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Blow up for a pseudo-parabolic equation with variable nonlinearity depending on (x,t) and negative initial energy

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ABSTRACT

We study the Dirichlet problem for the pseudo-parabolic equation

$$u_t - \operatorname{div}\left(a(x,t)|\nabla u|^{p(x,t)-2}\nabla u\right) - \Delta u_t = b(x,t)|u|^{q(x,t)-2}u$$

in the cylinder $Q_T = \Omega \times (0,T)$, where $\Omega \subset \mathbb{R}^d$ is a sufficiently smooth domain. The positive coefficients a, b and the exponents $p \geq 2$, $q > 2$ are given Lipschitz-continuous functions. The functions a, p are monotone decreasing, and b, q are monotone increasing in t. It is shown that there exists a positive constant $M = M(|\Omega|, \sup_{(x,t)\in Q_T} p(x,t), \sup_{(x,t)\in Q_T} q(x,t))$, such if the initial energy is negative,

$$E(0) = \int_\Omega \left(\frac{a(x,0)}{p(x,0)}|\nabla u_0(x)|^{p(x,0)} - \frac{b(x,0)}{q(x,0)}|u_0(x)|^{q(x,0)}\right) dx < -M,$$

then the problem admits a local in time solution with negative energy $E(t)$. If p and q are independent of t, then $M = 0$. For the solutions from this class, sufficient conditions for the finite time blow-up are derived.

1. Introduction

We study the Dirichlet problem for the pseudo-parabolic equation

$$\begin{cases} u_t - \operatorname{div}\left(a(x,t)|\nabla u|^{p(x,t)-2}\nabla u\right) - \Delta u_t = b(x,t)|u|^{q(x,t)-2}u & \text{for } (x,t) \in Q_T, \\ u = 0 \text{ on } \partial\Omega \times (0,T), \qquad u(x,0) = u_0(x) \text{ in } \Omega. \end{cases} \tag{1.1}$$

Here $\Omega \subset \mathbb{R}^d$ is a bounded domain with the sufficiently smooth boundary $\partial\Omega$, the exponents $p(x,t)$, $q(x,t)$ and the coefficients $a(x,t)$, $b(x,t)$ are given functions whose properties will be specified later. We

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are interested in the questions of local in time existence of solutions and the conditions of finite time blow-up. In the recent years, both questions were intensively studied. We refer here to papers [1–7] where these issues were discussed for the model Eq. (1.1) with the constant coefficients $a = b = 1$ and independent of t exponents $p(x)$, $q(x)$. These assumptions allow one to apply the traditional method based on the analysis of the functionals

$$J(u) = \int_\Omega \left(\frac{|\nabla u|^{p(x)}}{p(x)} - \frac{|u|^{q(x)}}{q(x)} \right) dx, \qquad I(u) = \int_\Omega \left(|\nabla u|^{p(x)} - |u|^{q(x)} \right) dx.$$

A complete classification of behavior of the weak solutions of the model problem (1.1) in terms of $J(u_0)$, $I(u_0)$ and $\|u_0\|_{W_0^{1,2}(\Omega)}$ is given in [6].

In this work, we are interested in the situation where the coefficients and the exponents in Eq. (1.1) are allowed to vary with t. To the best of our knowledge, for Eq. (1.1) this case has not yet been studied. We refer to [8] for a discussion of these questions for parabolic equations with variable growth, blow-up in solutions of Sobolev type equations is studied in the monograph [9]. The local existence and blow-up of solutions of pseudo-hyperbolic equations were studied in [10,11].

This paper is organized as follows. In Section 2 we prove the existence of local solutions. The existence theorem is proven under two different assumptions on the initial function u_0. If $u_0 \in W_0^{1,p(\cdot,0)}(\Omega)$ with $\inf_{(x,t)\in Q_T} p(x,t) \geq 2$ and

$$\int_\Omega \left(\frac{a(x,0)}{p(x,0)} |\nabla u_0(x)|^{p(x,0)} - \frac{b(x,0)}{q(x,0)} |u_0(x)|^{q(x,0)} \right) dx < -M \tag{1.2}$$

with a positive constant M, see (2.27) and (2.28), depending on $\sup_{(x,t)\in Q_T} p(x,t)$, $\sup_{(x,t)\in Q_T} q(x,t)$ and $|\Omega|$, then problem (1.1) has a local solution provided that at every moment t the functions $p^-(t) = \inf_{x\in\Omega} p(x,t)$ and $q^+(t) = \sup_{x\in\Omega} q(x,t)$ are subject to some conditions. These conditions coincide with those known for the case of constant or independent of t exponents p and q. Unlike this situation, if $u_0 \in W_0^{2,2}(\Omega)$ and satisfies (1.2), then a local solution exists for p and q within the ranges which are defined by the space dimension d only, and do not depend on each other.

In Section 3 we derive sufficient conditions for finite time blow-up. Two different situations are considered: $p^- := \inf_{(x,t)\in Q_T} p(x,t) = 2$, i.e., the equation may become semi-linear, or $p^- > 2$, where the equation is quasi-linear on the whole of the domain. In both cases, it is assumed that $\sup_{x\in\Omega} p(x,t) < \inf_{x\in\Omega} q(x,t)$ for every $t \in (0,T)$. The proofs of the blow-up in the cases $p^- = 2$ and $p^- > 2$ are different. Moreover, in the case $p^- = 2$ the reaction term has to satisfy the additional condition $\inf_{(x,t)\in Q_T} q(x,t) > 4$.

2. Existence of a local solution

The solution of problem (1.1) will be sought as an element of a variable Sobolev space. The definition and a brief description of these spaces are given in Appendix. Given $\tau \in (0,T)$, we denote $Q_\tau = \Omega \times (0,\tau)$.

Definition 2.1. A function u(x,t) is called a local solution of problem (1.1) if there exists $\Theta > 0$ such that

(1) $u \in \mathbb{W}_{p(\cdot)}(Q_\Theta) \cap L^{q(\cdot)}(Q_\Theta)$, $u_t \in L^2(Q_\Theta)$, $\nabla u_t \in (L^2(Q_\Theta))^d$;
(2) for every test-function $\phi \in \mathbb{W}_{p(\cdot)}(Q_\Theta) \cap L^{q(\cdot)}(Q_\Theta)$

$$\int_{Q_\Theta} \left(u_t\phi + \nabla u_t \cdot \nabla\phi + a|\nabla u|^{p-2}\nabla u \cdot \nabla\phi - b|u|^{q-2}u\phi \right) dxdt = 0; \tag{2.1}$$

(3) for every $\phi \in C^2(\Omega)$ $(u(\cdot,t) - u_0(\cdot), \phi(\cdot))_{2,\Omega} \to 0$ as $t \to 0$.

It is assumed throughout the text that

$$\begin{cases} 0 < a^- \le a(x,t) \le a^+ < \infty,\ 0 < b^- \le b(x,t) \le b^+ < \infty, \\ p,\ q,\ a,\ b \in C^{0,1}(\overline{Q}_T) \text{ with the Lipschitz constants } L_p,\ L_q,\ L_a,\ L_b, \\ p_t \le 0,\ q_t \ge 0,\ a_t \le 0,\ b_t \ge 0 \quad \text{a.e. in } Q_T. \end{cases} \tag{2.2}$$

We consider two different situations that correspond to different assumptions on the smoothness of the initial function u_0. Let us accept the notation

$$p^-(t) = \inf_{x\in\Omega} p(x,t), \quad p^+(t) = \sup_{x\in\Omega} p(x,t), \quad q^-(t) = \inf_{x\in\Omega} q(x,t), \quad q^+(t) = \sup_{x\in\Omega} q(x,t), \quad t \in [0,T],$$

so that

$$p^- = \inf_{t\in(0,T)} p^-(t), \quad p^+ = \sup_{t\in(0,T)} p^+(t), \quad q^- = \inf_{t\in(0,T)} q^-(t), \quad q^+ = \sup_{t\in(0,T)} q^+(t).$$

By

$$2^* = \begin{cases} \frac{2d}{d-2} & \text{if } d > 2, \\ \infty & \text{if } d = 2, \end{cases} \qquad (p^-(t))^* = \begin{cases} \frac{dp^-(t)}{d-p^-(t)} & \text{if } d > p^-(t), \\ \infty & \text{if } p^-(t) \le d \end{cases}$$

we denote the critical Sobolev exponents for the embeddings $W_0^{1,2}(\Omega) \subset L^s(\Omega)$ and $W_0^{1,p^-(t)}(\Omega) \subset L^s(\Omega)$. We will use the shorthand notation

$$p_0 = p(x,0), \quad q_0 = q(x,0), \quad a_0 = a(x,0), \quad b_0 = b(x,0).$$

Theorem 2.1. *Assume that conditions (2.2) are fulfilled. Let the data of problem (1.1) satisfy one of the following conditions:*

