On convergence in distribution of fuzzy random variables

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Abstract We study whether convergence in distribution of fuzzy random variables, defined as the weak convergence of their probability distributions, is consistent with the additional structure of spaces of fuzzy sets. Positive results are obtained which reinforce the viability of that definition.

1 Introduction

Convergence in distribution is one of the most useful notions of convergence for random variables, notably because of its role in the central limit theorem. It is typically defined as follows: $\xi_n \to \xi$ in distribution when $F_{\xi_n}(x) \to F_\xi(x)$ (where F_{ξ_n} and F_ξ are the respective cumulative distribution functions) for each point of continuity x of F_ξ.

Throughout the years, a number of proposals trying to extend the notion of a cumulative distribution function to fuzzy random variables have been made. Without judging their usefulness for specific problems, it is fair to say (Terán, 2012) that they fail to have the theoretical properties that make the cumulative distribution function important in the case of random variables and random vectors. Specifically, they do not determine the probability distribution of the fuzzy random variable. Thus they are not useful to study convergence in distribution of fuzzy random variables.

Since a fuzzy random variable can be equivalently described (Krätschmer, 2001) as a random element of a metric space of fuzzy sets (endowed with any of the d_p-metrics), it is possible to study probability distributions of fuzzy

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random variables using the general theory of probability distributions in metric spaces (e.g., Billingsley (1968)). This approach was taken by the authors in recent papers (Alonso de la Fuente and Terán (2022), Alonso de la Fuente and Terán (202x)). In particular, it provides a way to define convergence in distribution as being tantamount to *weak convergence* of the probability distributions (since the Helly–Bray theorem and its converse establish the equivalence of those two notions for ordinary random variables).

The theoretical properties of weak convergence are well understood (see Billingsley (1968)). But spaces of fuzzy sets have more structure than a generic metric space, which raises the question whether defining convergence in distribution via weak convergence works well with that additional structure.

In this contribution, we show that convergence in distribution of fuzzy random variables with convex values can be studied using the support function embedding into an L^p-type space (i.e., convergence in distribution of the fuzzy random variables and of their support functions are equivalent). We also show that this type of convergence is consistent with some known structures in the space of fuzzy sets. A sequence of trapezoidal fuzzy random variables converges in distribution if and only if the vertices of the trapezoid converge jointly as a 4-dimensional random vector. A sequence of random vectors converges in distribution if and only if their indicator functions converge as fuzzy random variables. Finally, we show a consistency result between convergence and the sum and product by scalars which parallels the corresponding property of ordinary random variables.

2 Preliminaries

Let $\mathcal{F}_c(\mathbb{R}^d)$ be the space of fuzzy sets $U : \mathbb{R}^d \to [0,1]$ whose α-cuts U_α are non-empty compact convex subsets of $\mathbb{R}^d$. Every fuzzy set $U \in \mathcal{F}_c(\mathbb{R}^d)$ is uniquely determined by its support function

$$\begin{aligned} s_U : [0,1] \times \mathbb{S}^{d-1} &\to \mathbb{R} \\ (r,\alpha) &\mapsto s_U(r,\alpha) = \sup_{x \in U_\alpha} \langle r, x \rangle \end{aligned}$$

where $\mathbb{S}^{d-1}$ denotes the unit sphere in $\mathbb{R}^d$.

For each $p \in [1,\infty)$, the metric d_p in $\mathcal{F}_c(\mathbb{R}^d)$, introduced by Klement et al. (1986) and Puri and Ralescu (1986), is defined by

$$d_p(U,V) = \left[\int_{[0,1]} (d_H(U_\alpha, V_\alpha))^p \, d\alpha \right]^{1/p}.$$

The metric ρ_p is defined by

$$\rho_p(U,V) = \left[\int_{[0,1]}\int_{S^{d-1}} |s_U(r,\alpha) - s_V(r,\alpha)|^p dr d\alpha\right]^{1/p}.$$

Denote by $\mathcal{K}_c(\mathbb{R}^d)$ the space of non-empty compact convex subsets of $\mathbb{R}^d$. Given a probability space $(\Omega, \mathcal{A}, P)$, a mapping $X : \Omega \to \mathcal{K}_c(\mathbb{R}^d)$ is called *random set* (also a *random compact convex set* in the literature) if X is measurable with respect to the Borel σ-algebra $\mathcal{B}_{\mathcal{K}_c(\mathbb{R}^d)}$ generated by the topology of the Hausdorff metric.

