Computers and Mathematics with Applications 135 (2023) 19–30

Contents lists available at ScienceDirect

Computers and Mathematics with Applications

journal homepage: www.elsevier.com/locate/camwa

A DG method for a stress formulation of the elasticity eigenproblem with strongly imposed symmetry ☆

Salim Meddahi

Facultad de Ciencias, Universidad de Oviedo, Federico García Lorca, 18, 33007-Oviedo, Spain

ARTICLE INFO

Keywords:
Elasticity eigenproblem
Mixed DG methods
Spectral analysis
Error estimates

ABSTRACT

We introduce a pure–stress formulation of the elasticity eigenvalue problem with mixed boundary conditions. We propose an H(div)-based discontinuous Galerkin method that imposes strongly the symmetry of the stress for the discretization of the eigenproblem. Under appropriate assumptions on the mesh and the degree of polynomial approximation, we prove the spectral correctness of the discrete scheme and derive optimal rates of convergence for eigenvalues and eigenfunctions. Finally, we provide numerical examples in two and three dimensions.

1. Introduction

The finite element determination of the vibration characteristics (natural frequencies and mode shapes) of elastic bodies is of great interest in structural mechanics. For example, the knowledge of the eigenfrequencies keeps the forced oscillations safe from resonance regimes, and the eigenmodes can be used to expand the solution of transient elastodynamic problems in a Fourier series. We approach this topic from the perspective of the mixed formulation derived from the Hellinger-Reissner variational principle. Namely, we are interested in variational formulations in which the Cauchy stress tensor prevails as the main unknown. In addition to the fact that accurate approximations of the stress are of paramount importance in many applications, it is well known that mixed formulations are immune to locking in the case of nearly incompressible materials.

In recent years, the theory of Descloux–Nassif–Rappaz [11,12] for non-compact operators has been successfully applied to the mixed finite element analysis of eigenvalue problems in elasticity [26,23,25]. The same approach allowed to deal with mixed formulations of the Stokes eigenproblem formulated in terms of a pseudo-stress [27,22] or the Cauchy stress tensor [25]. The symmetry requirement for the stress tensor, which reflects the conservation of angular momentum, is a specific feature of the Hellinger-Reissner variational principle. The imposition of this restriction in association with H(div)-conformity gives rise to conforming Galerkin methods with a very large number of degrees of freedom, and which are difficult to implement [2,20]. A common practice to overcome this drawback consists in enforcing the symmetry constraint variationally through a Lagrange multiplier. In this context, [26,25] validated the use of the weakly symmetric mixed finite elements [5,4,9,18] for the stress formulation of the elasticity eigenproblem.

Motivated by the ability of DG methods to handle efficiently *hp*-adaptive strategies and to facilitate the implementation of high order methods, an H(div)-based interior penalty version of [26] (that retains the weak imposition of the symmetry) has been introduced in [23]. Nevertheless, on account of [3,19,33], it is known that relaxing H(div)-conformity by using non-conforming or DG approximations for the elasticity source problem allows the incorporation of the symmetry constraint in the energy space at a reasonable computational cost. To our knowledge, the eigenvalue numerical analysis of such non-conforming/DG mixed methods is not yet available. In this work, we investigate whether the symmetry restriction can be strongly imposed in a pure–stress DG-formulation of the elasticity eigenvalue problem. The motivation for such an approach is the exact compliance of the conservation of angular momentum at the discrete level and a reduction of the number of degrees of freedom compared to mixed finite element methods [26,23] which enforce weakly the symmetry.

We propose a symmetric interior penalty Galerkin method that approximates the stress by symmetric tensors with piecewise polynomial entries of degree $k \geq 1$, in 2D and 3D. We note that, the stress/displacement DG formulation introduced in [33] for the elasticity source problem relies on the same discrete space for the stress. However, the displacement field is not present as an independent variable in our DG formulation because it is eliminated via the momentum balance equation. The same equation can be used to post-process the displacement at the discrete level. The numerical results show that this DG-scheme

☆ This research was supported by Ministerio de Ciencia e Innovación, Spain, Project PID2020-116287GB-I00.
E-mail address: salim@uniovi.es.

https://doi.org/10.1016/j.camwa.2023.01.022
Received 13 May 2022; Received in revised form 12 January 2023; Accepted 16 January 2023
Available online 27 January 2023

(whose stress/displacement version [33] is stable and optimally convergent for the source problem) does not generally provide a correct spectral approximation of the elasticity eigenproblem.

The main contribution of this work is the proof that inf-sup stability of the Scott-Vogelius element [31] for the Stokes problem (see Assumption 2 below) is a sufficient condition for the spectral correctness of the pure–stress DG formulation of the elasticity eigenproblem. The main challenge in the convergence analysis is the verification of the well-known Descloux–Nassif–Rappaz [11] condition, see (5.1) below. The proof of this condition requires a careful construction of quasi-interpolation operators and the use of Assumption 2 to ensure symmetry preservation. Furthermore, we establish optimal error estimates for eigenvalues and eigenfunctions in the natural DG energy norm.

We finally highlight that, a further novelty with respect to previous works [26,23] on the elasticity eigenvalue problem with reduced symmetry is that the present analysis does not rely on any extra Sobolev regularity of an auxiliary elasticity source problem. This allows for a wide range of applicability, our approach can be applied to eigenproblems posed in general domains, regardless of the boundary conditions and with minimal requirements on material coefficients.

Outline. The contents of this paper have been organized in the following manner. The remainder of this section contains notational conventions and definitions of Sobolev spaces. Section 2 presents the pure–stress formulation of the elasticity eigenproblem and provides a characterization of its spectrum. Preliminary definitions and auxiliary tools related with H(div)-based discontinuous Galerkin methods are collected in Section 3. The definition of the DG method (with strong symmetry of the stress) is detailed in Section 4, where we also introduce a couple of operators that are useful in our analysis. The spectral correctness of the DG scheme is treated in Section 5, together with the deduction of optimal error estimates for eigenvalues and eigenspaces. Several numerical results are presented in Section 6, confirming the expected rates of convergence for different parameter sets including the nearly incompressible regime.

Notations and Sobolev spaces. We denote the space of real matrices of order $d\times d$ by $\mathbb{M}$ and let $\mathbb{S}:=\{\boldsymbol{\tau}\in\mathbb{M};\ \boldsymbol{\tau}=\boldsymbol{\tau}^{\mathrm{t}}\}$ be the subspace of symmetric matrices, where $\boldsymbol{\tau}^{\mathrm{t}}:=(\tau_{ji})$ stands for the transpose of $\boldsymbol{\tau}=(\tau_{ij})$. The component-wise inner product of two matrices $\boldsymbol{\sigma},\boldsymbol{\tau}\in\mathbb{M}$ is defined by $\boldsymbol{\sigma}:\boldsymbol{\tau}:=\sum_{i,j}\sigma_{ij}\tau_{ij}$. We also introduce $\mathrm{tr}\,\boldsymbol{\tau}:=\sum_{i=1}^{d}\tau_{ii}$ and denote by I the identity in $\mathbb{M}$. Along this paper we convene to apply all differential operators row-wise. Hence, given a tensorial function $\boldsymbol{\sigma}:\Omega\to\mathbb{M}$ and a vector field $\boldsymbol{u}:\Omega\to\mathbb{R}^d$, we set the divergence $\mathbf{div}\,\boldsymbol{\sigma}:\Omega\to\mathbb{R}^d$, the gradient $\nabla\boldsymbol{u}:\Omega\to\mathbb{M}$, and the linearized strain tensor $\boldsymbol{\varepsilon}(\boldsymbol{u}):\Omega\to\mathbb{S}$ as

$$(\mathbf{div}\,\boldsymbol{\sigma})_i:=\sum_j\partial_j\sigma_{ij},\quad(\nabla\boldsymbol{u})_{ij}:=\partial_j u_i,\quad\text{and}\quad\boldsymbol{\varepsilon}(\boldsymbol{u}):=\tfrac{1}{2}\left(\nabla\boldsymbol{u}+(\nabla\boldsymbol{u})^{\mathrm{t}}\right).$$

Let Ω be a polyhedral Lipschitz domain of $\mathbb{R}^d$ $(d=2,3)$, with boundary $\partial\Omega$. For $s\in\mathbb{R}$, $H^s(\Omega,E)$ stands for the usual Hilbertian Sobolev space of functions with domain Ω and values in E, where E is either $\mathbb{R}$, $\mathbb{R}^d$ or $\mathbb{M}$. In the case $E=\mathbb{R}$ we simply write $H^s(\Omega)$. The norm of $H^s(\Omega,E)$ is denoted $\|\cdot\|_{s,\Omega}$ and the corresponding semi-norm $|\cdot|_{s,\Omega}$, indistinctly for $E=\mathbb{R},\mathbb{R}^d,\mathbb{M}$. We use the convention $H^0(\Omega,E):=L^2(\Omega,E)$ and let $(\cdot,\cdot)$ be the inner product in $L^2(\Omega,E)$, for $E=\mathbb{R},\mathbb{R}^d,\mathbb{M}$, namely,

$$(\boldsymbol{u},\boldsymbol{v}):=\int_\Omega\boldsymbol{u}\cdot\boldsymbol{v}\quad\forall\boldsymbol{u},\boldsymbol{v}\in L^2(\Omega,\mathbb{R}^d),$$

$$(\boldsymbol{\sigma},\boldsymbol{\tau}):=\int_\Omega\boldsymbol{\sigma}:\boldsymbol{\tau}\quad\forall\boldsymbol{\sigma},\boldsymbol{\tau}\in L^2(\Omega,\mathbb{M}).$$

We consider the space $H(\mathbf{div},\Omega,E)$ of tensors $\boldsymbol{\tau}\in L^2(\Omega,E)$ satisfying $\mathbf{div}\,\boldsymbol{\tau}\in L^2(\Omega,\mathbb{R}^d)$, and denote the corresponding norm $\|\boldsymbol{\tau}\|^2_{H(\mathbf{div},\Omega)}:=\|\boldsymbol{\tau}\|^2_{0,\Omega}+\|\mathbf{div}\,\boldsymbol{\tau}\|^2_{0,\Omega}$, where E is either $\mathbb{M}$ or $\mathbb{S}$. Let $\boldsymbol{n}$ be the outward unit normal vector to $\partial\Omega$. Let $\boldsymbol{\tau}$ be a sufficiently regular symmetric tensor, Green's formula

$$(\boldsymbol{\tau},\boldsymbol{\varepsilon}(\boldsymbol{v}))+(\mathbf{div}\,\boldsymbol{\tau},\boldsymbol{v})=\int_{\partial\Omega}\boldsymbol{\tau}\boldsymbol{n}\cdot\boldsymbol{v}\qquad\boldsymbol{v}\in H^1(\Omega,\mathbb{R}^d),\tag{1.1}$$

can be used to extend the normal trace operator $\boldsymbol{\tau}\to(\boldsymbol{\tau}|_{\partial\Omega})\boldsymbol{n}$ to a linear continuous mapping $(\cdot|_{\partial\Omega})\boldsymbol{n}:\ H(\mathbf{div},\Omega,\mathbb{S})\to H^{-\frac{1}{2}}(\partial\Omega,\mathbb{R}^d)$, where $H^{-\frac{1}{2}}(\partial\Omega,\mathbb{R}^d)$ is the dual of $H^{\frac{1}{2}}(\partial\Omega,\mathbb{R}^d)$.

Throughout this paper, we shall use the letter C to denote a generic positive constant independent of the mesh size h, that may stand for different values at its different occurrences. Moreover, given any positive expressions X and Y depending on h, the notation $X\lesssim Y$ means that $X\le CY$.

2. A stress formulation of the elasticity eigenproblem

Our aim is to determine the natural frequencies $\omega\in\mathbb{R}$ of an elastic structure with mass density ρ and occupying a polyhedral Lipschitz domain $\Omega\subset\mathbb{R}^d$ $(d=2,3)$. This amounts to solve the eigenproblem

$$\mathbf{div}\,\boldsymbol{\sigma}+\omega^2\rho(\boldsymbol{x})\boldsymbol{u}=0\quad\text{in }\Omega,\tag{2.1}$$

$$\mathcal{A}(\boldsymbol{x})\boldsymbol{\sigma}=\boldsymbol{\varepsilon}(\boldsymbol{u})\quad\text{in }\Omega,\tag{2.2}$$

where $\boldsymbol{u}:\Omega\to\mathbb{R}^d$ is the displacement field and $\boldsymbol{\sigma}:\Omega\to\mathbb{S}$ is the Cauchy stress tensor. The symmetric and positive-definite 4^{th}-order tensor $\mathcal{A}(\boldsymbol{x}):\mathbb{S}\to\mathbb{S}$ involved in the linear material law (2.2) is known as the compliance tensor. We assume that there exist constants $a^+>a^->0$ such that

$$a^-\boldsymbol{\zeta}:\boldsymbol{\zeta}\le\mathcal{A}(\boldsymbol{x})\boldsymbol{\zeta}:\boldsymbol{\zeta}\le a^+\boldsymbol{\zeta}:\boldsymbol{\zeta}\quad\forall\boldsymbol{\zeta}\in\mathbb{S},\quad\text{a.e. in }\Omega.$$

We also suppose that there exists a polygonal/polyhedral partition $\{\bar\Omega_j,\ j=1,\dots,J\}$ of $\bar\Omega$ and constants $\varrho_j>0$ such that $\rho|_{\Omega_j}:=\varrho_j>0$ for all $j=1,\dots,J$. We let $\varrho^+:=\max_j\varrho_j$ and $\varrho^-:=\min_j\varrho_j$.

We impose the boundary condition $\boldsymbol{u}=\boldsymbol{0}$ on a subset $\Gamma_D\subset\Gamma:=\partial\Omega$ of positive surface measure and let the structure free of stress on $\Gamma_N:=\Gamma\backslash\Gamma_D$. Here, we opt for combining the equilibrium equation (2.1) with the constitutive law (2.2) to eliminate the displacement field $\boldsymbol{u}$ and impose $\boldsymbol{\sigma}$ as a primary variable. This procedure leads to the following eigensystem: find eigenmodes $0\ne\boldsymbol{\sigma}:\Omega\to\mathbb{S}$ and eigenfrequencies $\omega\in\mathbb{R}$ such that

$$\begin{aligned}-\boldsymbol{\varepsilon}\left(\tfrac{1}{\rho}\,\mathbf{div}\,\boldsymbol{\sigma}\right)&=\omega^2\mathcal{A}\boldsymbol{\sigma}\quad\text{in }\Omega,\\ \tfrac{1}{\rho}\,\mathbf{div}\,\boldsymbol{\sigma}&=0\quad\text{on }\Gamma_D,\\ \boldsymbol{\sigma}\boldsymbol{n}&=\boldsymbol{0}\quad\text{on }\Gamma_N,\end{aligned}\tag{2.3}$$

where $\boldsymbol{n}$ stands for the exterior unit normal vector on Γ.

