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Results in Mathematics

Reflection Formulas for Order Derivatives of Bessel Functions

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Abstract. From new integral representations of the n-th derivative of Bessel functions with respect to the order, we derive some reflection formulas for the first and second order derivatives of $J_\nu(t)$ and $Y_\nu(t)$ for integral order, and for the n-th order derivatives of $I_\nu(t)$ and $K_\nu(t)$ for arbitrary real order. As an application of the reflection formulas obtained for the first order derivative, we extend some formulas given in the literature to negative integral order. Also, as a by-product, we calculate an integral which does not seem to be reported in the literature.

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1. Introduction

Bessel functions are the canonical solutions $y(t)$ of Bessel's differential equation:

$$t^2 y'' + t\,y' + \left(t^2 - \nu^2\right) y = 0, \tag{1}$$

where ν denotes the order of the Bessel function. This equation arises when finding separable solutions of Laplace equation in cylindrical coordinates, as well as in Helmholtz equation in spherical coordinates [6, Chap. 6]. The general solution of (1) is a linear combination of the *Bessel functions of the first and second kind*, i.e. $J_\nu(t)$ and $Y_\nu(t)$ respectively. In the case of pure imaginary argument, the solutions to the Bessel equations are called *modified Bessel functions of the first and second kind*, $I_\nu(t)$ and $K_\nu(t)$ respectively. Despite the fact the properties of the Bessel functions have been studied extensively in the literature [1,10], studies about successive derivatives of the Bessel functions with respect to the order ν are relatively scarce. For nonnegative integral order $\nu = m$, we find in the literature the following expressions in terms of finite sums of Bessel functions [7, Eqn. 10.15.3&4]

$$\left.\frac{\partial J_{\nu}(t)}{\partial\nu}\right|_{\nu=m} = \frac{\pi}{2}Y_m(t) + \frac{m!}{2}\sum_{k=0}^{m-1}\frac{J_k(t)}{k!\,(m-k)}\left(\frac{t}{2}\right)^{k-m}, \tag{2}$$

and

$$\left.\frac{\partial Y_{\nu}(t)}{\partial\nu}\right|_{\nu=m} = -\frac{\pi}{2}J_m(t) + \frac{m!}{2}\sum_{k=0}^{m-1}\frac{Y_k(t)}{k!\,(m-k)}\left(\frac{t}{2}\right)^{k-m}. \tag{3}$$

For modified Bessel functions, we have [7, Eqn. 10.38.3&4]

$$\left.\frac{\partial I_{\nu}(t)}{\partial\nu}\right|_{\nu=m} = (-1)^m\left[-K_m(t) + \frac{m!}{2}\sum_{k=0}^{m-1}\frac{(-1)^k I_k(t)}{k!\,(m-k)}\left(\frac{t}{2}\right)^{k-m}\right], \tag{4}$$

and

$$\left.\frac{\partial K_{\nu}(t)}{\partial\nu}\right|_{\nu=m} = \frac{m!}{2}\sum_{k=0}^{m-1}\frac{K_k(t)}{k!\,(m-k)}\left(\frac{t}{2}\right)^{k-m}. \tag{5}$$

Also, for the n-th derivative of the Bessel function of the first kind with respect to the order, we find in [8] a more complex expression in series form.

Regarding integral representations of the derivative of $J_{\nu}(t)$ and $I_{\nu}(t)$ with respect to the order, we find in [2] $\forall\Re\nu > 0$,

$$\frac{\partial J_{\nu}(t)}{\partial\nu} = \pi\nu\int_0^{\pi/2}\tan\theta\; Y_0\left(t\sin^2\theta\right)J_{\nu}\left(t\cos^2\theta\right)d\theta, \tag{6}$$

and

$$\frac{\partial I_{\nu}(t)}{\partial\nu} = -2\nu\int_0^{\pi/2}\tan\theta\; K_0\left(t\sin^2\theta\right)I_{\nu}\left(t\cos^2\theta\right)d\theta. \tag{7}$$

Other integral representations of the order derivative of $J_{\nu}(z)$ and $Y_{\nu}(z)$ are given in [4] for $\nu > 0$ and $t \neq 0$, $|\arg t| \leq \pi$, which read as,

$$\frac{\partial J_{\nu}(t)}{\partial\nu} = \pi\nu\left[Y_{\nu}(t)\int_0^t\frac{J_{\nu}^2(z)}{t}dz + J_{\nu}(t)\int_t^{\infty}\frac{J_{\nu}(z)\,Y_{\nu}(z)}{z}dz\right], \tag{8}$$

and

$$\begin{aligned}\frac{\partial Y_{\nu}(t)}{\partial\nu} = \pi\nu\Bigg[&J_{\nu}(t)\left(\int_t^{\infty}\frac{Y_{\nu}^2(z)}{z}dz - \frac{1}{2\nu}\right)\\ &-Y_{\nu}(t)\int_t^{\infty}\frac{J_{\nu}(z)\,Y_{\nu}(z)}{z}dz\Bigg].\end{aligned} \tag{9}$$