(i) $u_0 \in W_0^{1,p(\cdot,0)}(\Omega)$, $\partial\Omega \in C^k$, $k \ge 1 + d\left(\frac{1}{2} - \frac{1}{p^+}\right)$, *and*

$$2 \le p^-(t), \quad 2 < q^-(t) \le q^+(t) < (p^-(t))^*, \qquad t \in [0,T]; \tag{2.3}$$

(ii) $u_0 \in W_0^{2,2}(\Omega)$, $\partial\Omega \in C^3$, *and*

$$2 \le p^- \le p^+ < 2^*, \qquad 2 < q^+ < \begin{cases} \dfrac{2(d-2)}{d-4} & \text{if } d > 4, \\ \infty & \text{if } d \le 4. \end{cases} \tag{2.4}$$

There exists a number $M = M(|\Omega|, p^+, q^+) > 0$ *such that if the initial energy is negative*

$$\int_\Omega \left(\frac{a_0|\nabla u_0|^{p_0}}{p_0} - \frac{b_0|u_0|^{q_0}}{q_0} \right) dx < -M, \tag{2.5}$$

then there is t^* *such that for every* $\Theta \in (0,t^*)$ *problem (1.1) has a solution in the sense of Definition 2.1. The solution satisfies the energy equality: for a.e.* $t \in (0,\Theta)$

$$\frac{1}{2}\frac{d}{dt}\left(\|u(\cdot,t)\|^2_{W^{1,2}(\Omega)}\right) + \int_\Omega \left(a(x,t)|\nabla u|^{p(x,t)} - b(x,t)|u|^{q(x,t)} \right) dx = 0. \tag{2.6}$$

The energy

$$E(t) = \int_\Omega \left(\frac{a(x,t)|\nabla u|^{p(x,t)}}{p(x,t)} - \frac{b(x,t)|u|^{q(x,t)}}{q(x,t)} \right) dx$$

remains negative for all $t \in (0,\Theta)$ *and*

$$\|u_\tau\|^2_{W^{1,2}(Q_t)} + E(t) \le 0, \qquad Q_t = \Omega \times (0,t). \tag{2.7}$$

It is worth noting that in the case (ii) the existence is proven without any assumption on the relation between the exponents $p(x,t)$ and $q(x,t)$. The upper bounds for the their admissible values depend only on the space dimension d.

A solution of problem (1.1) is constructed as the limit of the sequence of finite-dimensional approximations

$$u_\epsilon = \lim_{m\to\infty} u^{(m)}, \qquad u^{(m)}(x,t) = \sum_{i=1}^{m} u_{i,m}(t)\psi_i(x) \in \mathcal{N}_m,$$

where (ψ_i, λ_i) are the eigenfunctions and eigenvalues of problem (A.1), and $\mathcal{N}_m$ is defined at the end of Appendix. The coefficients $u_{i,m}(t)$ are defined as the solutions of the Cauchy problem for the system of m ordinary nonlinear differential equations

$$\begin{cases} (1+\lambda_i)u'_{i,m}(t) = -\int_\Omega a(x,t)|\nabla u^{(m)}|^{p(x,t)-2}\nabla u^{(m)}\cdot\nabla\psi_i\,dx + \int_\Omega b(x,t)|u^{(m)}|^{q(x,t)-2}u^{(m)}\psi_i\,dx, \\ u_{i,m}(0) = u_{0,i}, \qquad i=1,2,\dots,m, \end{cases} \tag{2.8}$$

where the constants $u_{0,i}$ are the Fourier coefficients of u_0 in the basis $\{\psi_i\}$. We may choose (see Appendix)

$$u_0^{(m)} = \sum_{i=1}^{m} u_{0,i}\psi_i(x) \to u_0(x) \qquad \begin{array}{ll} \text{in } W_0^{1,p(\cdot,0)}(\Omega) & \text{if } u_0\in W_0^{1,p(\cdot,0)}(\Omega), \\ \text{in } W_0^{2,2}(\Omega) & \text{if } u_0 \in W_0^{2,2}(\Omega). \end{array}$$

By the Peano theorem, for every natural m system (2.1) has a solution $(u_{1,m}(t),\dots,u_{m,m}(t))$ on an interval $(0,T_m)$.

2.1. A priori estimates

Lemma 2.1. *Let conditions* (i) *of Theorem 2.1 be fulfilled. There exists $t^*\in(0,T]$ such that for every $\Theta\in(0,t^*)$ the functions $u^{(m)}$ satisfy the estimate*

$$\sup_{t\in(0,\Theta)} \|u^{(m)}(\cdot,t)\|^2_{W^{1,2}(\Omega)} + \int_{Q_\Theta} |\nabla u^{(m)}(x,t)|^{p(x,t)}\,dxdt + \int_{Q_\Theta} |u^{(m)}(x,t)|^{q(x,t)}\,dxdt \le C.$$

The constant C depends on Θ and $\|u_0\|_{W^{1,2}(\Omega)}$ but does not depend on m.

Proof. Multiplying jth equation in (2.1) by $u_{j,m}$ and summing over $j=1,\dots,m$ we obtain the equality

$$\frac{1}{2}\frac{d}{dt}\left(\|u^{(m)}(\cdot,t)\|^2_{W^{1,2}(\Omega)}\right) + \int_\Omega a(x,t)|\nabla u^{(m)}(x,t)|^{p(x,t)}\,dx = \int_\Omega b(x,t)|u^{(m)}(x,t)|^{q(x,t)}\,dx. \tag{2.9}$$

To estimate the source term on the right-hand side of (2.9) we fix $t\in(0,T_m)$ and consider two cases.

1. Let $2 < q^+(t) \le 2^*$. By the Young inequality and the embedding theorem

$$\int_\Omega |v|^{q(x,t)}\,dx \le 1 + \|v\|^{q^+(t)}_{q^+(t),\Omega} \le 1 + C\|v\|^{q^+(t)}_{W^{1,2}(\Omega)}.$$

Notice that $(p^-(t))^* = 2^*$ if $p^-(t)=2$. Therefore, we get

$$\frac{1}{2}\frac{d}{dt}\left(\|u^{(m)}(\cdot,t)\|^2_{W^{1,2}(\Omega)}\right) \le b^+\left(1 + C\left(\|u^{(m)}(\cdot,t)\|^2_{W^{1,2}(\Omega)}\right)^{q^+/2}\right).$$

2. Let $2^* < q^+(t) < (p^-(t))^*$ and $p^-(t) > 2$. By the Gagliardo–Nirenberg inequality, for every $v\in W_0^{1,p(\cdot,t)}(\Omega)\subseteq W_0^{1,p^-(t)}(\Omega)$

$$\int_\Omega |v|^{q^+(t)}\,dx \le C\|\nabla v\|^{q^+(t)\theta(t)}_{p^-(t),\Omega}\|v\|^{q^+(t)(1-\theta(t))}_{2^*,\Omega} \le C'\|\nabla v\|^{q^+(t)\theta(t)}_{p^-(t),\Omega}\|v\|^{(1-\theta(t))q^+(t)}_{W^{1,2}(\Omega)}, \tag{2.10}$$

provided

$$\theta(t)=\frac{\frac{1}{2^*}-\frac{1}{q^+(t)}}{\frac{1}{2^*}+\frac{1}{d}-\frac{1}{p^-(t)}}\in(0,1)\quad\Leftarrow\quad\begin{cases} q^+(t)>2^*,\\ \frac{1}{p^-(t)}<\frac{1}{d}+\frac{d-2}{2d}\quad\Leftrightarrow\quad p^-(t)>2,\\ q^+(t)<\dfrac{dp^-(t)}{d-p^-(t)}\le(p^-(t))^*.\end{cases}$$