In the following, we write H for the space $L^2(\Omega,\mathbb{S})$ endowed with the $\mathcal{A}$-weighted inner product $(\boldsymbol{\sigma},\boldsymbol{\tau})_{\mathcal{A}}:=(\mathcal{A}\boldsymbol{\sigma},\boldsymbol{\tau})$ and denote the corresponding norm $\|\boldsymbol{\tau}\|^2_{\mathcal{A}}:=(\mathcal{A}\boldsymbol{\tau},\boldsymbol{\tau})$. The eigenfunctions $\boldsymbol{\sigma}$ will be sought in the closed subspace X of $H(\mathbf{div},\Omega,\mathbb{S})$ defined by

$$X:=\left\{\boldsymbol{\tau}\in H(\mathbf{div},\Omega,\mathbb{S});\quad\langle\boldsymbol{\tau}\boldsymbol{n},\boldsymbol{\phi}\rangle_\Gamma=0\quad\forall\boldsymbol{\phi}\in H^{1/2}(\Gamma,\mathbb{R}^d),\ \boldsymbol{\phi}|_{\Gamma_D}=\boldsymbol{0}\right\},$$

where $\langle\cdot,\cdot\rangle_\Gamma$ holds for the duality pairing between $H^{\frac{1}{2}}(\Gamma,\mathbb{R}^d)$ and $H^{-\frac{1}{2}}(\Gamma,\mathbb{R}^d)$. We introduce the symmetric and positive semidefinite bilinear form $c:\ X\times X\to\mathbb{R}$ given by

$$c(\boldsymbol{\sigma},\boldsymbol{\tau}):=\left(\tfrac{1}{\rho}\,\mathbf{div}\,\boldsymbol{\sigma},\mathbf{div}\,\boldsymbol{\tau}\right)$$

and endow X with the Hilbertian inner product $a(\boldsymbol{\sigma},\boldsymbol{\tau}):=(\boldsymbol{\sigma},\boldsymbol{\tau})_{\mathcal{A}}+c(\boldsymbol{\sigma},\boldsymbol{\tau})$. We denote the corresponding norm $\|\boldsymbol{\tau}\|^2_X:=a(\boldsymbol{\tau},\boldsymbol{\tau})$.

Testing the first equation of (2.3) with $\boldsymbol{\tau}\in X$ and applying Green's formula (1.1) we deduce, after a shift argument, the following pure–stress variational formulation of the eigenproblem, find $0\ne\boldsymbol{\sigma}\in X$ and $\kappa=1+\omega^2\in\mathbb{R}$ such that

$$a(\boldsymbol{\sigma},\boldsymbol{\tau})=\kappa(\boldsymbol{\sigma},\boldsymbol{\tau})_{\mathcal{A}}\quad\forall\boldsymbol{\tau}\in X.\tag{2.4}$$

We introduce the source operator $\tilde{T} : L^2(\Omega,\mathbb{S}) \to X$ corresponding to the variational eigenproblem (2.4), which is defined for any $\boldsymbol{f} \in L^2(\Omega,\mathbb{S})$ by

$$a\big(\tilde{T}\boldsymbol{f},\boldsymbol{\tau}\big) = (\boldsymbol{f},\boldsymbol{\tau})_{\mathcal{A}} \quad \forall \boldsymbol{\tau} \in X. \tag{2.5}$$

Obviously, $\tilde{T}$ is linear and bounded, actually it holds true that,

$$\big\|\tilde{T}\boldsymbol{f}\big\|_X \le \|\boldsymbol{f}\|_{\mathcal{A}} \quad \forall \boldsymbol{f} \in H. \tag{2.6}$$

We denote the H^1-Sobolev space with incorporated Dirichlet boundary conditions on either Γ_D or Γ_N by

$$H^1_\star(\Omega,\mathbb{R}^d) := \Big\{ \boldsymbol{v} \in H^1(\Omega,\mathbb{R}^d);\ v|_{\Gamma_\star} = \boldsymbol{0} \Big\}, \quad \star \in \{D,N\}.$$

It is important to notice that testing (2.5) with a tensor $\boldsymbol{\tau} : \Omega \to \mathbb{S}$ whose entries are indefinitely differentiable and compactly supported in Ω proves that $\boldsymbol{\varepsilon}(\frac{1}{\varrho}\,\mathbf{div}(\tilde{T}\boldsymbol{f})) = \mathcal{A}(\tilde{T}-I)\boldsymbol{f} \in L^2(\Omega,\mathbb{S})$. Hence, by virtue of Korn's inequality, $\frac{1}{\varrho}\,\mathbf{div}(\tilde{T}\boldsymbol{f}) \in H^1(\Omega,\mathbb{R}^d)$ and it follows readily from Green's formula (1.1) that $\frac{1}{\varrho}\,\mathbf{div}(\tilde{T}\boldsymbol{f})$ vanishes on Γ_D. In other words, $\frac{1}{\varrho}\,\mathbf{div}(\tilde{T}\boldsymbol{f}) \in H^1_D(\Omega,\mathbb{R}^d)$ and there exists $C>0$ such that

$$\Big\| \tfrac{1}{\varrho}\,\mathbf{div}(\tilde{T}\boldsymbol{f}) \Big\|_{1,\Omega} \le C\|\boldsymbol{f}\|_{\mathcal{A}} \quad \forall \boldsymbol{f} \in H. \tag{2.7}$$

The operator $T := \tilde{T}|_X : X \to X$ is relevant in our analysis because its eigenvalues and those of problem (2.4) are reciprocal to each other and the corresponding eigenfunctions are the same. A full description of the spectrum of T will then solve problem (2.4).

We consider the direct sum decomposition $X = K \oplus K^\perp$ into closed subspaces

$$K := \{\boldsymbol{\tau} \in X;\ \mathbf{div}\,\boldsymbol{\tau} = 0 \text{ in } \Omega\} \quad \text{and}$$

$$K^\perp := \{\boldsymbol{\sigma} \in X;\ a(\boldsymbol{\sigma},\boldsymbol{\tau}) = 0,\ \forall \boldsymbol{\tau} \in K\},$$

which are orthogonal with respect to both $(\cdot,\cdot)_{\mathcal{A}}$ and $a(\cdot,\cdot)$. It is clear that $\kappa = 1$ is an eigenvalue of (2.4) with associated eigenspace K. Consequently, as K is not a finite-dimensional subspace of X, T is not a compact operator.

Lemma 2.1. *The orthogonal projection P in X onto $K^\perp$ with respect to the inner product $a(\cdot,\cdot)$ is characterized, for any $\boldsymbol{\sigma} \in X$, by $P\boldsymbol{\sigma} := \widetilde{\boldsymbol{\sigma}}$ where $\widetilde{\boldsymbol{\sigma}} = \mathcal{A}^{-1}\boldsymbol{\varepsilon}(\widetilde{\boldsymbol{u}})$ and $\widetilde{\boldsymbol{u}} \in H^1_D(\Omega,\mathbb{R}^d)$ is the unique solution of*

$$\big(\mathcal{A}^{-1}\boldsymbol{\varepsilon}(\widetilde{\boldsymbol{u}}),\boldsymbol{\varepsilon}(\boldsymbol{v})\big) = -(\mathbf{div}\,\boldsymbol{\sigma},\boldsymbol{v}) \quad \forall \boldsymbol{v} \in H^1_D(\Omega,\mathbb{R}^d). \tag{2.8}$$

Proof. We first point out that Korn's inequality provides the stability estimate

$$\|\widetilde{\boldsymbol{u}}\|_{1,\Omega} \le C\|\mathbf{div}\,\boldsymbol{\sigma}\|_{0,\Omega}. \tag{2.9}$$

By definition, (2.9) also ensures that $\|P\boldsymbol{\sigma}\|_{0,\Omega} \le C_1\|\mathbf{div}\,\boldsymbol{\sigma}\|_{0,\Omega}$. Moreover, $\mathbf{div}\,P\boldsymbol{\sigma} = \mathbf{div}\,\boldsymbol{\sigma}$ by construction, which ensures that $P : X \to X$ is bounded. Moreover, it is clear that $P\circ P = P$ and $\ker P = K$. It remains to show that the range of P coincides with $K^\perp$. To this end, we notice that, for any $\boldsymbol{\sigma} \in X$,

$$(P\boldsymbol{\sigma},\boldsymbol{\tau})_{\mathcal{A}} = \big(\boldsymbol{\varepsilon}(\widetilde{\boldsymbol{u}}),\boldsymbol{\tau}\big) = (\nabla\widetilde{\boldsymbol{u}},\boldsymbol{\tau}) = 0 \quad \forall \boldsymbol{\tau} \in K,$$

which proves that $P(X) \subset K^\perp$. The reciprocal inclusion is a consequence of $K^\perp = P(K^\perp) + (I-P)K^\perp = P(K^\perp) \subset P(X)$, where we used that $(I-P)X \subset K$, and the result follows. □

Lemma 2.2. *The inclusions $P(X) \hookrightarrow H$ and $P(X)\cap T(X) \hookrightarrow X$ are compact.*

Proof. Let $\{\boldsymbol{\sigma}_n\}_n$ be a weakly convergent sequence in X. The continuity of $P : X \to X$ implies that the sequence $\{\widetilde{\boldsymbol{\sigma}}_n\}_n := \{P\boldsymbol{\sigma}_n\}_n$ is also weakly convergent in X. By definition, $\widetilde{\boldsymbol{\sigma}}_n = \mathcal{A}^{-1}\boldsymbol{\varepsilon}(\widetilde{\boldsymbol{u}}_n)$, where $\widetilde{\boldsymbol{u}}_n \in H^1_D(\Omega,\mathbb{R}^d)$ solves (2.8) with right-hand side $-\mathbf{div}\,\boldsymbol{\sigma}_n$. It follows from (2.9) that $\widetilde{\boldsymbol{u}}_n$ is bounded in $H^1_D(\Omega,\mathbb{R}^d)$ and the compactness of the embedding $H^1(\Omega,\mathbb{R}^d) \hookrightarrow L^2(\Omega,\mathbb{R}^d)$ implies that $\{\widetilde{\boldsymbol{u}}_n\}_n$ admits a subsequence (denoted again $\{\widetilde{\boldsymbol{u}}_n\}_n$) that converges strongly in $L^2(\Omega,\mathbb{R}^d)$. Next, we deduce from Green's identity

$$\big(\widetilde{\boldsymbol{\sigma}}_p - \widetilde{\boldsymbol{\sigma}}_q, \widetilde{\boldsymbol{\sigma}}_p - \widetilde{\boldsymbol{\sigma}}_q\big)_{\mathcal{A}} = \big(\boldsymbol{\varepsilon}(\widetilde{\boldsymbol{u}}_p - \widetilde{\boldsymbol{u}}_q), \widetilde{\boldsymbol{\sigma}}_p - \widetilde{\boldsymbol{\sigma}}_q\big) = -\big(\widetilde{\boldsymbol{u}}_p - \widetilde{\boldsymbol{u}}_q, \mathbf{div}(\widetilde{\boldsymbol{\sigma}}_p - \widetilde{\boldsymbol{\sigma}}_q)\big),$$

that $\{\widetilde{\boldsymbol{\sigma}}_n\}_n$ is a Cauchy sequence in H, which implies that the embedding $P(X) \hookrightarrow H$ is compact.

Finally, it follows from (2.7) that

$$T(X)\cap P(X) \subset \Big\{ \boldsymbol{\sigma} \in P(X);\ \tfrac{1}{\varrho}\,\mathbf{div}\,\boldsymbol{\sigma} \in H^1(\Omega,\mathbb{R}^d) \Big\},$$

and the compactness of the embedding $T(X)\cap P(X) \hookrightarrow X$ is a consequence of the fact that the inclusion $\Big\{ \boldsymbol{\sigma} \in P(X);\ \tfrac{1}{\varrho}\,\mathbf{div}\,\boldsymbol{\sigma} \in H^1(\Omega,\mathbb{R}^d) \Big\} \subset X$ is compact. □

We point out that $\tilde{T}$ is symmetric with respect to $(\cdot,\cdot)_{\mathcal{A}}$, which implies that $P(X) = K^\perp$ is T-invariant. Consequently, it holds true that $T(P(X)) \subset P(X)\cap T(X)$ and Lemma 2.2 implies that the $a(\cdot,\cdot)$-symmetric and positive definite operator $T : K^\perp \to K^\perp$ is compact. Therefore, we have the following characterization of the spectrum of T.

Theorem 2.1. *The spectrum $\mathrm{sp}(T)$ of T is given by $\mathrm{sp}(T) = \{0,1\}\cup\{\eta_k\}_{k\in\mathbb{N}}$, where $\{\eta_k\}_k \subset (0,1)$ is a sequence of finite-multiplicity eigenvalues of T that converges to 0. The ascent of each of these eigenvalues is 1 and the corresponding eigenfunctions lie in $P(X)$. Moreover, $\eta = 1$ is an infinite-multiplicity eigenvalue of T with associated eigenspace K and $\eta = 0$ is not an eigenvalue.*

3. Definitions and auxiliary results

We consider a sequence $\{\mathcal{T}_h\}_h$ of shape-regular simplicial meshes that subdivide the domain $\bar{\Omega}$ into simplices K of diameter h_K. The parameter $h := \max_{K\in\mathcal{T}_h}\{h_K\}$ represents the mesh size of $\mathcal{T}_h$. We assume that $\mathcal{T}_h$ is aligned with the partition $\bar{\Omega} = \cup_{j=1}^J \bar{\Omega}_j$ and that $\mathcal{T}_h(\Omega_j) := \big\{K \in \mathcal{T}_h;\ K \subset \Omega_j\big\}$ is a shape-regular mesh of $\bar{\Omega}_j$ for all $j = 1,\cdots,J$ and all h.

For all $s \ge 0$, we consider the broken Sobolev space

$$H^s(\cup_j\Omega_j) := \Big\{ v \in L^2(\Omega);\ v|_{\Omega_j} \in H^s(\Omega_j),\ \forall j = 1,\dots,J \Big\}$$

corresponding to the partition $\bar{\Omega} = \cup_{j=1}^J \bar{\Omega}_j$. Its vectorial and tensorial versions are denoted $H^s(\cup_j\Omega_j,\mathbb{R}^d)$ and $H^s(\cup_j\Omega_j,\mathbb{M})$, respectively. Likewise, the broken Sobolev space with respect to the subdivision of $\bar{\Omega}$ into $\mathcal{T}_h$ is

$$H^s(\mathcal{T}_h,E) := \big\{\boldsymbol{v} \in L^2(\Omega,E):\ \boldsymbol{v}|_K \in H^s(K,E) \quad \forall K \in \mathcal{T}_h\big\},$$

for $E \in \{\mathbb{R},\mathbb{R}^d,\mathbb{M}\}$.

For each $\boldsymbol{v} := \{\boldsymbol{v}_K\} \in H^s(\mathcal{T}_h,\mathbb{R}^d)$ and $\boldsymbol{\tau} := \{\boldsymbol{\tau}_K\} \in H^s(\mathcal{T}_h,\mathbb{M})$ the components $\boldsymbol{v}_K$ and $\boldsymbol{\tau}_K$ represent the restrictions $\boldsymbol{v}|_K$ and $\boldsymbol{\tau}|_K$. When no confusion arises, the restrictions of these functions will be written without any subscript.

Hereafter, given an integer $m \ge 0$ and a domain $D \subset \mathbb{R}^d$, $\mathcal{P}_m(D)$ denotes the space of polynomials of degree at most m on D. We introduce the space

$$\mathcal{P}_m(\mathcal{T}_h) := \{v \in L^2(\Omega):\ v|_K \in \mathcal{P}_m(K),\ \forall K \in \mathcal{T}_h\}$$

of piecewise polynomial functions relatively to $\mathcal{T}_h$. We also consider the space $\mathcal{P}_m(\mathcal{T}_h,E)$ of functions with values in E and entries in $\mathcal{P}_m(\mathcal{T}_h)$, where E is either $\mathbb{R}^d$, $\mathbb{M}$ or $\mathbb{S}$.

Let us introduce now notations related to DG approximations of $H(\mathrm{div})$-type spaces. We say that a closed subset $F \subset \bar{\Omega}$ is an interior edge/face if F has a positive $(d-1)$-dimensional measure and if there

are distinct elements K and K' such that $F = \bar{K} \cap \bar{K}'$. A closed subset $F \subset \overline{\Omega}$ is a boundary edge/face if there exists $K \in \mathcal{T}_h$ such that F is an edge/face of K and $F = \bar{K} \cap \Gamma$. We consider the set $\mathcal{F}_h^0$ of interior edges/faces, the set $\mathcal{F}_h^\partial$ of boundary edges/faces and let $\mathcal{F}(K) := \{F \in \mathcal{F}_h; \quad F \subset \partial K\}$ be the set of edges/faces composing the boundary of K. We assume that the boundary mesh $\mathcal{F}_h^\partial$ is compatible with the partition $\partial\Omega = \Gamma_D \cup \Gamma_N$ in the sense that, if $\mathcal{F}_h^D = \{F \in \mathcal{F}_h^\partial : F \subset \Gamma_D\}$ and $\mathcal{F}_h^N = \{F \in \mathcal{F}_h^\partial : F \subset \Gamma_N\}$, then $\Gamma_D = \cup_{F \in \mathcal{F}_h^D} F$ and $\Gamma_N = \cup_{F \in \mathcal{F}_h^N} F$. We denote

$$\mathcal{F}_h := \mathcal{F}_h^0 \cup \mathcal{F}_h^\partial \qquad \text{and} \qquad \mathcal{F}_h^* := \mathcal{F}_h^0 \cup \mathcal{F}_h^N.$$

Obviously, in the case $\Gamma_D = \Gamma$ we have that $\mathcal{F}_h^* = \mathcal{F}_h^0$.

