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On a Characterization of the Essential Spectra

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Abstract. In this paper, we devote our research to the essential spectra of linear relations defined on a Banach space. We extend the main results of paper [1] to linear relations.

1. Introduction

We adhered with the notation and terminology of the book [3, 16]. Let $X, Y, Z, ...$ denote vector spaces over $\mathbb{K} = \mathbb{R}$ or $\mathbb{C}$. A multi-valued linear operator or linear relation T from X to Y is a mapping from a subspace $D(T)$ of X, called the domain of T, into the collection of nonempty subsets of Y such that $T(\alpha x_1 + \beta x_2) = \alpha T(x_1) + \beta T(x_2)$ for all nonzero scalars α, β and $x_1, x_2 \in D(T)$. If T maps the point of its domain to singletons, then T is said to be a single valued or simply an operator. We denote the class of linear relation from X to Y be $LR(X, Y)$ and we write $LR(X) = LR(X, X)$. A linear relation $T \in LR(X, Y)$ is uniquely determined by its graph, $G(T)$, which is defined by

$$G(T) = \{(x, y) \in X \times Y : x \in D(T), y \in Tx\},$$

so that we can identify T with $G(T)$. The inverse of T is the linear relation T^{-1} defined by

$$G(T^{-1}) = \{(y, x) \in Y \times X \text{ such that } (x, y) \in G(T)\}.$$

The subspace $T^{-1}(0)$ denoted by $N(T)$ is called the null space of T and T is called injective if $N(T) = \{0\}$. The range of T is the subspace $T(D(T))$ and T is said to be surjective if $R(T) = Y$. We denote $\alpha(T) := \dim N(T)$, $\beta(T) := \dim Y/R(T)$, $\overline{\beta}(T) := \dim Y/\overline{R(T)}$ and the index of T is the quantity $i(T) := \alpha(T) - \beta(T)$ provided $\alpha(T)$ and $\beta(T)$ are not both infinite. Let M be a subspace of X such that $M \cap \mathcal{D}(T) \neq \emptyset$. The restriction of T to M, denoted $T_{|M}$ is defined by

$$G(T_{|M}) := \{(x, y) : \ x \in M, \ y \in Tx\}.$$

For $\lambda \in \mathbb{K}$, $T, \ S \in LR(X, Y)$, the linear relation $\lambda - T$ and $T + S$ are defined by

$$G(\lambda - T) := \{(x, \lambda x - y) : \ (x, y) \in G(T)\}.$$

$$G(T + S) := \{(x, y) : \ y = u + v \text{ with } (x, u) \in G(T), \ (x, v) \in G(S)\}.$$

We say that $S \subset T$ if $G(S) \subset G(T)$.

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Let $T \in LR(X,Y)$ and $S \in LR(Y,Z)$ where $R(T) \cap D(T) \neq \emptyset$. The product of ST is defined by

$$G(ST) := \{(x,z) \in X \times Z : (x,u) \in G(T) \text{ and } (u,z) \in G(S) \text{ for some } u \in Y\}.$$

Assume that X and Y are normed spaces and $T \in LR(X,Y)$. If M is a closed subspace of X then J_M is the inclusion of M into X and Q_M is the quotient map from X onto X/M (see[2]-[15]). We shall denote $Q_{\overline{T(0)}}$ by Q_T. It easy to see that $Q_T T$ is single valued so that we can define $\|Tx\| := \|Q_T Tx\|$; $x \in D(T)$ and $\|T\| := \|Q_T T\|$. We say that T is continuous if $\|T\| < \infty$, bounded if it is continuous with $D(T) = X$, open if T^{-1} is continuous and T is called closed if its graph is a closed subspace.

If M is a subspace of X and N is a subspace of X' where X' is the dual space of X, then

$$M^{\perp} := \{x' \in X' : x'(M) = 0\} \text{ and } N^{\,T} := \{x \in X : N(x) = 0\}.$$

The conjugate of $T \in LR(X,Y)$ is the linear relation T' defined by

$$G(T') := G(-T^{-1})^{\perp} \subset Y' \times X',$$

so that

$$(y',x') \in G(T') \text{ if and only if } y'(y) = x'(x) \text{ for all } (x,y) \in G(T).$$

Let $T \in LR(X,Y)$, we say that T is F_+ if there exists a finite codimensional subspace M of $D(T)$ for which $T_{|M}$ is injective and open $\big($that is, there exists $\alpha > 0$ such that $\alpha\|m\| \leq \|Tm\|,\ m \in M\big)$ and T is called precompact if $Q_T T B_X$ is totally bounded $\big($see [16, p. 134]$\big)$. A closed linear relation T from a Banach space X into a Banach space Y is said upper semi-Fredholm relation which we abbreviate as Φ_+, if T has closed range and $\alpha(T) < \infty$, we denoted by $T \in \Phi_+(X,Y)$. T is called lower semi-Fredholm relation which we abbreviate as Φ_-, if $R(T)$ is a closed finite codimensional subspace of Y. T is said semi-Fredholm (resp. Fredholm) relation if $T \in \Phi_+ \cup \Phi_-$, (resp. $\Phi_+ \cap \Phi_-$).

This paper deals the essential spectra of a closed linear relation on a Banach space. In Section 2, we recall some useful basic properties of linear relations. In Section 3, present some auxiliary results which are used in the following sections. In Section 4, we define the Gustafson, Weidman, Kato, Wolf, Schechter and Browder essential spectra of a linear relation and we generalize the Proposition 1.1 (*ii*), (*iii*) and Theorem 2.1 in [1]. In Section 5, we prove that the properties Rakočević and Schmoeger essential spectra of a linear relation. In Section 6, we extend the results of Theorem 1.1 and Lemma 1.1 of [1] to linear relations.

2. Remember some useful basic properties of linear relations

We list some algebraic properties of linear relations.

Lemma 2.1. *Let X and Y be vector spaces and let $T \in LR(X,Y)$. Then*

(*i*) $\big($[16, p. 2]$\big)$ $D(T^{-1}) = R(T);\ D(T) = R(T^{-1})$.

(*ii*) $\big($[16, p. 3 (9)]$\big)$ *T injective if and only if* $T^{-1}T = I_{D(T)}$.

(*iii*) $\big($[16, p. 7, I.2.4)]$\big)$ $T(0)$ *and* $T^{-1}(0)$ *are subspaces.*

(*iv*) $\big($[16, p. 7, I.2.6 and I.2.8 (ii)]$\big)$ $x \in N(T)$ *if and only if* $0 \in Tx$ *if and only if* $Tx = T(0)$.

(*v*) $\big($[16, p. 8, I.2.9]$\big)$ *T single valued if and only if* $T(0) = \{0\}$.

(*vi*) $\big($[16, p. 8, I.2.10]$\big)$ $TT^{-1}(0) = T(0)$ *and* $T^{-1}T(0) = T^{-1}(0)$.

(*vii*) $\big($[16, p. 8, I.2.14 (b)]$\big)$ *Let $S,\ T \in LR(X,Y)$ such that $D(T) = D(S)$ and $T(0) = S(0)$. Then $T = S$ or the graphs of T and S are incomparable.*

(*vii*) $\big($[16, p. 8, I.2.11 (b)]$\big)$ *If* $G(S) \subset G(T)$, *then* T *is an extension of* S $\big($*that is* $S = T_{|D(S)}\big)$ *if and only if* $S(0) = T(0)$. ◊

Lemma 2.2. $\big($[22, p. 481, Lemma 7.2]$\big)$ *Let* X *be a vector space,* $T \in LR(X)$ *and* $\lambda \in \mathbb{K}\backslash\{0\}$. *Then* $N(\lambda - T)^n \subset R(T^m)$ *for all* $n,\ m \in \mathbb{N} \cup \{0\}$. ◊

Lemma 2.3. $\big($[22, p. 487, Theorem 9.1]$\big)$ *Let* X *be a vector space and* $T \in LR(X)$.*Then*

(*i*) *If* $\alpha(T) \leq \beta(T)$ *and* $\alpha(T) < \infty$, *then there exists an everywhere defined single valued* B *with* $\dim R(B) \leq \alpha(T)$ *such that* $T - B$ *is injective.*

(*ii*) *If* $\beta(T) \leq \alpha(T)$ *and* $\beta(T) < \infty$, *then there exists an everywhere defined single valued* B *with* $\dim R(B) \leq \beta(T)$ *such that* $T - B$ *is surjective.*

(*iii*) *If* $\alpha(T) = \beta(T)$ *then there exists an everywhere defined single valued* B *with* $\dim R(B) \leq \alpha(T)$ *such that* $T - B$ *is bijective.* ◊

Lemma 2.4. ([23, p. 2167, Proposition 5.2]) *Let* $X,\ Y,\ Z$ *be vector spaces and let* $T \in LR(X, Y)$, $S \in LR(Y, Z)$. *Assume that* T *and* S *have finite indices, then*

$$i(ST) = i(S) + i(T) + \dim \frac{Y}{R(T) + D(S)} - \dim\{T(0) \cap N(S)\}.$$

Also

$$\alpha(ST) \leq \alpha(S) + \alpha(T) \text{ and } \beta(ST) \leq \beta(S) + \beta(T). \quad ◊$$

We list some useful properties of closed, continuous and open linear relations in normed spaces.

Lemma 2.5. *Let* $X,\ Y,\ Z$ *be normed vector spaces and let* $T \in LR(X, Y)$. *Then*

(*i*) $\big($[16, p. 43, II.5.1 and p. 44, II.5.3]$\big)$ T^{-1} *closed if and only if* T *closed if and only if* $Q_T T$ *closed single valued and* $T(0)$ *closed space.*

(*ii*) $\big($[16, p. 43 (6)]$\big)$ *If* T *is continuous,* $D(T)$ *and* $T(0)$ *are closed then* T *is closed.*

(*iii*) $\big($[16, p. 33, II.3.2 (b) and p. 31, II.2.5]$\big)$ T *open if and only if* $\gamma(T) > 0$ *where* $\gamma(T) := \sup\big\{\lambda \geq 0 :\ \lambda d(X, N(T)) \leq \|TX\|,\ X \in D(T)\big\}$ *which coincides with* $\dfrac{1}{\|T^{-1}\|}$ (*so that* $\gamma(T) > 0$ *if and only if* T^{-1} *is continuous*).

