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The blocks with four irreducible characters

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Abstract

Suppose that B is a Brauer p-block of a finite group with defect group D. If B exactly contains four ordinary irreducible characters, then we show that D has order four or five, assuming the Alperin–McKay conjecture holds for B.

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1 | INTRODUCTION

Suppose that G is a finite group, p is a prime, and B is a Brauer p-block with defect group D. The classification of blocks with a small number $k(B)$ of irreducible complex characters in B is a hard problem. It is well known that if $n = 1$ or 2, then $k(B) = n$ if and only if $|D| = n$ (see [15, Theorem 3.18] and [1]); for $n = 3$, this is known to be a consequence of the Alperin–McKay conjecture, but no proof is yet available. Although the cases where B is a principal block and $k(B) = 4$ or 5 have been recently solved in [9] and [19], the non-principal block cases remain open. It is well known that many blocks with $k(B) = 4$ have defect groups with $|D| = 4$ or 5 (for instance $2.\mathrm{A}_5$ for

$p = 5$, or $2.S_5$ for $p = 2$), but it is not known if these are the only possibilities, even assuming the Alperin–McKay conjecture. The following is the main result of this paper.

Theorem A. *Suppose that B is a Brauer p-block of a finite group G with defect group D. Assume that $k(B) = 4$. If the Alperin–McKay conjecture holds for B, then $|D| = 4$ or $|D| = 5$.*

We prove Theorem A by studying finite groups with a small number of projective characters (in the sense of Schur), a problem of interest on its own. This constitutes the main part of this paper.

Finally, we would like to remark that our result can be seen as a contribution to Brauer's Problem 21, which asks whether or not, for a fixed integer n there are finitely many isomorphism classes of groups of prime-power order that can occur as a defect group of blocks containing exactly n irreducible ordinary characters. We care to remark that for p-solvable groups, this problem was already solved by Külshammer in [11] but without giving the exact bound on $|D|$. In this paper, we give this bound for $n = 4$.

2 | THE THEOREM

We denote by $\mathrm{Lin}(G)$, the group of linear characters of a finite group G. If $N \lhd G$ and $\lambda \in \mathrm{Irr}(N)$, we denote by $\mathrm{Irr}(G|\lambda)$ the set of characters $\chi \in \mathrm{Irr}(G)$ such that λ is a constituent of the restriction χ_N. If $\alpha \in \mathrm{Irr}(N)$ then $\mathrm{IBr}(G|\alpha)$ denotes the set of Brauer characters $\varphi \in \mathrm{IBr}(G)$ such that α is a constituent of the restriction φ_N.

Lemma 1. *Let G be a finite group. Suppose that $N \lhd G$ and assume $\lambda \in \mathrm{Irr}(N)$ is G-invariant and linear. Let $o(\lambda)$ be the order of λ as an element of $\mathrm{Lin}(N)$. If every Sylow p-subgroup of G/N has trivial Schur multiplier whenever p divides $o(\lambda)$ then λ extends to G.*

Proof. This is [7, Theorems 6.26 and 11.7]. □

Lemma 2 (Higgs). *Let G be a finite group, $N \lhd G$ and let $\theta \in \mathrm{Irr}(N)$ be G-invariant. If $\mathrm{Irr}(G|\theta) = \{\alpha, \beta\}$ then $\alpha(1) = \beta(1)$ and G/N is solvable.*

Proof. See [6]. □

It is worth mentioning that Lemma 2 depends on the Classification of Finite Simple Groups.

Lemma 3. *Let $Z \lhd G$ and let $\lambda \in \mathrm{Irr}(Z)$ be G-invariant. Suppose that $\lambda^G = e_1\chi_1 + e_2\chi_2$ for some $\chi_1, \chi_2 \in \mathrm{Irr}(G)$ and $e_1, e_2 \in \mathbb{N}$. If p is an odd prime dividing the order of G/Z and $Q/Z \in \mathrm{Syl}_p(G/Z)$, then $\lambda^Q = d\eta$ for some $\eta \in \mathrm{Irr}(Q)$ and $d \in \mathbb{N}$. In particular, Q is non-abelian.*