Definition 1 A mapping $X : (\Omega, \mathcal{A}, P) \to \mathcal{F}_c(\mathbb{R}^p)$ is called a *fuzzy random variable* if $X_\alpha : \omega \mapsto X(\omega)_\alpha$ is a random compact set for each $\alpha \in [0,1]$.

Denote by σ_L the smallest σ-algebra that makes the mappings $U \in \mathcal{F}_c(\mathbb{R}^d) \mapsto U_\alpha \in \mathcal{K}_c(\mathbb{R}^d)$ measurable. Thus a fuzzy random variable is the same thing as a $(\mathcal{A}, \sigma_L)$-measurable mapping. A sequence of probability measures $\{P_n\}_n$ on σ_L is said to *converge weakly in d_p* to a probability measure P if

$$\int f dP_n \to \int f dP$$

for every $f : \mathcal{F}_c(\mathbb{R}^d) \to \mathbb{R}$ which is d_p-continuous and bounded. A sequence $\{X_n\}_n$ of fuzzy random variables converges *weakly* or *in distribution in d_p* to a fuzzy random variable X if their distributions P_{X_n} converge weakly to P_X, namely

$$E[f(X_n)] \to E[f(X)]$$

for each bounded d_p-continuous function $f : \mathcal{F}_c(\mathbb{R}^d) \to \mathbb{R}$.

The Lebesgue measure in $[0,1]$ will be denoted by ℓ. The following results will be used in the sequel.

Lemma 1 *(Billingsley, 1968, Theorem 2.1) Let $\mathbb{E}$ be a metric space, P a probability measure and $\{P_n\}_n$ a sequence of probabilities in $(\mathbb{E}, \mathcal{B}_{\mathbb{E}})$. Then $P_n \to P$ weakly if and only if for every open set G we have* $\liminf_{n\to\infty} P_n(G) \geq P(G)$.

Lemma 2 *(Alonso de la Fuente and Terán, 2022, Theorem 3.5) Let $p \in [1, \infty)$. Let P_n, P be probability measures on σ_L, such that $P_n \to P$ weakly. Then there exist fuzzy random variables $X_n, X : ([0,1], \mathcal{B}_{[0,1]}, \mathbb{P}) \to (\mathcal{F}_c(\mathbb{R}^d), d_p)$, such that*

(a)The distributions of X_n and X are P_n and P, respectively.
(b)$X_n(t) \to X(t)$ in d_p for every $t \in [0,1]$.

Lemma 3 *(Alonso de la Fuente and Terán, 2022, Theorem 5.1) Let X_n and X be fuzzy random variables such that $X_n \to X$ in distribution in d_p. If $f : \mathcal{F}_c(\mathbb{R}^d) \to \mathcal{F}_c(\mathbb{R}^d)$ is a P_X-almost surely continuous function, then $f(X_n) \to f(X)$ weakly in d_p.*

Lemma 4 *(Parthasarathy, 1967, Corollary 3.3, p. 22) If $\mathbb{E}$ is a Borel subset of a complete separable metric space X and φ is a one-one measurable map of $\mathbb{E}$ into a separable metric space Y, then $\varphi(\mathbb{E})$ is a Borel subset of Y, $\mathbb{E}$ and $\varphi(\mathbb{E})$ are isomorphic as measurable spaces and φ is an isomorphism.*

Recall that a trapezoidal fuzzy number $Tra(a,b,c,d)$ has the following expression:

$$U(x) = \begin{cases} 0 & \text{if } x < a \\ \frac{x-a}{b-a} & \text{if } a \le x < b \\ 1 & \text{if } b \le x \le c \\ \frac{d-x}{d-c} & \text{if } c < x \le d \\ 0 & \text{if } x > d \end{cases}$$

We will denote the space of trapezoidal fuzzy numbers by $\mathcal{F}_c^{tra}(\mathbb{R})$.