We will need the space given on the skeletons of the triangulations $\mathcal{T}_h$ by $L^2(\mathcal{F}_h^*) := \bigoplus_{F \in \mathcal{F}_h^*} L^2(F)$. Its vector valued version is denoted $L^2(\mathcal{F}_h^*, \mathbb{R}^d)$. Here again, the components $\boldsymbol{v}_F$ of $\boldsymbol{v} := \{\boldsymbol{v}_F\} \in L^2(\mathcal{F}_h^*, \mathbb{R}^d)$ coincide with the restrictions $\boldsymbol{v}|_F$. We endow $L^2(\mathcal{F}_h^*, \mathbb{R}^d)$ with the inner product

$$(\boldsymbol{u}, \boldsymbol{v})_{\mathcal{F}_h^*} := \sum_{F \in \mathcal{F}_h^*} \int_F \boldsymbol{u}_F \cdot \boldsymbol{v}_F \quad \forall \boldsymbol{u}, \boldsymbol{v} \in L^2(\mathcal{F}_h^*, \mathbb{R}^d),$$

and denote the corresponding norm $\|\boldsymbol{v}\|_{0,\mathcal{F}_h^*}^2 := (\boldsymbol{v}, \boldsymbol{v})_{\mathcal{F}_h^*}$. From now on, $h_{\mathcal{F}} \in L^2(\mathcal{F}_h^*)$ is the piecewise constant function defined by $h_{\mathcal{F}}|_F := h_F$ for all $F \in \mathcal{F}_h^*$ with h_F denoting the diameter of edge/face F. By virtue of our hypotheses on ϱ and on the triangulation $\mathcal{T}_h$, we may consider that ϱ is an element of $\mathcal{P}_0(\mathcal{T}_h)$ and denote $\varrho_K := \varrho|_K$ for all $K \in \mathcal{T}_h$. We introduce $\varrho_{\mathcal{F}} \in L^2(\mathcal{F}_h^*)$ defined by $\varrho_F := \min\{\varrho_K, \varrho_{K'}\}$ if $K \cap K' = F$ and $\varrho_F := \varrho_K$ if $F \cap K \in \mathcal{F}_h^N$.

Given $\boldsymbol{v} \in H^s(\mathcal{T}_h, \mathbb{R}^d)$ and $\boldsymbol{\tau} \in H^s(\mathcal{T}_h, \mathbb{M})$, with $s > \frac{1}{2}$, we define averages $\{\boldsymbol{v}\} \in L^2(\mathcal{F}_h^*, \mathbb{R}^d)$ and jumps $[\![\boldsymbol{\tau}]\!] \in L^2(\mathcal{F}_h^*, \mathbb{R}^d)$ by

$$\{\boldsymbol{v}\}_F := (\boldsymbol{v}_K + \boldsymbol{v}_{K'})/2 \quad \text{and} \quad [\![\boldsymbol{\tau}]\!]_F := \boldsymbol{\tau}_K \boldsymbol{n}_K + \boldsymbol{\tau}_{K'} \boldsymbol{n}_{K'}$$
$$\forall F \in \mathcal{F}(K) \cap \mathcal{F}(K'),$$

with the conventions

$$\{\boldsymbol{v}\}_F := \boldsymbol{v}_K \quad \text{and} \quad [\![\boldsymbol{\tau}]\!]_F := \boldsymbol{\tau}_K \boldsymbol{n}_K \quad \forall F \in \mathcal{F}(K),\ F \in \mathcal{F}_h^N,$$

where $\boldsymbol{n}_K$ is the outward unit normal vector to ∂K.

For any $k \geq 1$, we let $X^{DG}(h) := X + X_h^{DG}$, with $X_h^{DG} := \mathcal{P}_k(\mathcal{T}_h, \mathbb{S})$. Given $\boldsymbol{\tau} \in X_h^{DG}$, we define $\mathbf{div}_h \boldsymbol{\tau} \in L^2(\Omega, \mathbb{R}^d)$ by $\mathbf{div}_h \boldsymbol{\tau}|_K := \mathbf{div}\, \boldsymbol{\tau}_K$ for all $K \in \mathcal{T}_h$ and endow $X^{DG}(h)$ with the norm

$$|||\boldsymbol{\tau}|||^2 := \|\boldsymbol{\tau}\|_{\mathcal{A}}^2 + \left\| \tfrac{1}{\sqrt{\varrho}} \mathbf{div}_h \boldsymbol{\tau} \right\|_{0,\Omega}^2 + \left\| \varrho_{\mathcal{F}}^{-\frac{1}{2}} h_{\mathcal{F}}^{-\frac{1}{2}} [\![\boldsymbol{\tau}]\!] \right\|_{0,\mathcal{F}_h^*}^2.$$

If it happens that $\mathbf{div}_h \boldsymbol{\tau} \in H^s(\mathcal{T}_h, \mathbb{R}^d)$ with $s > \frac{1}{2}$, we also introduce

$$|||\boldsymbol{\tau}|||_*^2 := |||\boldsymbol{\tau}|||^2 + \left\| \varrho_{\mathcal{F}}^{\frac{1}{2}} h_{\mathcal{F}}^{\frac{1}{2}} \{\tfrac{1}{\varrho} \mathbf{div}_h \boldsymbol{\tau}\} \right\|_{0,\mathcal{F}_h^*}^2.$$

It is important to notice that $|||\boldsymbol{\tau}||| = \|\boldsymbol{\tau}\|_X$ for all $\boldsymbol{\tau} \in X$.

The following discrete trace inequality is useful in our analysis.

Lemma 3.1. *There exists a constant $C_{\mathrm{tr}} > 0$ independent of h and ϱ such that*

$$\left\| \varrho_{\mathcal{F}}^{\frac{1}{2}} h_{\mathcal{F}}^{\frac{1}{2}} \{\tfrac{1}{\sqrt{\varrho}} \boldsymbol{v}\} \right\|_{0,\mathcal{F}_h^*} \leq C_{\mathrm{tr}} \left\| \tfrac{1}{\sqrt{\varrho}} \boldsymbol{v} \right\|_{0,\Omega} \quad \forall \boldsymbol{v} \in \mathcal{P}_k(\mathcal{T}_h, \mathbb{R}^d). \tag{3.1}$$

Proof. By definition of $\varrho_{\mathcal{F}}$, for any $\boldsymbol{v} \in \mathcal{P}_k(\mathcal{T}_h, \mathbb{R}^d)$, it holds true that

$$\left\| \varrho_{\mathcal{F}}^{\frac{1}{2}} h_{\mathcal{F}}^{\frac{1}{2}} \{\tfrac{1}{\varrho} \boldsymbol{v}\} \right\|_{0,\mathcal{F}_h^*}^2 = \sum_{F \in \mathcal{F}_h^*} h_F \left\| \varrho_F^{\frac{1}{2}} \{\tfrac{1}{\varrho} \boldsymbol{v}\}_F \right\|_{0,F}^2 \leq \sum_{F \in \mathcal{F}_h^*} h_F \left\| \{\tfrac{1}{\sqrt{\varrho}} \boldsymbol{v}\}_F \right\|_{0,F}^2$$
$$\lesssim \sum_{K \in \mathcal{T}_h} h_K \left\| \tfrac{1}{\sqrt{\varrho}} \boldsymbol{v}_K \right\|_{0,\partial K}^2.$$

Applying in the last inequality the well-known estimate (see for example [13])

$$h_K^{\frac{1}{2}} \|\phi\|_{0,\partial K} \leq C \|\phi\|_{0,K} \quad \forall \phi \in \mathcal{P}_k(K), \tag{3.2}$$

where $C > 0$ is independent of h, we obtain the result. □

For all $\boldsymbol{\sigma}, \boldsymbol{\tau} \in X^{DG}(h)$ and for a large enough given parameter $\mathtt{a} > 0$, we consider the symmetric bilinear form

$$c_h(\boldsymbol{\sigma}, \boldsymbol{\tau}) := \left(\tfrac{1}{\varrho} \mathbf{div}_h \boldsymbol{\sigma}, \mathbf{div}_h \boldsymbol{\tau} \right) + \mathtt{a} \left(\varrho_{\mathcal{F}}^{-1} h_{\mathcal{F}}^{-1} [\![\boldsymbol{\sigma}]\!], [\![\boldsymbol{\tau}]\!] \right)_{\mathcal{F}_h^*}$$
$$- \left(\{\tfrac{1}{\varrho} \mathbf{div}_h \boldsymbol{\sigma}\}, [\![\boldsymbol{\tau}]\!] \right)_{\mathcal{F}_h^*} - \left(\{\tfrac{1}{\varrho} \mathbf{div}_h \boldsymbol{\tau}\}, [\![\boldsymbol{\sigma}]\!] \right)_{\mathcal{F}_h^*}$$

and let

$$a_h(\boldsymbol{\sigma}, \boldsymbol{\tau}) := (\mathcal{A}\boldsymbol{\sigma}, \boldsymbol{\tau}) + c_h(\boldsymbol{\sigma}, \boldsymbol{\tau}).$$

For all $\boldsymbol{\sigma}, \boldsymbol{\tau} \in X^{DG}(h)$ satisfying $\mathbf{div}_h \boldsymbol{\sigma}, \mathbf{div}_h \boldsymbol{\tau} \in H^s(\mathcal{T}_h, \mathbb{R}^d)$ with $s > 1/2$, a straightforward application of the Cauchy-Schwarz inequality gives

$$|a_h(\boldsymbol{\sigma}, \boldsymbol{\tau})| \leq 2 |||\boldsymbol{\sigma}|||_* \, |||\boldsymbol{\tau}|||_*.$$

Moreover, if we take in the last estimate $\boldsymbol{\tau} = \boldsymbol{\tau}_h \in X_h^{DG}$, we deduce from Lemma 3.1 that,

$$|a_h(\boldsymbol{\sigma}, \boldsymbol{\tau}_h)| \leq M \, |||\boldsymbol{\sigma}|||_* \, |||\boldsymbol{\tau}_h|||, \tag{3.3}$$

with $M := 2\sqrt{1 + C_{\mathrm{tr}}^2}$.

The bilinear form $c_h(\cdot,\cdot)$ and the DG-norm $|||\cdot|||$ are designed in such a way that the coercivity of the bilinear form $a_h(\cdot,\cdot)$ on X_h^{DG} can be achieved with a stability parameter $\mathtt{a}$ that is independent of the material coefficients, as shown in the following result.

Proposition 3.1. *There exists a constant $\mathtt{a}^* > 0$, independent of ϱ and $\mathcal{A}$, such that if $\mathtt{a} \geq \mathtt{a}^*$, then*

$$c_h(\boldsymbol{\tau}, \boldsymbol{\tau}) \geq \frac{1}{2} \left(\left\| \varrho^{-\frac{1}{2}} \mathbf{div}_h \boldsymbol{\tau} \right\|_{0,\Omega}^2 + \left\| \delta_{\mathcal{F}}^{-\frac{1}{2}} h_{\mathcal{F}}^{-\frac{1}{2}} [\![\boldsymbol{\tau}]\!] \right\|_{0,\mathcal{F}_h^*}^2 \right) \quad \forall \boldsymbol{\tau} \in X_h^{DG}. \tag{3.4}$$

Proof. By definition, we have

$$c_h(\boldsymbol{\tau}, \boldsymbol{\tau}) = \left\| \varrho^{-\frac{1}{2}} \mathbf{div}_h \boldsymbol{\tau} \right\|_{0,\Omega}^2 + \mathtt{a} \left\| \varrho_{\mathcal{F}}^{-\frac{1}{2}} h_{\mathcal{F}}^{-\frac{1}{2}} [\![\boldsymbol{\tau}]\!] \right\|_{0,\mathcal{F}_h^*}^2 - 2 \left(\{\varrho^{-1} \mathbf{div}_h \boldsymbol{\tau}\}, [\![\boldsymbol{\tau}]\!] \right)_{\mathcal{F}_h^*} \tag{3.5}$$

Using the Cauchy-Schwarz inequality, Young's inequality together with the discrete trace inequality (3.1) we obtain the estimate

$$2 \left| \left(\{\varrho^{-1} \mathbf{div}_h \boldsymbol{\tau}\}, [\![\boldsymbol{\tau}]\!] \right)_{\mathcal{F}_h^0} \right| \leq 2 \left\| \gamma_{\mathcal{F}}^{\frac{1}{2}} h_{\mathcal{F}}^{\frac{1}{2}} \{\varrho^{-1} \mathbf{div}_h \boldsymbol{\tau}\} \right\|_{0,\mathcal{F}_h^*} \left\| \gamma_{\mathcal{F}}^{-\frac{1}{2}} h_{\mathcal{F}}^{-\frac{1}{2}} [\![\boldsymbol{\tau}]\!] \right\|_{0,\mathcal{F}_h^*}$$
$$\leq 2 C_{\mathrm{tr}} \left\| \varrho^{-\frac{1}{2}} \mathbf{div}\, \boldsymbol{\tau} \right\|_{0,\Omega} \left\| \gamma_{\mathcal{F}}^{-\frac{1}{2}} h_{\mathcal{F}}^{-\frac{1}{2}} [\![\boldsymbol{\tau}]\!] \right\|_{0,\mathcal{F}_h^*}$$
$$\leq \frac{1}{2} \left\| \varrho^{-\frac{1}{2}} \mathbf{div}\, \boldsymbol{\tau} \right\|_{0,\Omega}^2 + 2 C_{\mathrm{tr}}^2 \left\| \gamma_{\mathcal{F}}^{-\frac{1}{2}} h_{\mathcal{F}}^{-\frac{1}{2}} [\![\boldsymbol{\tau}]\!] \right\|_{0,\mathcal{F}_h^*}^2. \tag{3.6}$$

Combining (3.6) and (3.5) gives the result with $\mathtt{a}^* := 2C_{\mathrm{tr}}^2 + \frac{1}{2}$. □

4. The pure–stress DG scheme

We are now in a position to introduce the following mixed DG discretization of (2.4): Find $0 \neq \boldsymbol{\sigma}_h \in X_h^{DG}$ and $\kappa_h \in \mathbb{R}$ such that

$$a_h(\boldsymbol{\sigma}_h, \boldsymbol{\tau}) = \kappa_h (\boldsymbol{\sigma}_h, \boldsymbol{\tau})_{\mathcal{A}} \quad \forall \boldsymbol{\tau} \in X_h^{DG}. \tag{4.1}$$

Remark 4.1. We are only considering via (4.1) the symmetric interior penalty DG method (SIP) because the non-symmetric DG versions, known in the literature as NIP and IIP, have sub-optimal rates of convergence for the eigenvalues [1,22].

In all what follows, we make the following stability assumption.

Assumption 1. The parameter $\mathtt{a}$ is greater than or equal to $\mathtt{a}^*$: $\mathtt{a} \geq \mathtt{a}^* := 2C_{\rm tr}^2 + \frac{1}{2}$.