Proof. Since character triple isomorphisms preserve the number $|\mathrm{Irr}(G|\lambda)|$ and the structure of G/N, using [16, Corollary 5.9], there is no loss in assuming Z central. Since $\chi_1(1) = \chi_2(1)$ by Lemma 2 and $(\chi_i)_Z = e_i\lambda$ we have that $e_1 = \chi_1(1) = \chi_2(1) = e_2$ so $\lambda^G = e_1(\chi_1 + \chi_2)$. Also observe that $|G : Z| = \lambda^G(1) = 2e_1\chi_1(1) = 2\chi_1(1)^2$. Now, write $\psi = \chi_1 + \chi_2$. Since ψ vanishes on $G \setminus Z$ we have that $\psi_Q = d\lambda^Q$ where

$$d = \frac{2\chi_1(1)}{|Q : Z|}.$$

If $\eta \in \mathrm{Irr}(Q|\lambda)$ then

$$[\psi_Q, \eta] = [d\lambda^Q, \eta] = d\eta(1) \in \mathbb{Z},$$

so $d^2\eta(1)^2 \in \mathbb{Z}$. Now,

$$d^2\eta(1)^2 = \frac{4\chi_1(1)^2}{|Q:Z|^2}\eta(1)^2 = \frac{2|G:Q|\eta(1)^2}{|Q:Z|} \in \mathbb{Z}$$

and we conclude that $|Q:Z|$ divides $\eta(1)^2$. By [7, Corollary 2.30] we have that $\eta(1)^2 \leqslant |Q:Z|$ so $\eta(1)^2 = |Q:Z|$, and this implies that $\mathrm{Irr}(Q|\lambda) = \{\eta\}$ as wanted. □

Let V be the Galois field $\mathbb{F}_{q^m}$ for some prime power q. Then V is a vector space over $\mathbb{F}_q$ of dimension m. The semilinear group $\Gamma(V)$ is defined by

$$\Gamma(V) = \{x \mapsto ax^\sigma \mid a \in V \setminus \{0\}, \sigma \in \mathrm{Gal}(\mathbb{F}_{q^m}/\mathbb{F}_q)\}.$$

Note that $\Gamma(V)$ is a metacyclic group. Indeed, it contains the normal subgroup

$$\Gamma_0(V) = \{x \mapsto ax \mid a \in V \setminus \{0\}\},$$

which is cyclic and isomorphic to the multiplicative group of V, and

$$\Gamma(V)/\Gamma_0(V) \cong \mathrm{Gal}(\mathbb{F}_{q^m}/\mathbb{F}_q)$$

is cyclic of order m.

Whenever V is a $\mathbb{F}_p$-vector space of dimension a for some prime p, we may use the notation $\Gamma(p^a) = \Gamma(V)$.

Recall that a prime t is called a primitive prime divisor for (p, a) if t divides $p^a - 1$ but t does not divide $p^j - 1$ for $1 \leqslant j < a$. By a well-known result by Zsigmondy (see [14, Theorem 6.2] for instance), such a prime always exists except when $a = 6$ and $p = 2$, or $a = 2$ and $p + 1$ is a power of 2.

Lemma 4. *Let K be a finite group and let $Z \subseteq \mathbf{Z}(K)$. Suppose that there exist $H/Z \leqslant K/Z$ with $|K:H| = 2$ and $A/Z \leqslant K/Z$ isomorphic to a subgroup of a semilinear group $\Gamma(V)$, where V is a $\mathbb{F}_p$-vector space ($p \notin \{2,3\}$ prime) of dimension a, such that $H/Z = A/Z \times B/Z$ and $(A/Z)^g = B/Z$ for every $g \in K \setminus H$. Suppose further that A/Z acts transitively on $V \setminus \{0\}$. Then there is an odd prime divisor t of $p^a - 1$ such that K has abelian Sylow t-subgroups.*

Proof. We use the bar notation, so write $\overline{K} = K/Z$, $\overline{A} = A/Z$, $\overline{B} = B/Z$, and so on. Note that since $\overline{A}$ acts transitively on $V \setminus \{0\}$, we have that $p^a - 1$ divides $|\overline{A}|$, and so does every prime divisor of $p^a - 1$.

Suppose there is a primitive prime divisor t of $p^a - 1$. Let $\overline{T_1}$ be a Sylow t-subgroup of $\overline{A}$ and let $\overline{z} \in \mathbf{Z}(\overline{T_1})$ of order t. Let $\overline{T_0} = \langle \overline{z} \rangle$. By [2, Lemma 3.7] we have that $\overline{T_1} \leqslant \mathbf{C}_{\overline{A}}(\overline{T_0}) = \overline{A} \cap \Gamma_0(V) = \mathbf{F}(\overline{A})$, where $\mathbf{F}(\overline{A})$ denotes the Fitting subgroup of $\overline{A}$. Since $\Gamma_0(V)$ is cyclic, we have that $\overline{T_1}$ is cyclic and normal in $\overline{A}$.