(*iv*) $\big($[16, p. 47, II.5.16]$\big)$ *If* $S,\ T \in LR(X, Y)$ *such that* T *is closed and* S *is a continuous single valued, then* $T + S$ *is closed.*

(*v*) $\big($[16, p. 38, II.3.13 and II.3.14]$\big)$ *If* $T \in LR(X, Y)$ *and* $S \in LR(Y, Z)$ *with* $T(0) \subset D(S)$ *then* $\|STx\| \leq \|S\|\|Tx\|$, $x \in D(ST)$ *and* $\|ST\| \leq \|S\|\|T\|$.

(*vi*) $\big($[16, p. 146, v.2.9]$\big)$ *If* $T \in LR(X, Y)$ *and* $S \in LR(Y, Z)$ *such that* $\overline{T(0)} \subset D(S)$ *and* S *is a continuous, then* $Q_{ST}ST = Q_{ST}SQ_T^{-1}Q_T T$.

(*vii*) $\big($[16, p. 47, II.5.17]$\big)$ *If* $T \in LR(X, Y)$ *and* $S \in LR(Y, Z)$ *such that* T *closed,* $\alpha(S) < \infty$, $\gamma(S) > 0$ *and* $R(S)$ *closed then* ST *is closed.*

(*viii*) $\big($[16, p. 34, II.3.4]$\big)$ $N(T) \subset N(Q_T T)$ *and* $\gamma(T) \leq \gamma(Q_T T)$ *with equality if* $T(0)$ *is closed in* $R(T)$.

(*ix*) $\big($[16, p. 36, II.3.9]$\big)$ *If* T *is open and* $N(T)$ *is closed, then* $N(T) = N(Q_T T)$ *and* $\gamma(T) = \gamma(Q_T T)$.

(*x*) $\big($[16, p. 48, II.6.1]$\big)$ *If* M *is a subspace of* $D(T)$, *then* $\gamma(T) \leq \gamma(T|_{N(T)+M})$. ◊

We recall some useful properties of the conjugate T' of T.

Lemma 2.6. *Let X and Y be normed spaces and let $T \in LR(X,Y)$. Then*

(*i*) $\big($[16, p. 56, III.1.2 and III.1.4]$\big)$ *T' is closed such that $T'(0) = D(T)^{\perp}$; $N(T') = R(T)^{\perp}$ and if T is closed then $T(0) = D(T')^{\perp}$.*

(*ii*) $\big($[16, p. 57, III.1.5 (b)]$\big)$ *$S \in LR(X,Y)$ continuous such that $D(T) \subset D(S)$ then $(T+S)' = T' + S'$.*

(*iii*) $\big($[16, p. 72, III.4.6 (a),(c)]$\big)$ *T continuous if and only if $T(0)^{\perp} = D(T')$. In such case T' is continuous and $\|T\| = \|T'\|$.*

(*iv*) $\big($[16, p. 72, III.4.6 (b),(d)]$\big)$ *T open if and only if $R(T') = N(T)^{\perp}$. In such case $\gamma(T) = \gamma(T')$.*

(*v*) $\big($[16, p. 58, III.1.6]$\big)$ *Let $T \in LR(X,Y)$ and $S \in LR(Y,Z)$. Then $T'S' \subset (ST)'$.* ◊

Lemma 2.7. *Let X and Y be Banach spaces and let $T \in LR(X,Y)$ be closed. Then T open $\Leftrightarrow$ T' open $\Leftrightarrow$ $R(T)$ closed $\Leftrightarrow$ $R(T')$ closed.* ◊

Proof. T open $\Leftrightarrow$ $R(T)$ closed, follows immediately from $\big($[16, p. 71, III.4.2 (b)]$\big)$.

T open $\Leftrightarrow$ T' open, follows immediately from $\big($[16, p. 76, III.5.3 (a)]$\big)$.

T' open $\Leftrightarrow$ $R(T')$ closed, as T open $\Leftrightarrow$ $R(T)$ closed.

Lemma 2.8. *Let X be a normed space and let $T \in LR(X)$. If $0 < |\lambda| < \gamma(T)$. Then*

(*i*) $\big($[16, p. 81, III.7.4]$\big)$ *$\alpha(\lambda - T) \leq \alpha(T)$; $\overline{\beta}(\lambda - T) \leq \overline{\beta}(T)$.*

(*ii*) $\big($[16, p. 82, III.7.5]$\big)$ *T open with dense range then $\lambda - T$ has dense range.*

(*iii*) $\big($[16, p. 82, III.7.6]$\big)$ *T open and injective then $\lambda - T$ open and $\overline{\beta}(\lambda - T) = \overline{\beta}(T)$, here $\lambda - T := \lambda I_{D(T)} - T$.* ◊

3. Auxiliary results

Lemma 3.1. *Let M be a closed subspace of a normed space X and let N be a subspace of X containing M. Then*

(*i*) *N closed if and only if (N/M) closed.*

(*ii*) *If N is closed then $(X/N) \equiv (X/M)\big/(N/M)$ and $Q_N = Q_{N/M}Q_M$ where $\equiv$ is a canonical isometry.*

Proof. (*i*) it is elementary.

(*ii*) $\big($see [16, p. 124, IV.5.2]$\big)$.

Lemma 3.2. *Let X and Y be normed spaces and let $T \in LR(X,Y)$ be closed. Then $R(T)$ closed if and only if $R(Q_T T)$ closed. In such case $\beta(T) = \beta(Q_T T)$.* ◊

Proof. By Lemma 2.5 (*i*) $Q_T T$ is a closed, $Q_T T$ is a closed operator with $T(0)$ a closed subspace and hence

$$R(T) + \overline{T(0)}/\overline{T(0)} = R(T)/T(0),$$

clearly $T(0) \subset R(T)$ and thus by Lemma 3.1 (*i*) we have that $R(T)$ closed if and only if $R(Q_T T)$ closed. In such case we have by Lemma 3.1 (*ii*) that

$$Y/T(0)/R(T)/T(0) \equiv Y/R(T)$$

so that

$$\beta(T) := \dim Y/R(T) = \beta(Q_T T) := \dim Y/T(0)/R(Q_T T).$$

We shall use the following result of duality

Lemma 3.3. *Let X and Y be normed spaces and let $T \in LR(X, Y)$ be closed. Then*

(i) *T is not F_+ if and only if there is an infinite dimensional subspace M of $D(T)$ such that $T_{|M}$ is precompact.*

Now, assume that X and Y are Banach spaces and that T is closed. Then

(ii) *T_+ is F_+ if and only if T is Φ_+.*

(iii) *$T \in \Phi_+ \Leftrightarrow Q_T T \in \Phi_+ \Leftrightarrow T' \in \Phi_-$. In such case $i(T) = i(Q_T T) = -i(T')$.*

(iv) *$T \in \Phi_- \Leftrightarrow Q_T T \in \Phi_- \Leftrightarrow T' \in \Phi_+$. In such case $i(T) = i(Q_T T) = -i(T')$.* ◊

Proof. (i) See $\Big($[16, p. 136, V.1.6]$\Big)$.

(ii) See $\Big($[16, p. 138, V.1.7]$\Big)$.

(iii) By Lemma 2.5 (i), $Q_T T$ is a closed operator and $T(0)$ is a closed subspace. $T \in \Phi_+$, then $R(T)$ closed, by Lemma 3.2 we have $R(Q_T T)$ closed and applying Lemma 2.5 ($viii$), we obtain $N(T) = N(Q_T T)$. Since $\dim N(T) < \infty$ and $\beta(T) = \beta(Q_T T)$ we deduce that $Q_T T \in \Phi_+\big(X, Y/T(0)\big)$ with the same index that T. That $Q_T T \in \Phi_+ \Rightarrow T \in \Phi_+$, it is obtained similarly.

$T \in \Phi_+ \Leftrightarrow T' \in \Phi_-$. In effect we have: By Lemma 2.7, we have $R(T)$ closed $\Leftrightarrow R(T')$ closed and applying Lemma 2.6 (i), we obtain $N(T') = R(T)^\perp$ which implies that $\alpha(T') = \overline{\beta}(T)$. On the other hand, by Lemma 2.7, we have $R(T)$ closed $\Leftrightarrow$ T open and applying Lemma 2.6 (iv), we infer $R(T') = N(T)^\perp$ which implies that $\beta(T') = \alpha(T)$ if $R(T)$ is closed. From all the above, we deduce that

$$T \in \Phi_+ \Leftrightarrow T' \in \Phi_- \text{ and in that case } i(T) = -i(T').$$

(iv) With a reasoning similar to that (i).

In the following lemma, we give small perturbation of semiFredholm relations.

Lemma 3.4. *Let X be a Banach space and $T \in LR(X)$ be closed. Assume that $0 < |\lambda| < \gamma(T)$. Then*

(i) *$T \in \Phi_+ \Rightarrow \lambda - T \in \Phi_+$ and $i(\lambda - T) = i(T)$.*

(ii) *$T \in \Phi_- \Rightarrow \lambda - T \in \Phi_-$ and $i(\lambda - T) = i(T)$.* ◊

Proof. T closed then $Q_T T$ is a closed operator and by Lemma 2.5 (i) we have $T(0)$ closed, so that X and $Y/T(0)$ are Banach spaces. Assume $T \in \Phi_+ \cup \Phi_-$ then $R(T)$ is closed, by Lemma 3.2, we deduce $R(Q_T T)$ is closed. Assume that $0 < |\lambda| < \gamma(T)$ and that $T \in \Phi_+ \cup \Phi_-$. Then, applying Lemma 2.7 and Lemma 2.5 ($viii$), we obtain $0 < \gamma(T) = \gamma(Q_T T)$ and clearly we have

$$\|\lambda Q_T\| = |\lambda|\|Q_T\| \leq |\lambda|.$$

Moreover, by Lemma 2.5 (iv), we note that $\lambda - T$ is closed. By virtue of Lemma 2.1 (iii) and (v), we obtain that

$$\lambda - T(0) = \lambda(0) - T(0) = \{0\} - T(0) = T(0),$$

so that

(a) $Q_{\lambda - T} = Q_T$. Since T is closed, we deduce from Lemma 2.5 (i), (iv) and ($viii$) that:

(b) $Q_T T$ is closed with $\gamma(T) \leq \gamma(Q_T T)$, $T(0)$ is closed, $Y/T(0)$ is a Banach space and $\lambda - T$ is closed.