The second condition in (2.3) yields the inequality $\frac{\theta(t)q^+(t)}{p^-(t)}<1$. By Young's inequality, we deduce from (2.10) that for every $\epsilon>0$

$$\begin{aligned}\int_\Omega b|u^{(m)}|^{q^+(t)}\,dx&\le Cb^+\left(\int_\Omega|\nabla u^{(m)}|^{p^-(t)}\,dx\right)^{\frac{\theta(t)q^+(t)}{p^-(t)}}\|u^{(m)}\|_{W^{1,2}(\Omega)}^{q^+(t)(1-\theta(t))}\\ &\le\epsilon a^-\int_\Omega|\nabla u^{(m)}|^{p^-(t)}\,dx+C'\|u^{(m)}\|_{W^{1,2}(\Omega)}^{(1-\theta(t))\frac{p^-(t)q^+(t)}{p^-(t)-\theta(t)q^+(t)}}\\ &\le\epsilon\int_\Omega a|\nabla u^{(m)}|^p\,dx+C+C'\|u^{(m)}\|_{W^{1,2}(\Omega)}^{(1-\theta(t))\frac{p^-(t)q^+(t)}{p^-(t)-\theta(t)q^+(t)}}.\end{aligned}\tag{2.11}$$

Set

$$Y(t)=\|u^{(m)}(\cdot,t)\|^2_{W^{1,2}(\Omega)}+\int_0^t\int_\Omega a(x,\tau)|\nabla u^{(m)}(x,\tau)|^{p(x,\tau)}\,dxd\tau.$$

For a sufficiently small $\epsilon>0$, from (2.9) and (2.11) we obtain the inequality

$$\begin{cases} Y'(t)\le C'+C''Y^\gamma(t),\quad t\in(0,T_m),\\ Y(0)=\|u^{(m)}(\cdot,0)\|^2_{W^{1,2}(\Omega)}\le\|u_0\|^2_{W^{1,2}(\Omega)}:=\sigma,\qquad \gamma=\sup_{t\in(0,T)}\frac{q^+(t)p^-(t)(1-\theta(t))}{2(p^-(t)-\theta(t)q^+(t))}>1.\end{cases}\tag{2.12}$$

Consider the function

$$Z(t)=\frac{1}{\left(Z_0^{1-\gamma}-2C''(\gamma-1)t\right)^{\frac{1}{\gamma-1}}},\qquad Z_0=\max\left\{\sigma,\left(C'C''^{-1}\right)^{\frac{1}{\gamma}}\right\},\tag{2.13}$$

which solves the problem

$$\begin{cases} Z'(t)=2C''Z^\gamma(t)\quad\text{for } 0<t<t^*=\dfrac{1}{2C''Z_0^{\gamma-1}},\\ Z(0)=Z_0\ge\sigma.\end{cases}\tag{2.14}$$

Since $C''Z^\gamma(t)>C''Z_0^\gamma\ge C'$, $Z(t)$ satisfies the differential inequality

$$Z'(t)>C'+C''Z^\gamma(t)\quad\text{in }(0,t^*).\tag{2.15}$$

Summing $(2.12)_1$ and $-(2.15)$, and using the Lagrange mean value theorem, we obtain the differential inequality for the function $X(t)\equiv Y(t)-Z(t)$:

$$X'(t)<\gamma C''\int_0^1(\theta Z(t)+(1-\theta)Y(t))^{\gamma-1}\,d\theta X(t)\quad\text{for } 0<t<\min\{T_m,t^*\},\qquad X(0)\le 0.$$

By the Grönwall lemma $X(t)=Y(t)-Z(t)\le 0$ for $0<t<\min\{T_m,t^*\}$. If $T_m<t^*$, then $Y(T_m)<Z(T_m)<\infty$. System (2.1) can be solved then on an interval (T_m,T_m+h) with the initial data taken at the moment T_m. Since the majorant function $Z(t)$ does not change, the solution $u^{(m)}$ continues to the interval $(0,t^*)$ and the estimate $Y(t)\le Z(t)$ holds for all $t\in(0,t^*)$. □

Corollary 2.1. *Inequality (2.12) yields the uniform lower bound of the time interval where $\|u^{(m)}(\cdot,t)\|^2_{W^{1,2}(\Omega)}$ remain bounded. If $Y(t)\to\infty$ as $t\to t^*$, it is necessary that $\int_\sigma^\infty\dfrac{ds}{C'+C''s^\gamma}\le t^*$.*

Let us agree to use the shorthand notation $|v_{xx}|^2 = \sum_{i,j=1}^{d}(D^2_{x_i x_j} v)^2$.

Lemma 2.2. *Assume that conditions* (ii) *of Theorem 2.1 are fulfilled. Then there exists* t^* *such that for every* $\Theta \in (0, t^*)$ *the functions* $u^{(m)}$ *satisfy the uniform estimate*

$$\sup_{t\in(0,\Theta)} \|u^{(m)}(\cdot,t)\|^2_{W^{2,2}(\Omega)} + \int_{Q_\Theta} |\nabla u^{(m)}|^{p(x,t)-2} |u^{(m)}_{xx}|^2 \, dxdt + \int_{Q_\Theta} |\nabla u^{(m)}|^{p(x,t)} \, dxdt \\ \leq C\left(1 + \|u_0\|^2_{W^{2,2}(\Omega)}\right). \tag{2.16}$$

Moreover, for $t \in (0, \Theta)$

$$\int_\Omega |\nabla u^{(m)}|^{p(x,t)} \, dx + \int_\Omega |u^{(m)}|^{q(x,t)} \, dx \leq C'$$

with an independent of m *constant* C'.

Proof. Multiplying the jth equation of (2.1) by $-\lambda_j u^{(m)}_j$, summing over $j = 1, \dots, m$, and using the embedding inequality $\|v\|_{2(q^+-1),\Omega} \leq C\|\Delta v\|_{2,\Omega}$ $\forall v \in W^{2,2}_0(\Omega)$, we obtain

$$\frac{1}{2}\frac{d}{dt}\left(\|\nabla u^{(m)}\|^2_{2,\Omega} + \|\Delta u^{(m)}\|^2_{2,\Omega}\right) + \int_\Omega \operatorname{div}\left(a(x,t)|\nabla u^{(m)}|^{p(x,t)-2}\nabla u^{(m)}\right)\Delta u^{(m)} \, dx = \\ = -\int_\Omega b(x,t)|u^{(m)}|^{q(x,t)-2}u^{(m)}\Delta u^{(m)} \, dx \leq C\left(1 + \|u^{(m)}\|^{q^+-1}_{2(q^+-1),\Omega}\right)\|\Delta u^{(m)}\|_{2,\Omega} \\ \leq C\left(1 + \|\Delta u^{(m)}\|^2_{2,\Omega}\right) + C'\|\Delta u^{(m)}\|^{q^+}_{2,\Omega}. \tag{2.17}$$

Following the proof of [12, Lemma 3.2], we rewrite the second term on the left-hand side of (2.17) as follows:

$$-\int_\Omega \operatorname{div}\left(a|\nabla u^{(m)}|^{p(x,t)-2}\nabla u^{(m)}\right)\Delta u^{(m)} \, dx = -\int_\Omega a|\nabla u^{(m)}|^{p(x,t)-2}|u^{(m)}_{xx}|^2 \, dx + J_1 + J_2 + J_a + J_{\partial\Omega}, \tag{2.18}$$

whence

$$J_1 = \int_\Omega a(2 - p(x,t))|\nabla u^{(m)}|^{p(x,t)-4}\left(\sum_{k=1}^{d}\left(\nabla u^{(m)} \cdot \nabla (u^{(m)})_{x_k}\right)^2\right) dx,$$

$$J_2 = -\sum_{i,k=1}^{d}\int_\Omega a u^{(m)}_{x_i x_k} u^{(m)}_{x_i} |\nabla u^{(m)}|^{p-2} p_{x_k} \ln|\nabla u^{(m)}| \, dx,$$

$$J_a = -\int_\Omega |\nabla u^{(m)}|^{p(x,t)-2}\sum_{i,k=1}^{d} u^{(m)}_{x_i x_k} u^{(m)}_{x_i} a_{x_k} \, dx,$$

$$J_{\partial\Omega} = -\int_{\partial\Omega} a|\nabla u^{(m)}|^{p-2}\left(\Delta u^{(m)}(\nabla u^{(m)} \cdot \mathbf{n}) - \nabla u^{(m)} \cdot \nabla(\nabla u^{(m)} \cdot \mathbf{n})\right) dS.$$

