Multivariate OWA functions

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Abstract

Ordered Weighted Averaging (OWA) functions are a popular tool for the aggregation of real values and have been used successfully in several fields of application. The extension of these OWA functions to the multivariate setting is not unique and has been addressed separately by different disciplines. In this paper, we introduce a unifying perspective by presenting under a common framework different classes of multivariate OWA functions and discuss the main fulfilled properties by each of these classes.
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1. Introduction

Aggregation functions [27] as functions allowing to aggregate (i.e., combine) several real values into a single one have been studied for centuries and have been proven to be very useful in many fields of application [7]. Most commonly, those functions are required to be idempotent and monotone, thus resulting in the so-called averaging functions or means [29]. Among all families of averaging functions, Ordered Weighted Averaging (OWA) [55] functions undoubtedly are one of the most popular options due to their flexibility (since the OWA family accommodates a wide range of functions such as the mean, median, minimum and maximum) and desirable properties (idempotence, monotonicity, symmetry, continuity, equivariance under linear transformations, etc.). A recent survey on the topic can be found in [13].

In case the aim is no longer to aggregate real values but, instead, to aggregate data points in a higher dimension, the problem becomes more involved. Admittedly, there exists a very rich literature on the topic for the special case of the median [52] and the topic has recently gained some traction since brought to the field of aggregation theory by Gagolewski [23,24]; however the literature on extensions of OWA functions to the multivariate setting is still scattered and fragmented by field of research. For instance, the field of aggregation theory – typically focusing on properties based on preservation of an underlying notion of order – proposes to find linear extensions of the product order on $\mathbb{R}^m$ in order to define multivariate OWA functions [15], even though the resulting properties are in general not very appealing [46]. The field of multivariate statistics typically abandons the notion of order when dealing with more than one dimension focusing more strongly on properties such as equivariance under affine transformations [22,

38], which eventually implies that the notions of minimum and maximum are abandoned and the data points are simply sorted from most central to most extreme. A totally different perspective is pursued in classical multiobjective optimization [31], where criteria for selecting a multivariate minimum/maximum are developed.

In this paper, we aim at unifying the different points of view taken by the different fields. For this purpose, we firstly recall the classical notion of OWA function for the univariate setting in Section 2 and secondly present a common framework for the definition of multivariate OWA functions in Section 3. Some classes of multivariate OWA functions are next listed in Section 4. The work ends with a discussion on the properties fulfilled by each of the different classes of multivariate OWA functions in Section 5 and some conclusions in Section 6.

2. The univariate setting

2.1. Aggregation functions

Aggregation functions [27] are mathematical constructs that allow to formalize the process of combining several inputs into a single output that acts as some sort of representative. In the following, we present several properties that are of interest for such functions.

Definition 1. Consider $n \in \mathbb{N}$. A function $f : \mathbb{R}^n \to \mathbb{R}$ is said:

- to be internal within the points if, for any $(x_1, \ldots, x_n) \in \mathbb{R}^n$, it holds that

$$f(x_1, \ldots, x_n) \in \{x_1, \ldots, x_n\};$$

- to be compensative (or internal) if, for any $(x_1, \ldots, x_n) \in \mathbb{R}^n$, it holds that

$$\min(x_1, \ldots, x_n) \leq f(x_1, \ldots, x_n) \leq \max(x_1, \ldots, x_n);$$

- to be idempotent if, for any $x \in \mathbb{R}$, it holds that

$$f(x, \ldots, x) = x;$$

- to be monotone (increasing) if, for any $(x_1, \ldots, x_n), (y_1, \ldots, y_n) \in \mathbb{R}^n$ such that $x_i \leq y_i$ for any $i \in \{1, \ldots, n\}$, it holds that

$$f(x_1, \ldots, x_n) \leq f(y_1, \ldots, y_n);$$

- to be symmetric if, for any $(x_1, \ldots, x_n) \in \mathbb{R}^n$ and any permutation σ of $\{1, \ldots, n\}$, it holds that

$$f(x_1, \ldots, x_n) = f(x_{\sigma(1)}, \ldots, x_{\sigma(n)});$$

- to be continuous in each variable if, for any $(x_1, \ldots, x_n) \in \mathbb{R}^n$ and any $i \in \{1, \ldots, n\}$, it holds that

$$\lim_{y \to x_i} f(x_1, \ldots, x_{i-1}, y, x_{i+1}, \ldots, x_n) = f(x_1, \ldots, x_n);$$