Let $x \in T_1$ such that $\langle \overline{x} \rangle = \overline{T_1}$ and let $\overline{g} \in \overline{K} - \overline{H}$, so $\overline{g}$ is an involution. Let $\overline{y} = \overline{x}^{\overline{g}} \in \overline{B}$ and note that $\langle \overline{y} \rangle = \overline{T_2}$ is the Sylow t-subgroup of $\overline{B}$. Now, $\overline{T} = \overline{T_1} \times \overline{T_2}$, the Sylow t-subgroup of $\overline{H}$, is abelian. Write $\overline{T} = T/Z$, so $T' \subseteq Z$. We note that it is enough to show that T is abelian. Suppose to the contrary that T is not abelian. We have $1 \neq [x, y] \in T' \subseteq Z$, but since $\overline{g}$ is an involution and $x^g = yz_1$, $y^g = xz_2$ for some $z_1, z_2 \in Z$, then

$$[x, y]^g = [x^g, y^g] = [y, x] = [x, y]^{-1} \neq [x, y],$$

a contradiction. Thus, we may assume that there does not exist a primitive prime divisor for $p^a - 1$. Since $p \notin \{2, 3\}$, we have that $(p, a) = (2^m - 1, 2)$ for some integer $m > 2$. Then

$$p^2 - 1 = (p + 1)(p - 1) = 2^{m+1}(2^{m-1} - 1)$$

and hence there is an odd prime q dividing $p^2 - 1$. Since $|\overline{A} : \overline{A} \cap \Gamma_0(V)| \leqslant 2$, we have that t divides $|\overline{A} \cap \Gamma_0(V)|$. And hence $\overline{A}$ has a normal and cyclic Sylow t-subgroup. Now, we proceed as above. □

Given a finite group G, we denote by $\mathrm{Bl}(G)$ its set of p-blocks. If $B \in \mathrm{Bl}(G)$, we denote by $l(B)$ the number of irreducible Brauer characters in B.

Lemma 5 (Brauer's formula). *Suppose that $x_1, \ldots, x_k$ are representatives of the noncentral conjugacy classes of p-elements in G. Let B be a p-block of G. Then*

$$k(B) = |\mathbf{Z}(G)|_p\, l(B) + \sum_{i=1}^{k} \sum_{b \in \mathrm{Bl}(\mathbf{C}_G(x_i)) b^G = B} l(b).$$

Proof. See [15, Theorem 5.12]. □

In the following proof, we say that a primitive group G is affine if its socle V is p-elementary abelian. If $H \leqslant G$ is the stabilizer of the zero vector of V (viewed as an $\mathbb{F}_p$-vector space), then the number of orbits of H on V is the rank of G. Note that if the rank of G is 2 then G is doubly-transitive on V (see [8, Lemma 8.2]).

Theorem 6. *Let G be a finite group, B a p-block of G with $k(B) = 4$. Let D be a defect group of B and assume $D \lhd G$. Then D is isomorphic to $\mathsf{C}_4, \mathsf{C}_2 \times \mathsf{C}_2$ or C_5.*

Proof. We divide the proof in steps.

Step 0. If there exists $1 < N < D$ *with* $N \lhd G$, then $p = 2, 3$; *and if $\overline{B}$ is a block of G/N dominated by B, then $k(\overline{B}) = 2, 3$*

Let $N \lhd G$, with $N < D$ and let $\overline{B}$ be a block of G/N dominated by B with defect group D/N ([15, Theorem 9.9]). Then $k(\overline{B}) \leqslant 4$. If $k(\overline{B}) = 1$ then $N = D$ by [15, Theorem 3.18]. If $k(\overline{B}) = 4$, then [15, Theorem 6.10] implies $N = 1$. Hence, if $1 < N < D$, it is necessary that $k(\overline{B}) \in \{2, 3\}$. By [1, Theorem A] and [12, Theorem 4.1], this forces $p = 2, 3$.

Step 1. We may assume that $D \in \mathrm{Syl}_p(G)$ and that D is p-elementary abelian. In particular, G is p-solvable. Furthermore, we may assume $p \neq 2, 3$.