3 Main results

Our first result states that d_p-convergence in distribution of fuzzy random variables is equivalent with the convergence obtained by embedding them into an L^p-type space. Note that this is not an immediate consequence of the embedding.

Theorem 1 *Let $p \in [1,\infty)$. Let X_n, X be fuzzy random variables. Then the following conditions are equivalent.*

1. $X_n \to X$ in distribution in $(\mathcal{F}_c(\mathbb{R}^d), d_p)$.
2. $s_{X_n} \to s_X$ in distribution in $L^p(\mathbb{S}^{d-1} \times [0,1], \lambda \otimes \ell)$,

where λ denotes the uniform measure in $\mathbb{S}^{d-1}$.

Proof Denote by φ the mapping given by

$$\varphi : (\mathcal{F}_c(\mathbb{R}^d), \rho_p) \to L^p(\mathbb{S}^{d-1} \times [0,1], \lambda \otimes \ell)$$
$$U \mapsto s_U.$$

By, e.g., (Krätschmer, 2006, p. 444), φ is an isometry.

Let $Y_n, Y : ([0,1], \mathcal{B}_{[0,1]}, \ell) \to (\mathcal{F}_c(\mathbb{R}^d), d_p)$ be the fuzzy random variables given by Lemma 2. We have $Y_n(t) \to Y(t)$ in d_p for all $t \in [0,1]$, $P_{X_n} = \ell_{Y_n}$ and $P_X = \ell_Y$. Since d_p and ρ_p are topologically equivalent (Diamond and Kloeden, 1994, p. 65, Proposition 7.4.5), $\rho_p(Y_n(t), Y(t)) \to 0$ for every $t \in [0,1]$.

Set $s_{Y_n} = \varphi \circ Y_n$ and $s_Y = \varphi \circ Y$. Since φ is an isometry, we have $s_{Y_n}(t) \to s_Y(t)$. By (Alonso de la Fuente and Terán, 2022, Proposition 5.4), $(\mathcal{F}_c(\mathbb{R}^d), d_p)$ is a Lusin space, hence it is Borel measurable in every metric space it embeds into (see (Frolík, 1970, Proposition 7.11)). There follows that φ is Borel measurable and thus s_{Y_n} and s_Y are random elements of $L^p(\mathbb{S}^{d-1} \times [0,1], \lambda \otimes \ell)$.

We need to check $\ell_{s_{Y_n}} = P_{s_{X_n}}$. For any measurable subset A of $L^p(\mathbb{S}^{d-1} \times [0,1], \lambda \otimes \ell)$,

$$P_{s_{X_n}}(A) = P(\{\omega \in \Omega : s_{X_n}(\omega) \in A\}) = P(\{\omega \in \Omega : (\varphi \circ X_n)(\omega) \in A\})$$

$$= P(\{\omega \in \Omega : X_n(\omega) \in \varphi^{-1}(A)\}) = \ell(\{t \in [0,1] : Y_n(t) \in \varphi^{-1}(A)\})$$

$$= \ell(\{t \in [0,1] : (\varphi \circ Y_n)(t) \in A\}) = \ell(\{t \in [0,1] : s_{Y_n}(t) \in A\}) = \ell_{s_{Y_n}}(A).$$

Analogously, $\ell_{s_Y} = P_{s_X}$. Since $s_{Y_n} \to s_Y$ almost surely, by (Kallenberg, 2002, Lemma 4.2) almost sure convergence implies convergence in probability and by (Kallenberg, 2002, Lemma 4.7) convergence in probability implies weak convergence $\ell_{s_{Y_n}} \to \ell_{s_Y}$. In conclusion, $P_{s_{X_n}} \to P_{s_X}$ weakly, that is, $s_{X_n} \to s_X$ in distribution.