- to be equivariant under linear transformations if, for any $(x_1, \ldots, x_n) \in \mathbb{R}^n$, any $t \in \mathbb{R}$ and any $s > 0$, it holds that

$$f(s x_1 + t, \ldots, s x_n + t) = s f(x_1, \ldots, x_n) + t;$$

- to be equivariant under translations (translation equivariant) if, for any $(x_1, \ldots, x_n) \in \mathbb{R}^n$ and any $t \in \mathbb{R}$, it holds that

$$f(x_1 + t, \ldots, x_n + t) = f(x_1, \ldots, x_n) + t;$$

- to be equivariant under changes of scale (scale equivariant) if, for any $(x_1, \ldots, x_n) \in \mathbb{R}^n$ and any $s > 0$, it holds that

$$f(s x_1, \ldots, s x_n) = s f(x_1, \ldots, x_n).$$

Remark 2. Internality within the points implies compensativity, which at the same time implies idempotence. Under (increasing) monotonicity, compensativity and idempotence are equivalent (see Proposition 2.54 in [27]) and continuity in each variable is equivalent to classical continuity (see Proposition 2.8 in [27]). Obviously, equivariance under linear transformations implies both translation equivariance and scale equivariance.

Idempotent aggregation functions, called means [29] when also fulfilling the monotonicity property, are the most prominent type of aggregation functions and have been used largely by practitioners [6]. In this paper, and even though aggregation functions that are not idempotent have also attracted the attention of the scientific community [28], we will only deal with idempotent aggregation functions.

2.2. OWA functions

Back in 1988, Yager [55] proposed to consider linear combinations of the ordered inputs as a tool in decision making, introducing the notion of an Ordered Weighted Averaging (OWA) function. An OWA function characterizes and is characterized by a weighting vector $\mathbf{w} = (w_1, \ldots, w_n) \in [0, 1]^n$ verifying that $\sum_{i=1}^{n} w_i = 1$.

Definition 3. Consider $n \in \mathbb{N}$. The OWA function associated with the weighting vector $\mathbf{w} = (w_1, \ldots, w_n)$ is the function $f_{\mathbf{w}} : \mathbb{R}^n \to \mathbb{R}$ defined as:

$$f_{\mathbf{w}}(x_1, \ldots, x_n) = \sum_{i=1}^{n} w_i \, x_{(i)} \,,$$

where $x_{(i)}$ denotes the i-th largest value among $x_1, \ldots, x_n$.

OWA functions are known to fulfill all properties listed in Definition 1 (see, e.g., [40]), with the exception of internality within the points which is only assured to be fulfilled if the weighting vector is formed by a one and $n - 1$ zeros.

The most classical examples of OWA functions are associated with the following weighting vectors:

- The maximum ($f_{\max}$): $\mathbf{w}^{\max} = (1, 0, \ldots, 0)$.
- The minimum ($f_{\min}$): $\mathbf{w}^{\min} = (0, \ldots, 0, 1)$.
- The arithmetic mean (f_{am}): $\mathbf{w}^{\mathrm{am}} = (\frac{1}{n}, \ldots, \frac{1}{n})$.
- The median (f_{med}): $\mathbf{w}^{\mathrm{med}}$ is defined by $w_i^{\mathrm{med}} = 0$ for any $i \neq \frac{n+1}{2}$ and $w_{\frac{n+1}{2}}^{\mathrm{med}} = 1$ if n is odd; and by $w_i^{\mathrm{med}} = 0$ for any $i \notin \{\frac{n}{2}, \frac{n}{2} + 1\}$ and $w_{\frac{n}{2}}^{\mathrm{med}} = w_{\frac{n}{2}+1}^{\mathrm{med}} = \frac{1}{2}$ if n is even.

A prominent family of OWA functions is that of centered OWA functions [57], where the weighting vector is symmetric, i.e., $w_i = w_{n-i+1}$, for any $i \in \{1, \ldots, n\}$. Other prominent families of OWA functions are S-OWA functions, step OWA functions and window OWA functions [56]. From a similar perspective, linear combinations of order statistics (not necessarily convex) had already been explored in the field of statistics under the name of L-estimators [30], yielding popular notions such as Winsorized and trimmed means [17].