(i) Assume that $T \in \Phi_+(X, Y)$. Then, by Lemma 3.3 (iii) we have $Q_T T \in \Phi_+\big(X, Y/T(0)\big)$ and clearly, we obtain

$$\|\lambda Q_T\| \leq |\lambda|,$$

It follows from (a), (b) and [17, p. 112, V.1.6] that

$$Q_T(\lambda - T) \in \Phi_+\big(X, Y/T(0)\big),$$

with the same index that $Q_T T$, in this situation the desired assertion (*i*) follows immediately again from Lemma 3.3 (*iii*).

(*ii*) The proof is similar to that of (*i*) by using (*a*), (*b*) and [17, p. 112, V.1.6] and Lemma 3.3 (*iv*).

Remark 3.5. We note that the above Lemma 3.4 was proved by Cross [16, p. 205-206, V.15.6 and V.15.7] using operational quantities which are not introduced in this draft. Our proof is different, it is more short that the proof of Cross. ◊

We now present an useful result of perturbation of semi-Fredholm relations with compact relations. For this, we first prove the following lemma:

Lemma 3.6. *Let X is a normed space and Y is a Banach space. Let $T \in LR(X,Y)$ be closed and $S \in LR(X,Y)$ be continuous such that $S(0) \subset T(0)$ and $D(T) \subset D(S)$. Then $T+S$ is closed.* ◊

Proof. Case 1: T and S single valued. Let $(x_n) \subset D(T+S) = D(T) \cap D(S) = D(T)$, $x_n \to x$ and $(T+S)x_n \to y$. Then

$$\begin{aligned} \|Tx_n - Tx_m\| &\leq \|(T+S)(x_n - x_m)\| + \|S(x_n - x_m)\| \\ &\leq \|(T+S)(x_n - x_m)\| + \|S\|\|x_n - x_m\|, \end{aligned}$$

therefore (Tx_n) is a Cauchy sequence in Y Banach space, so that there exists $z \in Y$ such that $Tx_n \to z$ and since T is closed, $z = Tx, x \in D(T)$ and since $x_n \to x \in D(T) \subset D(S)$ and S is continuous, $Sx_n \to Sx$. Thus $(T+S)x_n \to Tx + Sx = y$. Hence $T+S$ closed.

Case 2: T and S linear relations. As $S(0) \subset T(0)$ we have that $(T+S)(0) = T(0) + S(0) = T(0)$ closed, hence $Q_{T+S} = Q_T$, applying Lemma 2.5 (*i*), we obtain $Q_T T$ closed. Since $S(0) \subset T(0)$, again applying Lemma 2.5 (*i*) we have $\overline{S(0)} \subset \overline{T(0)} = T(0)$, so that by Lemma 3.1 we infer that

$$Y/\overline{S(0)}\Big/T(0)/\overline{S(0)} \equiv Y/T(0),$$

with $Q_T = Q_{T(0)/\overline{S(0)}} Q_S$ and as S continuous. Therefore, by Lemma 2.5 (*ii*) we obtain $Q_S S$ is continuous. We have that $Q_T S$ is a continuous single valued. In this situation by the case 1, we obtain that

$Q_{T+S}(T+S) = Q_T(T+S) = Q_T T + Q_T S$ is closed. Again applying Lemma 2.5 (*i*) we have that $T+S$ closed, as desired.

Lemma 3.7. *Let X and Y be Banach spaces and let $T \in LR(X,Y)$ be closed. Then for any compact linear relation $K \in LR(X,Y)$ satisfying $D(T) \subset D(K)$ and $K(0) \subset T(0)$ we have that:*

(*i*) *$T \in \Phi_+$ then $T+K \in \Phi_+$ with $i(T+K) = i(T)$.*

(*ii*) *$T \in \Phi_-$ then $T+K \in \Phi_-$ with $i(T+K) = i(T)$.* ◊

Proof. By Lemma 3.6, $T+K$ is closed. Moreover, as $K(0) \subset T(0)$ we have $Q_{T+S} = Q_T$. Using Lemma 2.5 (*i*), $Q_T T$ is closed with $T(0)$ closed, so that X and $Y/T(0)$ are Banach spaces. Moreover, $K(0) \subset T(0)$ hence $\overline{K(0)} \subset \overline{T(0)} = T(0)$. Therefore, applying Lemma 3.1 (*ii*) we obtain $Q_T = Q_{T(0)/\overline{K(0)}} Q_K$ and hence $Q_T K$ is a compact operator (and hence strictly singular) with $D(Q_T T) = D(T) \subset D(K) = D(Q_T K)$. Then

(*i*) $T \in \Phi_+$, hence by Lemma 3.6 we obtain $Q_T T \in \Phi_+$ and using [17, p. 117, V.2.1] we have that

$$Q_{T+K}(T+K) = Q_T(T+K) = Q_T(T) + Q_T(K) \in \Phi_+,$$

with the same index that $Q_T T$. Again by Lemma 3.3 (iii), we conclude that $T+K \in \Phi_+$ with the same index that T.

(*ii*) Since $T \in \Phi_-$ then applying Lemma 3.3 (iv), we have $T' \in \Phi_+$. We shall prove that

(*a*) $K'(0) \subset T'(0)$. Indeed, since $D(T) \subset D(K)$ hence by Lemma 2.6 (*i*)

$$K'(0) = D(K)^{\perp} \subset D(T)^{\perp} = T'(0)$$

(b) $D(T') \subset D(K')$. Indeed, since $K(0) \subset T(0)$ then applying Lemma 2.6 (i) and (iii), we have

$$\overline{D(T')} = \left(D(T')^{\top}\right)^{\perp} = T(0)^{\perp} \subset K(0)^{\perp} = K'(0).$$

(c) K' compact. By [16, p. 163, V.5.15], so that by (i), we have that $T' + K' \in \Phi_+$ with $i(T' + K') = i(T')$. But $T' + K' = (T + K)'$ $\big($Lemma 2.6 (ii) since K compact hence K continuous [K compact hence K continuous $Q_K K$ compact operator]$\big)$. Therefore, using Lemma 3.3 (iv), we obtain $(T + K)' \in \Phi_+ \Longleftrightarrow T + K \in \Phi_-$ and $i(T + K) = i(T)$.

Remark 3.8. When, in the above Lemma 6.2, $T = \lambda I$, then $T(0) = \{0\}$ and so $K(0) \subset T(0)$ then $K(0) = \{0\}$, that is, K single valued. Hence, for $T = \lambda I$ the above Lemma 6.2 is a very know result of operators. ◇

4. Gustafson, Weidman, Kato, Wolf, Schechter and Browder essential spectra of a linear relation

The main purpose of this Section is to show that the properties of $\sigma_{ei}(.)$, $i = 1, 2, 3, 4, 5, 6$ for closed densely defined operators on Banach spaces obtained in [1] for Proposition 1.1 and Theorems 2.1 and 2.2 remain valid in the context of linear relations.

In this Section T will denote a closed linear relation in a complex Banach space X.

Definition 4.1. We define $\rho(T)$ the resolvent set of T, and the essential resolvent sets of T where we denote by $\rho_{ei}(T)$, $i = 1, 2, 3, 4, 5, 6$ as follows:

(i) $\rho(T) := \{\lambda \in \mathbb{C};\ \lambda - T \text{ bijective}\}$.

(ii) $\rho_{e1}(T) := \{\lambda \in \mathbb{C};\ \lambda - T \in \Phi_+\}$.

(iii) $\rho_{e2}(T) := \{\lambda \in \mathbb{C};\ \lambda - T \in \Phi_-\}$.

(iv) $\rho_{e3}(T) := \{\lambda \in \mathbb{C};\ \lambda - T \in \Phi_+ \cup \Phi_-\}$.

(v) $\rho_{e4}(T) := \{\lambda \in \mathbb{C};\ \lambda - T \in \Phi_+ \cap \Phi_-\}$.

(vi) $\rho_{e5}(T) := \{\lambda \in \mathbb{C};\ \lambda - T \in \Phi_+ \cap \Phi_- \text{ and } i(\lambda - T) = 0\}$.

(vii) $\rho_{e6}(T) := \{\lambda \in \mathbb{C};\ \lambda \in \rho_{e5}(T) \text{ such that all scalars near } \lambda \text{ are in } \rho(T)\}$. ◇

It is usual $\lambda - T := \lambda I_{D(T)} - T$. The spectrum of T, $\sigma(T)$, and the essential spectra, $\sigma_{ei}(T)$, $i = 1, 2, 3, 4, 5, 6$ are the respective complements of $\rho(T)$ and $\rho_{ei}(T)$ respectively: $\sigma(T) := \mathbb{C}\backslash\rho(T)$ and $\sigma_{ei}(T) := \mathbb{C}\backslash\rho_{ei}(T)$, $i = 1, 2, 3, 4, 5, 6$.

We denote that in [16, p. 220, V.I.1.2], for $S \in LR(X)$ where X complex normed space define $\rho(S) := \{\lambda \in \mathbb{C} : \lambda - S \text{ bijective, open with dense range}\}$, so that by virtue Lemma 2.7, we have that the notion of $\rho(S)$ when S is closed and X is complete coincides with our notion of resolvent set.

Proposition 4.2. *Let X be a Banach space and $T \in LR(X)$. Then*

$$\sigma_{ei}(T),\ i = 1, 2, 3, 4, 5, 6, \textit{ are closed.}$$ ◇

Proof. The case $i = 4$ is a generalization of Proposition 1.1 (i) in [1]. Let $\lambda \in \rho_{ei}(T)$, $i = 1, 2, 3, 4, 5, 6$. Then $R(\lambda - T)$ is closed and since $\lambda - T$ is closed in X $\big($see Lemma 2.5 (iv)$\big)$. By Lemma 2.7, we have that $\gamma(\lambda - T) > 0$. If $\lambda - T \in \Phi_+$ $\big($resp. $\Phi_-\big)$ and $|\eta - \lambda| < \gamma(T - \lambda)$ $\big($resp. $|\eta - \lambda| < \gamma(\lambda - T)\big)$ then, by Lemma 3.4 (i) $\big($resp. (ii)$\big)$, we have $\eta - T = (\eta - \lambda) - (\lambda - T) \in \Phi_+$ $\big($resp. $\Phi_-\big)$ with the same index that $\lambda - T$. Therefore $\rho_{ei}(T)$, $i = 1, 2, 3, 4, 5$ are open. Furthermore, since each component of $\rho_{e5}(T)$ is open, so $\rho_{e6}(T)$ is open.