For the converse, notice that $P_{X_n} = P_{s_{X_n}} \circ \varphi$ and $P_X = P_{s_X} \circ \varphi$. Since φ is an isometry, for any open set G of $\mathcal{F}_c(\mathbb{R}^d)$ there exists an open set $\mathbf{G}$ of $L^p(\mathbb{S}^{d-1} \times [0,1], \lambda \otimes \ell)$ such that $\varphi(G) = \mathbf{G} \cap \varphi(\mathcal{F}_c(\mathbb{R}^d))$. Then

$$\liminf_{n\to\infty} P_{s_{X_n}} \circ \varphi(G) = \liminf_{n\to\infty} P_{s_{X_n}}(\varphi(G)) = \liminf_{n\to\infty} P_{s_{X_n}}(\mathbf{G} \cap \varphi(\mathcal{F}_c(\mathbb{R}^d)))$$

$$= \liminf_{n\to\infty} P_{s_{X_n}}(\mathbf{G}) \geq P_{s_X}(\mathbf{G}) = P_{s_X}(\mathbf{G} \cap \varphi(\mathcal{F}_c(\mathbb{R}^d))) = P_{s_X}(\varphi(G)) = P_{s_X} \circ \varphi(G)$$

by Lemma 1 and knowing that s_{X_n} and s_X take on values in $\varphi(\mathcal{F}_c(\mathbb{R}^d))$. Again by Lemma 1, $\liminf_{n\to\infty} P_{X_n} \circ \varphi(G) \geq P_X \circ \varphi(G)$ yields $X_n \to X$ in distribution in d_p. □

Therefore this type of convergence can indeed be studied using support functions. Another question concerns the relationship between convergence of fuzzy random variables taking on values in parametric families of fuzzy sets (in this case, trapezoidal fuzzy sets but the study could be extended to other families) and convergence in distribution of their defining parameters. The content of the following lemma is intuitively clear although its proof is not trivial. For space reasons we skip the proof, which may appear elsewhere.

Lemma 5 *Let $p \in [1,\infty)$. Let $U_n, U \in \mathcal{F}_c^{tra}(\mathbb{R})$. If $U_n \to U$ in d_p, then the sequence $\{\|(U_n)_0\|\}_n$ is bounded.*

Theorem 2 *Let $p \in [1,\infty)$. Let X_n be $Tra(X_{n,1}, X_{n,2}, X_{n,3}, X_{n,4})$ where $X_{n,1} \leq X_{n,2} \leq X_{n,3} \leq X_{n,4}$ are random variables, and analogously $X = Tra(X_1, X_2, X_3, X_4)$. Then $X_n \to X$ in distribution in d_p if and only if, as random vectors in $\mathbb{R}^4$, $(X_{n,1}, X_{n,2}, X_{n,3}, X_{n,4}) \to (X_1, X_2, X_3, X_4)$ in distribution.*

Proof Set $A = \{(u_1, u_2, u_3, u_4) \in \mathbb{R}^4 : u_1 \leq u_2 \leq u_3 \leq u_4\}$. The mapping

$$\varphi : A \to (\mathcal{F}_c(\mathbb{R}), d_p)$$
$$(u_1, u_2, u_3, u_4) \mapsto Tra(u_1, u_2, u_3, u_4)$$

is injective. Let us show that φ is continuous. Let $(u_{n,1}, u_{n,2}, u_{n,3}, u_{n,4}) \to (u_1, u_2, u_3, u_4)$ in $\mathbb{R}^4$. Denote by U_n the fuzzy set $Tra(u_{n,1}, u_{n,2}, u_{n,3}, u_{n,4})$ and by U the fuzzy set $Tra(u_1, u_2, u_3, u_4)$.