Each of the terms J_1, J_2, J_a, $J_{\partial\Omega}$ can be bounded by the quantities depending only on the data. Because of the assumption $p \geq 2$, we get $J_1 \leq 0$. By Young's inequality, for every $\delta > 0$

$$|J_2| = \left|\sum_{i,k=1}^{d}\int_\Omega a u^{(m)}_{x_i x_k} u^{(m)}_{x_i}|\nabla u^{(m)}|^{p-2} p_{x_k} \ln|\nabla u^{(m)}| \, dx\right| \\ \leq C\int_\Omega \left(|\nabla u^{(m)}|^{\frac{p-2}{2}}|u^{(m)}_{xx}|\right)\left(|\nabla u^{(m)}|^{\frac{p}{2}}|\ln|\nabla u^{(m)}||\right) dx \\ \leq \delta\int_\Omega |\nabla u^{(m)}|^{p-2}|u^{(m)}_{xx}|^2 \, dx + C\int_\Omega |\nabla u^{(m)}|^p \ln^2|\nabla u^{(m)}| \, dx \tag{2.19}$$

with $C = C(d, L_p, a^{\pm}, \delta)$. If $\delta > 0$ is sufficiently small, the first term on the right-hand side of (2.19) is absorbed in the first term of (2.18). For every $\rho > 0$, the last term on the right-hand side of (2.19) is estimated by $C\int_{\Omega} |\nabla u^{(m)}|^{p^{+}+\rho}\,dx + C'$ with a constant C' depending on ρ. By the embedding theorem, for $u^{(m)} \in W_0^{2,2}(\Omega)$

$$\int_{\Omega} |\nabla u^{(m)}|^{p^{+}+\rho}\,dx \le C\|\Delta u^{(m)}\|_{2,\Omega}^{p^{+}+\rho}, \tag{2.20}$$

with $\rho > 0$ sufficiently small to fulfill the first condition of (2.4). The estimate on $J_{\partial\Omega}$ follows from [12, Lemma 4.4] and (2.20):

$$\begin{aligned}|J_{\partial\Omega}| &\le \delta \int_{\Omega} |\nabla u^{(m)}|^{p-2}|u_{xx}^{(m)}|^2\,dx + C\left(1 + \int_{\Omega} |\nabla u^{(m)}|^{p}\,dx\right)\\ &\le \delta \int_{\Omega} |\nabla u^{(m)}|^{p-2}|u_{xx}^{(m)}|^2\,dx + C\left(1 + \|\Delta u^{(m)}\|_{2,\Omega}^{p^{+}+\rho}\right).\end{aligned}$$

By using Young's inequality we estimate

$$|J_a| = \left|\int_{\Omega} |\nabla u^{(m)}|^{p(x,t)-2} \sum_{i,k=1}^{d} u_{x_i x_k}^{(m)} u_{x_i}^{(m)} a_{x_k}\,dx\right| \le \lambda \int_{\Omega} |\nabla u^{(m)}|^{p-2}|u_{xx}^{(m)}|^2\,dx + C\int_{\Omega} |\nabla u^{(m)}|^{p}\,dx$$

with an arbitrary $\lambda > 0$ and a constant $C = C(d, \lambda, L_a)$. Set $S_m(t) = \|\nabla u^{(m)}(\cdot,t)\|_{2,\Omega}^2 + \|\Delta u^{(m)}(\cdot,t)\|_{2,\Omega}^2$. Plugging the above estimates into (2.17) and dropping the nonnegative term on the left-hand side, we arrive at the ordinary differential inequality for $S_m(t)$:

$$S_m'(t) \le C + C'S_m(t) + C''S_m^{\frac{q^{+}}{2}}(t) + C'''S_m^{\frac{p^{+}+\rho}{2}}(t) \le C_1 + C_2 S_m^{\mu}(t), \quad t \in (0, T_m), \tag{2.21}$$

with the exponent $2\mu = \max\{p^{+}+\rho, q^{+}\} > 2$. The existence of a barrier of the form (2.13) on an independent of m interval $(0, t^*)$ follows as in the proof of Lemma 2.1. Inequalities (2.4), estimate (2.16) and the embedding theorems yield the continuous inclusions $W_0^{2,2}(\Omega) \subset L^{2(q^{+}-1)}(\Omega) \subset L^{q^{+}}(\Omega)$, $W_0^{2,2}(\Omega) \subset L^{p^{+}}(\Omega)$ and the last inequality of Lemma 2.2. □

Remark 2.1. If p is independent of x, the condition on p^{+} in (2.4) can be omitted because $J_2 = 0$.

Corollary 2.2. *It follows from (2.21) that the functions $S_m(t)$ remain bounded on the interval $(0, t^*)$, where t^* is independent of m and satisfies the inequality*

$$\int_{S^*}^{\infty} \frac{ds}{C_1 + C_2 s^{\mu}} \le t^*, \quad \mu > 1,$$

whence $S^* = \|\nabla u_0\|_{2,\Omega}^2 + \|\Delta u_0\|_{2,\Omega}^2$. By the embedding theorem, $\|u^{(m)}(\cdot,t)\|_{W^{1,2}(\Omega)}^2 \le CS_m(t)$ for $t \in (0, t^*)$.

Lemma 2.3. *Assume that either the conditions of Lemma 2.1, or Lemma 2.2 are fulfilled. There is a positive constant $M \equiv M(|\Omega|, p^{+}, q^{+})$ such that if the initial function satisfies (2.5), then for every $\Theta \in (0, t^*)$*

$$\|u_t^{(m)}\|_{W^{1,2}(Q_\Theta)}^2 + \sup_{t\in(0,\Theta)} \int_{\Omega} |\nabla u^{(m)}|^{p(x,t)}\,dx + \sup_{t\in(0,\Theta)} \int_{\Omega} |u^{(m)}|^{q(x,t)}\,dx \le C_* \tag{2.22}$$

with a constant C_* depending on $\|u_0\|_{W^{1,p(\cdot,0)}(\Omega)}$, or $\|u_0\|_{W^{2,2}(\Omega)}$, and Θ, but independent of m. Moreover,

$$E_m(t) \equiv \int_{\Omega} \left(\frac{a|\nabla u^{(m)}|^{p}}{p} - \frac{b|u^{(m)}|^{q}}{q}\right)dx < 0. \tag{2.23}$$

Proof. Multiplying jth equation in (2.1) by $u'_{j,m}$, summing the results over $j=1,\ldots,m$, and integrating over the interval $(0,t)$, we obtain the equality

$$\int_0^t \|u_\tau^{(m)}(\cdot,\tau)\|^2_{W^{1,2}(\Omega)}\,d\tau + \mathcal{I}_a + \mathcal{I}_b + \mathcal{I}_p + \mathcal{I}_q + E_m(t) = E_m(0), \tag{2.24}$$

where

$$E_m(0) = \int_\Omega \left(\frac{a_0|\nabla u_0^{(m)}|^{p_0}}{p_0} - \frac{b_0|u_0^{(m)}|^{q_0}}{q_0}\right)dx,\ \mathcal{I}_a = -\int_0^t\int_\Omega a_\tau \frac{|\nabla u^{(m)}|^p}{p}\,dxd\tau,\ \mathcal{I}_b = \int_0^t\int_\Omega b_\tau \frac{|u^{(m)}|^q}{q}\,dxd\tau,$$

$$\mathcal{I}_p = \int_0^t\int_\Omega a|\nabla u^{(m)}|^p\left(\frac{1}{p^2} - \frac{\ln|\nabla u^{(m)}|}{p}\right)p_\tau\,dxd\tau,\ \mathcal{I}_q = -\int_0^t\int_\Omega b|u^{(m)}|^q\left(\frac{1}{q^2} - \frac{\ln|u^{(m)}|}{q}\right)q_\tau\,dxd\tau.$$

For a given $u^{(m)}$ and a fixed $\tau \in (0,t)$ we split the domain Ω into the subdomains

$$\Omega_q^+(\tau) = \left\{x\in\Omega : \ln|u^{(m)}(x,\tau)| > \frac{1}{q(x,\tau)}\right\},$$
$$\Omega_q^-(\tau) = \left\{x\in\Omega : \ln|u^{(m)}(x,\tau)| \le \frac{1}{q(x,\tau)}\right\}.$$