Now, we try to generalize Proposition 1.1 (ii) and (iii).

Proposition 4.3. *Let $T \in \Phi_+ \cap \Phi_-$. Then there exists $\eta > 0$ such that $\alpha(\lambda - T)$ and $\beta(\lambda - T)$ are constant whenever $0 < |\lambda| < \eta$.* ◇

Proof. Case 1: $T \in \Phi_+$, let $\lambda \in \mathbb{C}\backslash\{0\}$. Then

(a) By Lemma 2.5 (iv) we have $\lambda - T$ is closed.

(b) Applying Lemma 2.2, we obtain $N(\lambda - T) \subset R^\infty(T) := \bigcap_{n\in\mathbb{N}} R(T^n)$.

(c) $R^\infty(T)$ closed. We prove by induction that $R(T^n)$ is closed. For $n = 1$, it is clear by the hypothesis $T \in \Phi_+$. Assume that $R(T^n)$ is closed. Then $N(T) + R(T^n)$ is closed (as $\dim N(T) < \infty$) and thus since T is closed, we have that $T_0 := T_{|N(T)+R(T^n)}$ is also closed and applying Lemmas 2.7 and 2.5 (x), we have $0 < \gamma(T) < \gamma(T_0)$. Hence, T_0 is open and so, by Lemma 2.7, we have $R(T_0)$ is closed. But

$$\begin{aligned} R(T_0) &:= T\Big(N(T) + R(T^n)\Big) \\ &:= T\Big(N(T)\Big) + T\Big(R(T^n)\Big), \quad \Big(\text{see [16, p. 9, I.3.1]}\Big) \\ &:= TT^{-1}(0) + R(T^{n+1}) \\ &:= T(0) + R(T^{n+1}), \quad \Big(\text{see Lemma 2.1 (vi)}\Big) \\ &:= R(T^{n+1}), \quad \Big(\text{clearly } T(0) \subset R(T^p), \text{ for all } p \in \mathbb{N}\Big). \end{aligned}$$

Therefore (c) is true.

Now, we define T_∞ the restriction of T to $R^\infty(T)$. Then

(d) T_∞ is closed, $T_\infty : R^\infty(T) \longrightarrow R^\infty(T)$ is surjective. Indeed, T_∞ is closed. Follows immediately observing that T is closed and by (c) we have $R^\infty(T)$ is closed. Let $m \geq n$, then $N(T) \cap R(T^m) \subset N(T) \cap R(T^n)$ and since $\dim\Big(N(T) \cap R(T^p)\Big) < \infty$ for all $p \in \mathbb{N}$, then we have that the sequence $\Big(N(T) \cap R(T^n)\Big)_{n\in\mathbb{N}}$ is a stationary sequence for n large enough. This shows that there exists $d \in \mathbb{N}$ for which

$$N(T) \cap R(T^m) = N(T) \cap R(T^d), \quad n \geq d. \tag{1}$$

This property (1) helps to see that T_∞ is surjective. In fact, let $z \in R^\infty(T)$, then for each $n \in \mathbb{N}$, there exists $(x_n) \in D(T) \cap R(T^n)$ such that $z \in Tx_n$ hence $0 = z - z \in Tx_n - Tx_m = T(x_n - x_m)$, so that by Lemma 2.1 ($iv$) we have $x_n - x_d \in N(T)$ and if $n \geq d$ then $x_n - x_d \in N(T) \cap R(T^d)$ which coincides with $N(T) \cap R^\infty(T) \subset R^\infty(T)$, $\Big($using Eq. (1)$\Big)$ and hence $x_d \in R^\infty(T) \cap D(T)$ and since $z \in Tx_d$ we obtain that $z \in T(R^\infty(T) \cap D(T))$. Hence $R^\infty(T) \subset T(R^\infty(T) \cap D(T))$. Clearly $T(R^\infty(T) \cap D(T)) \subset R^\infty(T)$. Therefore ($d$) is true.

(e) T_∞ is open. We can see by (d) and Lemma 2.7.

(f) $N(\lambda - T) = N(\lambda_\infty - T_\infty)$; $\beta(\lambda_\infty - T_\infty) = \beta(T_\infty) = 0$ if $0 < |\lambda| < \gamma(T_\infty)$. λ_∞ denotes the restriction of λI to $R^\infty(T) \cap D(T)$. By Lemma 2.8 (i) and the assertion (d) we have $\overline{\beta}(\lambda_\infty - T_\infty) \leq \overline{\beta}(T_\infty) = 0$, $\alpha(\lambda_\infty - T_\infty) \leq \alpha(T_\infty)$. Furthermore, by Eq. (1) we have $N(T_\infty) := N(T) \cap R^\infty(T) = N(T) \cap R(T^d)$ finite dimensional so that $i(T_\infty) = \alpha(T_\infty) < \infty$ and also applying the assertion (b) we obtain $N(\lambda_\infty - T_\infty) = N(\lambda - T) \cap R^\infty(T) = N(\lambda - T)$. On the other hand, by Lemma 3.4 (i) we infer $i(\lambda_\infty - T_\infty) = i(T_\infty)$ and $\lambda_\infty - T_\infty \in \Phi_+$ (in particular $R(\lambda_\infty - T_\infty)$ is closed). In short, (f) is true.

(g) There exists $\varepsilon > 0$ such that if $0 < |\lambda| < \varepsilon$ then by Lemma 3.4 (i) we obtain $\lambda - T \in \Phi_+$ with $i(\lambda - T) = i(T)$. Now, (f) and (g) implies the desired result if $T \in \Phi_+$.

Case 2: $T \in \Phi_-$. The result is obtained applying the case 1 to the conjugate T' of T. Indeed, by Lemma 2.5 (iv) we have $(\lambda - T)' = \lambda - T'$ and therefore $T \in \Phi_- \Longleftrightarrow T' \in \Phi_+$, in such case, applying Lemma 3.3 (iv), we obtain $i(T) = i(T')$.

Now, we can generalize the Proposition 1.1 (ii) and (iii) in [1].

Proposition 4.4. *Let $T \in LR(X)$ and C denotes a component of $\rho_{ei}(T)$, $i = 1,2,3,4$. Then*

(*i*) *$\alpha(\lambda - T)$ and $\beta(\lambda - T)$ have constant values n_1 and n_2, respectively, except perhaps at isolated points where $n_1 < \alpha(\lambda - T)$ and $n_2 < \beta(\lambda - T)$.*

(*ii*) *The index is constant in C.* ◊

Remark 4.5. In the assertion (*i*) $\big($resp. (*ii*)$\big)$, the case $i = 4$ generalizes Proposition 1.1 (*iii*) $\big($resp. (*ii*)$\big)$ in [1]. ◊

Proof. Since any component of an open set of $\mathbb{C}$, is open, we have that C is open. Define $\alpha(T) := \alpha(\lambda - T)$ and choose λ_0 such that $\alpha(\lambda_0) := n_1$ is the smallest nonnegative integer attained by $\alpha(\lambda)$ on C. Suppose $\alpha(\lambda') \neq n_1$. Owing to the connectivity of C, there exists an arc Γ lying in C with endpoints λ_0 and λ'. It follows from the above Proposition 4.3 and the fact that C is open, that about each $\eta \in \Gamma$ there exists an open ball $B(\eta, r) \subset C$ such that $\alpha(\lambda)$ is constant on $B(\eta, r)\backslash\{\eta\}$. Since Γ is compact and connected, there exists $\lambda_1, ..., \lambda_n = \lambda'$ such that

$$B(\eta_1, r_1), ..., B(\eta_n, r_n) \text{ cover } \Gamma \text{ and } B(\eta_i, r_i) \cap B(\eta_{i+1}, r_{i+1}) \neq \emptyset,\ \ 0 \leq i \leq n-1. \tag{2}$$

We assert from Lemma 3.4 (small perturbation) that $\alpha(\lambda) \leq \alpha(\lambda_0)$ for λ sufficiently close to λ_0. Thus , since $\alpha(\lambda_0)$ is the minimum value attained by $\alpha(\lambda)$ on C, it follows that $\alpha(\lambda) = \alpha(\lambda_0)$ for λ sufficiently close to λ_0. Since $\alpha(\lambda)$ is constant for all $\lambda \neq \lambda_0$ in $B(\lambda_0, r_0)$, this constant must be $\alpha(\lambda_0)$. Similarly $\alpha(\lambda)$ is constant on $B(\lambda_i, r_i)\backslash\{\lambda_i\}$, $1 \leq i \leq n$. Thus by Eq. (2) we have $\alpha(\lambda) = \alpha(\lambda_0)$ for all $\lambda \in B(\lambda', r')\backslash\{\lambda'\}$ and $\alpha(\lambda') > \eta_1$. To see that the result is true for $\beta(\lambda - T)$, we pass to the conjugate T' of T and apply the above using the equality $\alpha(\lambda - T') = \beta(\lambda - T)$.

(*ii*) If λ and λ' are distinct points in C and Γ is an arc in C with endpoints λ and λ', then by Lemma 3.4, there exists $\varepsilon > 0$ such that $i(\eta - T) = i(\lambda - T)$ for any $0 < |\eta| < \varepsilon$. Clearly the open balls $B(\lambda, .)$, $\lambda \in \Gamma$ cover C which is compact and thus a finite number of these balls overlap, it follows that $i(\lambda - T) = i(\lambda' - T)$, are desired.

Now, we prove the homologous of Theorem 2.1 in [1].

Proposition 4.6. *If $\rho_{e4}(T)$ is connected and $\rho(T) \neq \emptyset$. Then*

$$\sigma_{e4}(T) = \sigma_{e5}(T).$$ ◊

Proof. Clearly, $\sigma_{e4}(T) \subset \sigma_{e5}(T)$. We prove that $\sigma_{e5}(T) \subset \sigma_{e4}(T)$ showing that $\rho_{e4}(T) \cap \sigma_{e5}(T) = \emptyset$. Assume that there exists

$$\lambda_0 \in \rho_{e4}(T) \cap \sigma_{e5}(T), \tag{3}$$

and let $\lambda_1 \in \rho(T)$ (so that $\lambda_1 - T \in \Phi_+ \cap \Phi_-$ with $i(\lambda_1 - T) = 0$). Since $\rho_{e4}(T)$ is connected, it follows from the above Proposition 4.4 (*ii*) that $i(\lambda - T)$ is constant on any component of $\rho_{e4}(T)$. Therefore $i(\lambda_1 - T) = i(\lambda_0 - T) = 0$, hence $\lambda_0 \notin \sigma_{e5}(T)$ which contradicts for Eq. (3).
In the following, we prove the homologous of Theorem 2.2 for $i = 1,2,3,4,5$ in [1].