The first part is [18, Theorem 6]. Let $\Phi(D) \lhd G$ be the Frattini subgroup of D and assume $\Phi(D) > 1$. Let $\overline{B}$ be the unique block of $G/\Phi(D)$ dominated by B. By Step 0 we have $p \in \{2,3\}$ and $k(\overline{B}) \in \{2,3\}$. By [1, Theorem A] and [12, Theorem 4.1] we have that $|D/\Phi(D)| = p$ and hence D is cyclic [8, Problem 1D.9]. Now the result follows easily by applying [3, Theorem 1]. Hence, we may assume $\Phi(D) = 1$ so D is p-elementary abelian.

Suppose that $p = 2$. Again by Theorem A of [1] we know that $|D| > 2$ and by Corollary 1.3 (iii) of [13] we obtain that $|D| = 4$ and then $D = \mathsf{C}_2 \times \mathsf{C}_2$, and we are done. Now, suppose that $p = 3$. In this case, by [13, Corollary 1.6] we would have a contradiction.

Step 2. D is a minimal normal subgroup of G.

If there exists a minimal normal subgroup of G strictly contained in D, then by Step 0 we have that $p = 2, 3$, contradicting Step 1. Hence, D is a minimal normal subgroup of G.

Step 3. We may assume that $\mathbf{Z}(G) = \mathbf{O}_{p'}(G)$.

Let $N = \mathbf{O}_{p'}(G)$ and let $\lambda \in \mathrm{Irr}(N)$ be such that if $b = \{\lambda\} \in \mathrm{Bl}(N)$ then B covers b. By the Fong–Reynolds correspondence [15, Theorem 9.14], we may assume that b is G-invariant, and hence λ is G-invariant. Now (G, N, λ) is an ordinary-modular character triple (see [15, Problem 8.10]). By [15, Problem 8.13], we know that there exists an isomorphic ordinary-modular character triple (H, M, φ) with M a p'-group and φ linear and faithful (in particular, M is central). Moreover, since $G/N \cong H/M$, we have that $M = \mathbf{O}_{p'}(H)$ and that H is also p-solvable. Now let B_1 be a p-block of H covering $b_1 = \{\varphi\}$. By [15, Theorem 10.20], we have that

$$|\mathrm{Irr}(B)| = |\mathrm{Irr}(G|\lambda)| = |\mathrm{Irr}(H|\varphi)| = |\mathrm{Irr}(B_1)|$$

and if D_1 is a defect group of B_1 then $D_1 \in \mathrm{Syl}_p(H)$. We claim that if D_1 is one of $\mathsf{C}_4, \mathsf{C}_5, \mathsf{C}_2 \times \mathsf{C}_2$, then so is D. Indeed, if $Q/M \in \mathrm{Syl}_p(H/M)$ then $Q = D_1 \times M$. Since $G/N \cong H/M$, we have that $D \cong DN/N \cong D_1M/M \cong D_1$ and D has one of the desired structures. Thus, by working with (H, M, φ), we may assume that λ is linear, faithful, and N is central.

Write $Z = \mathbf{Z}(G)$. Next we prove that $N = Z$. To do so, since $N \subseteq Z$, we just need to show that $|Z|_p = 1$. Assume by way of contradiction that $|Z|_p > 1$. It follows from Brauer's formula (see Lemma 5) that $k(B) \geqslant |Z|_p l(B)$. Since $k(B) = 4$ this forces $p = 2, 3$, contradicting Step 1. Thus, $N = Z$.

Step 4. Let $\{x_1, \ldots, x_t\}$ be a set of representatives of the G-conjugacy classes of p-elements of G. Then $t = 2$ or $t = 3$.

We may assume that $x_1 = 1$, so $\{x_2, \ldots, x_t\}$ are a set of representatives of the non-trivial G-conjugacy classes of p-elements of G. By Brauer's formula (Lemma 5) and Step 3 we have

$$k(B) = l(B) + \sum_{i=2}^{t} \sum_{b \in \mathrm{Bl}(\mathbf{C}_G(x_i)), b^G = B} l(b). \tag{2.1}$$

The case where $l(B) = 1$ is done in [10] (in a wider context), so we may assume $l(B) \geqslant 2$. Since $b^G = B$ for some $b \in \mathrm{Bl}(\mathbf{C}_G(x_i))$ (see [15, Theorem 4.14]), we have that either $t = 2$ or $t = 3$, as desired.