Recently, in [5], we find the following integral representations of the derivatives of the modified Bessel functions $I_\nu(t)$ and $K_\nu(t)$ with respect to the order for $\nu > 0$ and $t \neq 0$, $|\arg t| \leq \pi$,

$$\frac{\partial I_\nu(t)}{\partial \nu} = -2\nu \left[I_\nu(t) \int_t^\infty \frac{K_\nu(z) I_\nu(z)}{z} dz + K_\nu(t) \int_0^t \frac{I_\nu^2(z)}{z} dz \right], \tag{10}$$

and

$$\frac{\partial K_\nu(t)}{\partial \nu} = 2\nu \left[K_\nu(t) \int_t^\infty \frac{I_\nu(z) K_\nu(z)}{z} dz - I_\nu(t) \int_t^\infty \frac{K_\nu^2(z)}{z} dz \right]. \tag{11}$$

The great advantage of the integral expressions (8)–(11) is that the integrals involved in them can be calculated in closed-form [5]. Also, $\forall \nu \notin \mathbb{Z}$, expressions in closed-form for the second and third derivatives with respect to the order are found in [3], but these expressions are extraordinarily complex, above all for the third derivative.

In view of the literature commented above, the goal of this article is twofolded. On the one hand, in Sect. 2, we obtain simple integral representations for the n-th derivatives of the Bessel functions with respect to the order. The great advantage of these expressions is that its numerical evaluation is quite rapid and straightforward. As a by-product, we obtain the calculation of an integral which does not seem to be reported in the literature.

On the other hand, in Sect. 3, we derive some reflection formulas for the first and second order derivative of $J_\nu(t)$ and $Y_\nu(t)$ for integral order, from the expressions obtained in Sect. 2. Also, we derive reflection formulas for the n-th order derivative of $I_\nu(t)$ and $K_\nu(t)$ for arbitrary real order. As an application of the reflection formulas obtained for the first order derivative, we extend formulas (2)–(5) to negative integral orders.

Finally, we collect our conclusions in Sect. 4.

2. Integral Representations of n-th Order Derivatives

In order to perform the n-th derivatives of Bessel and modified Bessel functions with respect to the order, first we state the following n-th derivatives, that can be proved easily by induction and using the binomial theorem.

Proposition 1. *The n-th derivative of the functions*

$$\begin{aligned} f_1(\nu) &= \cos(t \sin x - \nu x), \\ f_2(\nu) &= \sin(t \sin x - \nu x), \\ f_3(\nu) &= e^{-\nu x} \sin \pi\nu = \Im\left(e^{(i\pi - x)\nu}\right), \\ f_4(\nu) &= e^{-\nu x} \cos \pi\nu = \Re\left(e^{(i\pi - x)\nu}\right), \end{aligned}$$

with respect to the order ν *are given by*

$$f_1^{(n)}(\nu) = x^n \cos(t\sin x - \nu x - n\pi/2), \tag{12}$$

$$f_2^{(n)}(\nu) = x^n \sin(t\sin x - \nu x - n\pi/2), \tag{13}$$

$$\begin{aligned} f_3^{(n)}(\nu) &= e^{-\nu x}\Im\left[(i\pi - x)^n e^{i\pi\nu}\right] \\ &= e^{-\nu x}\left[p_n(x)\sin\pi\nu + q_n(x)\cos\pi\nu\right], \end{aligned}$$

$$f_4^{(n)}(\nu) = e^{-\nu x}\Re\left[(i\pi - x)^n e^{i\pi\nu}\right] \tag{14}$$

$$= e^{-\nu x}\left[p_n(x)\cos\pi\nu - q_n(x)\sin\pi\nu\right]. \tag{15}$$

where we have set the polynomials

$$p_n(x) = \Re\left[(i\pi - x)^n\right] = \sum_{k=0}^{\lfloor n/2\rfloor} \binom{n}{2k} (-1)^{n+k} \pi^{2k} x^{n-2k}, \tag{16}$$

$$q_n(x) = \Im\left[(i\pi - x)^n\right] = \sum_{k=0}^{\lfloor (n-1)/2\rfloor} \binom{n}{2k+1} (-1)^{n+k+1} \pi^{2k+1} x^{n-2k-1}. \tag{17}$$

Next, we present the integral representations of the n-th order derivatives of the Bessel functions from the integral representations of these functions and the results stated in Proposition 1.

Theorem 2. *For* $n = 0, 1, \ldots$, *the following integral representation holds true:*

$$\begin{aligned} \frac{\partial^n}{\partial\nu^n} J_\nu(t) = \frac{1}{\pi}\int_0^\pi x^n \cos\left(t\sin x - \nu x - \frac{\pi}{2}n\right) dx \\ - \frac{1}{\pi}\int_0^\infty e^{-t\sinh x - \nu x}\left[p_n(x)\sin\pi\nu + q_n(x)\cos\pi\nu\right] dx. \end{aligned} \tag{18}$$