Proposition 4.7. *If $0 \in \rho(T)$ then for $\lambda \in \mathbb{C}\backslash\{0\}$, we have*

$$\lambda \in \sigma_{ei}(T) \text{ if and only if } \frac{1}{\lambda} \in \sigma_{ei}(T^{-1}),\ i = 1,2,3,4,5.$$ ◊

Proof. (*a*) T^{-1} is a bounded single valued and closed. Indeed, T closed and injective, so by Lemma 2.1 (*v*) we have $N(T) := T^{-1}(0) = \{0\}$ equivalently T^{-1} is a single valued. Moreover, applying Lemma 2.1 (*i*) and since T is surjective then we obtain $D(T^{-1}) = R(T) = X$. By Lemmas 2.5 (*iii*) and 2.7 then T^{-1} is continuous. Again, applying Lemma 2.5 (*i*) T closed hence T^{-1} closed.

(b) $\lambda - T = -\lambda(\lambda^{-1} - T^{-1})T$. Indeed, we pose $A := \lambda - T := \lambda I_{D(T)} - T$, $S := -\lambda(\lambda^{-1} - T^{-1})$ and $B := -\lambda(\lambda^{-1} - T^{-1})T := ST$, then $D(A) = D(B)$. Indeed, $D(A) = D(\lambda - T) = D(\lambda) \cap D(T) = D(T)$ and

$$\begin{aligned} D(B) = D(ST) &:= \{x \in D(T);\ Tx \cap D(S) \neq \emptyset\}, \Big(\text{see [16, p. 2, I.1.3]}\Big) \\ &:= D(T), \Big(\text{as } D(S) = D(\lambda^{-1} - T^{-1}) = \\ &\quad\ D(T^{-1}) = R(T) = X\Big) \\ &:= D(T). \end{aligned}$$

$B = A$. Indeed, by $\Big($[16, p. 11, I.4.2 (d)]$\Big)$ we have

$$\begin{aligned} (\lambda^{-1} - T^{-1})T &\subset \lambda^{-1}T - T^{-1}T. \\ &= \lambda^{-1}T - I_{D(T)} \ \Big(\text{see Lemma 2.1 } (ii)\Big) \\ -\lambda(\lambda^{-1} - T^{-1})T &\subset -\lambda\lambda^{-1}T + \lambda I_{D(T)} \\ &= \lambda - T := A \end{aligned}$$

$B(0) \subset A(0)$. Since $B \subset A$, it is clear that $B(0) = A(0)$. Furthermore, $A(0) := (\lambda - T)(0) = -T(0) = T(0)$ $\Big($as $T(0)$ is a subspace Lemma 2.1 (iii)$\Big)$. Let $z \in T(0)$ then $\lambda^{-1}z - T^{-1} = \lambda^{-1}z$ since $T^{-1}z \in T^{-1}T(0) = T^{-1}(0) = \{0\}$ $\Big(T^{-1}$ is single valued by (a) $\Big)$. Hence, if $z \in T(0)$ then $-\lambda\Big(\lambda^{-1} - T^{-1}\Big)T(0) := B(0)$, thus $A(0) \subset B(0)$. In consequence $A(0) = B(0)$. Now, it follows from Lemma 2.1 (vii) that $A = B$, so that (b) is true.

(c) $N(\lambda - T) = N(\lambda^{-1} - T^{-1})$ and $R(\lambda - T) = R(\lambda^{-1} - T^{-1})$. In fact, applying ($b$) we have $R(\lambda - T) = R\Big(\lambda^{-1} - T^{-1})T\Big) = (\lambda^{-1} - T^{-1})R(T) = R(\lambda^{-1} - T^{-1})$ $\Big($as $R(T) = X$ by (a)$\Big)$. Moreover, $N(\lambda - T) = N(\lambda^{-1} - T^{-1})$. Indeed,

$$\begin{aligned} x \in N(\lambda - T) &\iff \begin{cases} x \in D(\lambda - T) = D(T) \\ (\lambda - T)x = (\lambda - T)(0) \ \Big(\text{see Lemma 2.1 } (iv)\Big) \end{cases} \\ &\iff \lambda x - Tx = -T(0) = T(0) \ \Big(T(0) \text{ is a subspace } \Big(\text{see Lemma 2.1 } (iii)\Big)\Big) \\ &\iff Tx = \lambda x - T(0) \ \Big(\text{since } T^{-1}T = I_{D(T)}\Big(\text{see Lemma 2.1 } (ii)\Big) \text{ and} \\ &\qquad \text{applying Lemma 2.1 } (vi) \text{ and } (a)\ T^{-1}T(0) = T^{-1}(0) = \{0\}\Big) \\ &\iff x = T^{-1}Tx = \lambda T^{-1}x - T^{-1}T(0) = \lambda T^{-1}x \\ &\iff 0 = x - \lambda T^{-1}x \\ &\iff 0 = \lambda^{-1}x - T^{-1}x = (\lambda^{-1} - T^{-1})x \\ &\iff x \in N(\lambda^{-1} - T^{-1}). \end{aligned}$$

Therefore (a) and (c) ensures the desired Proposition 4.7.

5. Rakočević and Schmoeger Essential Spectra of a linear relation

The main purpose of this section is to prove that the properties of $\sigma_{e7}(.)$ and $\sigma_{e8}(.)$ for closed densely defined operators in Banach spaces obtained in [1, 18–20] are valid for closed linear relations. Perhaps we must write $\sigma_{e7}(.) := \sigma_{eap}(.)$ and $\sigma_{e8}(.) := \sigma_{e\delta}(.)$. In this Section T will denote a closed linear relation on a complex Banach space X.

Definition 5.1. We define $\sigma_{e7}(T) := \bigcap\limits_{K \in \Re_T} \sigma_{ap}(T + K)$ and $\sigma_{e8}(T) := \bigcap\limits_{K \in \Re_T} \sigma_\delta(T + K)$, with $\Re_T := \Big\{K \in LR(X) :$ K compact, $D(T) \subset D(K)$, $K(0) \subset T(0)\Big\}$, $\sigma_{ap}(T) := \{\lambda \in \mathbb{C} :\ \lambda - T$ not bounded below$\}$, where bounded below is injective and open and $\sigma_\delta(T) := \{\lambda \in \mathbb{C} :\ \lambda - T$ is not surjective$\}$. ◇

We now prove the analogous of Proposition 1.3 in [1].

Proposition 5.2. *Characterization of* $\sigma_{e7}(.)$ *and* $\sigma_{e8}(.)$.

(i) $\lambda \notin \sigma_{e7}(T)$ *if and only if* $\lambda - T \in \Phi_+$ *and* $i(\lambda - T) \leq 0$.

(ii) $\lambda \notin \sigma_{e8}(T)$ *if and only if* $\lambda - T \in \Phi_-$ *and* $i(\lambda - T) \geq 0$. ◊

Proof. We note that $\lambda - T$ is closed by Lemma 2.5 (*iv*).

(*i*) ($\Rightarrow$) Let $\lambda \notin \sigma_{e7}(T)$. Then there exist $K \in K_T$ such that $\lambda - T - K = (\lambda - T) - K = \lambda - (T + K)$ is bounded below. Then, by Lemma 3.6 we have $\lambda - (T + K)$ is closed, and applying Lemma 2.7 we infer that $\lambda - (T + K)$ is injective with closed range, since is closed and open. Hence $\lambda - (T + K) \in \Phi_+$ and $i((\lambda - T) - K) := \alpha(\lambda - T - K) - \beta(\lambda - T - K) = 0 - \beta(\lambda - T - K) \leq 0$. Therefore

(*a*) There exists $K \in \Re_T$ such that $\lambda - (T + K)$ is injective, open and with index ≤ 0. Moreover

(*b*) If $S := (\lambda - T) - (K - K)$ then $Q_T = Q_{\lambda - T} = Q_S$ and $Q_T(\lambda - T) = Q_T S = Q_S S$. Indeed, $K(0) \subset T(0)$, T is closed and by Lemma 2.5 (*i*) we have that $\overline{K(0)} \subset \overline{T(0)} = T(0)$. Note that $T(0)$ and $K(0)$ are subspaces $\big($see Lemma 2.1 (*iii*)$\big)$, we have that $S(0) := (\lambda - T)(0) + (K - K)(0) = \lambda(0) - T(0) + K(0) - K(0) = T(0) + K(0) = T(0)$, so that $\overline{S(0)} = \overline{T(0)} = T(0)$ and thus $Q_S = Q_T$ and as $(\lambda - T)(0) = -T(0) = T(0)$ is $Q_T = Q_{\lambda - T}$. Moreover, applying Lemma 3.1 (*ii*) we obtain $Q_T = Q_{T(0)/\overline{K(0)}} Q_K$ and hence $Q_T(\lambda - T) = Q_T S$, since, clearly $Q_K\big((\lambda - T) + (K - K)\big) = Q_K(\lambda - T) + Q_K K - Q_K K = Q_K(\lambda - T)$, because $Q_K K$ is an operator. Hence (*b*) is true.

Now, we have the following chain of implications: (*a*) $\Rightarrow$ $\lambda - (T + K) \in \Phi_+$ $\Rightarrow$ $\lambda - (T + K) + K \in \Phi_+$ $\Rightarrow$ $(\lambda - T) + (K - K) \in \Phi_+$ with the same index that $\lambda - (T + K)$ (see Lemma 6.2. Therefore, applying (*b*) we infer that $Q_T(\lambda - T) = Q_T(S) = Q_S(S) \in \Phi_+$ and by Lemma 3.3 (*iii*), we have the same index that S, $\lambda - T \in \Phi_+$ and $i(\lambda - T) = i(Q_T(\lambda - T)) = i(S)$ which coincides on the index of $\lambda - (T + K)$ which is ≤ 0 by (*a*).