By [15, Theorem 10.20] we have that $\mathrm{Irr}(B) = \mathrm{Irr}(G|\lambda)$, where $\lambda \in \mathrm{Irr}(Z)$ is the character from Step 3. From now on, we denote by b the unique block of ZD covered by B and note that $1_D \times \lambda = \hat{\lambda} \in \mathrm{Irr}(b)$ is G-invariant (by Step 3) so b is G-invariant.

Step 5. Suppose that $|D| = p^2$. Then $l(B) = 2$ and G acts on $D \setminus \{1\}$ transitively.

Let $u \in D, u \neq 1$. Let $C = \mathbf{C}_G(u)$ and let $b_u \in \mathrm{Bl}(C|D)$ with $b^G = B$ by [15, Theorem 4.14]. Since $k(B) = 4$ and $1 < l(B) < 4$, Equation (2.1) forces either $l(b_u) = 1$ or $l(b_u) = 2$.

Suppose that $l(b_u) = 1$. Note that since $D \lhd G$ and $D \subseteq C$, we have that D is a defect group of b_u by [15, Theorem 4.8]. By [15, Theorem 9.10], b_u dominates a unique block $\overline{b_u} \in \mathrm{Bl}(C/\langle u\rangle)$ with defect group $D/\langle u\rangle$, which is cyclic since $|D| = p^2$. By [15, Theorem 9.10] we have $l(\overline{b_u}) = l(b_u)$. By [15, Theorem 11.13], $\overline{b_u}$ has inertial index $l(b_u)$, and so does b_u. Note that $b \in \mathrm{Bl}(\mathbf{C}_G(D))$ is a root of b_u, so that $b^C = b_u$. Recall that b is G-invariant because λ is G-invariant. In particular, b is C-invariant and we have $|C : \mathbf{C}_G(D)| = l(b_u) = 1$. Hence the action of G/ZD is Frobenius on D. This implies that the Sylow subgroups of G/ZD are cyclic or generalized quaternion, by [8, Theorems 6.10 and 6.11], so they have trivial Schur multiplier. By Lemma 1, $\hat{\lambda}$ extends to G and so does λ. By Gallagher's theorem, $|\mathrm{Irr}(G|\lambda)| = |\mathrm{Irr}(G/Z)|$ so G/Z has exactly four conjugacy classes. Using that $p \neq 2, 3$ by Step 1, we get that G/Z is isomorphic to the dihedral group of order 10, and $D \cong \mathsf{C}_5$ as desired.

Then we may assume that $l(b_u) = 2$ and hence, by Brauer's formula 2.1 we have that $t = 2$ and $l(B) = 2$.

Step 6. We have that G/Z is an affine primitive permutation group of rank 2 or 3.

By the Schur–Zassenhaus theorem there is $K \leqslant G$ such that $G = KD$ and $K \cap D = 1$. Write $\overline{G} = G/Z$, $\overline{D} = DZ/Z$ and $\overline{K} = K/Z$. We have that $\overline{G}$ acts on $\Omega = \{\overline{K}d \mid d \in D\}$ transitively via the action of right multiplication. Note that $\overline{K}$ is the stabilizer of the trivial class in Ω. If $K < L < G$, then $L = KU$ where $U = L \cap D$. Since $D \lhd G$, we have $U \lhd L$ so K normalizes U and then $U \lhd G$, a contradiction with Step 2. Thus, $\overline{K}$ is maximal in $\overline{G}$, we have that this action is primitive (see [8, Corollary 8.14], for instance) and $\overline{G}$ is a primitive permutation group with socle $\overline{D}$.

Since the action of $\overline{K}$ on Ω has the same number of orbits as the action by conjugation of $\overline{G}$ on $\overline{D}$, by Step 4 it has 2 or 3 orbits. Hence, the rank of $\overline{G}$ is 2 or 3, as wanted.

Step 7. We may assume the rank of $\overline{G}$ is 3.

Suppose that $\overline{G}$, and hence $\overline{K}$, has rank 2, so that $t = 2$. By the main result of [17] and using that $p \neq 2, 3$ by Step 1, either $\overline{K}$ is isomorphic to a subgroup of the semilinear group $\Gamma(p^d)$ or

$$p^d \in \{5^2, 7^2, 11^2, 19^2, 23^2, 29^2, 59^2\},$$

where $|D| = p^d$.

We assume first that $\overline{K}$ is isomorphic to a subgroup of $\Gamma(p^d)$. In this case, we know that there exists $Z \leqslant H \lhd K$ with H/Z cyclic (and hence, H abelian) with $|H : Z| = s \mid p^d - 1$ and index $|K : H| = t \mid d$. Since there are just two orbits of p-elements in G, we have that G/ZD acts transitively on $D \setminus \{1\}$ and $p^d - 1 \mid |G : ZD|$.