Proof. Perform the n-th derivative w.r.t the order in Schläfli integral representation of $J_\nu(t)$ [7, Eqn. 10.9.6], i.e. $\forall\, \Re t > 0$,

$$J_\nu(t) = \frac{1}{\pi}\int_0^\pi \cos(t\sin x - \nu x)\, dx - \frac{\sin\nu\pi}{\pi}\int_0^\infty e^{-t\sinh x - \nu x} dx, \tag{19}$$

and apply (12) and (14) to obtain (18), as we wanted to prove. □

Theorem 3. *For* $n = 0, 1, \ldots$, *the following integral representation holds true:*

$$\begin{aligned} \frac{\partial^n}{\partial\nu^n} Y_\nu(t) = \frac{1}{\pi}\int_0^\pi x^n \sin\left(t\sin x - \nu x - \frac{\pi}{2}n\right) dx \\ - \frac{1}{\pi}\int_0^\infty e^{-t\sinh x}\left(x^n e^{\nu x} + e^{-\nu x}\left[p_n(x)\cos\pi\nu - q_n(x)\sin\pi\nu\right]\right) dx. \end{aligned} \tag{20}$$

Proof. Perform the n-th derivative w.r.t the order in the following integral representation of the Bessel function of the second kind [7, Eqn. 10.9.7], i.e. $\forall\, \Re t > 0$,

$$Y_\nu(t) = \frac{1}{\pi}\int_0^\pi \sin(t\sin x - \nu x)\, dx - \frac{1}{\pi}\int_0^\infty e^{-t\sinh x}\left(e^{\nu x} + e^{-\nu x}\cos\nu\pi\right) dx, \tag{21}$$

and apply formulas (13) and (14), to obtain (20), as we wanted to prove. □

Theorem 4. *For $n = 0, 1, \ldots$, the following integral representation holds true:*

$$\frac{\partial^n}{\partial\nu^n} I_\nu(t) = \frac{1}{\pi}\int_0^\pi x^n e^{t\cos x}\cos\left(\nu x + \frac{\pi}{2}n\right) dx - \frac{1}{\pi}\int_0^\infty e^{-t\cosh x - \nu x}\left[p_n(x)\sin\pi\nu + q_n(x)\cos\pi\nu\right] dx. \tag{22}$$

Proof. Perform the n-th derivative w.r.t the order in the following integral representation [7, Eqn. 10.32.4], i.e. $\forall\, \Re t > 0$,

$$I_\nu(t) = \frac{1}{\pi}\int_0^\pi e^{t\cos x}\cos\nu x\ dx - \frac{\sin\nu\pi}{\pi}\int_0^\infty e^{-t\cosh x - \nu x} dx. \tag{23}$$

and apply (12) with $t = 0$ and (14) to obtain (22), as we wanted to prove. □

Theorem 5. *For $n = 0, 1, \ldots$, the following integral representation holds true:*

$$\frac{\partial^n}{\partial\nu^n} K_\nu(t) = \frac{1}{2}\int_0^\infty x^n e^{-t\cosh x}\left[e^{\nu x} + (-1)^n e^{-\nu x}\right] dx \tag{24}$$

$$= \frac{1}{2}\int_{-\infty}^\infty x^n e^{\nu x - t\cosh x} dx. \tag{25}$$

Proof. Perform the n-th derivative w.r.t the order in the following integral representations of the Macdonald function [6, Eqn. 5.10.23], i.e. $\forall\, \Re t > 0$,

$$K_\nu(t) = \int_0^\infty e^{-t\cosh x}\cosh\nu x\, dx \tag{26}$$

$$= \frac{1}{2}\int_{-\infty}^\infty e^{\nu x - t\cosh x}\ dx. \tag{27}$$

□

For $n = 1$, the above integral (25) is calculated in [5] in closed-form, thus $\forall \nu \in \mathbb{R}\setminus\{-1/2, -3/2, \ldots\}$, $\Re t > 0$,

$$\int_{-\infty}^\infty x\, e^{\nu x - t\cosh x} dx = \nu\left[\frac{K_\nu(z)}{\sqrt{\pi}} G^{3,1}_{2,4}\left(z^2 \middle| \begin{array}{c} 1/2, 1 \\ 0, 0, \nu, -\nu \end{array}\right) - \sqrt{\pi} I_\nu(z)\, G^{4,0}_{2,4}\left(z^2 \middle| \begin{array}{c} 1/2, 1 \\ 0, 0, \nu, -\nu \end{array}\right)\right], \tag{28}$$

where the function $G_{p,q}^{m,n}$ denotes the Meijer-G function [7, Eqn. 16.17.1]. If $2\nu \notin \mathbb{Z}$, the above expression is reduced in terms of generalized hypergeometric functions ${}_pF_q$ [7, Eqn. 16.2.1] as [3],

$$\begin{aligned}
&\int_{-\infty}^{\infty} x\, e^{\nu x - t\cosh x} dx \\
&= \pi \csc \pi\nu \left\{ \pi \cot \pi\nu\, I_\nu(z) - \left[I_\nu(z) + I_{-\nu}(z) \right] \right. \\
&\quad \left. \left[\frac{z^2}{4(1-\nu^2)}\, {}_3F_4\left(\left. \begin{array}{c} 1, 1, \frac{3}{2} \\ 2, 2, 2-\nu, 2+\nu \end{array} \right| z^2 \right) + \log\left(\frac{z}{2}\right) - \psi(\nu) - \frac{1}{2\nu} \right] \right\} \\
&\quad + \frac{1}{2} \left\{ I_{-\nu}(z)\, \Gamma^2(-\nu) \left(\frac{z}{2}\right)^{2\nu} {}_2F_3\left(\left. \begin{array}{c} \nu, \frac{1}{2}+\nu \\ 1+\nu, 1+\nu, 1+2\nu \end{array} \right| z^2 \right) \right. \\
&\quad \left. - I_\nu(z)\, \Gamma^2(\nu) \left(\frac{z}{2}\right)^{-2\nu} {}_2F_3\left(\left. \begin{array}{c} -\nu, \frac{1}{2}-\nu \\ 1-\nu, 1-\nu, 1-2\nu \end{array} \right| z^2 \right) \right\}. \qquad (29)
\end{aligned}$$