Taking into account the assumption $q_t \ge 0$, we may estimate

$$\begin{aligned}
\mathcal{I}_q &= \int_0^t\int_\Omega b|u^{(m)}|^q\left(-\frac{1}{q^2} + \frac{\ln|u^{(m)}|}{q}\right)q_\tau\,dxd\tau = \\
&= \int_0^t\int_{\Omega_q^+(\tau)} b|u^{(m)}|^q\left|-\frac{1}{q^2} + \frac{\ln|u^{(m)}|}{q}\right|q_\tau\,dxd\tau \\
&\quad - \int_0^t\int_{\Omega_q^-(\tau)} b|u^{(m)}|^q\left|-\frac{1}{q^2} + \frac{\ln|u^{(m)}|}{q}\right|q_\tau\,dxd\tau \ge \\
&- \int_0^t\int_{\Omega_q^-(\tau)} b|u^{(m)}|^q\left|-\frac{1}{q^2} + \frac{\ln|u^{(m)}|}{q}\right|q_\tau\,dxd\tau \\
&\quad \ge -\int_0^t\int_\Omega b|u^{(m)}|^q\left|-\frac{1}{q^2} + \frac{\ln|u^{(m)}|}{q}\right|q_\tau\,dxd\tau.
\end{aligned} \tag{2.25}$$

Split $\Omega = \Omega_p^+(\tau)\cup\Omega_p^-(\tau)$, $\tau\in(0,t)$, with

$$\Omega_p^+(\tau) = \left\{x\in\Omega : \ln|\nabla u^{(m)}(x,\tau)| > \frac{1}{p(x,\tau)}\right\},$$
$$\Omega_p^-(\tau) = \left\{x\in\Omega : \ln|\nabla u^{(m)}(x,\tau)| \le \frac{1}{p(x,\tau)}\right\}.$$

Proceeding in the same way as in estimating $\mathcal{I}_q$ and using the assumption $p_t \le 0$, we obtain

$$\begin{aligned}
\mathcal{I}_p &= \int_0^t\int_\Omega a|\nabla u^{(m)}|^p\left(\frac{1}{p^2} - \frac{\ln|\nabla u^{(m)}|}{p}\right)p_\tau\,dxd\tau = \\
&= \int_0^t\int_\Omega a|\nabla u^{(m)}|^p\left(\frac{\ln|\nabla u^{(m)}|}{p} - \frac{1}{p^2}\right)|p_\tau|\,dxd\tau = \\
&= \int_0^t\int_{\Omega_p^+(\tau)} a|\nabla u^{(m)}|^p\left(\frac{\ln|\nabla u^{(m)}|}{p} - \frac{1}{p^2}\right)|p_\tau|\,dxd\tau \\
&- \int_0^t\int_{\Omega_p^-(\tau)} a|\nabla u^{(m)}|^p\left|\frac{\ln|\nabla u^{(m)}|}{p} - \frac{1}{p^2}\right||p_\tau|\,dxd\tau \\
&\ge -\int_0^t\int_{\Omega_p^-(\tau)} a|\nabla u^{(m)}|^p\left|\frac{\ln|\nabla u^{(m)}|}{p} - \frac{1}{p^2}\right||p_\tau|\,dxd\tau \ge
\end{aligned}$$

$$
\begin{aligned}
-\int_0^t\int_\Omega a|\nabla u^{(m)}|^p\left|\frac{\ln|\nabla u^{(m)}|}{p}-\frac{1}{p^2}\right||p_\tau|\,dxd\tau= \\
=\int_0^t\int_\Omega a|\nabla u^{(m)}|^p\left|\frac{\ln|\nabla u^{(m)}|}{p}-\frac{1}{p^2}\right|p_\tau\,dxd\tau.
\end{aligned}
\tag{2.26}
$$

Set

$$
\begin{aligned}
\widehat{M}&=b^+\sup\left\{|\theta|^q\left|-\frac{1}{q^2}+\frac{\ln|\theta|}{q}\right|:\quad q\in[q^-,q^+],\quad |\theta|\le \mathrm{e}^{\frac{1}{q^-}}\right\},\\
\widetilde{M}&=a^+\sup\left\{|\theta|^p\left|-\frac{1}{p^2}+\frac{\ln|\theta|}{p}\right|:\quad p\in[p^-,p^+],\quad |\theta|\le \mathrm{e}^{\frac{1}{p^-}}\right\}.
\end{aligned}
$$

Gathering (2.25), (2.26), we may write

$$
\begin{aligned}
-\mathcal{I}_q-\mathcal{I}_p&\le\int_0^t\int_\Omega(\widehat{M}q_\tau-\widetilde{M}p_\tau)\,dxd\tau=\int_\Omega\left(\widehat{M}(q(x,t)-q(x,0))-\widetilde{M}(p(x,t)-p(x,0))\right)\,dx\\
&\le\int_\Omega\left(\widehat{M}q(x,t)+\widetilde{M}p(x,0)\right)\,dx\le|\Omega|\left(\widehat{M}q^++\widetilde{M}p^+\right)=:M.
\end{aligned}
\tag{2.27}
$$

Let us claim that

$$
E(0)+M<0. \tag{2.28}
$$

Since $E_m(0)\to E(0)$ as $m\to\infty$, it follows from the choice of the sequence $\{u_0^{(m)}\}$ that $E_m(0)+M<0$ for the sufficiently large m. Moreover, since $a_t\le 0$ and $b_t\ge 0$, it follows from (2.24) and (2.28) that

$$
\int_0^t\|u_\tau^{(m)}(\cdot,\tau)\|^2_{W_0^{1,2}(\Omega)}\,d\tau+E_m(t)\le -M
$$

for the sufficiently large m. Hence, $E_m(t)\le 0$ for all $t\in(0,t^*)$.

Let the data satisfy the conditions of Lemma 2.1. By using (2.11) we deduce that

$$
\int_\Omega a|\nabla u^{(m)}|^p\,dx\le\int_\Omega b|u^{(m)}|^q\,dx\le\epsilon\int_\Omega a|\nabla u^{(m)}|^p\,dx+C+C'\|u^{(m)}\|^\lambda_{2,\Omega}\le\epsilon\int_\Omega a|\nabla u^{(m)}|^p\,dx+C''
$$

with an arbitrary $\epsilon>0$ and a constant C'' depending on $\|u_0\|_{W^{1,p(\cdot,0)}(\Omega)}$ and ϵ but independent of m. Thus, for every $t\in(0,\Theta)$ both terms of $E_m(t)$ are bounded by an independent of m constant.

If the conditions of Lemma 2.2 are fulfilled, then both terms of $E_m(t)$ are uniformly bounded by virtue of the embeddings $W_0^{2,2}(\Omega)\subset L^{q+}(\Omega)$, $W_0^{2,2}(\Omega)\subset W_0^{1,p^+}(\Omega)$. $\square$

2.2. Proof of Theorem 2.1

The uniform estimates of Lemmas 2.1, 2.2, 2.3 allow one to extract a subsequence with the following convergence properties: there exist $u(x,t)$ and η such that

$u^{(m)}\to$ u in $C^0([0,\Theta];L^2(\Omega))$and a.e. in$Q_\Theta$(Aubin–Lions Lemma), $u_t^{(m)}\rightharpoonup u_t$ in $L^2(0,\Theta;W_0^{1,2}(\Omega))$,
$\nabla u^{(m)}\to\nabla$ u in $C^{\frac12}([0,\Theta];(L^2(\Omega))^d)$, $\nabla u^{(m)}\rightharpoonup\nabla$ u in $(L^{p(\cdot)}(Q_\Theta))^d$,
$a(x,t)|\nabla u^{(m)}|^{p(x,t)-2}\nabla u^{(m)}\rightharpoonup\eta$ in $(L^{p'(\cdot)}(Q_\Theta))^d$,
$b(x,t)|u^{(m)}|^{q(x,t)-2}u^{(m)}\rightharpoonup b(x,t)|u|^{q(x,t)-2}$u in $L^{q'(\cdot)}(Q_\Theta)$ [13, Ch.1,Lemma 1.3].

By the method of construction, for every m and $k\le m$

$$
\int_{Q_\Theta}\left(u_t^{(m)}\xi_k+\nabla u_t^{(m)}\cdot\nabla\xi_k+a|\nabla u^{(m)}|^{p-2}\nabla u^{(m)}\cdot\nabla\xi_k-b|u^{(m)}|^{q-2}u^{(m)}\xi_k\right)\,dxdt=0
$$

for every $\xi_k \in \mathcal{N}_k$. By letting $m \to \infty$ we obtain the equality

$$\int_{Q_\Theta} \left(u_t \xi_k + \nabla u_t \cdot \nabla \xi_k + \eta \cdot \nabla \xi_k - b|u|^{q-2} u \xi_k \right) dxdt = 0. \tag{2.29}$$

Since $\mathbb{W}_{p(\cdot)}(Q_\Theta) = \bigcup_{k=1}^{\infty} \mathcal{N}_k$ (see Appendix), the same equality holds for every $\xi \in \mathbb{W}_{p(\cdot)}(Q_\Theta)$. For the proof we take a sequence $\xi_k \to \xi$ in $\mathbb{W}_{p(\cdot)}(Q_\Theta)$ and pass to the limit in (2.29) as $k \to \infty$. The limit η is identified by the standard monotonicity arguments. The initial condition is fulfilled by continuity.