Recall that $\mathrm{IBr}(B) = \mathrm{IBr}(G|\lambda)$. Since $l(B) \leqslant 3$, we have that $\mathrm{IBr}(HD|\lambda)$ has at most three orbits, and each of the orbits is of size at most $|G : HD| = |K : H| = t \leqslant d$. Hence $|\mathrm{IBr}(HD|\lambda)| \leqslant 3d$. Note that $\mathrm{IBr}(HD) = \mathrm{IBr}(HD/D) = \mathrm{Irr}(HD/D)$ by [15, Lemma 2.32] and hence $\mathrm{IBr}(HD|\lambda) = \mathrm{Irr}(HD|\hat{\lambda})$ where again $\hat{\lambda} = 1_D \times \lambda$ is the canonical extension of λ to ZD. Since $\hat{\lambda}$ is invariant and HD/ZD is cyclic, $\hat{\lambda}$ extends to HD and hence $|\mathrm{IBr}(HD|\lambda)| = |HD/ZD| = |H : Z| = s$. Then $s \leqslant 3d$. Now,

$$p^d - 1 \leqslant |G : ZD| = st \leqslant 3dt \leqslant 3d^2.$$

Note that, if $d \in \{1, 2, 3\}$, we would have $p \in \{2, 3\}$ that is a contradiction. Hence, we may assume $d > 3$ and $p > 3$. But then we have a contradiction since in this case $3d^2 < p^d - 1$.

Hence, we may assume that we are in one of the exceptions listed above, so in particular $|D| = p^2$. By Step 5, we have that $l(B) = 2$ and G acts transitively on $D \setminus \{1\}$. Let $u \in D \setminus \{1\}$ and let $b_u \in \mathrm{Bl}(\mathbf{C}_G(u)|D)$ inducing B. By the argument in Step 5, b_u has inertial index 2, so $|\mathbf{C}_G(u) : \mathbf{C}_G(D)| = 2$, and $l(b_u) = 2$. Since G acts transitively on $D \setminus \{1\}$, we have that

$$|G : ZD| = |G : \mathbf{C}_G(D)| = 2|G : \mathbf{C}_G(u)| = 2(p^2 - 1).$$

Since we are dealing with the case that G/ZD is not a subgroup of the semilinear group, using [21, Table 15.1] and the fact that $|G : ZD| = 2(p^2 - 1)$, we have that $G/Z = \mathrm{PrimitiveGroup}(r, i)$ with

$$(r, i) \in \{(5^2, 18), (11^2, 42), (29^2, 110)\}$$

in [5]. These groups have a normal subgroup $N/Z \lhd G/Z$ of index 2 and such that for $q \neq p$, the Sylow q-subgroups of N/Z are either cyclic or quaternion (in any case they have trivial Schur multiplier). In particular, $\hat{\lambda}$ extends to N by Lemma 1. Now, $|\operatorname{Irr}(G|\hat{\lambda})| = 4$ and the G-orbits of $\operatorname{Irr}(N|\hat{\lambda})$ have size at most $|G : N| = 2$, so we have $|\operatorname{Irr}(N|\hat{\lambda})| \leqslant 8$ but by Gallagher's theorem $|\operatorname{Irr}(N|\hat{\lambda})| = |\operatorname{Irr}(N/ZD)| = k(N/ZD)$. If $p \neq 5$, we have $k(N/ZD) > 8$ so these cases are impossible. If $p = 5$ then $N/ZD \cong \mathrm{SL}(2, 3)$ and $\operatorname{Irr}(N|\hat{\lambda})$ contains three characters of degree 1, three characters of degree 2 and a character of degree 3 by Gallagher's theorem. Since $|G : N| = 2$, the G-orbits in $\operatorname{Irr}(N|\hat{\lambda})$ have size at most 2. This yields at least 5 G-orbits in $\operatorname{Irr}(N|\hat{\lambda})$, which is a contradiction.

Final Step.

By Step 7, the rank of $\overline{G}$ is 3. Note that in this case we have $|\operatorname{Irr}(K|\lambda)| = l(B) = 2$. By Lemma 2, K/Z is solvable, and so is G/Z. By Step 5, we may assume that $d \neq 2$. By the main result of [4] (and taking into account that $\overline{K} = K/Z$ is a p'-group, $p \neq 2, 3$ and $d \neq 2$), we are in one of the following situations:

Case 1: $\overline{K} \leqslant \Gamma(p^d)$.