It is worth noting that the numerical evaluation of the integral representations of the n-th order derivatives of the Bessel functions given in (18), (20), (22), and (25) is quite efficient if we use a "double exponential" strategy [9].

3. Reflection Formulas

3.1. Bessel Functions

Theorem 6. *$\forall t \in \mathbb{C}$ and $m = 0, 1, \ldots,$ the following reflection formula holds true:*

$$\left. \frac{\partial J_\nu(t)}{\partial \nu} \right|_{\nu=m} + (-1)^m \left. \frac{\partial J_\nu(t)}{\partial \nu} \right|_{\nu=-m} = \pi\, Y_m(t). \qquad (30)$$

Proof. From the integral representation (18) for $n = 1$, we have

$$\begin{aligned}
\left. \frac{\partial J_\nu(t)}{\partial \nu} \right|_{\nu=\pm m} &= \frac{1}{\pi} \int_0^\pi x \sin(t \sin x \mp m x)\, dx \\
&\quad + (-1)^{m+1} \int_0^\infty e^{-t \sinh x} e^{\mp m x} dx. \qquad (31)
\end{aligned}$$

On the one hand, consider $m = 2k$, thus, according to (31), the LHS of (30) becomes

$$\begin{aligned}
\left. \frac{\partial J_\nu(t)}{\partial \nu} \right|_{\nu=2k} + \left. \frac{\partial J_\nu(t)}{\partial \nu} \right|_{\nu=-2k} &= \frac{2}{\pi} \int_0^\pi x \sin(t \sin x) \cos(2kx)\, dx \\
&\quad - \int_0^\infty e^{-t\sinh x} \left(e^{2kx} + e^{-2kx} \right) dx. \qquad (32)
\end{aligned}$$

To calculate the first integral on the RHS of (32), perform the substitution $\xi = x - \pi/2$, eliminate the term that vanishes by parity, and undo the change of variables to obtain

$$\frac{2}{\pi}\int_0^\pi x\sin\left(t\sin x\right)\cos\left(2kx\right)dx = \int_0^\pi \sin\left(t\sin x\right)\cos\left(2kx\right)dx. \tag{33}$$

Applying the trigonometric identity $\sin\left(\alpha+\beta\right) = \sin\alpha\cos\beta + \sin\beta\cos\alpha$, rewrite (33) as

$$\frac{2}{\pi}\int_0^\pi x\sin\left(t\sin x\right)\cos\left(2kx\right)dx = \int_0^\pi \sin\left(t\sin x + 2kx\right)dx - \int_0^\pi \cos\left(t\sin x\right)\sin\left(2kx\right)dx. \tag{34}$$

The second integral on the RHS of (34) vanishes by parity performing the substitution $\xi = x - \pi/2$. Therefore, (32) becomes

$$\left.\frac{\partial J_\nu\left(t\right)}{\partial\nu}\right|_{\nu=2k} + \left.\frac{\partial J_\nu\left(t\right)}{\partial\nu}\right|_{\nu=-2k} = \int_0^\pi \sin\left(t\sin x + 2kx\right)dx - \int_0^\infty e^{-t\sinh x}\left(e^{2kx}+e^{-2kx}\right)dx. \tag{35}$$

According to the integral representation (21) and the property [7, Eqn. 10.4.1]

$$Y_{-m}\left(t\right) = \left(-1\right)^m Y_m\left(t\right), \tag{36}$$

rewrite (35) as (30) for $m = 2k$. This completes the proof for $m = 2k$.

On the other hand, consider $m = 2k+1$, thus, according to (31), the LHS of (30) becomes

$$\begin{aligned}\left.\frac{\partial J_\nu\left(t\right)}{\partial\nu}\right|_{\nu=2k+1} &- \left.\frac{\partial J_\nu\left(t\right)}{\partial\nu}\right|_{\nu=-2k-1} \\ &= \frac{-2}{\pi}\int_0^\pi x\cos\left(t\sin x\right)\sin\left(\left(2k+1\right)x\right)dx \\ &\quad + \int_0^\infty e^{-t\sinh x}\left(e^{(2k+1)x}-e^{-(2k+1)x}\right)dx.\end{aligned} \tag{37}$$

Following the same steps as in (33)–(35), we calculate the first integral on the RHS of (37) as,

$$\frac{-2}{\pi}\int_0^\pi x\cos\left(t\sin x\right)\sin\left(\left(2k+1\right)x\right)dx = -\int_0^\pi \sin\left(t\sin x + \left(2k+1\right)x\right)dx. \tag{38}$$