($\Leftarrow$) Let $\lambda \in \mathbb{C}$ such that $\lambda - T \in \Phi_+$ and $i(\lambda - T) \leq 0$. Then by Lemma 2.3 (*i*) we can construct an everywhere defined single valued B such that $\dim R(B) \leq \alpha(\lambda - T)$ and $(\lambda - T) - B$ is injective. Furthermore, see the proof of Lemma 2.3 (*i*). B is defined by

$$B : X \longrightarrow Bx := \sum_{i=1}^{n} x'_i(x) y_i, \quad x \in X.$$

Where $\{x_1, ..., x_n\}$ is a basis of $N(\lambda - T)$. Choose linear functionals $x'_1, ..., x'_n$ such that $x'_i(x_j) = \delta_{ij}$ and choose $y_1, ..., y_n \in X$ such that $[y_1], ..., [y_n] \in X/R(\lambda - T)$ are linearly independent $\big($such elements exist since $n \leq \beta(\lambda - T)\big)$. Hence, it is clear that B is continuous, so that B is a bounded finite rank operator and so it is clear that $B \in \Re_T$ and also $(\lambda - T) - B = \lambda - (T + B)$ is injective. Since $\lambda - T \in \Phi_+$ we have by Lemma 3.6 (*i*) that $\lambda - (T + B) \in \Phi_+$ with the same index that $\lambda - T$. In this situation we have that $B \in \Re_T$, $\lambda - (T + B)$ is injective, closed with closed range then by Lemma 2.7 $\lambda - (T + B)$ is open, so that $\lambda \in \rho_{ap}(T + K)$. Therefore $\lambda \notin \sigma_{e7}(T)$.

(*ii*) ($\Rightarrow$) Similar to (*i*) ($\Rightarrow$) then using Lemmas 3.3 (*ii*) and 6.2 (*ii*).

($\Leftarrow$) Similar to (*i*) ($\Leftarrow$) using Lemma 2.3 (*ii*) hence

$$B : X \longrightarrow Bx := \sum_{i=1}^{n} x'_i(x) y_i, \quad x \in X.$$

Let $1 \leq q := \beta(\lambda - T)$ where let $x_1, ..., x_q$ be a set of q linearly independent elements of $N(\lambda - T)$ $\big($such elements exist because $q \leq \alpha(\lambda - T)\big)$. Choose linear functionals $x'_1, ..., x'_q$ such that $x'_i(x_j) = \delta_{ij}$, $1 \leq i, j \leq q$ and choose $y_1, ..., y_q$ such that $[y_1], ..., [y_q] \in X/R(\lambda - T)$ determine a basis of $X/R(\lambda - T)$.

Now, we prove the analogous Theorem 2.2 (*i*) in [1] of $i = 7, 8$.

Proposition 5.3. *If $0 \in \rho(T)$ then for $\lambda \in \mathbb{C}\backslash\{0\}$ we have*

$$\lambda \in \sigma_{e7}(T) \Big(resp.\ \lambda \in \sigma_{e8}(T)\Big)\ if\ and\ only\ if\ \frac{1}{\lambda} \in \sigma_{e7}(T^{-1}) \Big(resp.\ \lambda \in \sigma_{e8}(T^{-1})\Big). \qquad \diamond$$

Proof. Arguing as in Proposition 4.7 we have that

(a) T^{-1} is a bounded, closed and single valued.

(b) $N(\lambda - T) = N(\lambda^{-1} - T^{-1})$ and $R(\lambda - T) = R(\lambda^{-1} - T^{-1})$.

Now, the result follows immediately from the above Proposition 5.3.

In following, we gives other properties of $\sigma_{e7}(.)$ and $\sigma_{e8}(.)$.

Proposition 5.4. *$\sigma_{e7}(T) = \sigma_{e8}(T')$ and $\sigma_{e7}(T') = \sigma_{e8}(T)$.* $\diamond$

Proof. $\lambda \notin \sigma_{e7}(T)$. Then by Proposition 5.2 (i), we have $\lambda - T \in \Phi_+$ with $i(\lambda - T) \leq 0$, hence by Lemma 3.3 (iii), we infer $(\lambda - T)' \in \Phi_-$ with $i(\lambda - T) = -i\Big((\lambda - T)'\Big)$ and since by Lemma 2.6 we have $(\lambda - T)' = \lambda - T'$ we infer $(\lambda - T)' \in \Phi_-$ with $i(\lambda - T') \geq 0$, by Proposition 5.2, we have $\lambda \notin \sigma_{e8}(T')$. Hence $\sigma_{e7}(T) = \sigma_{e8}(T')$.

Let $\lambda \notin \sigma_{e8}(T)$. Then by Proposition 5.2 (ii), we have $\lambda - T \in \Phi_-$ with $i(\lambda - T) \geq 0$, hence by Lemma 3.3 (iv), we infer $(\lambda - T)' \in \Phi_+$ with $i(\lambda - T) = -i(\lambda - T)'$ and since by Lemma 2.6 we have $(\lambda - T)' = \lambda - T'$ we infer $(\lambda - T)' \in \Phi_-$ with $i(\lambda - T') \leq 0$, by Proposition 5.2 (ii), we have $\lambda \notin \sigma_{e8}(T')$. Therefore $\sigma_{e7}(T') = \sigma_{e8}(T)$.

Proposition 5.5. *$\sigma_{e7}(T)$ and $\sigma_{e8}(T)$ are closed.* $\diamond$

Proof. Since, by Proposition 5.4 $\sigma_{e8}(T) = \sigma_{e7}(T')$ it is enough to show that $\sigma_{e7}(T)$ is closed. For this, it is enough to see that $\sigma_{ap}(T)$ is closed by the definition of $\sigma_{e7}(T)$. Let $\lambda \in \rho_{ap}(T)$, that is, $\lambda - T$ is bounded below, that is $\lambda - T$ is injective and open so that $\gamma(\lambda - T) > 0$. Assume that $\eta \in \mathbb{C}$ such that $|\eta - \lambda| < \gamma(\lambda - T)$. Then it follows from Lemma 2.8 (iii) that $\eta - T = (\eta - \lambda) + (\lambda - T)$ is open and it follows from Lemma 2.8 (i) that $\alpha(\eta - T) \leq \alpha(\lambda - T) = 0$. Hence $\eta - T$ is bounded below. Therefore $\rho_{ap}(T)$ is open, as desired.

6. Fredholm and semi-Fredholm perturbation

In this Section, we extend the results of Theorem 1.1 and Lemma 1.1 of [1] to linear relations.

6.1. Product of semi-Fredholm linear relations

Proposition 6.1. *Let X, Y and Z be Banach spaces and let $S \in LR(X, Y)$ and $T \in LR(Y, Z)$ be closed. Then*

(i) $S,\ T \in \Phi_+ \implies TS \in \Phi_+$.

(ii) $S,\ T \in \Phi_-$ *and* TS *closed* $\implies TS \in \Phi_-$.

(iii) $S,\ T \in \Phi \implies TS \in \Phi$ *and* $i(TS) = i(T) + i(S) + \dim\Big(Y/R(S) + D(T)\Big) - \dim\{S(0) \cap N(T)\}$. $\diamond$

Proof. (i) (a) TS closed. It follows immediately from Lemmas 2.7 and 2.5 (vii).

(b) $\alpha(TS) < \infty$. By Lemma 2.4, $\alpha(TS) \leq \alpha(T) + \alpha(S)$. Hence, it only remains to see that.

(c) $R(TS)$ closed. In fact, by (a) we have TS is closed, with X and Z Banach spaces, the Lemma 2.7 say that $R(TS)$ closed, hence TS is open. We now prove that TS is open, we define $T_0 := T_{|N(T)+R(S)}$. Then

(d) T_0 closed. It is obvious since T is closed and $N(T) + R(S)$ is closed because $\dim N(T) < \infty$ and $R(T)$ is closed.

The above Proposition generalizes Theorem 1.1 (iii) in [1], since S and T are bounded operators on a Banach space X, then it is clear that both are closed, that TS is bounded and hence closed and that by Lemma 2.1 (v), we have $T(0) = S(0) = \{0\}$. We observe that for linear relations in general, the index of the product is not the sum of the index of the both linear relations.

Now, we generalize the Theorem 1.1 (*i*), (*ii*) and (*iv*) in [1]. For this end, we first prove the following elementary lemma.

Lemma 6.2. *Let X and Y be Banach spaces and let $T \in LR(X,Y)$ be closed. Then T is continuous therefore $D(T)$ is closed.* ◊

Proof. Case 1: T single valued. It is very known $\big((\Rightarrow)$ it is clear: $x \in \overline{D(T)}$ hence there exist $x_n \subset D(T)$, $x_n \to x$ so that $\|Tx_n - Tx_m\| \leq \|T\|\|x_n - x_m\|$, $n,\ m \in \mathbb{N}$, therefore $Tx_n \subset Y$ cauchy with Y complete therefore $y \in Y$, $Tx_n \to y$ and thus as T is closed we have that $x \in D(T)$ and $Tx = y$. $(\Leftarrow)$ It follows by the closed graph theorem for operators$\big)$.

Case 2: T linear relation. Passing to $Q_T T$ and using the case 1. Indeed, $Q_T T$ is closed with $T(0)$ closed $\big($see Lemma 2.5 (*i*)$\big)$, X and $Y/T(0)$ are complete. Then T continuous, hence by Lemma 2.5 (*ii*), we have $\|T\| := \|Q_T T\| < \infty$, therefore, by the case 1 we obtain $D(Q_T T) = D(T)$ is closed.

In particular we have "If X is a Banach space and $T \in LR(X)$ be closed and everywhere defined, then T is bounded."

Proposition 6.3. *Let X, Y and Z be Banach spaces and let $S \in LR(X,Y)$, $T \in LR(Y,Z)$ be closed and everywhere defined such that $TS \in \Phi_+(X,Z)$. Then $S \in \Phi_+(X,Y)$.* ◊

Proof. (*a*) S and T are bounded. By the above consequence of Lemma 3.4.

(*b*) $Q_{TS}TS = (Q_{TS}TQ_S^{-1})(Q_S S)$. Indeed, by Lemma 2.5 (*i*) we have $S(0) = \overline{S(0)} \subset D(T) = Y$, so that (*b*) follows from Lemma 2.5 (*vi*).