In this case, we have a subgroup $Z \leqslant H \leqslant K$ with H/Z cyclic (and hence, H abelian) with $|H : Z| = s \mid p^d - 1$ and index $|K : H| = t \mid d$. Since $|\mathrm{Irr}(K|\lambda)| = 2$ and $|\mathrm{Irr}(H|\lambda)| = |H/Z| = s$ we obtain that

$$s = |\mathrm{Irr}(H|\lambda)| \leqslant 2t \leqslant 2d.$$

Now, G/ZD acts on D in two nontrivial conjugacy classes. Hence

$$p^d - 1 \leqslant 2|G : DZ| = 2st \leqslant 4d^2.$$

If $d = 1$, we obtain $p = 5$ and we are done. In the other case, we have $d > 3$ and $p > 3$, but this is a contradiction.

Case 2: $\overline{K}$ imprimitive.

We have that K/Z is an imprimitive linear group with imprimitivity spaces V_1, V_2, where $D = V_1 \oplus V_2$ and $|V_1| = |V_2| = p^a$. Write $\overline{H} = \mathbf{N}_{\overline{K}}(V_1) = \mathbf{N}_{\overline{K}}(V_2)$. Then $\overline{K} \cong (\overline{A} \times \overline{B}) \rtimes \mathsf{C}_2$ where $\overline{H} \cong \overline{A} \times \overline{B}$ and $\overline{A} \cong \overline{H}/\mathbf{C}_{\overline{H}}(V_1) \cong \overline{B}$. By part 3 of Theorem 1.1 of [4], we have that $\overline{A}$ is a solvable linear group transtive on $V_1 \setminus \{0\}$, so by Huppert's theorem (see [14, Theorem 6.8] for instance) we have that either $\overline{A} \leqslant \Gamma(p^a)$ or $p^a \in \{5^2, 7^2, 11^2, 23^2\}$ (recall that $p \neq 3$ by Step

1). Since $|\operatorname{Irr}(K|\lambda)| = 2$, by Lemmas 3 and 4 and Step 1 we are in the latter situation. Note that $|\operatorname{Irr}(K|\lambda)| = 2$ forces $|\operatorname{Irr}(H|\lambda)| \in \{1, 4\}$.

Since we are in one of the exceptions of Huppert's theorem we have that $\overline{A}$ contains a normal subgroup $\overline{N}$ isomorphic to SL(2, 3) (see [21, Table 15.1] for more details on the structure of these groups). Write $N/Z = \overline{N}$. Then λ extends to N by Lemma 1 and by Gallagher's theorem the degrees of the irreducible characters of $\operatorname{Irr}(N|\lambda)$ are $\{1, 2, 3\}$, where the degrees 1 and 2 appear three times each and the degree 3 appears once. Let $\xi, \gamma, \delta \in \operatorname{Irr}(N|\lambda)$ with $\xi(1) = 1$, $\gamma(1) = 2$ and $\delta(1) = 3$. Observe that δ is H-invariant.

Suppose first that $|\operatorname{Irr}(H|\delta)| = 1$. Then δ is fully ramified in H and hence, $|H/N|$ is a square, but this is not possible since in all the possible cases $|H/N|_3 \in \{3, 27\}$. Therefore, $|\operatorname{Irr}(H|\delta)| > 1$. Then $|\operatorname{Irr}(H|\lambda)| = 4$ and by Clifford's theorem we deduce that ξ and γ lie under a unique irreducible character in $\operatorname{Irr}(H|\lambda)$, so they are fully ramified in their stabilizers H_ξ and H_γ. Again, by Clifford's theorem it follows that $|H : H_\xi| = |H : H_\gamma| = 3$ and hence $|H : N|_2 = |H_\xi : N|_2$. Now $|H : N| = |\overline{A}|^2/|\overline{N}|$, and since $|\overline{N}|_2 = 8$ it follows that $|H : N|_2$ is never a square. We conclude that $1 < |H_\xi : N|_2$ is not a square, so $|H_\xi : N|$ is not a square, yielding a contradiction with the fact that ξ is fully ramified in H_ξ.

Case 3: $p^d = 7^4$.