Therefore, inserting (38) in (37) and taking into account the integral representation (21) and the property (36), rewrite (38) as

$$\left.\frac{\partial J_\nu\left(t\right)}{\partial\nu}\right|_{\nu=2k+1} - \left.\frac{\partial J_\nu\left(t\right)}{\partial\nu}\right|_{\nu=-2k-1} = \pi\, Y_{2k+1}\left(t\right),$$

which completes the proof. □

Corollary 7. *We can extend the formula given in* (2) *to negative integral orders with the aid of the reflection formula* (30)*, resulting in*

$$\left.\frac{\partial J_\nu(t)}{\partial \nu}\right|_{\nu=\pm m} = (\pm 1)^m \left[\frac{\pi}{2} Y_m(t) \pm \frac{m!}{2} \sum_{k=0}^{m-1} \frac{J_k(t)}{k!\,(m-k)} \left(\frac{t}{2}\right)^{k-m}\right]. \tag{39}$$

Theorem 8. $\forall t \in \mathbb{C}$ *and* $m = 0, 1, \ldots$*, the following reflection formula holds true:*

$$\left.\frac{\partial^2 J_\nu(t)}{\partial \nu^2}\right|_{\nu=m} + (-1)^{m+1} \left.\frac{\partial^2 J_\nu(t)}{\partial \nu^2}\right|_{\nu=-m} = 2\pi \left.\frac{\partial Y_\nu(t)}{\partial \nu}\right|_{\nu=m} + \pi^2 J_m(t). \tag{40}$$

Proof. From the integral representation (18) for $n = 2$, we have

$$\begin{aligned}\left.\frac{\partial^2 J_\nu(t)}{\partial \nu^2}\right|_{\nu=\pm m} &= \frac{-1}{\pi} \int_0^\pi x^2 \cos(t \sin x \mp mx)\, dx \\ &\quad + 2(-1)^m \int_0^\infty e^{-t \sinh x} x\, e^{\mp mx} dx.\end{aligned}$$

Therefore,

$$\begin{aligned}&\left.\frac{\partial^2 J_\nu(t)}{\partial \nu^2}\right|_{\nu=m} + (-1)^{m+1} \left.\frac{\partial^2 J_\nu(t)}{\partial \nu^2}\right|_{\nu=-m} \\ &\quad = \frac{-1}{\pi} \int_0^\pi x^2 \left[\cos(t \sin x - mx) + (-1)^{m+1} \cos(t \sin x + mx)\right] dx \\ &\qquad + 2 \int_0^\infty e^{-t \sinh x} x \left[(-1)^m e^{-mx} - e^{mx}\right] dx.\end{aligned} \tag{41}$$

On the one hand, calculate the first integral on the RHS of (41) for $m = 2k$,

$$\begin{aligned}&\frac{1}{\pi} \int_0^\pi x^2 \left[\cos(t \sin x + 2kx) - \cos(t \sin x - 2kx)\right] dx \\ &\quad = \frac{-2}{\pi} \int_0^\pi x^2 \sin(t \sin x) \sin 2kx, dx.\end{aligned}$$

Performing the change of variables $\xi = x - \pi/2$ and cancelling the corresponding terms by parity, we arrive at

$$2(-1)^k \int_{-\pi/2}^{\pi/2} \xi \sin(t \cos \xi) \sin(-2k\xi)\, d\xi, \tag{42}$$

Applying the trigonometric identity $\cos(\alpha + \beta) = \cos\alpha \cos\beta - \sin\alpha \sin\beta$, eliminating one of the resulting integrals by parity, and undoing the substitution performed, we rewrite (42) as

$$
\begin{aligned}
&-2\int_0^{\pi}\left(x-\frac{\pi}{2}\right)\cos\left(t\sin x-2kx\right)dx \\
&\qquad = -2\int_0^{\pi} x\cos\left(t\sin x-2kx\right)dx+\pi^2 J_{2k}\left(t\right),
\end{aligned}
\tag{43}
$$

where we have applied the integral representation (19).

On the other hand, calculate the first integral on the RHS of (41) for $m=2k+1$,

$$
\begin{aligned}
&\frac{-1}{\pi}\int_0^{\pi} x^2\left[\cos\left(t\sin x-\left(2k+1\right)x\right)+\cos\left(t\sin x+\left(2k+1\right)x\right)\right]dx \\
&\qquad = \frac{-2}{\pi}\int_0^{\pi} x^2\cos\left(t\sin x\right)\cos\left(\left(2k+1\right)x\right)dx.
\end{aligned}
$$

Following the same steps as in (42)–(43), but applying the trigonometric identity $\sin\left(\alpha+\beta\right)=\sin\alpha\cos\beta+\sin\beta\cos\alpha$, we rewrite the first integral on the LHS of (41) as

$$
-2\int_0^{\pi} x\cos\left(t\sin x-\left(2k+1\right)x\right)dx+\pi^2 J_{2k+1}\left(t\right). \tag{44}
$$

