1}(r)$ or $\mathrm{Sp}_{2n}(r)$, we have Sylow p- and q-subgroups are already Sylow subgroups of $\mathrm{GL}_{2n+1}(r)$ (and hence of $\mathrm{GL}_{2n}(r)$) when d is even, and are Sylow subgroups of the naturally-embedded $\mathrm{GL}_n(r)$ if d is odd. For these cases, let $e := d_p(r^2) = d_q(r^2)$, write $n = ew + b$ with $0 \leq b < e$, and let $w = a_1 p^{t_1} + a_2 p^{t_2} + \cdots + a_k p^{t_k} = b_1 q^{m_1} + \cdots + b_{k'} q^{m_{k'}}$ be the p-adic and q-adic expansions of w as before. Again without loss, we assume $a_1 p^{t_1} < b_1 q^{m_1}$. In particular, $\widetilde{P}$ and $\widetilde{Q}$ are again isomorphic to Sylow subgroups of H^*_{ew} and of the form $\widetilde{P} \cong P_1^{a_1} \times \cdots \times P_k^{a_k}$, where each P_i is a Sylow p-subgroup of $\mathrm{GL}_{dp^{t_i}}(r)$ and can be identified with a Sylow p-subgroup of $H^*_{ep^{t_i}}$, and similar for $\widetilde{Q}$. As before, let $x \in \mathbf{Z}(P_1)$ with no eigenvalues equal to 1 and let $s = (x, \ldots, x, 1, \ldots 1) \in \mathbf{Z}(\widetilde{P})$ with a_1 copies of x. Then we can see from the centralizer structure of semisimple elements that $\mathbf{C}_{H^*}(s) \cong \mathbf{C}_{H^*_{a_1 ep^{t_1}}}(x, \ldots, x) \times H^*_{n-ea_1 p^{t_1}}$. Since the Sylow q-subgroups of $H^*_{a_1 ep^{t_1}}$ and $H^*_{n-ea_1 p^{t_1}}$ can be identified with Sylow subgroups of linear groups in an analogous way as for H^*, depending on whether d was even or odd, we have $|\widetilde{Q}| \nmid |\mathbf{C}_{H^*}(s)|$ for the same reason as in the case of linear groups above.

If $H^* = \mathrm{SO}^{\epsilon}_{2n}(r)$, then we have embeddings $\mathrm{SO}_{2n-1}(r) \leq H^* \leq \mathrm{SO}_{2n+1}(r)$, and $\widetilde{P}$ and $\widetilde{Q}$ are both Sylow subgroups of either $\mathrm{SO}_{2n-1}(r)$ or $\mathrm{SO}_{2n+1}(r)$. In this case, letting $m \in \{n, n-1\}$ so that $\widetilde{P}$, $\widetilde{Q}$ are Sylow subgroups of $\mathrm{SO}_{2m+1}(r)$ and now writing $m = ew + b$ with w written with p- and q-adic expansions as before, $\widetilde{P}$ can again be written $\widetilde{P} \cong P_1^{a_1} \times \cdots \times P_k^{a_k}$ with each P_i a Sylow subgroup of $\mathrm{GL}_{dp^{t_i}}(r)$, which in this case can also be identified with a Sylow p-subgroup of either $\mathrm{SO}^+_{2ep^{t_i}}(r)$ or $\mathrm{SO}^-_{2ep^{t_i}}(r)$. From here, arguing similar to before, we obtain an element $s \in \mathbf{Z}(\widetilde{P})$ with $\mathbf{C}_{H^*}(s)$ isomorphic to a subgroup of $\mathbf{C}_{\mathrm{GO}^{\pm}_{2a_1 ep^{t_1}}(r)}(\mathrm{diag}(x, \ldots, x)) \times \mathrm{GO}^{\pm}_{2(n-ea_1 p^{t_1})}(r)$. Since q is odd, we again see in the same way as above that $|\widetilde{Q}|$ does not divide $|\mathbf{C}_{H^*}(s)|$.

We are finally left with the case that $2 \in \{p, q\}$. Let $\hat{H}^*$ denote the group $\mathrm{GO}_{2n+1}(r)$, $\mathrm{Sp}_{2n}(r)$, or $\mathrm{GO}^{\epsilon}_{2n}(r)$ respectively, so that $[\hat{H}^* : H^*]$ divides 2. Note that if $p = 2$ and $H^* \neq \hat{H}^*$, then $\widetilde{P}$ is index-2 in a Sylow 2-subgroup $\hat{P}$ of $\hat{H}^*$, which are again described by Carter–Fong [6]. Here in the case of $\mathrm{GO}_{2n+1}(r)$ or $\mathrm{Sp}_{2n}(r)$, writing $n = 2^{t_1} + \cdots + 2^{t_k}$ for the 2-adic expansion with $t_1 < \cdots < t_k$, we have $\hat{P} \cong P_1 \times \cdots \times P_k$, where P_i is a Sylow 2-subgroup of $\mathrm{GO}_{2\cdot 2^{t_i}+1}(r)$, respectively, $\mathrm{Sp}_{2\cdot 2^{t_i}}(r)$. In the case $\hat{H}^* = \mathrm{GO}^{\epsilon}_{2n}(r)$, we have $\hat{P}$ is either a Sylow 2-subgroup of $\mathrm{GO}_{2n+1}(r)$, embedded as before, or of the form $P_0 \times C_2 \times C_2$, where P_0 is a Sylow 2-subgroup of $\mathrm{GO}_{2n-1}(r)$. From here, we may argue analogously to before, keeping in mind that when $p = 2$, choosing $x \in \mathbf{Z}(P_1)$ to have 2^{t_1+1} eigenvalues -1 yields an element of determinant 1, and hence an element of $\widetilde{P} = \hat{P} \cap H^*$. □

Conjecture C for simple groups (and Theorem E) now follows from Propositions 4.1–4.6.

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