To prove identity (2.6) we fix $t, t+h \in (0, t^*)$, $h > 0$, and choose u for the test-function in identity (2.1). Dividing by h we have

$$\frac{1}{h} \int_t^{t+h} \int_\Omega \left(u_\tau u + \nabla u_\tau \cdot \nabla u + a|\nabla u|^p - b|u|^q \right) dx d\tau = 0.$$

By the Lebesgue differentiation theorem, for a.e. t each term of this equality has a limit as $h \to 0$, whence (2.6). Inequality (2.7) follows from (2.23) and the Fatou lemma, the inequality $E(t) < 0$ is an immediate consequence of (2.7).

3. Blow up of a local solution

Let $u(x,t)$ be a local solution of problem (1.1). Introduce the function

$$f(t) = \frac{1}{2} \int_0^t \|u(\cdot,\tau)\|^2_{W^{1,2}(\Omega)} d\tau \tag{3.1}$$

and assume that the solution $u(x,t)$ satisfies inequality (2.7). By choosing u for the test-function in (2.1) we conclude that the energy equality (2.6) is fulfilled. By virtue of this inequality, for every $t \in (0, t^*)$

$$f'(t) = \frac{1}{2} \|u(\cdot,t)\|^2_{W^{1,2}(\Omega)} = \frac{1}{2} \|u_0\|^2_{W^{1,2}(\Omega)} + \int_0^t \int_\Omega \left(b|u|^q - a|\nabla u|^p \right) dx d\tau \geq 0. \tag{3.2}$$

It follows then from (2.7) that for a.e. $t \in (0, t^*)$

$$f''(t) = \int_\Omega \left(u u_t + \nabla u \cdot \nabla u_t \right) dx = \int_\Omega \left(b|u|^q - a|\nabla u|^p \right) dx. \tag{3.3}$$

Theorem 3.1. *Let $u(x,t)$ we a local solution of problem (1.1) such that inequality (2.7) is fulfilled. Assume that the exponents $p(x,t)$, $q(x,t)$ satisfy one of the following conditions:*

(i) *there is $\delta > 0$ such that*

$$2 \leq p^- \leq p^+(t) \leq \max\{4(1+\delta), p^+(t)\} < q^-(t), \quad t \in [0,T]; \tag{3.4}$$

(ii)

$$2 < p^-(t) \leq p^+(t) < q^-(t), \quad t \in [0,T]. \tag{3.5}$$

Then the local solution $u(x,t)$ blows-up in a finite time: there exists $T^* < \infty$ such that

$$\|u(\cdot,t)\|^2_{W^{1,2}(\Omega)} \nearrow \infty, \quad \textit{as } t \nearrow T^*.$$

Remark 3.1. The assertions of Theorem 3.1 are independent of the conditions of the existence Theorem 2.1 and apply to every local solution of problem (1.1), provided (2.7) is fulfilled. The conditions of these theorems are compatible. Assumptions (3.4) agree with the assumptions (ii) of Theorem 2.1 if $u_0 \in W_0^{2,2}(\Omega)$, $\partial\Omega \in C^3$, and (2.4) holds. Under assumptions (3.5) problem (1.1) admits a local solution if $u_0 \in W_0^{1,p(\cdot,0)}(\Omega)$, $\partial\Omega \in C^k$, and the second inequality of (2.3) is fulfilled.

Remark 3.2. Under the conditions of the existence theorem, the lower bounds for the blow-up moment T^* follow from Corollaries 2.1, 2.2.

3.1. Proof of Theorem 3.1(i)

Assumption (3.4) allows one to find a positive function $\lambda(t)$ such that

$$\frac{1}{q^-(t)} < \lambda(t) < \frac{1}{\max\{4(1+\delta), p^+(t)\}}. \tag{3.6}$$

We multiply equality (3.3) by $\lambda(t)$ and add the resulting equality to inequality (2.7) integrated over the interval $(0,t)$:

$$E(t) + \lambda(t)\int_\Omega (-a|\nabla u|^p + b|u|^q)\,dx + \int_0^t \|u_\tau(\cdot,\tau)\|^2_{W^{1,2}(\Omega)}\,d\tau \le \lambda(t)f''(t).$$

This inequality can be continued as follows:

$$\int_0^t \|u_\tau(\cdot,\tau)\|^2_{W^{1,2}(\Omega)}\,d\tau + \int_\Omega \left(a^-\left(\frac{1}{p^+(t)} - \lambda(t)\right)|\nabla u|^p + b^-\left(\lambda(t) - \frac{1}{q^-(t)}\right)|u|^q\right)dx \le \lambda(t)f''(t). \tag{3.7}$$

We adapt the method from [8, pp. 258–261], which was applied to parabolic equations with variable nonlinearity. Let us denote by T^* the time of existence of the solution u:

$$T^* = \sup\{t > 0 :\ f'(s) < \infty \text{ for } s < t\}.$$

Since $u_t \in L^2(0,\Theta; W_0^{1,2}(\Omega))$ for some $\Theta > 0$, and $f'(0) < \infty$, it is necessary that $T^* > 0$. The solution $u(x,t)$ blows-up in finite time if T^* is finite. By virtue of (3.6) and (3.7)

$$0 < \int_0^t \|u_\tau(\cdot,\tau)\|^2_{W^{1,2}(\Omega)}\,d\tau \le \lambda(t)f''(t). \tag{3.8}$$

Using Hölder's inequality and (3.8), we obtain the following chain of relations:

$$\begin{aligned}
(f'(t) - f'(0))^2 &= \left(\int_0^t \frac{d}{d\tau}\left(\frac{1}{2}\|u(\cdot,\tau)\|^2_{W^{1,2}(\Omega)}\right)d\tau\right)^2 = \left(\int_0^t \left(\int_\Omega (uu_\tau + \nabla u\cdot\nabla u_\tau)\,dx\right)d\tau\right)^2 \\
&\le \left(\int_0^t \left(\|u\|_{2,\Omega}\|u_\tau\|_{2,\Omega} + \|\nabla u\|_{2,\Omega}\|\nabla u_\tau\|_{2,\Omega}\right)d\tau\right)^2 \le \left(\int_0^t (2f'(\tau))^{\frac{1}{2}}\left(\|u_\tau\|_{2,\Omega} + \|\nabla u_\tau\|_{2,\Omega}\right)d\tau\right)^2 \\
&\le 2f(t)\int_0^t \left(\|u_\tau\|_{2,\Omega} + \|\nabla u_\tau\|_{2,\Omega}\right)^2 d\tau \le 4f(t)\int_0^t \|u_\tau(\cdot,\tau)\|^2_{W^{1,2}(\Omega)}\,d\tau \le 4\lambda(t)f(t)f''(t) \\
&\le \frac{1}{1+\delta}f(t)f''(t).
\end{aligned} \tag{3.9}$$

The last inequality leads to the second-order differential inequality for $f(t)$:

$$(1+\delta)\left(f'(t) - f'(0)\right)^2 \le f(t)f''(t),\ \ f(0) = 0,\ \ f'(t) > 0,\ \ f''(t) > 0. \tag{3.10}$$

We want to prove that the function $f'(t)$ becomes unbounded at a finite moment. Let us show first that if $f'(t)$ exists for all $t > 0$, it is necessary that $f'(t) \nearrow \infty$ as $t \to \infty$. Assume the contrary: there exists a positive constant L such that $0 \le f'(t) \le L$ for all $t > 0$. The function f is strictly positive by definition, f'' is strictly positive and increasing by virtue of (3.8), while f' is strictly positive and monotone increasing due to the Lagrange intermediate value theorem. Fix an arbitrary $\tau > 0$. By assumption $f(t) \le f(0) + Lt = Lt$, and by virtue of (3.10)

$$\frac{1+\delta}{Lt} \le \frac{f''(t)}{\left(f'(t) - f'(0)\right)^2} \text{ in } (\tau, t).$$