In this case G/Z is a subgroup of one of the groups of part 2(d) of [4, Theorem 1.1], and has degree 7^4. By the main result of [20], the primitive permutation groups of degree 7^4 are classified and this guarantees that the [5] library of such groups is complete. By using [5] we obtain that G/Z is one of the groups $G/Z = \text{Primitive Group}(7^4, i)$ where $i \in \{774, 775\}$.

In the case $i = 774$, we can find a normal subgroup $Z \subseteq N \lhd G$ with $|G : N| = 4$ and all Sylow q-subgroups of N, for $q \neq p$, are cyclic, so they have trivial Schur multiplier. In this case, we have that λ extends to N by Lemma 1. Then λ extends to $N \cap K$ and $|\operatorname{Irr}(N \cap K|\lambda)| = |\operatorname{Irr}(N \cap K/Z)| = |\operatorname{Irr}(N/DZ)| = 192$. However, since $|\operatorname{Irr}(K|\lambda)| = l(B) = 2$ and using $|K : K \cap N| = |G : N| = 4$ we have that $|\operatorname{Irr}(N \cap K|\lambda)| \leqslant 8$, a contradiction.

In the case $i = 775$, we can find a normal subgroup N/Z of order $3 \cdot 7^4$. Again, λ extends to N and if c is the unique block of N covered by B we have that $\operatorname{Irr}(c) = \operatorname{Irr}(N|\lambda)$. Then there are at most 4 orbits in $\operatorname{Irr}(N|\lambda)$ of size dividing 640 and the sum of the sizes is 803. By an easy counting argument we exclude the possibility that there are either two or three orbits, and if there are 4 orbits, then they have sizes $\{1, 2, 160, 640\}$, in particular there is an orbit of size 1. Let θ be the G-invariant irreducible character of c. Now, since $\mathbf{C}_G(D) \subseteq N$ we have that B is the unique block of G covering c ([15, Theorem 9.19 and Lemma 9.20]) and hence there is just one irreducible character in G lying over θ (the one lying in B). This means that there exists $\theta \in \operatorname{Irr}(N|\lambda)$ that is fully ramified and by [16, Problem 8.3] we have that there is no self-centralizing cyclic subgroup in G/N. However, if $P/N \in \operatorname{Syl}_5(G/N)$ and $C/N = \mathbf{C}_{G/N}(P/N)$ then C/N is cyclic of order 10 and $\mathbf{C}_{G/N}(C/N) = C/N$. □

In the following, we write $k_0(B)$ for the number of height zero characters in the p-block B.

Corollary 7. *Let G be a finite group, B a p-block with $k(B) = 4$ and D a defect group of B. Assume $k_0(B) = k_0(b)$ where $b \in \operatorname{Bl}(\mathbf{N}_G(D))$ is the Brauer correspondent of B in $\mathbf{N}_G(D)$. Then D is isomorphic to* $\mathsf{C}_4, \mathsf{C}_2 \times \mathsf{C}_2, \mathsf{C}_5$.

Proof. By Theorem 9.9(b) of [15], b dominates some block $\overline{b} \in \operatorname{Bl}(\mathbf{N}_G(D)/\Phi(D))$ with defect group $D/\Phi(D)$. We have $k(\overline{b}) = k_0(\overline{b}) \leqslant k_0(b)$ by [18, Theorem 6]. We explore all the possibilities for $k(\overline{b})$.

If $k(\overline{b}) = 1$ then $D/\Phi(D) = 1$, which is impossible. If $k(\overline{b}) = 2$, then by the main result of [1] we have $|D/\Phi(D)| = 2$ and then D is cyclic and $p = 2$. Using [3, Theorem 1] we conclude that $|D| = 4$ and we are done. If $k(\overline{b}) = 3$, then by [12, Theorem 4.1] we have $D/\Phi(D)$ is cyclic of order 3, so D is a cyclic 3-group. Then using [3, Theorem 1] we get a contradiction. Finally, if $k(\overline{b}) = 4$ then $D/\Phi(D)$ is one of the groups from Theorem 6, and it must be elementary abelian. If $|D/\Phi(D)| = 5$ then D is cyclic and then by [3, Theorem 1] we have $D \cong \mathsf{C}_5$. Otherwise, $D/\Phi(D) \cong \mathsf{C}_2 \times \mathsf{C}_2$ and $p = 2$. Since $k(\overline{b}) \leqslant k_0(B)$ we have $k(B) = k_0(B)$ and we apply Corollary 1.3 (iii) of [13] to conclude that $|D| = 4$. □

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