From (43) and (44), Eq. (41) becomes

$$
\begin{aligned}
&\left.\frac{\partial^2 J_\nu\left(t\right)}{\partial\nu^2}\right|_{\nu=m}+\left(-1\right)^{m+1}\left.\frac{\partial^2 J_\nu\left(t\right)}{\partial\nu^2}\right|_{\nu=-m} \\
&\qquad = -2\int_0^{\pi} x\cos\left(t\sin x-mx\right)dx+\pi^2 J_m\left(t\right) \\
&\qquad\quad +2\int_0^{\infty} e^{-t\sinh x}x\left[\left(-1\right)^m e^{-mx}-e^{mx}\right]dx.
\end{aligned}
\tag{45}
$$

Finally, take into account the integral representation (20) for $n=1$, to express (45) as (40), as we wanted to prove. □

Corollary 9. *Taking into account* (3), *Eq.* (40) *becomes*

$$
\left.\frac{\partial^2 J_\nu\left(t\right)}{\partial\nu^2}\right|_{\nu=m}+\left(-1\right)^{m+1}\left.\frac{\partial^2 J_\nu\left(t\right)}{\partial\nu^2}\right|_{\nu=-m}=\pi\, m!\sum_{k=0}^{m-1}\frac{Y_k\left(t\right)}{k!\left(m-k\right)}\left(\frac{t}{2}\right)^{k-m}. \tag{46}
$$

Similar derivations to those obtained above in Theorems 6 and 8, and Corollaries 7 and 9, can be obtained for the Bessel function of the second kind. Next, we present these results, omitting details.

Theorem 10. $\forall t\in\mathbb{C}$ *and* $m=0,1,\ldots$, *the following reflection formula holds true:*

$$
\left.\frac{\partial Y_\nu\left(t\right)}{\partial\nu}\right|_{\nu=m}+\left(-1\right)^{m}\left.\frac{\partial Y_\nu\left(t\right)}{\partial\nu}\right|_{\nu=-m}=-\pi\, J_m\left(t\right). \tag{47}
$$

Corollary 11. *We extend the formula given in (3) to negative integral orders with the aid of the reflection formula (47), resulting in*

$$\left.\frac{\partial Y_{\nu}(t)}{\partial \nu}\right|_{\nu=\pm m} = (\pm 1)^{m}\left[-\frac{\pi}{2}J_{m}(t) \pm \frac{m!}{2}\sum_{k=0}^{m-1}\frac{Y_{k}(t)}{k!\,(m-k)}\left(\frac{t}{2}\right)^{k-m}\right]. \quad (48)$$

Theorem 12. *$\forall t \in \mathbb{C}$ and $m = 0, 1, \ldots$, the following reflection formula holds true:*

$$\left.\frac{\partial^{2} Y_{\nu}(t)}{\partial \nu^{2}}\right|_{\nu=m} + (-1)^{m+1}\left.\frac{\partial^{2} Y_{\nu}(t)}{\partial \nu^{2}}\right|_{\nu=-m} = -2\pi\left.\frac{\partial J_{\nu}(t)}{\partial \nu}\right|_{\nu=m} + \pi^{2} Y_{m}(t). \quad (49)$$

Corollary 13. *Taking into account (2), Eq. (49) becomes*

$$\left.\frac{\partial^{2} Y_{\nu}(t)}{\partial \nu^{2}}\right|_{\nu=m} + (-1)^{m+1}\left.\frac{\partial^{2} Y_{\nu}(t)}{\partial \nu^{2}}\right|_{\nu=-m} = -\pi\, m!\sum_{k=0}^{m-1}\frac{J_{k}(t)}{k!\,(m-k)}\left(\frac{t}{2}\right)^{k-m}. \quad (50)$$

Remark 1. We can prove Theorems 6, 8, 10 and 12 in an alternatively way, considering the following function proportional to the first Hankel function and defined as:

$$\begin{aligned} F_{\nu}(t) = {} & \frac{1}{\pi}\int_{0}^{\pi} e^{i(t\sin x-\nu x)}dx - \frac{ie^{-i\nu\pi}}{\pi}\int_{0}^{\infty} e^{-i\sinh x-\nu x}dx \\ & -\frac{i}{\pi}\int_{0}^{\infty} e^{-i\sinh x+\nu x}dx, \end{aligned}$$

since

$$\Re\left(F_{\nu}(t)\right) = J_{\nu}(t), \quad \Im\left(F_{\nu}(t)\right) = Y_{\nu}(t).$$

3.2. Modified Bessel functions

Theorem 14. *$\forall t \in \mathbb{C}$ and $\mu \in \mathbb{R}$, the following reflection formula holds true:*

$$\begin{aligned} & \left.\frac{\partial^{n} I_{\nu}(t)}{\partial \nu^{n}}\right|_{\nu=\mu} + (-1)^{n+1}\left.\frac{\partial^{n} I_{\nu}(t)}{\partial \nu^{n}}\right|_{\nu=-\mu} \\ & \quad = \frac{-2}{\pi}\sin\pi\mu\sum_{k=0}^{\lfloor n/2\rfloor}\binom{n}{2k}\left(-\pi^{2}\right)^{k}\frac{\partial^{n-2k}K_{\mu}(t)}{\partial\mu^{n-2k}} \\ & \quad\quad - 2\cos\pi\mu\sum_{k=0}^{\lfloor (n-1)/2\rfloor}\binom{n}{2k+1}\left(-\pi^{2}\right)^{k}\frac{\partial^{n-2k-1}K_{\mu}(t)}{\partial\mu^{n-2k-1}}. \end{aligned} \quad (51)$$