(*c*) TQ_S^{-1} and $Q_{TS}TQ_S^{-1}$ are continuous. Indeed, $Q_S : Y \longrightarrow Y/S(0)$ surjective single valued, hence by Lemma 2.7 Q_S is open and $\gamma(Q_S) > 0$, by Lemma 2.5 (*iii*) we have Q_S^{-1} is continuous. Since $Q_S^{-1}(0) := N(Q_S) = \overline{S(0)} = S(0) \subset D(T) = Y$, it follows from Lemma 2.5 (*v*) that TQ_S^{-1} is continuous and since $D(Q_{TS}) = Z$ it follows again Lemma 2.5 (*v*) that $Q_{TS}TQ_S^{-1}$ is continuous.

(*d*) $Q_{TS}TQ_S^{-1}$ is single valued. Indeed, $Q_{TS}TQ_S^{-1}$ is single valued, applying Lemma 2.5 (*i*) we have $Q_{TS}TQ_S^{-1}(0) = \{0\}$. But $Q_{TS}TQ_S^{-1}(0) := Q_{TS}TN(Q_S) = Q_{TS}\overline{S(0)} = Q_{TS}TS(0)$, so S closed, applying Lemma 2.5 (*i*) we have $S(0)$ closed. Therefore $Q_{TS}TQ_S^{-1} = \{0\}$ since $Q_{TS}TS$ is single valued.

(*e*) $Q_{TS}TQ_S^{-1}$ is everywhere defined. It is clear by [16, p. 3, (2)] $D(Q_{TS}TQ_S^{-1}) = Q_S D(Q_{TS}T)$ and

$$\begin{aligned} D(Q_{TS}T) &:= \{y \in D(T); Ty \cap D(Q_{TS}) \neq \emptyset\} \\ &:= \{y \in D(T) = Y;\ Ty \neq \emptyset\} \\ &:= D(T) = Y. \end{aligned}$$

Therefore, $Q_{TS}TQ_S^{-1}$ and $Q_S S$ are bounded operators in Banach spaces such that $Q_{TS}TS = (Q_{TS}TQ_S^{-1})(Q_S S)$ and by Lemma 3.3 (*iii*) we have $Q_{TS}TS \in \Phi_+$, hence $TS \in \Phi_+$. In this situation by [21] we have that $Q_S S$ is a Φ_+-operator and thus $S \in \Phi_+$ by virtue of Lemma 3.3 (*iii*).

Now, we extend Theorem 1.1 (*ii*) to linear relations. But, here appear many problems, for example:

(*a*) For bounded operators S and T is $(TS)' = S'T'$, but for linear relations in general only is true that $S'T' \subset (TS)'$ $\big($see Lemma 2.6 (*v*)$\big)$.

(*b*) T is a bounded operator hence T' is a bounded operator. However we have that, if T is a bounded closed not single valued hence T' continuous single valued and $D(T') \neq Y'$ (so that T' is not bounded).

Proof. (*b*) $D(T') \neq Y'$. Indeed, if $D(T') = Y'$, then $D(T')^T = (Y')^T = \{0\}$ and since T is continuous, hence applying Lemma 2.6 (*iii*) we have $T(0)^\top = D(T')$, therefore, we deduce that if T is closed (as $T(0)$ is closed), continuous and $D(T') = Y'$, then $T(0) = \overline{T(0)} = (T(0)^\perp)^\top = D(T')^\top = \{0\}$, that is, $T(0) = \{0\}$ equivalently, by

Lemma 2.1 (v) we have T is single valued. Furthermore, $D(T) = X$ hence by Lemma 2.6 (i) $T'(0) = D(T)^{\perp} = \{0\}$ therefore applying Lemma 2.1 (v) T' single valued. Also T continuous therefore by Lemma 2.6 (iii) we have T' continuous.

It seems that there are many problems to extend Theorem 1.1 (ii), but we give the similar result in the following proposition.

Proposition 6.4. *Let X, Y and Z be Banach spaces and let $S \in LR(X,Y)$ and $T \in LR(Y,Z)$ be closed with $D(T) = Y$ and S continuous. If $TS \in \Phi_-(X,Z)$ then $T \in \Phi_-(Y,Z)$.* ◊

Proof. (a) $D(TS) = D(S)$, $T'(0) = \{0\}$ and $S'T'(0) = S'(0) = (TS)'(0)$. Indeed,

$$\begin{aligned} D(TS) &:= \{x \in D(S);\ Sx \cap D(T) \neq \emptyset\} \\ &= \{x \in D(S);\ Sx \neq \emptyset\} \text{ (as } D(T) = Y) \\ &:= D(S). \end{aligned}$$

Applying Lemma 2.6 (i), we have $T'(0) = D(T)^{\perp} = \{0\}$ (as $D(T) = Y$), again by Lemma 2.6 (i), we obtain $(TS)'(0) = D(TS)^{\perp} = D(S)^{\perp} = S'(0) = S'T'(0)$. Hence ($a$) holds.

(b) $(TS)'$ is an extension of $S'T'$. Indeed, by Lemma 2.6 (v), we have $S'T' \subset (TS)'$, by (a) we have $S'T'(0) = (TS)'(0)$. Now, applying Lemma 2.1 (viii), (b) is holds.

(c) $S'T'$ is F_+. Assume $S'T'$ is not F_+. Then by Lemma 3.3 (i), there exists an infinite dimensional subspace M of $(S'T')$ such that $S'T'_{|M}$ is precompact and so by (b), $(TS)'_{|M}$ is precompact which implies by Lemma 3.3 (i) that $(TS)'$ is not F_+, but as $TS \in \Phi_+$, in particular TS is closed. Applying Lemma 3.3 (iv), we obtain $(TS)' \in \Phi_+$, by Lemma 3.3 (ii) we have $(TS)' \in F_+$. Since, applying Lemma 2.6 (i), we infer $(TS)'$ is closed.

(d) $T' \in \Phi_+$. In fact, by (c), we have $S'T' \in F_+$, so by the definition of F_+ there is a finite codimensional subspace M of $D(S'T')$) for which $\alpha\|m\| \leq \|S'T'm\|$, $m \in M$, for some $\alpha > 0$. Since S continuous, applying Lemma 2.6 (iii), we have S' continuous. By (a) we obtain $T'(0) = \{0\} \subset D(S')$. We deduce from Lemma 2.5 (v) that

$$\alpha\|m\| \leq \|S'T'm\| \leq \|S'\|\|T'm\|.$$

Therefore

$$\frac{\alpha}{\|S'\|} \leq \|T'm\|,\ m \in M.$$

Hence $T' in F_+$ and since T is closed, applying Lemma l11 (ii) and (iii) we have that $T \in \Phi_-$, as desired.

Now, this suggests that we try to extend Theorem 1.1 (iv) (only (iv) because (iii) is generalized in Proposition 6.1).

Proposition 6.5. *Let X be a Banach space and let $S, T \in LR(X)$ be closed and everywhere defined. If TS and ST are Fredholm then T and S are Fredholm.* ◊

Proof. (a) S and T are Φ_+. By Proposition 6.3.

(b) $\beta(T) < \infty$ and $\beta(S) < \infty$. Indeed, as $ST \in \Phi_-$ is $R(ST)$ a closed finite codimensional subspace of X and by (a), we have $R(S)$ is closed then by Lemma 3.1 (ii), we obtain that

$$X/R(ST)\Big/R(S)/R(ST) \equiv Y/R(S),$$

and consequently $R(S)$ is finite codimensional and closed. Hence, $S \in \Phi_-$. since $TS \in \Phi_-$ with a similar argument we obtain that $T \in \Phi_-$. The proof is now complete.

In the above Proposition 6.5 we have that

$$\begin{aligned} i(TS) &= i(T) + i(S) - \dim[S(0) \cap N(T)] \\ i(ST) &= i(T) + i(S) - \dim[S(0) \cap N(T)], \end{aligned}$$

so that $i(TS) = i(ST) \leq i(T) + i(S)$.

In short, our extension of Theorem 1.1 in [1] is the following theorem

Theorem 6.6. *Let X be a Banach space and let $S, T \in LR(X)$ be closed Then*

(i) *If $S, T \in \Phi_+$ then $ST \in \Phi_+$ and $TS \in \Phi_+$.*

(ii) *If $S, T \in \Phi_-$ with TS (resp. ST) is closed then $TS \in \Phi_-$ (resp. $ST \in \Phi_-$).*

(iii) *If $S, T \in \Phi$ then $TS \in \Phi$ and* $i(TS) = i(T) + i(S) + \dim X\big/R(S) + D(T) - \dim[S(0) \cap N(T)]$.

(iv) *If S and T everywhere defined and $TS \in \Phi_+$ then $S \in \Phi_+$.*

(v) *If S and T everywhere defined such that $TS \in \Phi$ and $ST \in \Phi$ then $S \in \Phi$ and $T \in \Phi$.* ◊

6.2. Semi-Fredholm perturbation classes

Our Lemma in Section 3, Definition 1.1 and Lemma 1.1 in [1] suggest the following notion.

Definition 6.7. Let X be a Banach space and let $S \in LR(X)$ be continuous. Then

(i) S is called a Fredholm perturbation if $T + S \in \Phi$ whenever $T \in \Phi$ with $D(T) \subset D(S)$ and $S(0) \subset T(0)$.

(ii) S is called an upper semi-Fredholm perturbation if $T + S \in \Phi_+$ whenever $T \in \Phi_+$ with $D(T) \subset D(S)$ and $S(0) \subset T(0)$.

(iii) S is called an lower semi-Fredholm perturbation if $T + S \in \Phi_-$ whenever $T \in \Phi_-$ with $D(T) \subset D(S)$ and $S(0) \subset T(0)$. ◊

The Lemma 6.2 shows that if $K \in LR(X)$ is compact then K is a Φ_+, Φ_- and Φ-perturbation and moreover the index is stable. We now try to extend Lemma 1.1 in [1] concerning the stability of the index under Φ_+, Φ_- and Φ-perturbation (in the sense of Definition 6.7). For this end, we shall use the following elementary Lemma.