Now let $\alpha \in [0,1]$,

$$d_H(U_{n_\alpha}, U_\alpha) = \max\{|\inf(U_n)_\alpha - \inf U_\alpha|, |\sup(U_n)_\alpha - \sup U_\alpha|\}$$

$$= \max\{|(1-\alpha)\inf(U_n)_0 + \alpha \inf(U_n)_1 - (1-\alpha)\inf U_0 - \alpha \inf U_1|,$$

$$|\alpha \sup(U_n)_1 + (1-\alpha)\sup(U_n)_0 - \alpha \sup U_1 - (1-\alpha)\sup U_0|\}$$

$$= \max\{|(1-\alpha)(u_{n,1} - u_1) + \alpha(u_{n,2} - u_2)|, |\alpha(u_{n,3} - u_3) + (1-\alpha)(u_{n,4} - u_4)|\}$$

$$\leq \max\{|u_{n,1} - u_1|, |u_{n,2} - u_2|, |u_{n,3} - u_3|, |u_{n,4} - u_4|\}.$$

Since the last term is the max distance between both vectors in $\mathbb{R}^4$ and is independent of α, indeed it bounds $d_p(U_n, U)$, making φ be d_p-continuous.

We will establish now two further facts which will be used in the proof. Firstly, since A is closed in $\mathbb{R}^4$, it is complete and separable. Moreover $(\mathcal{F}_c(\mathbb{R}), d_p)$ is separable, hence by Lemma 4 the image $\varphi(A) = \mathcal{F}_c^{tra}(\mathbb{R})$ is Borel measurable.

Secondly, set

$$A_{a,b} = \{(u_1, u_2, u_3, u_4) \in \mathbb{R}^4 : a \leq u_1 \leq u_2 \leq u_3 \leq u_4 \leq b\} \subseteq A$$

for each $a, b \in \mathbb{R}$. Since $A_{a,b}$ is compact, φ is continuous and $\varphi(A) = \mathcal{F}_c^{tra}(\mathbb{R})$ is a Hausdorff space, the restriction $\varphi|_{A_{a,b}}$ is a homeomorphism (Joshi, 1983, Corollary 2.4, p. 169).

($\Rightarrow$) By Lemma 2, there exist fuzzy random variables Y_n, Y such that $Y_n(t) \to Y(t)$ in d_p for each $t \in [0,1]$, $\ell_{Y_n} = P_{X_n}$ and $\ell_Y = P_X$. Since

$$\ell_{Y_n}(\mathcal{F}_c^{tra}(\mathbb{R})) = P_{X_n}(\mathcal{F}_c^{tra}(\mathbb{R})) = 1,$$

Y_n and Y are almost surely trapezoidal fuzzy sets. For clarity, we assume without loss of generality that all $Y_n(t), Y(t)$ are trapezoidal fuzzy sets (otherwise it would suffice to modify the value of those variables in a null set, which would not change their probability distributions). Set $Tra(Y_{n,1}, Y_{n,2}, Y_{n,3}, Y_{n,4}) = Y_n$, $Tra(Y_1, Y_2, Y_3, Y_4) = Y$ and let us show that $(Y_{n,1}, Y_{n,2}, Y_{n,3}, Y_{n,4})$ converges in distribution to (Y_1, Y_2, Y_3, Y_4).

By Lemma 5, each sequence $\{\|(Y_n)_0(t)\|\}_n$ is bounded by some constant M_t. Therefore $(Y_{n,1}(t), Y_{n,2}(t), Y_{n,3}(t), Y_{n,4}(t)) \in A_{-M_t, M_t}$ for all $t \in [0,1]$. By the homeomorphism between A_{-M_t, M_t} and $\varphi(A_{-M_t, M_t})$, the 4-dimensional vector converges to $(Y_1(t), Y_2(t), Y_3(t), Y_4(t))$. Almost sure convergence of those vectors implies their convergence in distribution. To finish the proof, we just need to check $\ell_{(Y_{1_n}, \ldots, Y_{4_n})} = P_{(X_{1_n}, \ldots, X_{4_n})}$. For any Borel subset $B \subseteq \mathbb{R}^4$,