3. Extension of OWA operators to the multivariate framework

In this paper, we are interested in how the notion of an OWA function can be extended from the univariate setting to the multivariate setting, i.e., the goal is no longer to aggregate real numbers in $\mathbb{R}$ but to aggregate real data points in $\mathbb{R}^m$ with $m \geq 2$. The problem setting, as brought to the field of aggregation theory in [23,24], is formalized as follows. Consider n data points $\mathbf{x}_1, \ldots, \mathbf{x}_n \in \mathbb{R}^m$, with $m \geq 2$. All $\mathbf{x}_i$ are treated as column vectors and $(\mathbf{x}_1, \ldots, \mathbf{x}_n)$ is treated as a matrix with m rows and n columns. The j-th component of the data point $\mathbf{x}_i$ is denoted as $\mathbf{x}_i(j)$. The product order on $\mathbb{R}^m$ is defined as $\mathbf{x}_{i_1} \leq_m \mathbf{x}_{i_2}$ if $\mathbf{x}_{i_1}(j) \leq \mathbf{x}_{i_2}(j)$ for any $j \in \{1, \ldots, m\}$. A multivariate (aggregation) function is a function $f : (\mathbb{R}^m)^n \to \mathbb{R}^m$, for which some desirable properties are listed right after.

Definition 4. Consider $n, m \in \mathbb{N}$ with $m \geq 2$. A function $f : (\mathbb{R}^m)^n \to \mathbb{R}^m$ is said:

(IWP) to be internal within the points if, for any $(\mathbf{x}_1, \ldots, \mathbf{x}_n) \in (\mathbb{R}^m)^n$, it holds that

$$f(\mathbf{x}_1, \ldots, \mathbf{x}_n) \in \{\mathbf{x}_1, \ldots, \mathbf{x}_n\};$$

(ICH) to be internal within the convex hull (CH-internal) if, for any $(\mathbf{x}_1, \ldots, \mathbf{x}_n) \in (\mathbb{R}^m)^n$, it holds that

$$f(\mathbf{x}_1, \ldots, \mathbf{x}_n) \in \mathrm{CH}(\mathbf{x}_1, \ldots, \mathbf{x}_n),$$

where $\mathrm{CH}(\mathbf{x}_1, \ldots, \mathbf{x}_n)$ represents the convex hull of $\mathbf{x}_1, \ldots, \mathbf{x}_n$ defined as $\mathrm{CH}(\mathbf{x}_1, \ldots, \mathbf{x}_n) = \big\{\mathbf{x} = \sum_{i=1}^n \lambda_i \mathbf{x}_i \in \mathbb{R}^m \mid \lambda_i \geq 0, \sum_{i=1}^n \lambda_i = 1\big\}$;

(IBB) to be internal within the bounding box (BB-internal) if, for any $(\mathbf{x}_1, \ldots, \mathbf{x}_n) \in (\mathbb{R}^m)^n$, it holds that

$$f(\mathbf{x}_1, \ldots, \mathbf{x}_n) \in \mathrm{BB}(\mathbf{x}_1, \ldots, \mathbf{x}_n),$$

where $\mathrm{BB}(\mathbf{x}_1, \ldots, \mathbf{x}_n)$ represents the bounding box of $\mathbf{x}_1, \ldots, \mathbf{x}_n$ defined as $\mathrm{BB}(\mathbf{x}_1, \ldots, \mathbf{x}_n) = \big[\min_{i=1}^n \mathbf{x}_i(1), \max_{i=1}^n \mathbf{x}_i(1)\big] \times \cdots \times \big[\min_{i=1}^n \mathbf{x}_i(m), \max_{i=1}^n \mathbf{x}_i(m)\big]$;

(ID) to be idempotent if, for any $\mathbf{x} \in \mathbb{R}^m$, it holds that

$$f(\mathbf{x}, \ldots, \mathbf{x}) = \mathbf{x};$$

(M) to be $\leq_m$-monotone (increasing) if, for any $(\mathbf{x}_1, \ldots, \mathbf{x}_n), (\mathbf{y}_1, \ldots, \mathbf{y}_n) \in (\mathbb{R}^m)^n$ such that $\mathbf{x}_i \leq_m \mathbf{y}_i$ for any $i \in \{1, \ldots, n\}$, it holds that

$$f(\mathbf{x}_1, \ldots, \mathbf{x}_n) \leq_m f(\mathbf{y}_1, \ldots, \mathbf{y}_n);$$

(S) to be symmetric if, for any $(\mathbf{x}_1, \ldots, \mathbf{x}_n) \in (\mathbb{R}^m)^n$ and any permutation σ of $\{1, \ldots, n\}$, it holds that

$$f(\mathbf{x}_1, \ldots, \mathbf{x}_n) = f(\mathbf{x}_{\sigma(1)}, \ldots, \mathbf{x}_{\sigma(n)});$$