A straightforward integration of this inequality over the interval (τ, t) leads to the contradiction:

$$L \geq f'(t) - f'(0) \geq \frac{f'(\tau) - f'(0)}{1 - \dfrac{1+\delta}{L}(f'(\tau) - f'(0)) \ln \dfrac{t}{\tau}} \geq L + 1,$$

provided that $t \geq \tau \exp\left(\frac{L}{1+\delta}\left(\frac{1}{f'(\tau)-f'(0)} - \frac{1}{L+1}\right)\right)$. Thus, if $f'(t)$ exists for all $t > 0$, then $f'(t) \nearrow \infty$ as $t \to \infty$. It follows that there exists a moment t_0 and a constant $1 < \nu < 1 + \delta$ such that

$$(f'(t) - f'(0))^2 \geq \frac{\nu}{1+\delta}(f'(t))^2 \text{ for } t \geq t_0.$$

This observation allows one to continue (3.9) as follows:

$$\nu(f'(t))^2 \leq (1+\delta)\left(f'(t) - f'(0)\right)^2 \leq f''(t)f(t) \text{ for } t \geq t_0.$$

Assuming that $f'(t)$ remains bounded for all finite t, we rewrite this inequality in the form

$$\frac{\nu f'(t)}{f(t)} \leq \frac{f''(t)}{f'(t)}, \quad t > t_0.$$

Integrating the last inequality over the interval (t_0, t), we derive:

$$\frac{f'(t)}{f^\nu(t)} \geq K, \quad K = \frac{f'(t_0)}{f^\nu(t_0)}, \quad \nu > 1. \tag{3.11}$$

Integration of (3.11) yields the inequality

$$f^{\nu-1}(t) \geq \frac{f^{\nu-1}(t_0)}{1 - (\nu-1)\dfrac{f'(t_0)}{f(t_0)}(t - t_0)} \nearrow \infty \text{ as } t \nearrow T^* \leq t_0 + \frac{f(t_0)}{(\nu-1)f'(t_0)}.$$

Substituting it into (3.11), we conclude that $f'(t)$ becomes infinite as $t \nearrow T^*$.

3.2. Proof of Theorem 3.1(ii)

Combining (2.7) with (3.3), and using $a(x,t)^{\frac{q^-(t)-p(x,t)}{p(x,t)}} \geq a^{-\frac{q^-(t)-p^+(t)}{p^+(t)}}$, we obtain the inequality

$$\begin{aligned} f''(t) &= \int_\Omega \frac{qb|u|^q}{q}\,dx - \int_\Omega a|\nabla u|^p\,dx \geq q^-(t)\int_\Omega \frac{b|u|^q}{q}\,dx - \int_\Omega a|\nabla u|^p\,dx \pm q^-(t)\int_\Omega \frac{a|\nabla u|^p}{p}\,dx = \\ &= q^-(t)\int_\Omega \left(\frac{b|u|^q}{q} - \frac{a|\nabla u|^p}{p}\right)dx + \int_\Omega a\frac{(q^-(t)-p)}{p}|\nabla u|^p\,dx \\ &\geq q^-(t)\int_\Omega \left(\frac{b|u|^q}{q} - \frac{a|\nabla u|^p}{p}\right)dx + a^- \frac{q^-(t)-p^+(t)}{p^+(t)}\int_\Omega |\nabla u|^p\,dx =: I_1(t) + I_2(t). \end{aligned} \tag{3.12}$$

Since $E(t) \leq 0$, then $I_1(t) = -q^-(t)E(t) \geq 0$, and (3.12) takes on the form

$$f''(t) \geq a^- \frac{q^-(t) - p^+(t)}{p^+(t)}\int_\Omega |\nabla u|^p\,dx. \tag{3.13}$$

Splitting Ω into the subsets $\Omega^+(t) = \{x \in \Omega : |\nabla u| > 1\}$, $\Omega^-(t) = \Omega \setminus \Omega^+(t)$, and using Hölder's inequality, we estimate:

$$\left(\int_{\Omega^\pm(t)} |\nabla u|^2\,dx\right)^{\frac{p^\mp(t)}{2}} \leq \left(\int_{\Omega^\pm(t)} |\nabla u|^{p^\mp(t)}\,dx\right)|\Omega|^{\frac{p^\mp(t)}{2}-1} \leq \left(\int_{\Omega^\pm(t)} |\nabla u|^p\,dx\right)|\Omega|^{\frac{p^\mp(t)}{2}-1}.$$

Inequality (3.13) leads then to the inequality

$$\begin{aligned} f''(t) &\geq a^- \frac{q^-(t)-p^+(t)}{p^+(t)} \left(\int_{\Omega^-(t)} |\nabla u|^{p^+(t)}\,dx + \int_{\Omega^+(t)} |\nabla u|^{p^-(t)}\,dx \right) \\ &\geq a^- \frac{q^-(t)-p^+(t)}{p^+(t)} \left(|\Omega|^{1-\frac{p^+(t)}{2}} \|\nabla u\|_{2,\Omega^-(t)}^{p^+(t)} + |\Omega|^{1-\frac{p^-(t)}{2}} \|\nabla u\|_{2,\Omega^+(t)}^{p^-(t)} \right) \\ &\geq a^- \frac{q^-(t)-p^+(t)}{p^+(t)} \min\left\{ |\Omega|^{1-\frac{p^-(t)}{2}}, |\Omega|^{1-\frac{p^+(t)}{2}} \right\} \left(\|\nabla u\|_{2,\Omega^-(t)}^{p^+(t)} + \|\nabla u\|_{2,\Omega^+(t)}^{p^-(t)} \right) \\ &\geq C_* \left(\|\nabla u\|_{2,\Omega^-(t)}^{p^+(t)} + \|\nabla u\|_{2,\Omega^+(t)}^{p^-(t)} \right) \end{aligned} \tag{3.14}$$

with the constant $C_* = a^- \inf_{t\in(0,T)} \frac{q^-(t)-p^+(t)}{p^+(t)} \inf_{t\in(0,T)} \min\left\{ |\Omega|^{1-\frac{p^-(t)}{2}}, |\Omega|^{1-\frac{p^+(t)}{2}} \right\}$. Inequality (3.14) yields the inequalities

$$(f''(t))^{\frac{2}{p^+(t)}} \geq C_*^{\frac{2}{p^+(t)}} \int_{\Omega^-(t)} |\nabla u|^2\,dx, \quad (f''(t))^{\frac{2}{p^-(t)}} \geq C_*^{\frac{2}{p^-(t)}} \int_{\Omega^+(t)} |\nabla u|^2\,dx,$$

whence

$$(f''(t))^{\frac{2}{p^+(t)}} + (f''(t))^{\frac{2}{p^-(t)}} \geq C^* \int_{\Omega} |\nabla u|^2\,dx, \quad C^* = \inf_{t\in(0,T)} \min\left\{ C_*^{\frac{2}{p^-(t)}}, C_*^{\frac{2}{p^+(t)}} \right\}. \tag{3.15}$$

By the Poincaré inequality $\|u\|_{2,\Omega}^2 \leq \frac{1}{\lambda_1}\|\nabla u\|_{2,\Omega}^2$ where λ_1 is the least eigenvalue of problem (A.1), it follows that

$$f'(t) = \frac{1}{2}\int_{\Omega} \left(u^2 + |\nabla u|^2\right) dx \leq \widehat{C} \int_{\Omega} |\nabla u|^2\,dx, \quad \widehat{C} = \frac{1}{2}\left(1 + \frac{1}{\lambda_1}\right),$$

and (3.15) takes on the form

$$(f''(t))^{\frac{2}{p^+(t)}} + (f''(t))^{\frac{2}{p^-(t)}} \geq \widetilde{C} f'(t), \qquad \widetilde{C} = C^* \widehat{C}^{-1}.$$