Proof. Applying the integral representation (22), we arrive at

$$\left.\frac{\partial^n I_\nu(t)}{\partial\nu^n}\right|_{\nu=\mu} + (-1)^{n+1} \left.\frac{\partial^n I_\nu(t)}{\partial\nu^n}\right|_{\nu=-\mu}$$
$$= -\frac{1}{\pi}\int_0^\infty e^{-t\cosh x}\Big\{p_n(x)\sin\pi\mu\left[e^{-\mu x}+(-1)^n e^{\mu x}\right]$$
$$+ q_n(x)\sin\pi\mu\left[e^{-\mu x}+(-1)^{n+1}e^{\mu x}\right]\Big\}dx,$$

wherein the first integral of (22) vanishes. Taking into account the definitions of the polynomials p_n and q_n given in (16) and (17),

$$\left.\frac{\partial^n I_\nu(t)}{\partial\nu^n}\right|_{\nu=\mu} + (-1)^{n+1} \left.\frac{\partial^n I_\nu(t)}{\partial\nu^n}\right|_{\nu=-\mu}$$
$$= \frac{-1}{\pi}\sin\pi\mu\sum_{k=0}^{\lfloor n/2\rfloor}\binom{n}{2k}\left(-\pi^2\right)^k$$
$$\int_0^\infty e^{-t\cosh x}x^{n-2k}\left[e^{-\mu x}(-1)^n+e^{\mu x}\right]dx$$
$$-\cos\pi\mu\sum_{k=0}^{\lfloor(n-1)/2\rfloor}\binom{n}{2k+1}\left(-\pi^2\right)^k$$
$$\int_0^\infty e^{-t\cosh x}x^{n-2k-1}\left[e^{-\mu x}(-1)^{n+1}+e^{\mu x}\right]dx. \tag{52}$$

Finally, taking into account the integral representation (24), rewrite (52) as (51), as we wanted to prove. □

From the general expression (51), we obtain some interesting particular cases. For non-negative integral orders $\mu = m = 0, 1, \ldots$, Eq. (51) is reduced to

$$\left.\frac{\partial^n I_\nu(t)}{\partial\nu^n}\right|_{\nu=m} + (-1)^{n+1} \left.\frac{\partial^n I_\nu(t)}{\partial\nu^n}\right|_{\nu=-m}$$
$$= 2(-1)^{m+1}\sum_{k=0}^{\lfloor(n-1)/2\rfloor}\binom{n}{2k+1}\left(-\pi^2\right)^k\left.\frac{\partial^{n-2k-1}K_\nu(t)}{\partial\nu^{n-2k-1}}\right|_{\nu=m}. \tag{53}$$

Taking $n = 0$ in (53), we obtain the well-known expression given in the literature [7, Eqn. 10.27.1],

$$I_m(t) - I_{-m}(t) = 0. \tag{54}$$

Taking $n = 1, 2$ in (53), we obtain respectively:

$$\left.\frac{\partial I_\nu(t)}{\partial\nu}\right|_{\nu=m} + \left.\frac{\partial I_\nu(t)}{\partial\nu}\right|_{\nu=-m} = 2(-1)^{m+1}K_m(t), \tag{55}$$

and

$$\left.\frac{\partial^2 I_\nu(t)}{\partial\nu^2}\right|_{\nu=m} - \left.\frac{\partial^2 I_\nu(t)}{\partial\nu^2}\right|_{\nu=-m} = 4(-1)^{m+1}\left.\frac{\partial K_\nu(t)}{\partial\nu}\right|_{\nu=m}. \tag{56}$$

The above results (55) and (56) allow us to state the following Corollaries.

Corollary 15. *We extend the formula given in* (4) *to negative integral orders with the aid of the reflection formula* (55)*, resulting in*

$$\left.\frac{\partial I_\nu(t)}{\partial\nu}\right|_{\nu=\pm m} = (-1)^m\left[-K_m(t) \pm \frac{m!}{2}\sum_{k=0}^{m-1}\frac{(-1)^k I_k(t)}{k!\,(m-k)}\left(\frac{t}{2}\right)^{k-m}\right]. \tag{57}$$

Corollary 16. *Taking into account* (5)*, Eq. (56) becomes*

$$\left.\frac{\partial^2 I_\nu(t)}{\partial\nu^2}\right|_{\nu=m} - \left.\frac{\partial^2 I_\nu(t)}{\partial\nu^2}\right|_{\nu=-m} = 2(-1)^{m+1} m!\sum_{k=0}^{m-1}\frac{K_k(t)}{k!\,(m-k)}\left(\frac{t}{2}\right)^{k-m}. \tag{58}$$