Under this assumption, Proposition 3.1 permits us to guarantee the well–posedness of the discrete source operator $\tilde{T}_h : H \to X_h^{DG}$ given, for any $\boldsymbol{f} \in H$, by

$$a_h(\tilde{T}_h \boldsymbol{f}, \boldsymbol{\tau}_h) = \big(\boldsymbol{f}, \boldsymbol{\tau}_h\big)_{\mathcal{A}} \quad \forall \boldsymbol{\tau}_h \in X_h^{DG}. \tag{4.2}$$

Actually, $\tilde{T}_h$ is uniformly bounded, namely,

$$|||\tilde{T}_h \boldsymbol{f}||| \leq 2\|\boldsymbol{f}\|_{\mathcal{A}} \quad \forall \boldsymbol{f} \in H. \tag{4.3}$$

Similarly to the continuous case, we observe that $(1 \neq \kappa_h, \boldsymbol{\sigma}_h) \in \mathbb{R} \times X_h^{DG}$ is a solution of problem (4.1) if and only if $(\frac{1}{\kappa_h}, \boldsymbol{\sigma}_h)$ is an eigenpair of $T_h := \tilde{T}_h|_{X_h^{DG}}$, i.e., $T_h \boldsymbol{\sigma}_h = \frac{1}{\kappa_h}\boldsymbol{\sigma}_h$. Moreover, it is clear that $\kappa_h = 1$ is an eigenvalue common to (4.1) and T_h with corresponding eigenspace

$$K_h := \big\{\boldsymbol{\tau}_h \in X_h^{DG};\ c_h(\boldsymbol{\tau}_h, \boldsymbol{\tau}_h) = 0\big\}. \tag{4.4}$$

The following result establishes a Céa estimate for the DG approximation (4.2) of (2.5).

Theorem 4.1. *Under Assumption 1, for all $\boldsymbol{f} \in H$, it holds true that*

$$|||(\tilde{T} - \tilde{T}_h)\boldsymbol{f}||| \leq (1 + 2M) \inf_{\boldsymbol{\tau}_h \in X_h^{DG}} |||\tilde{T}\boldsymbol{f} - \boldsymbol{\tau}_h|||_*, \tag{4.5}$$

with M as in (3.3).

Proof. We already know from (2.7) that $\boldsymbol{u} := \frac{1}{\varrho}\,\mathbf{div}(\tilde{T}\boldsymbol{f}) \in H_D^1(\Omega, \mathbb{R}^d)$. Hence, using the integration by parts (1.1) elementwise in

$$a(\tilde{T}\boldsymbol{f}, \boldsymbol{\tau}) = (\boldsymbol{f}, \boldsymbol{\tau})_{\mathcal{A}} \quad \forall \boldsymbol{\tau} \in X$$

gives

$$\big(\tilde{T}\boldsymbol{f} - \boldsymbol{f}, \boldsymbol{\tau}\big)_{\mathcal{A}} = (\boldsymbol{\varepsilon}(\boldsymbol{u}), \boldsymbol{\tau}) = -\big(\boldsymbol{u}, \mathbf{div}_h\,\boldsymbol{\tau}\big) + (\{\boldsymbol{u}\}, [\![\boldsymbol{\tau}]\!])_{\mathcal{F}_h^*} \quad \forall \boldsymbol{\tau} \in X_h^{DG}.$$

Substituting back $\boldsymbol{u} = \frac{1}{\varrho}\,\mathbf{div}(\tilde{T}\boldsymbol{f})$ into the last expression we get

$$\big(\tilde{T}\boldsymbol{f}, \boldsymbol{\tau}\big)_{\mathcal{A}} + \Big(\tfrac{1}{\varrho}\,\mathbf{div}(\tilde{T}\boldsymbol{f}), \mathbf{div}_h\,\boldsymbol{\tau}\Big) - \Big(\{\tfrac{1}{\varrho}\,\mathbf{div}(\tilde{T}\boldsymbol{f})\}, [\![\boldsymbol{\tau}]\!]\Big)_{\mathcal{F}_h^*} = (\boldsymbol{f}, \boldsymbol{\tau})_{\mathcal{A}}$$
$$\forall \boldsymbol{\tau} \in X_h^{DG}.$$

Combining the last identity with (4.2) yields the following consistency property

$$a_h((\tilde{T} - \tilde{T}_h)\boldsymbol{f}, \boldsymbol{\tau}_h) = 0 \quad \forall \boldsymbol{\tau}_h \in X_h^{DG} \quad \forall \boldsymbol{f} \in H. \tag{4.6}$$

Now, by virtue of (3.3), (3.4) and (4.6), it holds true that

$$\tfrac{1}{2}|||\tilde{T}_h\boldsymbol{f} - \boldsymbol{\tau}_h|||^2 \leq a_h(\tilde{T}_h\boldsymbol{f} - \boldsymbol{\tau}_h, \tilde{T}_h\boldsymbol{f} - \boldsymbol{\tau}_h) = a_h(\tilde{T}\boldsymbol{f} - \boldsymbol{\tau}_h, \tilde{T}_h\boldsymbol{f} - \boldsymbol{\tau}_h)$$
$$\leq M\,|||\tilde{T}\boldsymbol{f} - \boldsymbol{\tau}_h|||_*|||\tilde{T}_h\boldsymbol{f} - \boldsymbol{\tau}_h||| \quad \forall \boldsymbol{\tau}_h \in X_h^{DG},$$

and the result follows from the triangle inequality. □

4.1. The operator $\mathcal{J}_h^s$

For technical reasons, we want to consider here the $H(\mathbf{div}, \Omega, \mathbb{S})$-conforming finite element space given by $X_h^c := \mathcal{P}_h(\mathcal{T}_h, \mathbb{S}) \cap X$. The goal of this section is to prove that, under certain conditions on the mesh and on the polynomial degree k, it holds true that

$$\inf_{\boldsymbol{\tau}_h \in X_h^c} \|\boldsymbol{\sigma} - \boldsymbol{\tau}_h\|_X \longrightarrow 0, \quad \text{when } h \to 0, \quad \forall \boldsymbol{\sigma} \in X. \tag{4.7}$$

The main obstacle in performing this task is the symmetry constraint. Let us ignore this constraint and discuss, in a first step, approximation properties of the Brezzi-Douglas-Marini (BDM) mixed finite element discretization of

$$H_N(\mathbf{div}, \Omega, \mathbb{M})$$
$$:= \Big\{\boldsymbol{\tau} \in H(\mathbf{div}, \Omega, \mathbb{M});\quad \langle \boldsymbol{\tau}\boldsymbol{n}, \boldsymbol{\phi}\rangle_\Gamma = 0 \quad \forall \boldsymbol{\phi} \in H^{1/2}(\Gamma, \mathbb{R}^d),\ \boldsymbol{\phi}|_{\Gamma_D} = \mathbf{0}\Big\}.$$

Given $s > 1/2$ the tensorial version of the canonical BDM finite element interpolant

$$\Pi_h^{\rm BDM} : H_N(\mathrm{div}, \Omega, \mathbb{M}) \cap H^s(\cup_j \Omega_j, \mathbb{M}) \to H_N(\mathrm{div}, \Omega, \mathbb{M}) \cap \mathcal{P}_k(\mathcal{T}_h, \mathbb{M})$$

satisfies the following classical error estimate, [7, Proposition 2.5.4],

$$\Big\|\boldsymbol{\tau} - \Pi_h^{\rm BDM}\boldsymbol{\tau}\Big\|_{0,\Omega} \leq Ch^{\min\{s,k+1\}} \sum_{j=1}^{J} \|\boldsymbol{\tau}\|_{s,\Omega_j}$$
$$\forall \boldsymbol{\tau} \in H_N(\mathrm{div}, \Omega, \mathbb{M}) \cap H^s(\cup_j \Omega_j, \mathbb{M}), \quad s > 1/2, \tag{4.8}$$

Moreover, we have the well-known commutativity property,

$$\mathbf{div}\,\Pi_h^{\rm BDM}\boldsymbol{\tau} = Q_h^{k-1}\,\mathbf{div}\,\boldsymbol{\tau} \quad \forall \boldsymbol{\tau} \in H_N(\mathrm{div}, \Omega, \mathbb{M}) \cap H^s(\cup_j \Omega_j, \mathbb{M}), \quad s > 1/2, \tag{4.9}$$

where Q_h^{k-1} stands for the $L^2(\Omega, \mathbb{R}^d)$-orthogonal projection onto $\mathcal{P}_{k-1}(\mathcal{T}_h, \mathbb{R}^d)$. Therefore, if $\mathbf{div}\,\boldsymbol{\tau} \in H^s(\cup_j \Omega_j, \mathbb{R}^d)$, we obtain

$$\|\mathbf{div}(\boldsymbol{\tau} - \Pi_h^{\rm BDM}\boldsymbol{\tau})\|_{0,\Omega} = \|\mathbf{div}\,\boldsymbol{\tau} - Q_h^{k-1}\,\mathbf{div}\,\boldsymbol{\tau}\|_{0,\Omega} \leq Ch^{\min\{s,k\}} \sum_{j=1}^{J} \|\mathbf{div}\,\boldsymbol{\tau}\|_{s,\Omega_j}. \tag{4.10}$$

We point out that one can actually extend the domain of the canonical interpolation operator $\Pi_h^{\rm BDM}$ to $H_N(\mathbf{div}, \Omega, \mathbb{M}) \cap H^s(\Omega, \mathbb{M})$, for any $s > 0$. In the case of a constant function ϱ and a constant tensor $\mathcal{A}$, classical regularity results [10,17] ensure the existence of $\hat{s} \in (0, 1]$ (depending on Ω on the boundary conditions and on the Lamé coefficients) such that the solution $\widetilde{\boldsymbol{u}}$ of problem (2.8) belongs to $H^{1+s}(\Omega, \mathbb{R}^d) \cap H_D^1(\Omega, \mathbb{R}^d)$ for all $s \in (0, \hat{s})$. However, our aim here is to avoid relying on regularity results that may be difficult to establish for the elasticity system in the case of general domains, boundary conditions and material properties. For this reason, we resort to the following smoothed projector recently introduced by Licht [21, Theorem 6.3].

Theorem 4.2. *There exists a bounded and linear operator $\mathcal{J}_h : L^2(\Omega, \mathbb{M}) \to H_N(\mathbf{div}, \Omega, \mathbb{M}) \cap \mathcal{P}_k(\mathcal{T}_h, \mathbb{M})$ such that*

i) There exists $C > 0$ independent of h such that

$$\|\boldsymbol{\sigma} - \mathcal{J}_h\boldsymbol{\sigma}\|_{0,\Omega} \leq C \inf_{\boldsymbol{\tau}_h \in H_N(\mathbf{div}, \Omega, \mathbb{M}) \cap \mathcal{P}_k(\mathcal{T}_h, \mathbb{M})} \|\boldsymbol{\sigma} - \boldsymbol{\tau}_h\|_{0,\Omega} \quad \forall \boldsymbol{\sigma} \in L^2(\Omega, \mathbb{M})$$

ii) $\mathbf{div}\,\mathcal{J}_h\boldsymbol{\sigma} = Q_h^{k-1}\,\mathbf{div}\,\boldsymbol{\sigma}$ for all $\boldsymbol{\sigma} \in H_N(\mathbf{div}, \Omega, \mathbb{M})$.

The operator $\mathcal{J}_h$ doesn't preserve symmetry. To remedy this drawback, we follow [15,18,32] and use a symmetrization procedure that requires the stability the Scott-Vogelius element [31] for the Stokes problem. We refer to [14, Section 55.3] for a detailed account on the conditions (on the mesh $\mathcal{T}_h$ and k) under which this stability property is guaranteed in 2D and 3D. The analysis that follows from now on is based on the following assumption.

Assumption 2. The pair $\{\mathcal{P}_{k+1}(\mathcal{T}_h,\mathbb{R}^d)\cap H^1_N(\Omega,\mathbb{R}^d),\mathcal{P}_k(\mathcal{T}_h)\}$ is stable for the Stokes problem on the mesh $\mathcal{T}_h$: there exists $\beta>0$ independent of h such that

$$\sup_{\boldsymbol{v}_h\in\mathcal{P}_{k+1}(\mathcal{T}_h,\mathbb{R}^d)\cap H^1_N(\Omega,\mathbb{R}^d)}\frac{(\operatorname{div}\boldsymbol{v}_h,\phi_h)}{\|\boldsymbol{v}_h\|_{1,\Omega}}\geq\beta\|\phi_h\|_{0,\Omega}\quad\forall\phi_h\in\mathcal{P}_k(\mathcal{T}_h).\tag{4.11}$$

Lemma 4.1. *Under Assumption 2, there exists a linear operator*

$$S_h:\ \mathcal{P}_k(\mathcal{T}_h,\mathbb{M})\cap H_N(\mathbf{div},\Omega,\mathbb{M})\to\mathcal{P}_k(\mathcal{T}_h,\mathbb{S})\cap X$$

such that, for all $\boldsymbol{\tau}_h\in\mathcal{P}_k(\mathcal{T}_h,\mathbb{M})\cap H_N(\mathbf{div},\Omega,\mathbb{M})$,

i) $\mathbf{div}(\boldsymbol{\tau}_h-S_h\boldsymbol{\tau}_h)=\mathbf{0}$ *in* Ω,
ii) and $\|\boldsymbol{\tau}_h-S_h\boldsymbol{\tau}_h\|_{0,\Omega}\leq C\left\|\boldsymbol{\tau}_h-\boldsymbol{\tau}_h^{\mathrm{t}}\right\|_{0,\Omega}$, *with* $C>0$ *independent of* h.

Proof. We only sketch the proof given in [32, Lemma 5.2] and adapt it to our boundary conditions, see also [15,18]. In the case $d=2$, given $\boldsymbol{\tau}_h\in\mathcal{P}_k(\mathcal{T}_h,\mathbb{M})\cap H_N(\mathbf{div},\Omega,\mathbb{M})$, it follows from Assumption 2 that there exists $\boldsymbol{w}_h\in H^1_N(\Omega,\mathbb{R}^d)\cap\mathcal{P}_{k+1}(\mathcal{T}_h,\mathbb{R}^d)$ satisfying $\operatorname{div}\boldsymbol{w}_h=\tau_{h,21}-\tau_{h,12}$ and

$$\|\boldsymbol{w}_h\|_{1,\Omega}\lesssim\|\tau_{h,12}-\tau_{h,21}\|_{0,\Omega}\lesssim\|\boldsymbol{\tau}_h-\boldsymbol{\tau}_h^{\mathrm{t}}\|_{0,\Omega}.\tag{4.12}$$

We recall that all differential operators are applied row-wise and let $S_h\boldsymbol{\tau}_h:=\boldsymbol{\tau}_h+\nabla^\perp\boldsymbol{w}_h$, where $\nabla^\perp:=(-\partial_2,\partial_1)^{\mathrm{t}}$ is the rotated gradient. By construction, $\mathbf{div}(\boldsymbol{\tau}_h-S_h\boldsymbol{\tau}_h)=\mathbf{0}$ and thanks to (4.12) it holds true that $\|\boldsymbol{\tau}_h-S_h\boldsymbol{\tau}_h\|_{0,\Omega}\lesssim\|\boldsymbol{\tau}_h-\boldsymbol{\tau}_h^{\mathrm{t}}\|_{0,\Omega}$. Moreover, it is easy to check that $S_h\boldsymbol{\tau}_h=(S_h\boldsymbol{\tau}_h)^{\mathrm{t}}$. It remains to show that S_h preserves the boundary condition on Γ_N. This follows from the fact that $\nabla^\perp\boldsymbol{w}_h\boldsymbol{n}=(\partial_\tau w_{h,1},\partial_\tau w_{h,2})^{\mathrm{t}}$ and the tangential derivatives $\partial_\tau w_{h,j}:=\partial_1 w_{h,j}n_2-\partial_2 w_{h,j}n_1$, $j=1,2$, vanish on Γ_N since $\boldsymbol{w}_h\in H^1_N(\Omega,\mathbb{R}^d)$. This finishes the proof of the result in the two dimensional case.