Lemma 6.8. *Let X be a space complete and S, $T \in LR(X)$ such that T is closed, S is continuous, $D(T) \subset D(S)$ and $S(0) \subset T(0)$. Then T' is closed, S' is continuous, $D(T') \subset D(S')$ and $S'(0) \subset T'(0)$.* ◊

Proof. We note by Lemma 2.6 (i) that T' is closed. Since S is continuous then applying Lemma 2.6 (iii), we have that S' is continuous and $D(S') = S(0)^\perp$, so that

$$\begin{aligned} D(S') &= S(0)^\perp \supset T(0)^\perp \\ &= \left(D(T')^\top\right)^\perp \ \left(\text{see Lemma 2.6 } (i)\right) \\ &\supset \overline{D(T')} \\ &\supset D(T'). \end{aligned}$$

Hence $D(T') \subset D(S')$. Finally, by Lemma 2.6 (i) $S'(0) = D(S)^\perp \subset D(T)^\perp$ $\big($as again Lemma 2.6 (i) we have that $D(T) \subset D(S) = T'(0)\big)$. Therefore $S'(0) \subset T'(0)$.

The following proposition is the generalization of Lemma 1.1 in [1].

Proposition 6.9. *Let X and Y be Banach spaces and let $T \in LR(X, Y)$ be closed and $S \in LR(X, Y)$ be continuous such that $D(T) \subset D(S)$ and $S(0) \subset T(0)$. Then*

(i) *If $T \in \Phi_+$ and S is a Φ_+-perturbation, then $T + S \in \Phi_+$ and $i(T + S) = i(T)$.*

(ii) *If $T \in \Phi_-$ and S is a Φ_--perturbation, then $T + S \in \Phi_-$ and $i(T + S) = i(T)$.*

(iii) *If $T \in \Phi$ and S is a Φ-perturbation, then $T + S \in \Phi$ and $i(T + S) = i(T)$.* ◊

Proof. We first note that

(a) $(T+S)(0) = T(0)$ closed and $Q_{T+S}(T+S) = Q_T T + Q_R Q_S S$ is closed where $R := T(0)/\overline{S(0)}$. Indeed, $(T+S)(0) = T(0) + S(0) = T(0)$ (as $S(0) \subset T(0)$) which is closed, since T closed then applying Lemma 2.5 (i) we have $T(0)$ closed. Hence $Q_{T+S} = Q_T$. Moreover, by [16, p. 11, I.4.2 (e)] we obtain $Q_{T+S} = Q_T(T+S) = Q_T T + Q_T S$ and since $S(0) \subset T(0)$ then $\overline{S(0)} \subset \overline{T(0)} = T(0)$. Furthermore, applying Lemma 3.1 (ii) we have

$$Y/\overline{S(0)} \Big/ T(0)/\overline{S(0)} \equiv Y/T(0),$$

and $Q_T = Q_R Q_S$ where $R := T(0)/\overline{S(0)}$. Therefore ($a$) is true.

(b) $T+S$ is closed. Indeed, T is closed hence, by Lemma 2.5 (i) we have $Q_T T$ is closed single valued. Clearly $Q_R Q_S S$ is single valued continuous, so applying 2.5 (iv), we infer that $Q_T + Q_R Q_S S$ is closed. Since, $Q_{T+S}(T+S) = Q_T + Q_R Q_S S$ then by (a), we have that $Q_{T+S}(T+S)$ is a closed single valued and $(T+S)(0)$, so that by 2.5 (i) we obtain that $T+S$ is closed.

(i) Assume that $T \in \Phi_+$. Then T is closed with closed range in Banach spaces, hence by Lemma 2.7 $\gamma(T) > 0$ and $\dim N(T) < \infty$, hence $N(T)$ is closed, we can apply Lemma 2.5 (ix) to say that $N(T) = N(Q_T T)$ and $0 < \gamma(T) < \gamma(Q_T T)$. Let us consider two cases for S:

Case 1: $\|S\| < \gamma(T)$. We note by Lemma 3.6 (i), we have that $T \in \Phi_+$ if and only if $Q_T T$ is single valued upper semiFredholm. $D(Q_T T) = D(T) \subset D(S) = D(Q_R Q_S S)$ and $\|Q_R Q_S S\| \leq \|Q_S S\| := \|S\| < \gamma(T) = \gamma(Q_T T)$. Then, by virtue of [17, p. 112, v.1.6 (e)], we have that $Q_T + Q_R Q_S S \in \Phi_+$ and $i(Q_T + Q_R Q_S S) = i(Q_T T)$. Now, applying ($a$), we obtain that $Q_{T+S}(T+S) \in \Phi_+$ with $i(Q_T T) = i\big(Q_{T+S}(T+S)\big)$ and by (b), we have $T+S$ is closed, we can apply Lemma 3.6 (i) to conclude that $T+S \in \Phi_+$ and $i(T) = i(T+S)$.

Case 2: S is a Φ_+-perturbation. It is clear that

$$\|\lambda S\| := \|Q_{\lambda S} \lambda S\| = |\lambda| \|Q_S S\| := |\lambda| \|S\|,$$

since, by Lemma 2.1 (iii) we have $\lambda S(0) = S(0)$ which is a subspace and $D(\lambda S) = D(S)$ which is a subspace, for all $\lambda \in \mathbb{K}$, hence λS is a Φ_+−perturbation and so by Definition 6.7, we have that

(c) $T + \lambda S \in \Phi_+$ for all $\lambda \in \mathbb{K}$, so that, we can consider $i(T+\lambda S)$. Let $\mathbb{I} := [0,1]$ which its usual topology and let $\mathbf{Z} := \mathbb{Z} \cup \{-\infty\}$ ($\mathbb{Z}$:= integers) with the discrete topology. We shall prove that the map $\Theta : \mathbb{I} \longrightarrow \mathbf{Z}$ defined by $\Theta(\lambda) = i(T+\lambda S)$, $\lambda \in \mathbb{I}$, is continuous. For this,

(d) Let $\lambda_0 \in \mathbb{I}$ arbitrary but fixed. Then, for $\lambda \in \mathbb{I}$ such that $|\lambda - \lambda_0| < \gamma(T+\lambda S)\Big/\|S\|$ we have that $T + \lambda S + \lambda_0 S - \lambda_0 S \in \Phi_+$ and $i(T+\lambda S) = i(T + \lambda S + \lambda_0 S - \lambda_0 S)$.

Indeed, by (c), $T + \lambda_0 S \in \Phi_+$, in particular $\gamma(T+\lambda_0 S) > 0$ and since $0 < \|S\| < \infty$, we have $\gamma(T+\lambda_0 S)\Big/\|S\| > 0$. Let $\lambda \in \mathbb{I}$ such that $|\lambda - \lambda_0| < \gamma(T+\lambda_0 S)\Big/\|S\|$, then $\|(\lambda - \lambda_0)S\| = \|\lambda S - \lambda_0 S\| = |(\lambda - \lambda_0)| \|S\| < \gamma(T + \lambda_0 S)$ and thus by substituting $T + \lambda_0 S$ for T and $(\lambda - \lambda_0)S$ for S in the case 1, it follows the property (d).

(e) Let λ, λ_0, T and S satisfy the hypothesis in (d). Then

$$i(T+\lambda S) = i(T+\lambda_0 S).$$

Indeed, since $S(0) \subset T(0)$ we have that

$$Q_T = Q_{T+\lambda S} = Q_{T+\lambda_0 S} = Q_{T+\lambda S+\lambda_0 S-\lambda_0 S}.$$

We write $A := T + \lambda S + \lambda_0 S - \lambda_0 S$ and $Q_T = Q_R Q_S$ $\big($as in (a)$\big)$. We have that

$$\begin{aligned} Q_S(T + \lambda S + \lambda_0 S - \lambda_0 S) &= Q_S(T+\lambda S) + \lambda_0 Q_S S - \lambda_0 Q_S S \\ &= Q_S(T+\lambda S). \end{aligned}$$

So that $Q_T A = Q_A A = Q_R Q_S A = Q_R Q_S(T+\lambda S) = Q_T(T+\lambda S) = QT + \lambda S(T+\lambda S)$, that is $Q_A A = Q_{T+\lambda S}(T+\lambda S)$ with $A \in \Phi_+$ by $\big((d)\big)$ and $T + \lambda S \in \Phi_+$ by $\big((c)\big)$ Then by Lemma 2.3 (i) we have $i(Q_A A) = i(A)$ and also $i\big(Q_{T+\lambda S}(T + \lambda S) = i(T + \lambda S)$ and since $i(A) = i(T + \lambda_0 S)$ by (d) we conclude that

$$i(T + \lambda_0 S) = i(T + \lambda S),$$

as desired.

(f) The map

$$\begin{array}{rcl} \Theta : \mathbb{I} & \longrightarrow & \mathbb{Z} \\ \lambda & \longrightarrow & \Theta(\lambda) := i(T + \lambda S) \end{array}$$

is continuous. Indeed, let $\varepsilon > 0$, applying (e), we have that there exists $\delta := \gamma(T + \lambda_0)/\|S\|$ such that, if $\lambda \in \mathbb{I}$ with $|\lambda - \lambda_0| < \delta$, then $|i(T + \lambda S) - i(T + \lambda_0 S)| = |0| = 0 < \varepsilon$, so that Θ is continuous.

(g) $i(T) = i(T + S)$. By (f) Θ is continuous, so that $\Theta(\mathbb{I})$ is a connected set which therefore consists of only one point. It follows that $i(T) := \Theta(0) = \Theta(1) := i(T + S)$. Therefore (g) is true.

This completes the proof of (i).

(ii) By definition of Φ_-perturbation, we have clearly that for $\lambda \in \mathbb{K}\backslash\{0\}$, $T + \lambda S \in \Phi_-$ equivalently $(T + \lambda S)' \in \Theta_+$ and since $(T + \lambda S)' = T' + (\lambda S)' = T' + \lambda S'$ $\big($obvious that $(\lambda S)' = \lambda S'$, $\lambda \neq 0\big)$. Hence $T' + \lambda S' \in \Phi_+$. Furthermore by Lemma 6.8, S' is continuous then $\lambda S'$ is continuous $\lambda s'(0) = S'(0) \subset T'(0)$ and $D(T') \subset D(S') = D(\subset \lambda S')$ so that, we can apply (i) obtaining that $i(T') = i(T' + S')$ with T' and $T' + S' \in \Phi_+$ and thus

$$-i(T) = i(T') = i(T' + S') = i((T + S)') = -i(T + S).$$

Therefore $i(T) = i(T + S)$, as required.

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