For positive half-integral orders, i.e. $\mu = m + \frac{1}{2}$ with $\mu = m = 0, 1, \ldots$ in (51), we have

$$\left.\frac{\partial^n I_\nu(t)}{\partial\nu^n}\right|_{\nu=m+1/2} + (-1)^{n+1}\left.\frac{\partial^n I_\nu(t)}{\partial\nu^n}\right|_{\nu=-m-1/2} \tag{59}$$

$$= \frac{2}{\pi}(-1)^{m+1}\sum_{k=0}^{\lfloor n/2\rfloor}\binom{n}{2k}\left(-\pi^2\right)^k\left.\frac{\partial^{n-2k} K_\nu(t)}{\partial\nu^{n-2k}}\right|_{\nu=m+1/2}. \tag{60}$$

Taking $n = 0, 1$ in (59), we have respectively

$$I_{m+1/2}(t) - I_{-m-1/2}(t) = \frac{2}{\pi}(-1)^{m+1} K_{m+1/2}(t),$$

which is given in the literature [7, Eqn. 10.47.11], and

$$\left.\frac{\partial I_\nu(t)}{\partial\nu}\right|_{\nu=m+1/2} + \left.\frac{\partial I_\nu(t)}{\partial\nu}\right|_{\nu=-m-1/2} = \frac{2}{\pi}(-1)^{m+1}\left.\frac{\partial K_\nu(t)}{\partial\nu}\right|_{\nu=m+1/2}, \tag{61}$$

which is fulfilled for $m = 0$ by the formulas found in [7, Eqn. 10.38.6&7].

Theorem 17. $\forall t \in \mathbb{C}$ *and* $m = 0, 1, \ldots,$ *the following reflection formula holds true:*

$$\left.\frac{\partial^n K_\nu(t)}{\partial\nu^n}\right|_{\nu=\mu} + (-1)^{n+1}\left.\frac{\partial^n K_\nu(t)}{\partial\nu^n}\right|_{\nu=-\mu} = 0. \tag{62}$$

Proof. First, consider the case $n = 2k$ and take into account the integral representation (11), thereby (62) reads as

$$\left.\frac{\partial^{2k} K_\nu(t)}{\partial\nu^{2k}}\right|_{\nu=\mu} - \left.\frac{\partial^{2k} K_\nu(t)}{\partial\nu^{2k}}\right|_{\nu=-\mu} = \int_{-\infty}^{\infty} x^{2k} e^{-t\cosh x}\sinh(\mu x)\,dx = 0,$$

which vanishes by parity. Second, consider the case $n = 2k+1$ and the integral representation (11), thus (62) becomes

$$\left.\frac{\partial^{2k+1} K_\nu(t)}{\partial \nu^{2k+1}}\right|_{\nu=\mu} + \left.\frac{\partial^{2k+1} K_\nu(t)}{\partial \nu^{2k+1}}\right|_{\nu=-\mu} = \int_{-\infty}^{\infty} x^{2k+1} e^{-t\cosh x} \cosh(\mu x)\, dx = 0,$$

which is also null by parity. This completes the proof. □

Corollary 18. *We extend the formula given in* (5) *to negative integral orders, taking* $n = 1$ *and* $\mu = m = 0, 1, \ldots$ *in* (62)*, resulting in*

$$\left.\frac{\partial K_\nu(t)}{\partial \nu}\right|_{\nu=\pm m} = \pm\frac{m!}{2}\sum_{k=0}^{m-1} \frac{K_k(t)}{k!\,(m-k)} \left(\frac{t}{2}\right)^{k-m}. \tag{63}$$

Corollary 19. *From* (25) *with* $n = 1$ *and* (63)*, we obtain,* $\forall m = 0, 1, \ldots$

$$\int_{-\infty}^{\infty} x\, e^{\pm m x - t\cosh x} dx = \pm m! \sum_{k=0}^{m-1} \frac{K_k(t)}{k!\,(m-k)} \left(\frac{t}{2}\right)^{k-m}. \tag{64}$$

4. Conclusions

On the one hand, we have obtained new integral expressions for the n-th derivatives of the Bessel functions with respect to the order in (18), (20), (22), and (25). As a by-product, we have calculated an integral, which does not seem to be reported in the literature, in three forms, i.e. (28), (29), and (64), depending on the parameter ν.

On the other hand, we have derived reflection formulas for the first and second order derivative of $J_\nu(t)$ in (30) and (46), and of $Y_\nu(t)$ in (47) and (50). Also, for arbitrary order ν, we have derived the reflection formula (51) for the n-th order derivative of $I_\nu(t)$, and the reflection formula (62) for the n-th order derivative of $K_\nu(t)$ in. As particular cases, for $I_\nu(t)$ and integral order m, we have obtained the reflection formula (55) for the first order derivative, and (58) for the second order derivative. Also, for half-integral order, we have obtained formula (61).

Finally, it is worth noting that the reflection formulas of the modified Bessel function given in (55) and (56) are obtained from the general formula (51). This is not the case of the reflection formulas given in (30), (40), (47) and (49) for the Bessel functions of first and second kinds, where we do not have a general formula as in the case of modified Bessel function. Moreover, in the case of the Macdonald function, we have obtained a reflection formula for arbitrary order in (62). This situation shows the different behavior of the order derivatives of Bessel functions with respect to order derivatives of the modified Bessel functions.

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