In the case $d=3$, we let $S_h\boldsymbol{\tau}_h:=\boldsymbol{\tau}_h+\nabla\times\boldsymbol{w}_h$, with $\boldsymbol{w}_h=\mathbf{z}_h^{\mathrm{t}}-(\operatorname{tr}\mathbf{z_h})I$, where the tensor $\mathbf{z}_h\in H^1_N(\Omega,\mathbb{M})\cap\mathcal{P}_{k+1}(\mathcal{T}_h,\mathbb{M})$ satisfies $\mathbf{div}\,\mathbf{z}_h=(\boldsymbol{\tau}_{h,23}-\boldsymbol{\tau}_{h,32},\boldsymbol{\tau}_{h,31}-\boldsymbol{\tau}_{h,13},\boldsymbol{\tau}_{h,12}-\boldsymbol{\tau}_{h,21})^{\mathrm{t}}$ and $\|\mathbf{z}_h\|_{1,\Omega}\lesssim\|\boldsymbol{\tau}_h-\boldsymbol{\tau}_h^{\mathrm{t}}\|_{0,\Omega}$. The existence of $\mathbf{z}_h$ is ensured by Assumption 2. In this way, we also have $\mathbf{div}(\boldsymbol{\tau}_h-S_h\boldsymbol{\tau}_h)=\mathbf{0}$ and $\|\boldsymbol{\tau}_h-S_h\boldsymbol{\tau}_h\|_{0,\Omega}\lesssim\|\boldsymbol{\tau}_h-\boldsymbol{\tau}_h^{\mathrm{t}}\|_{0,\Omega}$. The proof of the symmetry property $S_h\boldsymbol{\tau}_h=(S_h\boldsymbol{\tau}_h)^{\mathrm{t}}$ is a little more involved in this case, as shown in [32, Lemma 5.2]. Finally, we point out that $(\nabla\times\boldsymbol{w}_h)\boldsymbol{n}=(\operatorname{div}_\Gamma(\boldsymbol{w}_h^1\times\boldsymbol{n}),\operatorname{div}_\Gamma(\boldsymbol{w}_h^2\times\boldsymbol{n}),\operatorname{div}_\Gamma(\boldsymbol{w}_h^3\times\boldsymbol{n}))^{\mathrm{t}}$ on Γ, where $\boldsymbol{w}_h^j$, $j=1,2,3$ stand for the rows of $\boldsymbol{w}_h$ and $\operatorname{div}_\Gamma$ represents the divergence operator on the surface Γ. Taking into account that, $\boldsymbol{w}_h|_{\Gamma_N}=\mathbf{0}$, we deduce that $S_h\boldsymbol{\tau}_h$ belongs to $\mathcal{P}_k(\mathcal{T}_h,\mathbb{S})\cap X$ for all $\boldsymbol{\tau}_h\in\mathcal{P}_k(\mathcal{T}_h,\mathbb{M})\cap H_N(\mathbf{div},\Omega,\mathbb{M})$, which finishes the proof of the result in the tree-dimensional case. □

We are able to state now the counterpart of Theorem 4.2 for $\mathcal{J}_h^s:=S_h\circ\mathcal{J}_h:\ L^2(\Omega,\mathbb{M})\to X_h^c$.

Corollary 4.1. *Under Assumption 2,* $\mathcal{J}_h^s:\ L^2(\Omega,\mathbb{S})\to X_h^c$ *satisfies*

i) $\left\|\boldsymbol{\sigma}-\mathcal{J}_h^s\boldsymbol{\sigma}\right\|_{0,\Omega}\leq C\inf_{\boldsymbol{\tau}_h\in H_N(\mathbf{div},\Omega,\mathbb{M})\cap\mathcal{P}_k(\mathcal{T}_h,\mathbb{M})}\|\boldsymbol{\sigma}-\boldsymbol{\tau}_h\|_{0,\Omega}\quad\forall\boldsymbol{\sigma}\in L^2(\Omega,\mathbb{S})$ *with* C *independent of* h,
ii) and $\mathbf{div}\,\mathcal{J}_h^s\boldsymbol{\sigma}=Q_h^{k-1}\,\mathbf{div}\,\boldsymbol{\sigma}$ *for all* $\boldsymbol{\sigma}\in X$.

Proof. The commuting property for $\mathcal{J}_h^s$ follows from the corresponding property for $\mathcal{J}_h$ and from the fact that S_h preserves the divergence of tensors, as stated in Lemma 4.1 i). In addition, as a consequence the property given by Lemma 4.1 ii), for any $\boldsymbol{\sigma}\in L^2(\Omega,\mathbb{S})$, it holds true that

$$\begin{aligned}\left\|\boldsymbol{\sigma}-\mathcal{J}_h^s\boldsymbol{\sigma}\right\|_{0,\Omega}&\leq\|\boldsymbol{\sigma}-\mathcal{J}_h\boldsymbol{\sigma}\|_{0,\Omega}+\|\mathcal{J}_h\boldsymbol{\sigma}-S_h(\mathcal{J}_h\boldsymbol{\sigma})\|_{0,\Omega}\\&\lesssim\|\boldsymbol{\sigma}-\mathcal{J}_h\boldsymbol{\sigma}\|_{0,\Omega}+\|\boldsymbol{\sigma}-\mathcal{J}_h\boldsymbol{\sigma}-(\boldsymbol{\sigma}-\mathcal{J}_h\boldsymbol{\sigma})^{\mathrm{t}}\|_{0,\Omega}\\&\lesssim\|\boldsymbol{\sigma}-\mathcal{J}_h\boldsymbol{\sigma}\|_{0,\Omega},\end{aligned}\tag{4.13}$$

and the first statement of the Corollary follows from Theorem 4.2 i). □

Using the density of smooth functions in $H_N(\mathbf{div},\Omega,\mathbb{M})$ [21, Lemma 1.2] and the interpolation error estimates satisfied by the BDM projector, we deduce from Corollary 4.1 that, for all $\boldsymbol{\sigma}\in X$,

$$\begin{aligned}\inf_{\boldsymbol{\tau}_h\in X_h^c}\|\boldsymbol{\sigma}-\boldsymbol{\tau}_h\|_X&\leq\left\|\boldsymbol{\sigma}-\mathcal{J}_h^s\boldsymbol{\sigma}\right\|_X\\&\lesssim\inf_{\boldsymbol{\tau}_h\in H_N(\mathbf{div},\Omega,\mathbb{M})\cap\mathcal{P}_k(\mathcal{T}_h,\mathbb{M})}\|\boldsymbol{\sigma}-\boldsymbol{\tau}_h\|_{0,\Omega}+\|\mathbf{div}\,\boldsymbol{\sigma}-Q_h^{k-1}\,\mathbf{div}\,\boldsymbol{\sigma}\|_{0,\Omega}\to0\end{aligned}$$

when h goes to zero, which proves (4.7).

4.2. The operator P_h

In what follows, the norm of a linear and continuous operator $L:V_1\to V_2$ between two Hilbert spaces V_1 and V_2 is denoted $\|L\|_{\mathcal{L}(V_1,V_2)}:=\sup_{v\in V_1,\|v\|_{V_1}=1}\|Lv\|_{V_2}$.

Under Assumption 1, it follows from (4.4) and Proposition 3.1 that

$$\left\|\rho^{-\frac12}\,\mathbf{div}_h\,\boldsymbol{\tau}_h\right\|_{0,\Omega}^2+\left\|\delta_{\mathcal{F}}^{-\frac12}h_{\mathcal{F}}^{-\frac12}[\![\boldsymbol{\tau}_h]\!]\right\|_{0,\mathcal{F}_h^*}^2=0\quad\forall\boldsymbol{\tau}_h\in K_h.$$

This means that the elements of K_h have continuous normal components across the internal edges/faces, have a vanishing normal component on Γ_N and are globally divergence free. It follows that

$$K_h=\left\{\boldsymbol{\tau}_h\in X_h^c;\ \mathbf{div}\,\boldsymbol{\tau}_h=0\text{ in }\Omega\right\}\subset K.$$

This inclusion is crucial in our analysis. It is important to notice that the $a(\cdot,\cdot)$-orthogonal complement $K_h^\perp:=\left\{\boldsymbol{\sigma}_h\in X_h^c;\ a(\boldsymbol{\sigma}_h,\boldsymbol{\tau}_h)=0,\ \forall\boldsymbol{\tau}_h\in K_h\right\}$ is not a subset of $K^\perp=P(X)$. Let $P_h:X_h^c\to K_h^\perp$ be the X-orthogonal projection in X_h^c onto $K_h^\perp$. The following result provides an estimate for the operator $(P-P_h)|_{X_h^c}$.

Lemma 4.2. *Under Assumptions 1 and 2, it holds true that*

$$\|P-P_h\|_{\mathcal{L}(X_h^c,X)}\leq2\left\|(I-\mathcal{J}_h^s)P\right\|_{\mathcal{L}(X,H)}.$$

Proof. Let us first notice that, by definition of P and $\mathcal{J}_h^s$, for any $\boldsymbol{\tau}_h\in X_h^c$,

$$\begin{aligned}\mathbf{div}(\boldsymbol{\tau}_h-\mathcal{J}_h^sP\boldsymbol{\tau}_h)&=\mathbf{div}\,\boldsymbol{\tau}_h-\mathbf{div}\,\mathcal{J}_hP\boldsymbol{\tau}_h=\mathbf{div}\,\boldsymbol{\tau}_h-Q_h^{k-1}\,\mathbf{div}\,P\boldsymbol{\tau}_h\\&=\mathbf{div}\,\boldsymbol{\tau}_h-Q_h^{k-1}\,\mathbf{div}\,\boldsymbol{\tau}_h=\mathbf{0},\end{aligned}$$

which proves that $(I-\mathcal{J}_h^sP)X_h^c\subset K_h$. Hence, it follows from the triangle inequality that

$$\begin{aligned}\|(P-P_h)\boldsymbol{\tau}_h\|_X&\leq\left\|P_h\boldsymbol{\tau}_h-\mathcal{J}_h^sP\boldsymbol{\tau}_h\right\|_X+\left\|(I-\mathcal{J}_h^s)P\boldsymbol{\tau}_h\right\|_X\\&=\left\|P_h\boldsymbol{\tau}_h-\mathcal{J}_h^sP\boldsymbol{\tau}_h\right\|_{\mathcal{A}}+\left\|(I-\mathcal{J}_h^s)P\boldsymbol{\tau}_h\right\|_{\mathcal{A}},\quad\forall\boldsymbol{\tau}_h\in X_h^c,\end{aligned}\tag{4.14}$$

where we took into account that $P_h\boldsymbol{\tau}_h-\mathcal{J}_h^sP\boldsymbol{\tau}_h=\boldsymbol{\tau}_h-\mathcal{J}_h^sP\boldsymbol{\tau}_h-(\boldsymbol{\tau}_h-P_h\boldsymbol{\tau}_h)\in K_h$ and

$$P\boldsymbol{\tau}_h-\mathcal{J}_h^sP\boldsymbol{\tau}_h=\boldsymbol{\tau}_h-\mathcal{J}_h^sP\boldsymbol{\tau}_h-(\boldsymbol{\tau}_h-P\boldsymbol{\tau}_h)\in K.$$

To estimate the first term in the right-hand side of (4.14), we take advantage of the inclusion $K_h\subset K$ to write

$$\left(P_h\boldsymbol{\tau}_h-\mathcal{J}_h^sP\boldsymbol{\tau}_h,P_h\boldsymbol{\tau}_h-\mathcal{J}_h^sP\boldsymbol{\tau}_h\right)_{\mathcal{A}}=\left(P\boldsymbol{\tau}_h-\mathcal{J}_h^sP\boldsymbol{\tau}_h,P_h\boldsymbol{\tau}_h-\mathcal{J}_h^sP\boldsymbol{\tau}_h\right)_{\mathcal{A}}$$

and we deduce from the Cauchy-Schwarz inequality that

$$\left\|P_h\boldsymbol{\tau}_h-\mathcal{J}_h^sP\boldsymbol{\tau}_h\right\|_{\mathcal{A}}\leq\left\|(I-\mathcal{J}_h^s)P\boldsymbol{\tau}_h\right\|_{\mathcal{A}}.$$

Plugging the last estimate in (4.14) gives the result. □

Remark 4.2. A conforming approximation of problem (2.4) based on X_h^c is not useful in practice since it is not straightforward to construct an explicit basis of this finite element space.

5. Spectral correctness of the DG scheme and error estimates

5.1. The main result

Even in the case of conforming Galerkin approximations of eigenproblems, it is well-known [6] that when the source operator is not compact, a convergent discrete scheme for the source problem doesn't necessarily provide a correct approximation of the spectrum. A fortiori, in our case, Theorem 4.1 is not enough to prevent (4.1) from producing spurious eigenvalues. The procedure introduced in [11,12] to analyze the spectral approximation of non compact operators has been recently adapted in [23, Section 5] to a DG context (cf. also [8]). It is shown that the main ingredient to prove the spectral correctness of the method is the uniform convergence of $\tilde{T}_h$ to $\tilde{T}$ with respect to the following h–dependent norm,

$$\|\tilde{T}-\tilde{T}_h\|_h := \sup_{\boldsymbol{\tau}_h\in\boldsymbol{\mathcal{P}}_k(\mathcal{T}_h,\mathbb{S})} \frac{|||(\tilde{T}-\tilde{T}_h)\boldsymbol{\tau}_h|||}{|||\boldsymbol{\tau}_h|||} \longrightarrow 0, \quad \text{when } h\to 0. \tag{5.1}$$

We need the following technical result to prove (5.1).

Lemma 5.1. *Under Assumption 2, there exists a projector $\mathcal{X}_h^s : X_h^{DG} \to X_h^c$ such that*

$$h\left\|\mathbf{div}_h(\boldsymbol{\tau}_h-\mathcal{X}_h^s\boldsymbol{\tau}_h)\right\|_{0,\Omega} + \left\|\boldsymbol{\tau}_h-\mathcal{X}_h^s\boldsymbol{\tau}_h\right\|_{0,\Omega} \le Ch\left\|h_{\mathcal{F}}^{-1/2}[\![\boldsymbol{\tau}_h]\!]\right\|_{0,\mathcal{F}_h^*} \quad \forall\boldsymbol{\tau}_h\in X_h^{DG}, \tag{5.2}$$

with $C>0$ independent of h.

Proof. It is proved in [24, Proposition 5.2] that there exists a projector $\mathcal{X}_h : X_h^{DG} \to H_N(\mathbf{div},\Omega,\mathbb{M})\cap\mathcal{P}_k(\mathcal{T}_h,\mathbb{M})$ such that

$$h\|\mathbf{div}_h(\boldsymbol{\tau}_h-\mathcal{X}_h\boldsymbol{\tau}_h)\|_{0,\Omega} + \|\boldsymbol{\tau}_h-\mathcal{X}_h\boldsymbol{\tau}_h\|_{0,\Omega} \le Ch\left\|h_{\mathcal{F}}^{-1/2}[\![\boldsymbol{\tau}_h]\!]\right\|_{0,\mathcal{F}_h^*} \quad \forall\boldsymbol{\tau}_h\in X_h^{DG}. \tag{5.3}$$

By construction of S_h, the operator $\mathcal{X}_h^s := S_h\circ\mathcal{X}_h$ satisfies $\mathbf{div}\,\mathcal{X}_h^s\boldsymbol{\tau}_h = \mathbf{div}\,\mathcal{X}_h\boldsymbol{\tau}_h$. Moreover, using property ii) of Lemma 4.1 and reasoning as for estimate (4.13) yield

$$\left\|\boldsymbol{\tau}_h-\mathcal{X}_h^s\boldsymbol{\tau}_h\right\|_{0,\Omega} \lesssim \|\boldsymbol{\tau}_h-\mathcal{X}_h\boldsymbol{\tau}_h\|_{0,\Omega}.$$

It follows that (5.2) is a direct consequence of (5.3). □

We point out that the stability of $\mathcal{X}_h^s : X_h^{DG}\to X_h^c$ follows directly from the triangle inequality and (5.2), namely,

$$\left\|\mathcal{X}_h^s\boldsymbol{\tau}_h\right\|_{\mathcal{X}} \le C|||\boldsymbol{\tau}_h||| \quad \forall\boldsymbol{\tau}_h\in X_h^{DG}, \tag{5.4}$$

with $C>0$ independent of h.