$$\ell_{Y_{1,n}, \ldots, Y_{4,n}}(B) = \ell(\{t \in [0,1] : (Y_{1,n}, \ldots, Y_{4,n})(t) \in A \cap B\})$$

$$= \ell(\{t \in [0,1] : Y_n(t) \in \varphi(A \cap B)\}) = P(\{\omega \in \Omega : X_n(\omega) \in \varphi(A \cap B)\})$$
$$= P(\{\omega \in \Omega : (X_{1,n}, \ldots, X_{4,n})(\omega) \in A \cap B\}) = P_{X_{1,n},\ldots,X_{4,n}}(B).$$

Analogously, $\ell_{Y_1,\ldots,Y_4} = P_{X_1,\ldots,X_4}$.

($\Leftarrow$) By the Skorokhod representation theorem in $\mathbb{R}^4$, there exist random vectors $(Y_{n,1}, Y_{n,2}, Y_{n,3}, Y_{n,4}), (Y_1, Y_2, Y_3, Y_4)$ such that $\ell_{(Y_{n,1},Y_{n,2},Y_{n,3},Y_{n,4})} = P_{(X_{n,1},X_{n,2},X_{n,3},X_{n,4})}$, $\ell_{(Y_1,Y_2,Y_3,Y_4)} = P_{(X_1,X_2,X_3,X_4)}$ and $(Y_{n,1}, Y_{n,2}, Y_{n,3}, Y_{n,4})$ converges to (Y_1, Y_2, Y_3, Y_4) pointwise. Set

$$Y_n = Tra(Y_{n,1}, Y_{n,2}, Y_{n,3}, Y_{n,4}), Y = Tra(Y_1, Y_2, Y_3, Y_4).$$

By the continuity of φ, $Y_n(t) \to Y(t)$ in d_p for each $t \in [0,1]$. By Lemmas 4.2 and 4.7 in Kallenberg (2002), almost sure convergence implies convergence in distribution. Finally, one shows like before $\ell_{Y_n} = P_{X_n}$ and $\ell_Y = P_X$, whence $X_n \to X$ in distribution in d_p. □

Since a random variable ξ can be identified with the trapezoidal fuzzy set $Tra(\xi, \xi, \xi, \xi)$, which is the indicator function $I_{\{\xi\}}$, the following corollary holds.

Corollary 1 *Let ξ_n, ξ be random variables. Then $\xi_n \to \xi$ in distribution if and only if $I_{\{\xi_n\}} \to I_{\{\xi\}}$ in distribution in d_p.*

The following proposition is analogous to an important property of convergence in distribution for random variables. It states that convergence is compatible with the operations in $\mathcal{F}_c(\mathbb{R}^d)$.

Proposition 1 *Let X_n, X be fuzzy random variables such that $X_n \to X$ in distribution in d_p. Then*

1. *For every $U \in \mathcal{F}_c(\mathbb{R}^d)$, we have $X_n + U \to X + U$ in distribution in d_p.*
2. *For every $a \in \mathbb{R}$, we have $aX_n \to aX$ in distribution in d_p.*

Proof Since the mappings $V \in \mathcal{F}_c(\mathbb{R}^d) \mapsto V + U$ and $V \in \mathcal{F}_c(\mathbb{R}^d) \mapsto aV$ are d_p-continuous, we obtain the result with an application of the continuous mapping theorem (Lemma 3). □

Remark 1 It is not true, in general, that $X_n + Y \to X + Y$ in distribution in d_p provided $X_n \to X$ in distribution. That fails even for random variables.

We close the paper by pointing out another parallel with ordinary random variables: if the limit is a degenerate fuzzy random variable U, then $X_n \to U$ in distribution in d_p if and only if $X_n \to U$ in probability in d_p (by an application of (Kallenberg, 2002, Lemma 4.7)).

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