Since $f'(t) > f'(0) > 0$, it follows that

$$\begin{aligned} f''(t) &\geq \frac{1}{2} \min\left\{ \widetilde{C}^{\frac{p^+(t)}{2}}, \widetilde{C}^{\frac{p^-(t)}{2}} \right\} \min\left\{ (f'(t))^{\frac{p^+(t)}{2}}, (f'(t))^{\frac{p^-(t)}{2}} \right\} = \\ &= \frac{1}{2} \min\left\{ \widetilde{C}^{\frac{p^+(t)}{2}}, \widetilde{C}^{\frac{p^-(t)}{2}} \right\} \min\left\{ \left(\frac{f'(t)}{f'(0)}\right)^{\frac{p^+(t)}{2}} (f'(0))^{\frac{p^+(t)}{2}}, \left(\frac{f'(t)}{f'(0)}\right)^{\frac{p^-(t)}{2}} (f'(0))^{\frac{p^-(t)}{2}} \right\} \\ &\geq D \min\left\{ \left(\frac{f'(t)}{f'(0)}\right)^{\frac{p^+(t)}{2}}, \left(\frac{f'(t)}{f'(0)}\right)^{\frac{p^-(t)}{2}} \right\} \geq D \left(\frac{f'(t)}{f'(0)}\right)^{\frac{p^-}{2}} = \widetilde{D}\,(f'(t))^{\frac{p^-}{2}} \end{aligned} \tag{3.16}$$

with the constants $D := \frac{1}{2}\inf_{t\in(0,T)} \min\left\{ \widetilde{C}^{\frac{p^+(t)}{2}}, \widetilde{C}^{\frac{p^-(t)}{2}} \right\} \inf_{t\in(0,T)} \min\left\{ (f'(0))^{\frac{p^-(t)}{2}}, (f'(0))^{\frac{p^+(t)}{2}} \right\}$, $\widetilde{D} := \frac{D}{f'(0)^{\frac{p^-}{2}}}$. Integration of (3.16) leads to the estimate

$$f'(t) \geq \frac{f'(0)}{\left(1 - (\mu-1)t\widetilde{D}(f'(0))^{\mu-1}\right)^{\frac{1}{\mu-1}}}, \quad \mu = \frac{p^-}{2} > 1,$$

whence

$$f'(t) = \frac{1}{2}\|u(\cdot,t)\|_{W^{1,2}(\Omega)}^2 \nearrow \infty \text{ as } t \nearrow T^* \leq \frac{1}{(\mu-1)\widetilde{D}(f'(0))^{\mu-1}}.$$

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Appendix. The function spaces

The natural framework for the study of PDEs with variable nonlinearity is furnished by the variable Lebesgue and Sobolev spaces. A detailed exposition of the theory of these spaces can be found in a number of sources, see, e.g. [14]. Below we collect several basic facts on the variable spaces, which are used throughout the text. Let Ω be a Lipschitz domain and $p:\Omega\mapsto[p^-,p^+]\subset(1,\infty)$ be a given continuous function, $p^{\pm}$ are known constants. Let

$$L^{p(\cdot)}(\Omega)=\left\{u \text{ is measurable on } \Omega:\ \ \rho_{p(\cdot)}(u)\equiv\int_\Omega |u(x)|^{p(x)}\,dx<\infty\right\}.$$

The set $L^{p(\cdot)}(\Omega)$ equipped with the norm $\|u\|_{p(\cdot),\Omega}=\inf\left\{\lambda>0:\ \rho_{p(\cdot)}\left(\frac{u}{\lambda}\right)\le 1\right\}$ becomes a Banach space. The space $L^{p(\cdot)}(\Omega)$ is separable if $1<p^-\le p(x)\le p^+<\infty$. We will use the following properties of the spaces $L^{p(\cdot)}(\Omega)$.

- If $p_1,p_2\in C^0(\overline{\Omega})$ and $p_1(x)\ge p_2(x)$ in Ω, then $L^{p_1(\cdot)}(\Omega)\subset L^{p_1(\cdot)}(\Omega)$ and for every $u\in L^{p_1(\cdot)}(\Omega)$ $\|u\|_{p_2(\cdot),\Omega}\le C(|\Omega|,p_1^{\pm})\|u\|_{p_1(\cdot),\Omega}$.
- The generalized Hölder inequality: if $u\in L^{p(\cdot)}(\Omega)$, $v\in L^{p'(\cdot)}(\Omega)$, where $p'(x)=\frac{p(x)}{p(x)-1}$ is the conjugate of $p(x)$, then $\int_\Omega uv\,dx\le 2\|u\|_{p(\cdot),\Omega}\|v\|_{p'(\cdot),\Omega}$.
- The relations between the norm $\|\cdot\|_{p(\cdot),\Omega}$ and the modular $\rho_{p(\cdot)}(\cdot)$ are given by the inequalities $\min\left\{\rho_{p(\cdot)}^{\frac{1}{p^+}}(u),\rho_{p(\cdot)}^{\frac{1}{p^-}}(u)\right\}\le\|u\|_{p(\cdot),\Omega}\le\max\left\{\rho_{p(\cdot)}^{\frac{1}{p^+}}(u),\rho_{p(\cdot)}^{\frac{1}{p^-}}(u)\right\}$.

The variable Sobolev space $W^{1,p(\cdot)}(\Omega)$ is the set $\{u\in L^{p(\cdot)}(\Omega):\ |\nabla u|\in L^{p(\cdot)}(\Omega)\}$ equipped with the norm $\|u\|_{W^{1,p(\cdot)}(\Omega)}=\|u\|_{p(\cdot),\Omega}+\|\nabla u\|_{p(\cdot),\Omega}$, and $W_0^{1,p(\cdot)}(\Omega)=W^{1,p(\cdot)}(\Omega)\cap W_0^{1,1}(\Omega)$.

- If $p\in C^0(\overline{\Omega})$ and $u\in W_0^{1,p(\cdot)}(\Omega)$, then the Poincaré inequality holds, $\|u\|_{p(\cdot),\Omega}\le C\|\nabla u\|_{p(\cdot),\Omega}$, which makes $\|\nabla u\|_{p(\cdot),\Omega}$ the equivalent norm of $W_0^{1,p(\cdot)}(\Omega)$.
- If $p\in C_{log}(\Omega)$, i.e., is continuous with the logarithmic modulus of continuity, $|p(x)-p(y)|\le C\ln\dfrac{1}{|x-y|}$ for all $x,y\in\Omega$, $|x-y|<\frac12$, then the set $C_0^\infty(\Omega)$ (smooth functions with compact support) is dense in $W_0^{1,p(\cdot)}(\Omega)$. The space $W_0^{1,p(\cdot)}(\Omega)$ can be equivalently defined as the closure of $C_0^\infty(\Omega)$ with respect to the norm $\|\cdot\|_{W^{1,p(\cdot)}(\Omega)}$.
- Let $\{\psi_i\}$, $\{\lambda_i\}$ be the sequences of eigenfunctions and eigenvalues of the Dirichlet problem for the Laplace operator in Ω:
$$(\nabla\psi_i,\nabla\phi)_{2,\Omega}=\lambda_i(\psi_i,\phi)_{2,\Omega}\qquad\forall\phi\in W_0^{1,2}(\Omega).\tag{A.1}$$
Let us denote $\mathcal{P}_m=\operatorname{span}\{\psi_1,\dots,\psi_m\}$. If $p\in C_{log}(\overline{\Omega})$ and $\partial\Omega\in C^k$ with $k\ge 1+d\left(\frac12-\frac{1}{p^+}\right)$, then $\bigcup_{m=1}^\infty\mathcal{P}_m$ is dense in $W_0^{1,p(\cdot)}(\Omega)$ (see [15, Lemma 2.1] for the proof). If $\partial\Omega\in C^2$, the set $\{\psi_i\}$ is dense in $W_0^{2,2}(\Omega)$.
- Let $Q_T=\Omega\times(0,T)$ and $p:Q_T\mapsto[p^-,p^+]$ be a function from $C_{log}(\overline{Q}_T)$. We define the spaces of functions defined on Q_T
$$\mathbb{V}_{p(\cdot,t)}(\Omega)=\left\{u:u\in L^2(\Omega)\cap W_0^{1,1}(\Omega),\ |\nabla u|^{p(x,t)}\in L^1(\Omega)\right\},\quad t\in(0,T),$$

$$\mathbb{W}_{p(\cdot)}(Q_T) = \left\{ u : (0,T) \mapsto \mathbb{V}_{p(\cdot,t)}(\Omega) : u \in L^2(Q_T),\ |\nabla u|^{p(x,t)} \in L^1(Q_T) \right\},$$

and equip $\mathbb{W}_{p(\cdot)}(Q_T)$ with the norm $\|u\|_{\mathbb{W}_{p(\cdot)}(Q_T)} = \|u\|_{2,Q_T} + \|\nabla u\|_{p(\cdot),Q_T}$.

- The set $\bigcup_{m=1}^{\infty} \mathcal{N}_m$ is dense in $\mathbb{W}_{p(\cdot)}(Q_T)$, where $\mathcal{N}_m = \left\{ \sum_{i=1}^{m} \theta_i(t)\psi_i(x),\ \theta_i \in C^1[0,T] \right\}$.

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