We are now in a position to prove the main result of this article.

Theorem 5.1. *Under Assumptions 1 and 2, it holds true that*

$$\|\tilde{T}-\tilde{T}_h\|_h \le C\left(h+\left\|(I-\mathcal{J}_h^s)P\right\|_{\mathcal{L}(X,H)} + \|(I-\mathcal{J}_h^s)TP\|_{\mathcal{L}(X,X)}\right),$$

with C independent of h.

Proof. For any $\boldsymbol{\tau}_h\in X_h^{DG}$, we consider the splitting $\boldsymbol{\tau}_h = (I-\mathcal{X}_h^s)\boldsymbol{\tau}_h + P_h\mathcal{X}_h^s\boldsymbol{\tau}_h + (I-P_h)\mathcal{X}_h^s\boldsymbol{\tau}_h$ and exploit the fact that $(I-P_h)\mathcal{X}_h^s\boldsymbol{\tau}_h\in K_h\subset K$ is in the kernel of $(\tilde{T}-\tilde{T}_h)$ to obtain

$$\begin{aligned}(\tilde{T}-\tilde{T}_h)\boldsymbol{\tau}_h &= (\tilde{T}-\tilde{T}_h)(\mathcal{I}-\mathcal{X}_h^s)\boldsymbol{\tau}_h + (\tilde{T}-\tilde{T}_h)P_h\mathcal{X}_h^s\boldsymbol{\tau}_h\\ &= (\tilde{T}-\tilde{T}_h)(I-\mathcal{X}_h^s)\boldsymbol{\tau}_h + (\tilde{T}-\tilde{T}_h)(P_h-P)\mathcal{X}_h^s\boldsymbol{\tau}_h\\ &\quad + (\tilde{T}-\tilde{T}_h)P\mathcal{X}_h^s\boldsymbol{\tau}_h.\end{aligned}$$

It follows from the triangle inequality, (2.6) and (4.3) that

$$\begin{aligned}|||(\tilde{T}-\tilde{T}_h)\boldsymbol{\tau}_h||| &\le 3\left\|(\mathcal{I}-\mathcal{X}_h^s)\boldsymbol{\tau}_h\right\|_{\mathcal{A}} + 3\|(P-P_h)\mathcal{X}_h^s\boldsymbol{\tau}_h\|_{\mathcal{A}}\\ &\quad + \left|||(\tilde{T}-\tilde{T}_h)P\mathcal{X}_h^s\boldsymbol{\tau}_h|||\right|.\end{aligned} \tag{5.5}$$

Using (5.2), we can bound the first term in the right-hand side of (5.5) as follows,

$$\left\|(I-\mathcal{X}_h^s)\boldsymbol{\tau}_h\right\|_{\mathcal{A}} \le a^+\|(I-\mathcal{X}_h^s)\boldsymbol{\tau}_h\|_{0,\Omega} \le Ch|||\boldsymbol{\tau}_h|||. \tag{5.6}$$

For the second term, (5.4) and Lemma 4.2 yield

$$\left\|(P-P_h)\mathcal{X}_h^s\boldsymbol{\tau}_h\right\|_{\mathcal{A}} \le \|P-P_h\|_{\mathcal{L}(X_h^c,X)}\left\|\mathcal{X}_h^s\boldsymbol{\tau}_h\right\|_X \lesssim \left\|(I-\mathcal{J}_h^s)P\right\|_{\mathcal{L}(X,H)}|||\boldsymbol{\tau}_h|||. \tag{5.7}$$

To bound the third term in the right-hand side of (5.5), we begin by applying Céa estimate (4.5) to obtain

$$\left|||(\tilde{T}-\tilde{T}_h)P\mathcal{X}_h^s\boldsymbol{\tau}_h|||\right| \le (1+2M)\left|||(I-\mathcal{J}_h^s)TP\mathcal{X}_h^s\boldsymbol{\tau}_h|||\right|_*. \tag{5.8}$$

Let us introduce the notation $\boldsymbol{u} := \frac{1}{\varrho}\mathbf{div}(TP\mathcal{X}_h^s\boldsymbol{\tau}_h)\in H_D^1(\Omega,\mathbb{R}^d)$ and notice that, by virtue of (2.7) and (5.4),

$$\|\boldsymbol{u}\|_{1,\Omega} \le C\left\|P\mathcal{X}_h^s\boldsymbol{\tau}_h\right\|_{\mathcal{A}} \le C\left\|\mathcal{X}_h^s\boldsymbol{\tau}_h\right\|_X \le C_1|||\boldsymbol{\tau}_h|||. \tag{5.9}$$

Moreover, taking into account that ϱ is piecewise constant and using Corollary 4.1 ii) we can write $\frac{1}{\varrho}\mathbf{div}(I-\mathcal{J}_h^s)TP\mathcal{X}_h^s\boldsymbol{\tau}_h = \boldsymbol{u} - \frac{1}{\varrho}Q_h^{k-1}(\mathbf{div}\,TP\mathcal{X}_h^s\boldsymbol{\tau}_h) = (I-Q_h^{k-1})\boldsymbol{u}$ and it follows that

$$\begin{aligned}\left|||(I-\mathcal{J}_h^s)TP\mathcal{X}_h^s\boldsymbol{\tau}_h|||\right|_*^2 &= \left\|(I-\mathcal{J}_h^s)TP\mathcal{X}_h^s\boldsymbol{\tau}_h\right\|_X^2\\ &\quad + \left\|\sqrt{\varrho_{\mathcal{F}}h_{\mathcal{F}}}\{(I-Q_h^{k-1})\boldsymbol{u}\}\right\|_{0,\mathcal{F}_h^*}^2.\end{aligned} \tag{5.10}$$

Now, from the one hand,

$$\begin{aligned}\left\|(I-\mathcal{J}_h^s)TP\mathcal{X}_h^s\boldsymbol{\tau}_h\right\|_X &\le \|(I-\mathcal{J}_h^s)TP\|_{\mathcal{L}(X,X)}\left\|\mathcal{X}_h^s\boldsymbol{\tau}_h\right\|_X\\ &\lesssim \|(I-\mathcal{J}_h^s)TP\|_{\mathcal{L}(X,X)}|||\boldsymbol{\tau}_h|||\end{aligned} \tag{5.11}$$

and from the other hand, a classical scaling argument combined with (5.9) yields

$$\left\|\sqrt{h_{\mathcal{F}}}(\boldsymbol{u}-Q_h^{k-1}\boldsymbol{u})\right\|_{0,\mathcal{F}_h^*} \lesssim h|\boldsymbol{u}|_{1,\Omega} \le Ch|||\boldsymbol{\tau}_h|||. \tag{5.12}$$

Using (5.11) and (5.12) in (5.8) gives the estimate

$$\left|||(I-\mathcal{J}_h^s)TP\mathcal{X}_h^s\boldsymbol{\tau}_h|||\right|_* \le C\left(h+\|(I-\mathcal{J}_h^s)TP\|_{\mathcal{L}(X,X)}\right)|||\boldsymbol{\tau}_h|||. \tag{5.13}$$

Finally, plugging (5.6), (5.7), and (5.13) in (5.5) gives the result. □

Corollary 5.1. *Under Assumptions 1 and 2, it holds true that*

$$\lim_{h\to 0}\|\tilde{T}-\tilde{T}_h\|_h = 0.$$

Proof. The pointwise convergence of $I-\mathcal{J}_h^s : X\to X$ to zero (ensured by Corollary 4.1) and the compactness of $P : X\to H$ and $TP : X\to X$ imply that the operators $(I-\mathcal{J}_h^s)P : X\to H$ and $(I-\mathcal{J}_h^s)TP : X\to X$ are uniformly convergent to zero; namely,

$$\lim_{h\to 0}\left\|(I-\mathcal{J}_h^s)P\right\|_{\mathcal{L}(X,H)} = 0, \quad \text{and} \quad \lim_{h\to 0}\left\|(I-\mathcal{J}_h^s)TP\right\|_{\mathcal{L}(X,X)} = 0,$$

and the result follows directly from Theorem 5.1. □

5.2. Spectral correctness and convergence

For the sake of completeness, in the remainder of this section we show (by applying a number of results from [23, Section 5]) how to exploit property (5.1) to derive the correct spectral convergence of (4.1). Let us first introduce some notations. For $\boldsymbol{\sigma}\in X^{DG}(h)$ and E and F closed subspaces of $X^{DG}(h)$, we set $\delta(\boldsymbol{\sigma},E) := \inf_{\boldsymbol{\tau}\in E}|||\boldsymbol{\sigma}-\boldsymbol{\tau}|||$,

$\delta(E,F) := \sup_{\sigma\in E:\,|||\sigma|||=1} \delta(\boldsymbol{\sigma},F)$, and $\widehat{\delta}(E,F) := \max\{\delta(E,F),\delta(F,E)\}$, the latter being the so called *gap* between subspaces E and F.

Let $\Lambda \subset \mathbb{C}\setminus\{0,1\}$ be an arbitrary compact set with smooth boundary $\partial\Lambda$ satisfying $\partial\Lambda \cap \mathrm{sp}(T) = \emptyset$. We assume that there are m eigenvalues $\eta_1^\Lambda,\dots,\eta_m^\Lambda$ of T (repeated according to their algebraic multiplicities) inside $\partial\Lambda$. The following result shows that the resolvent $\left(zI-\tilde{T}\right)^{-1} : X^{DG}(h) \longrightarrow X^{DG}(h)$ is uniformly bounded with respect to h and $z\in\partial\Lambda$.

Lemma 5.2. *There exists a constant $C>0$ independent of h such that*

$$|||(zI-\tilde{T})\boldsymbol{\tau}||| \geq C\,|||\boldsymbol{\tau}||| \quad \forall \boldsymbol{\tau}\in X^{DG}(h),$$

for all $z\in\partial\Lambda$.

Proof. See [23, Lemma 3.2] □

We deduce from Lemma 5.2 that the operator $\mathcal{E} := \frac{1}{2\pi i}\int_{\partial\Lambda}\left(zI-\tilde{T}\right)^{-1} dz : X^{DG}(h) \longrightarrow X^{DG}(h)$ is well-defined and bounded uniformly in h. Moreover, $\mathcal{E}|_X : X\to X$ is a projector onto the finite dimensional space $\mathcal{E}(X)$ spanned by the generalized eigenfunctions associated with the finite set of eigenvalues of T contained in Λ. Actually, it is easy to check that $\tilde{T} : X^{DG}(h)\to X^{DG}(h)$ and $T : X\to X$ have the same eigenvalues in Λ and that $\mathcal{E}(X^{DG}(h)) = \mathcal{E}(X)$.

The next step consists in combining Lemma 5.2 and Theorem 5.1 to deduce that the discrete resolvent $\left(zI-\tilde{T}_h\right)^{-1} : X^{DG}(h) \longrightarrow X^{DG}(h)$ is also uniformly bounded, provided h is small enough, cf. [23, Lemma 5.1] for more details.

Lemma 5.3. *Under Assumptions 1 and 2, there exists $h_0>0$ such that for all $h\leq h_0$,*

$$|||(zI-\tilde{T}_h)\boldsymbol{\tau}||| \geq C\,|||\boldsymbol{\tau}||| \quad \text{for all } \boldsymbol{\tau}\in X^{DG}(h) \text{ and } z\in\partial\Lambda$$

with $C>0$ independent of h.

Here again, it follows from Lemma 5.3 that, for h small enough, the linear operator

$$\mathcal{E}_h := \frac{1}{2\pi i}\int_{\Lambda}\left(zI-\tilde{T}_h\right)^{-1} dz : X^{DG}(h) \longrightarrow X^{DG}(h)$$

is uniformly bounded in h. Likewise, $\mathcal{E}_h|_{X_h^{DG}} : X_h^{DG}\to X_h^{DG}$ is a projector onto the $\tilde{T}_h$-invariant subspace $\mathcal{E}_h(X^{DG}(h)) = \mathcal{E}_h(X_h^{DG})$ corresponding to the eigenvalues of $T_h : X_h^{DG}\to X_h^{DG}$ contained in Λ.

The approximation properties of the eigenfunctions of problem (2.4) by means of those of problem (4.1) are obtained as a consequence of the following estimate of the distance between $\mathcal{E}_h(X_h^{DG})$ and $\mathcal{E}(X)$, measured in terms of the gap $\widehat{\delta}$.

Theorem 5.2. *Under Assumptions 1 and 2, there exists $h_0>0$ such that for all $h\leq h_0$,*

$$\widehat{\delta}(\mathcal{E}(X),\mathcal{E}_h(X_h^{DG})) \leq C\left(\|\tilde{T}-T_h\|_h + \delta(\mathcal{E}(X),X_h^{DG})\right), \tag{5.14}$$

with $C>0$ independent of h.

Proof. See [23, Theorem 5.1]. □

We point out that since $\mathcal{E}(X)$ is a finite dimensional subspace of X, (4.7) and Theorem 5.1 ensure the convergence of $\widehat{\delta}(\mathcal{E}(X),\mathcal{E}_h(X_h^{DG}))$ to zero when $h\to 0$. This is the main ingredient in the proof of the following Theorem, cf. [23, Theorem 5.2] for more details.

Theorem 5.3. *Assume that Assumptions 1 and 2 are satisfied. Let $\Lambda\subset\mathbb{C}\setminus\{0,1\}$ be an arbitrary compact set with smooth boundary $\partial\Lambda$ satisfying $\partial\Lambda\cap\mathrm{sp}(T)=\emptyset$. We assume that there are m eigenvalues $\eta_1^\Lambda,\dots,\eta_m^\Lambda$ of T (repeated according to their algebraic multiplicities) contained in Λ. We also consider the eigenvalues $\eta_{1,h}^\Lambda,\dots,\eta_{m(h),h}^\Lambda$ of $T_h : X_h^{DG}\to X_h^{DG}$ lying in Λ and repeated according to their algebraic multiplicities. Then, there exists $h_0>0$ such that $m(h)=m$ for all $h\leq h_0$ and*

$$\lim_{h\to 0}\max_{1\leq i\leq m}|\eta_i^\Lambda-\eta_{i,h}^\Lambda| = 0.$$

Moreover, if $\mathcal{E}(X)$ is the T-invariant subspace of X spanned by the generalized eigenfunctions corresponding to the set of eigenvalues $\{\eta_i^\Lambda,\ i=1,\dots,m\}$ and $\mathcal{E}_h(X_h^{DG})$ is the T_h-invariant subspace of X_h^{DG} spanned by the eigenspaces corresponding to $\{\eta_{i,h}^\Lambda,\ i=1,\dots,m\}$ then $\widehat{\delta}(\mathcal{E}(X),\mathcal{E}_h(X_h^{DG}))\to 0$ as $h\to 0$.

5.3. Error estimates for eigenvalues and eigenfunctions

Theorem 5.3 guaranties that the discontinuous Galerkin scheme (4.1) does not pollute the spectrum of T with spurious modes. Moreover, it proves the convergence of eigenvalues and eigenfunctions with correct multiplicity. However, in practice the space $\mathcal{E}_\eta(X)$ of generalized eigenfunctions corresponding to a given isolated eigenvalue $\eta\neq 1$ enjoys individual smoothness properties and the term $\|\tilde{T}-T_h\|_h$ in (5.14) prevents from taking advantage of this specific regularity. For this reason, we are going to show now that the gap between the continuous and discrete eigenspaces corresponding to a particular eigenvalue $\eta\neq 1$ can be bounded only in terms of

$$\delta^*(\mathcal{E}(X),X_h^{DG}) := \sup_{\boldsymbol{\sigma}\in\mathcal{E}(X),\|\boldsymbol{\sigma}\|_X=1}\ \inf_{\boldsymbol{\tau}_h\in X_h^{DG}} |||\boldsymbol{\sigma}-\boldsymbol{\tau}_h|||_*.$$

Hence, hereafter we focus on a particular isolated eigenvalue $\eta\neq 1$ of T of algebraic multiplicity m and let $D_\eta\subset\mathbb{C}$ be a closed disk centered at η with boundary γ such that $D_\eta\cap\mathrm{sp}(T)=\{\eta\}$. We denote by $\mathcal{E}_\eta := \frac{1}{2\pi i}\int_\gamma\left(zI-\tilde{T}\right)^{-1} dz : X\to X$ the projector onto the eigenspace $\mathcal{E}_\eta(X)$ of η and we define, for h small enough, the projector by $\mathcal{E}_{\eta,h} := \frac{1}{2\pi i}\int_\gamma\left(zI-\tilde{T}_h\right)^{-1} dz : X_h^{DG}\to X_h^{DG}$ onto the T_h-invariant subspace $\mathcal{E}_{\eta,h}(X_h^{DG})$ corresponding to the m eigenvalues of $T_h : X_h^{DG}\to X_h^{DG}$ contained in γ. A straightforward adaptation of [23, Theorem 6.1] gives the following result.

Theorem 5.4. *Assume that Assumptions 1 and 2 are satisfied. For h small enough, there exists a constant C independent of h such that*

$$\widehat{\delta}\left(\mathcal{E}_\eta(X),\mathcal{E}_{\eta,h}(X_h^{DG})\right) \leq C\delta^*(\mathcal{E}_\eta(X),X_h^{DG}). \tag{5.15}$$

We conclude with the following rates of convergence for eigenfunctions and eigenvalues.

Theorem 5.5. *Assume that Assumptions 1 and 2 are satisfied. Let $r>0$ be such that $\mathcal{E}_\eta(X)\subset\left\{\boldsymbol{\tau}\in H^r(\cup_j\Omega_j,\mathbb{M});\ \mathbf{div}\,\boldsymbol{\tau}\in H^{1+r}(\cup_j\Omega_j,\mathbb{R}^d)\right\}$. Then, there exists $C>0$ independent of h such that, for h small enough,*

$$\widehat{\delta}\left(\mathcal{E}_\eta(X),\mathcal{E}_{\eta,h}(X_h^{DG})\right) \leq Ch^{\min\{r,k\}}. \tag{5.16}$$

Moreover, there exists $C'>0$ independent of h such that

$$\max_{1\leq i\leq m}|\kappa-\kappa_{i,h}| \leq C'\,h^{2\min\{r,k\}}, \tag{5.17}$$

where $\kappa := 1/\eta$ and $\kappa_{i,h} := 1/\eta_{i,h}$, $i=1,\dots,m$.

Proof. For any $\boldsymbol{\sigma}\in\mathcal{E}_\eta(X)$, taking into account (4.9), it holds true that

$$\inf_{\boldsymbol{\tau}_h\in X_h^{DG}}|||\boldsymbol{\sigma}-\boldsymbol{\tau}_h|||_* \leq \left|\left|\left|\boldsymbol{\sigma}-\Pi_h^{\mathrm{BDM}}\boldsymbol{\sigma}\right|\right|\right|_*$$

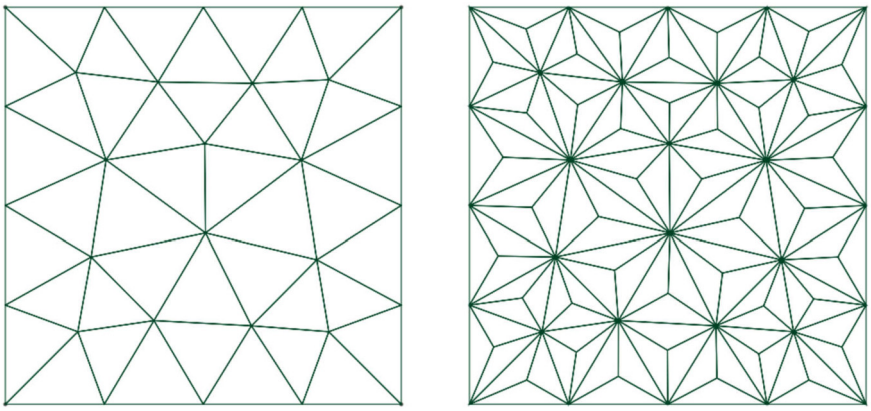

Fig. 6.1. An unstructured shape-regular mesh $\mathcal{T}_h$ of mesh size $h = 1/4$ (left) and the corresponding barycentric refinement $\mathcal{T}_h^{\text{bary}}$ (right).

$$= \left(\left\| (I - \Pi_h^{\text{BDM}})\boldsymbol{\sigma} \right\|_{\mathcal{A}}^2 + \left\| \tfrac{1}{\sqrt{\varrho}} (I - Q_h^{k-1}) \operatorname{\mathbf{div}} \boldsymbol{\sigma} \right\|_{0,\Omega}^2 \right.$$
$$\left. + \left\| \varrho_F^{\frac{1}{2}} h_F^{\frac{1}{2}} \{ \tfrac{1}{\varrho} (I - Q_h^{k-1}) \operatorname{\mathbf{div}} \boldsymbol{\sigma} \} \right\|_{\mathcal{F}_h^*}^2 \right)^{1/2}.$$

Using (4.8), (4.10) for the first and second terms of the last identity, respectively, and employing classical scaling arguments for the last one we deduce that

$$\inf_{\boldsymbol{\tau}_h \in X_h^{DG}} |||\boldsymbol{\sigma} - \boldsymbol{\tau}_h|||_* \lesssim h^{\min\{r,k\}} \left(\sum_{j=1}^{J} \|\boldsymbol{\sigma}\|_{r,\Omega_j}^2 + \|\operatorname{\mathbf{div}} \boldsymbol{\sigma}\|_{r,\Omega_j}^2 \right)^{1/2},$$

and (5.16) follows from the fact that $\mathcal{E}_\eta(X)$ is a finite dimensional subspace of X.

The deduction of (5.17) from (5.16) is obtained in the classical way, see [23, Theorem 6.2] for more details. □

Remark 5.1. The displacement field corresponding to a given $\boldsymbol{\sigma} \in \mathcal{E}_\eta(X)$ is $\boldsymbol{u} = -\frac{1}{\varrho(\kappa-1)} \operatorname{\mathbf{div}} \boldsymbol{\sigma}$. Let us associate to $\boldsymbol{u}$ the discrete displacement $\boldsymbol{u}_h := -\frac{1}{\varrho(\kappa_h-1)} \operatorname{\mathbf{div}}_h \boldsymbol{\sigma}_h$, where $\kappa_h := (\sum_{i=1}^m \kappa_{i,h})/m$. By virtue of the triangle inequality and Theorem 5.5, for h small enough, we have the error estimate

$$\|\boldsymbol{u} - \boldsymbol{u}_h\|_{0,\Omega} \le \frac{1}{\varrho} \Big(\frac{1}{\kappa - 1} \|\operatorname{\mathbf{div}}_h(\boldsymbol{\sigma} - \boldsymbol{\sigma}_h)\|_{0,\Omega}$$
$$+ \Big| \frac{1}{\kappa - 1} - \frac{1}{\kappa_h - 1} \Big| \|\operatorname{\mathbf{div}}_h \boldsymbol{\sigma}_h\|_{0,\Omega} \Big) \lesssim h^{\min\{r,k\}}.$$

6. Numerical results

We point out that the inf-sup condition (4.11) corresponding to the Scott–Vogelius element is only known to be satisfied under certain conditions on the mesh and the polynomial degree k. Namely, in two dimensions, Assumption 2 holds true on shape-regular triangulations with no singular vertices for $k \ge 3$, cf. [31] and in dimension three, it is satisfied on uniform simplicial meshes for $k \ge 5$ [36]. For lower values $1 \le k < 2d - 1$ of the polynomial degree, we can ensure Assumption 2 by considering shape-regular meshes with barycentric refinements (Alfeld splits), see [30,34]. Similar results have been proved for meshes of Powell–Sabin type [35,37], which will not be employed here.

To our knowledge, the stability results mentioned so far for the Scott–Vogelius element have only been obtained for homogeneous Dirichlet or Neumann boundary conditions on Γ. However, there is numerical evidence that the stability and optimal accuracy of this finite element method also occur on barycentric refinements of shape-regular meshes when mixed boundary conditions are imposed [29,16].

Table 6.1
Lowest natural frequencies on an unstructured shape-regular triangulation $\mathcal{T}_h$ of mesh size $h = 1/64$.

$k = 1$		
$\mathtt{a}_0 = 4$	$\mathtt{a}_0 = 8$	$\mathtt{a}_0 = 16$
0.678702	0.679772	0.680201
1.695659	1.697598	1.698374
1.816904	1.819964	1.821196
2.639076	2.940891	2.944729
2.837896	3.014975	3.016507
2.931343	**3.401037**	3.442011
3.011368	3.440911	4.138696
3.108155	**3.651704**	**4.157343**
3.417512	**4.007783**	**4.453139**
3.441640	4.134962	4.628728

In what follows, we say that $\mathcal{T}_h^{\text{bary}}$ is a simplicial barycentric $(d+1)$-sected mesh of size h if $\mathcal{T}_h^{\text{bary}}$ is obtained after refinement of a shape-regular simplicial mesh $\mathcal{T}_h$ of size h by subdividing each simplex in $\mathcal{T}_h$ into $d+1$ sub-simplices by connecting the barycenter with the $d+1$ vertices, see Fig. 6.1.

The numerical results presented in this section have been implemented using the finite element library `Netgen/NGSolve` [28]. In examples 1 and 2 below, we solve the classical displacement-based elasticity eigenproblem corresponding to each case in order to obtain reference values for the eigenfrequencies.

Example 1: Spectral correctness of the DG scheme in two dimensions. We assume that problem (2.3) is posed in the unit square $\Omega = (0,1)^2$ and let the compliance tensor $\mathcal{A}$ be given by Hooke's law

$$\mathcal{A}\boldsymbol{\tau} = \frac{1}{2\mu} \boldsymbol{\tau} - \frac{\lambda}{2\mu(d\lambda + 2\mu)} \operatorname{tr} \boldsymbol{\tau} I, \tag{6.1}$$

where λ and μ are the Lamé coefficients. We select in this example constant values for the mass density $\varrho = 1$, Young's modulus $E = 1$ and Poisson's ratio $\nu = 0.35$. We recall that the Lamé coefficients are related to E and ν by

$$\lambda := \frac{E\nu}{(1+\nu)(1-2\nu)} \quad \text{and} \quad \mu := \frac{E}{2(1+\nu)}.$$

We assume that the solid is fixed at the bottom side $\Gamma_D = (0,1) \times \{0\}$ of the square and free of stress on the remaining three sides $\Gamma_N = \Gamma \setminus \Gamma_D$.

We report in Table 6.1 the 10 smallest vibration frequencies $\omega_{hi} := \sqrt{\kappa_{hi} - 1}$ obtained by solving the DG scheme (4.1) on an unstructured shape-regular triangulation $\mathcal{T}_h$ of mesh size $h = 1/64$ at the lowest order $k = 1$ and for stability parameters $\mathtt{a} = \mathtt{a}_0 k^2$, with $\mathtt{a}_0 := 4, 8, 16$. We observe that spurious eigenvalues (the numbers in bold font) emerge at random positions, which indicates that the approximation is not spectrally correct. To ensure Assumption 2, we solve now (4.1) on

Table 6.2
Lowest natural frequencies on a barycentric refinement of a shape-regular triangulation $\mathcal{T}_h^{\text{bary}}$ of mesh size $h = 1/16$.

$k = 1$		
$\mathtt{a}_0 = 2$	$\mathtt{a}_0 = 4$	$\mathtt{a}_0 = 8$
0.677490	0.677848	0.679280
1.693684	1.694236	1.696863
1.811491	1.814970	1.819734
2.910592	2.925604	2.940824
3.002259	3.008271	3.014647
3.425010	3.432744	3.440763
4.091722	4.118359	4.137677
4.587870	4.609321	4.626700
4.745053	4.769440	4.786882
4.717068	4.740651	4.758608

Table 6.3
Lowest natural frequencies on an unstructured shape-regular triangulation $\mathcal{T}_h$ of mesh size $h = 1/16$.

$k = 2$		
$\mathtt{a}_0 = 2$	$\mathtt{a}_0 = 4$	$\mathtt{a}_0 = 8$
0.676422	0.678614	0.679493
1.691646	1.695661	1.697263
1.811504	1.817566	1.820003
2.915251	2.933614	2.941019
3.004017	3.011501	3.014480
3.428332	3.436800	3.440173
4.111159	4.128585	4.135536
4.561698	4.618693	4.625052
4.603006	4.749476	4.755874
4.733581	4.777890	4.783112

Table 6.4
Lowest natural frequencies on a barycentric refinement of a shape-regular triangulation $\mathcal{T}_h^{\text{bary}}$ of mesh size $h = 1/16$.

$k = 2$		
$\mathtt{a}_0 = 2$	$\mathtt{a}_0 = 4$	$\mathtt{a}_0 = 8$
0.679380	0.680102	0.680389
1.696791	1.698115	1.698640
1.818694	1.820700	1.821497
2.937186	2.943178	2.945560
3.013475	3.015922	3.016894
3.438312	3.441135	3.442255
4.131920	4.137557	4.139793
4.621689	4.627024	4.629140
4.752329	4.757616	4.759720
4.780796	4.785130	4.786851

a barycentric trisected mesh $\mathcal{T}_h^{\text{bary}}$ of size $h = 1/16$. The results displayed in Table 6.2 (for $k = 1$ and for stabilization parameters $\mathtt{a} = \mathtt{a}_0 k^2$, $\mathtt{a}_0 := 2, 4, 8$) provide correct eigenfrequencies.

We repeat the same experiment by solving (4.1) with quadratic polynomial order. We employ an unstructured shape-regular triangulation $\mathcal{T}_h$ of size $h = 1/16$ in Table 6.3 and a barycentric trisected mesh $\mathcal{T}_h^{\text{bary}}$ of size $h = 1/16$ in Table 6.4. Only one spurious eigenfrequency shows up in Table 6.3 among the first 10 eigenvalues in the case $\mathtt{a}_0 = 2$. The DG method seems to provide a spectrally correct quadratic approximation on $\mathcal{T}_h$ for $\mathtt{a}_0$ sufficiently large, even though Assumption 2 is not known to be satisfied on $\mathcal{T}_h$ for $k = 2$.

We finish this series of tests by reporting in Table 6.5 and Table 6.6 the eigenfrequencies obtain by solving (4.1) for $k = 3$ and $k = 4$, respectively. For these values of the polynomial order k, Assumption 2 is satisfied on unstructured shape-regular meshes $\mathcal{T}_h$. We take $h = 1/16$ and let $\mathtt{a} = \mathtt{a}_0 k^2$ for $\mathtt{a}_0 := 2, 4, 8$ in each case. As expected, all the computed eigenfrequencies are correct.

Table 6.5
Lowest natural frequencies on an unstructured shape-regular triangulation $\mathcal{T}_h$ of mesh size $h = 1/16$.

$k = 3$		
$\mathtt{a}_0 = 2$	$\mathtt{a}_0 = 4$	$\mathtt{a}_0 = 8$
0.676520	0.678355	0.679133
1.692090	1.695468	1.696897
1.812994	1.818054	1.820203
2.919900	2.935216	2.941725
3.004835	3.011233	3.013935
3.430379	3.437405	3.440382
4.115109	4.129772	4.135970
4.605120	4.619032	4.624917

Table 6.6
Lowest natural frequencies on an unstructured shape-regular triangulation $\mathcal{T}_h$ of mesh size $h = 1/16$.

$k = 4$		
$\mathtt{a}_0 = 2$	$\mathtt{a}_0 = 4$	$\mathtt{a}_0 = 8$
0.678431	0.679425	0.679859
1.695330	1.697156	1.697950
1.817152	1.819895	1.821092
2.932397	2.940705	2.944336
3.010780	3.014227	3.015725
3.436205	3.440011	3.441666
4.127128	4.135079	4.138538
4.616777	4.624336	4.627618

Table 6.7
Lowest natural frequencies on a barycentric refinement of a shape-regular triangulation of mesh size $h = 1/4$.

$k = 2$		
$\mathtt{a}_0 = 4$	$\mathtt{a}_0 = 8$	$\mathtt{a}_0 = 16$
4.459599	4.462669	4.463426
4.459644	4.462324	4.463592
4.459717	4.462452	4.463901
4.735649	4.783271	4.787701
4.856537	4.784776	4.789726
4.774039	4.785016	4.790044
4.772765	5.820485	5.831800
4.772953	5.834638	5.829005
5.188265	6.027316	6.030225
5.288405	6.015911	6.034207

Table 6.8
Lowest natural frequencies on a barycentric refinement of a shape-regular triangulation of mesh size $h = 1/4$.

$\mathtt{a}_0 = 8$	
$k = 3$	$k = 4$
4.460305	4.460220
4.460295	4.460222
4.460286	4.460221
4.770938	4.770735
4.770963	4.770734
4.770972	4.770732
5.805414	5.804214
5.881658	5.804351
6.014326	6.013368
6.014811	6.017531

Example 2: Spectral correctness of the DG scheme in three dimensions. We consider a solid represented by the unit cube $\Omega = (0,1)^3$ and impose a Dirichlet boundary condition on the whole boundary $\Gamma_D = \Gamma$. We maintain the same expression (6.1) for the compliance tensor $\mathcal{A}$

Table 6.9
Computed lowest eigenvalues ζ_{jh}, $j = 1, \ldots, 5$, of problem (6.2) and averaged rates of convergence for a set of unstructured shape-regular curved meshes $\widetilde{\mathcal{T}}_h$ with decreasing mesh sizes h and for polynomial degrees of approximation $k = 3, 4$. The exact eigenvalues are given by (6.3).

k	h	$3\zeta_{1h}$	$3\zeta_{2h}$	$3\zeta_{3h}$	$3\zeta_{4h}$	$3\zeta_{5h}$
3	1/2	14.685386398526	26.399150575171	26.399686773148	40.827777930236	40.828051578839
	1/4	14.682071040481	26.375381383674	26.375576473616	40.710758108868	40.711302839983
	1/8	14.681971611106	26.374624687636	26.374624985835	40.706510271654	40.706514798709
	1/16	14.681970657338	26.374616530921	26.374616534235	40.706466349720	40.706466354220
$\mathtt{avg}(\boldsymbol{r}^i_{h,3})$		**5.92**	**5.94**	**5.94**	**5.93**	**5.93**
4	1/2	14.682036415561	26.375811477238	26.375817494769	40.715004945300	40.715041953610
	1/4	14.681971020753	26.374623059451	26.374625770791	40.706530102238	40.706538022701
	1/8	14.681970642864	26.374616440514	26.374616441501	40.706465931297	40.706465952803
	1/16	14.681970642114	26.374616427187	26.374616427190	40.706465818502	40.706465821494
$\mathtt{avg}(\boldsymbol{r}^i_{h,4})$		**7.55**	**8.52**	**8.47**	**8.25**	**7.10**

and the same constant coefficients $\rho = 1$, $E = 1$ and $\nu = 0.35$ used in the previous example.

Only spurious eigenvalues appeared when solving problem (4.1) on unstructured shape-regular simplicial partitions $\mathcal{T}_h$ for $k = 2, 3, 4$, for a wide range of parameters $\mathtt{a} = \mathtt{a}_0 k^2$, and for the largest number of degrees of freedom allowed by our computational capacity in each case. To guarantee Assumption 2, we solved (4.1) on a quadrisected barycentric mesh $\mathcal{T}_h^{\mathtt{bary}}$ of size $h = 1/4$ and reported the results in Tables 6.7 and 6.8. The results displayed in Table 6.7 indicate the spectral correctness of the DG scheme of quadratic order for $\mathtt{a}$ large enough. Finally, we list in Table 6.8 the first natural frequencies obtained by solving (4.1) at cubic and quartic order with $\mathtt{a} = 8k^2$.

Example 3: Accuracy verification and stability in the nearly incompressible limit. We have seen through Example 1 and Example 2 that, if Assumption 1 and Assumption 2 are met, the DG scheme (4.1) does not pollute the spectrum of T with spurious modes. The aim now is to confirm that the eigenvalues converge at the expected rate and with correct multiplicity.

We take $\varrho = 1$, $E = 1$ and $\mathcal{A}$ given by (6.1) and we let $\Gamma_D = \Gamma$. We observe that in the limit $\lambda \to \infty$ (or $\nu \to 0.5$), the eigenvalues of (2.4) converge to the eigenvalues of the following perfectly incompressible elasticity eigenproblem (see [23, Appendix 9]): find eigenmodes $0 \neq \boldsymbol{\sigma}^\infty : \Omega \to \mathbb{S}$ and eigenvalues $\zeta \in \mathbb{R}$ such that,

$$\begin{aligned} -\boldsymbol{\varepsilon}(\mathbf{div}\,\boldsymbol{\sigma}^\infty) &= \frac{3\zeta}{2}(\boldsymbol{\sigma}^\infty)^{\mathtt{D}} \quad \text{in } \Omega, \\ \mathbf{div}\,\boldsymbol{\sigma}^\infty &= 0 \quad \text{on } \Gamma, \\ (\mathrm{tr}\,\boldsymbol{\sigma}^\infty, 1) &= 0, \end{aligned} \tag{6.2}$$

where $\boldsymbol{\tau}^{\mathtt{D}} := \boldsymbol{\tau} - \frac{1}{d}(\mathrm{tr}\,\boldsymbol{\tau})\,I$ is the deviatoric part of a tensor $\boldsymbol{\tau}$. Actually, (6.2) is the stress formulation of the Stokes eigenproblem [25, Section 6.2] with formal velocity and pressure fields given by $\boldsymbol{u}^\infty = -\frac{2\mu}{\zeta}\,\mathbf{div}\,\boldsymbol{\sigma}^\infty$ and $p^\infty := -\frac{1}{d}\,\mathrm{tr}\,\boldsymbol{\sigma}^\infty$, respectively.

It turns out that, on the unit disk, the eigenvalues of the Stokes eigenproblem (6.2) are given by the sequence $\left\{\frac{1}{2}J_{n\ell}^2\right\}_{n\geq 1,\,\ell\geq 1}$, where J_{nk} is the ℓ-th positive zero of the Bessel function J_n of the first kind of order n. Accurate approximations of the first five eigenvalues are given by

$$\begin{aligned} 3\zeta_1 &= J_{11}^2 \simeq 14.681970642124 \\ 3\zeta_2 = 3\zeta_3 &= J_{21}^2 \simeq 26.374616427163 \\ 3\zeta_4 = 3\zeta_5 &= J_{31}^2 \simeq 40.706465818200. \end{aligned} \tag{6.3}$$

To deal with the completely incompressible case $\nu = 0.5$, one can adapt the DG method (4.1) for problem (6.2) by changing the bilinear form $(\boldsymbol{\sigma}, \boldsymbol{\tau})_{\mathcal{A}}$ in (2.4) to $\frac{1}{2\mu}(\boldsymbol{\sigma}^{\mathtt{D}}, \boldsymbol{\tau}^{\mathtt{D}})$ and incorporating the constraint $(\mathrm{tr}\,\boldsymbol{\sigma}^\infty, 1) = 0$ into X, see [25, Section 6.2] for more details. Here, with the aim to test the performance of the scheme in the nearly incompressible case, we instead approximate the eigenvalues of (6.2) by solving the original DG method (4.1) with a Poisson's ratio $\nu = 0.5 - 10^{-13}$.

We denote by $\zeta_{ih} = \sqrt{3(\kappa_{ih} - 1)}$ the approximation of ζ_i computed by solving problem (4.1) on a series of exact meshes $\widetilde{\mathcal{T}}_h$ of $\bar{\Omega}$ with decreasing mesh sizes h, and for polynomial degrees $k = 3, 4$. The assembling of the generalized eigenproblems corresponding to (4.1) is performed thanks to the support of Netgen/NGSolve [28] for curved finite elements of arbitrary order. We present in Table 6.9 the first five computed eigenvalues and report the arithmetic mean $\mathtt{avg}(\boldsymbol{r}^i_{h,k})$ of the three experimental rates of convergence, which are obtained for each eigenvalue by mean of the formula

$$\boldsymbol{r}^i_{h,k} := \frac{\log\left(|\zeta_i - \omega^2_{ih}|/|\zeta_j - \omega^2_{i\hat{h}}|\right)}{\log(h/\hat{h})}, \quad i = 1, \ldots, 4, \quad k = 3, 4, \tag{6.4}$$

where h and $\hat{h}$ are two consecutive mesh sizes.

We observe that a convergence of order $2k$ is attained for each eigenvalue, as predicted by the error estimate (5.17). At the same time, this test shows that the DG-scheme (4.1) is inmune to locking in the nearly incompressible limit.

Data availability

Data will be made available on request.

References

[1] P.F. Antonietti, A. Buffa, I. Perugia, Discontinuous Galerkin approximation of the Laplace eigenproblem, Comput. Methods Appl. Mech. Eng. 195 (2006) 3483–3503.
[2] D.N. Arnold, G. Awanou, R. Winther, Finite elements for symmetric tensors in three dimensions, Math. Comput. 77 (2008) 1229–1251.
[3] D.N. Arnold, G. Awanou, R. Winther, Nonconforming tetrahedral mixed finite elements for elasticity, Math. Models Methods Appl. Sci. 24 (2014) 783–796.
[4] D.N. Arnold, F. Brezzi, J. Douglas, PEERS: a new mixed finite element method for plane elasticity, Jpn. J. Appl. Math. 1 (1984) 347–367.
[5] D.N. Arnold, R.S. Falk, R. Winther, Mixed finite element methods for linear elasticity with weakly imposed symmetry, Math. Comput. 76 (2007) 1699–1723.
[6] D. Boffi, Finite element approximation of eigenvalue problems, Acta Numer. 19 (2010) 1–120.
[7] D. Boffi, F. Brezzi, M. Fortin, Mixed Finite Element Methods and Applications, Springer Series in Computational Mathematics, vol. 44, Springer, Heidelberg, 2013.
[8] A. Buffa, I. Perugia, Discontinuous Galerkin approximation of the Maxwell eigenproblem, SIAM J. Numer. Anal. 44 (2006) 2198–2226.
[9] B. Cockburn, J. Gopalakrishnan, J. Guzmán, A new elasticity element made for enforcing weak stress symmetry, Math. Comput. 79 (2010) 1331–1349.
[10] M. Dauge, Elliptic boundary value problems on corner domains, in: Smoothness and Asymptotics of Solutions, in: Lecture Notes in Mathematics, vol. 1341, Springer-Verlag, Berlin, 1988.
[11] J. Descloux, N. Nassif, J. Rappaz, On spectral approximation. Part 1: the problem of convergence, RAIRO. Anal. Numér. 12 (1978) 97–112.
[12] J. Descloux, N. Nassif, J. Rappaz, On spectral approximation. Part 2: error estimates for the Galerkin method, RAIRO. Anal. Numér. 12 (1978) 113–119.
[13] D.N. Di Pietro, A. Ern, Mathematical Aspects of Discontinuous Galerkin Methods, Springer-Verlag, Berlin, Heidelberg, 2012.

[14] A. Ern, J.-L. Guermond, Finite Elements II—Galerkin Approximation, Elliptic and Mixed PDEs, Texts in Applied Mathematics, vol. 73, Springer, 2021.
[15] R.S. Falk, Finite element methods for linear elasticity, in: F. Brezzi, D. Boffi, L. Demkowicz, R.G. Durán (Eds.), Mixed Finite Elements, Compatibility Conditions, and Applications, Springer, Berlin, 2008, pp. 159–194.
[16] P.E. Farrell, L. Mitchell, L.R. Scott, F. Wechsung, A Reynolds-robust preconditioner for the Scott–Vogelius discretization of the stationary incompressible Navier-Stokes equations, SMAI J. Comput. Math. 7 (2021) 75–96.
[17] P. Grisvard, Problèmes aux limites dans les polygones. Mode d'emploi, EDF Bull. Direction Études, Rech. Sér. C Math. Inform. 3 (1) (1986) 21–59.
[18] J. Gopalakrishnan, J. Guzmán, A second elasticity element using the matrix bubble, IMA J. Numer. Anal. 32 (2012) 352–372.
[19] J. Gopalakrishnan, J. Guzmán, Symmetric nonconforming mixed finite elements for linear elasticity, SIAM J. Numer. Anal. 49 (2011) 1504–1520.
[20] J. Hu, Finite element approximations of symmetric tensors on simplicial grids in Rn: the higher order case, J. Comput. Math. 33 (2015) 283–296.
[21] M.W. Licht, Smoothed projections and mixed boundary conditions, Math. Comput. 88 (2019) 607–635.
[22] F. Lepe, D. Mora, Symmetric and non-symmetric discontinuous Galerkin methods for a pseudostress formulation of the Stokes spectral problem, SIAM J. Sci. Comput. 42 (2020) A698–A722.
[23] F. Lepe, S. Meddahi, D. Mora, R. Rodríguez, Mixed discontinuous Galerkin approximation of the elasticity eigenproblem, Numer. Math. 142 (2019) 749–786.
[24] A. Márquez, S. Meddahi, T. Tran, Analyses of mixed continuous and discontinuous Galerkin methods for the time harmonic elasticity problem with reduced symmetry, SIAM J. Sci. Comput. 37 (2015) 1909–1933.
[25] S. Meddahi, Variational eigenvalue approximation of non-coercive operators with application to mixed formulations in elasticity, SeMA J. 79 (2022) 139–164.
[26] S. Meddahi, D. Mora, R. Rodríguez, Finite element spectral analysis for the mixed formulation of the elasticity equations, SIAM J. Numer. Anal. 51 (2013) 1041–1063.
[27] S. Meddahi, D. Mora, R. Rodríguez, A finite element analysis of a pseudostress formulation for the Stokes eigenvalue problem, IMA J. Numer. Anal. 35 (2015) 749–766.
[28] Netgen/NGSolve, Finite element library, https://ngsolve.org.
[29] M.A. Olshanskii, L.G. Rebholz, Application of barycenter refined meshes in linear elasticity and incompressible fluid dynamics, Electron. Trans. Numer. Anal. 38 (2011) 258–274.
[30] J. Qin, On the convergence of some low order mixed finite elements for incompressible fluids, Thesis (Ph.D.)–The Pennsylvania State University, PA, ProQuest LLC, Ann Arbor, MI, 1994.
[31] L.R. Scott, M. Vogelius, Norm estimates for a maximal right inverse of the divergence operator in spaces of piecewise polynomials, RAIRO Modél. Math. Anal. Numér. 19 (1985) 111–143.
[32] F. Wang, S. Wu, J. Xu, A mixed discontinuous Galerkin method for linear elasticity with strongly imposed symmetry, J. Sci. Comput. 83 (2020) 2.
[33] S. Wu, S. Gong, J. Xu, Interior penalty mixed finite element methods of any order in any dimension for linear elasticity with strongly symmetric stress tensor, Math. Models Methods Appl. Sci. 27 (2017) 2711–2743.
[34] S. Zhang, A new family of stable mixed finite elements for the 3D Stokes equations, Math. Comput. 74 (2005) 543–554.
[35] S. Zhang, On the P1 Powell–Sabin divergence-free finite element for the Stokes equations, Comput. Math. 26 (2008) 456–470.
[36] S. Zhang, Divergence-free finite elements on tetrahedral grids for $k \geq 6$, Math. Comput. 80 (2011) 669–695.
[37] S. Zhang, Quadratic divergence-free finite elements on Powell–Sabin tetrahedral grids, Calcolo 48 (2011) 211–244.