(C) to be continuous in each variable if, for any $(\mathbf{x}_1, \ldots, \mathbf{x}_n) \in (\mathbb{R}^m)^n$ and any $i \in \{1, \ldots, n\}$, it holds that

$$\lim_{\mathbf{y} \to \mathbf{x}_i} f(\mathbf{x}_1, \ldots, \mathbf{x}_{i-1}, \mathbf{y}, \mathbf{x}_{i+1}, \ldots, \mathbf{x}_n) = f(\mathbf{x}_1, \ldots, \mathbf{x}_n);$$

(EA) to be equivariant under affine transformations (affine equivariant) if, for any $(\mathbf{x}_1, \ldots, \mathbf{x}_n) \in (\mathbb{R}^m)^n$, any invertible $\mathbf{A} \in \mathbb{R}^{m \times m}$ and any $\mathbf{t} \in \mathbb{R}^m$, it holds that

$$f(\mathbf{A}\mathbf{x}_1 + \mathbf{t}, \ldots, \mathbf{A}\mathbf{x}_n + \mathbf{t}) = \mathbf{A} f(\mathbf{x}_1, \ldots, \mathbf{x}_n) + \mathbf{t};$$

(EO) to be equivariant under orthogonal transformations (orthogonal equivariant) if, for any $(\mathbf{x}_1, \ldots, \mathbf{x}_n) \in (\mathbb{R}^m)^n$ and any orthogonal matrix $\mathbf{O} \in \mathbb{R}^{m \times m}$ (i.e., a square matrix such that $\mathbf{O}^T = \mathbf{O}^{-1}$), it holds that

$$f(\mathbf{O}\mathbf{x}_1, \ldots, \mathbf{O}\mathbf{x}_n) = \mathbf{O} f(\mathbf{x}_1, \ldots, \mathbf{x}_n);$$

(ET) to be equivariant under translations (translation equivariant) if, for any $(\mathbf{x}_1, \ldots, \mathbf{x}_n) \in (\mathbb{R}^m)^n$ and any $\mathbf{t} \in \mathbb{R}^m$, it holds that

$$f(\mathbf{x}_1 + \mathbf{t}, \ldots, \mathbf{x}_n + \mathbf{t}) = f(\mathbf{x}_1, \ldots, \mathbf{x}_n) + \mathbf{t};$$

(ES) to be equivariant under changes of scale (scale equivariant) if, for any $(\mathbf{x}_1, \ldots, \mathbf{x}_n) \in (\mathbb{R}^m)^n$ and any invertible diagonal matrix $\mathbf{S} \in \mathbb{R}^{m \times m}$, it holds that

$$f(\mathbf{S}\mathbf{x}_1, \ldots, \mathbf{S}\mathbf{x}_n) = \mathbf{S} f(\mathbf{x}_1, \ldots, \mathbf{x}_n),$$

(EUS) to be equivariant under uniform changes of scale (uniform-scale equivariant) if, for any $(\mathbf{x}_1, \ldots, \mathbf{x}_n) \in (\mathbb{R}^m)^n$ and any $s > 0$, it holds that

$$f(s\mathbf{x}_1, \ldots, s\mathbf{x}_n) = s f(\mathbf{x}_1, \ldots, \mathbf{x}_n).$$

Table 1
List of data points in $\mathbb{R}^2$, used for illustrative purposes.

i	1	2	3	4	5	6	7	8	9	10	11
$\mathbf{x}_i(1)$	0.3	0.4	0.4	0.6	0.7	0.8	0.2	0.5	0.8	0.4	0.5
$\mathbf{x}_i(2)$	0.3	0.4	0.5	0.5	0.7	0.3	0.5	0.3	0.5	0.1	0.7

Remark 5. Internality within the points implies CH-internality, which at the same time implies BB-internality, and, ultimately, BB-internality implies idempotence. Note that both CH-internality and BB-internality coincide if $m = 1$ and, under even weaker conditions than orthogonal equivariance, both CH-internatility and BB-internality coincide for $m \geq 2$ (see Lemma 11 in [24]). There exist many other definitions of monotonicity different than $\leq_m$-monotonicity for multivariate functions, a review can be found in [48]. Affine equivariance implies all four among orthogonal equivariance, translation equivariance, scale equivariance and uniform-scale equivariance, whereas scale equivariance implies uniform-scale equivariance. Note that there is some ambiguity on the term scale equivariance, sometimes used for the notion provided in Definition 4 and sometimes used for the here-called uniform-scale equivariance.

For defining an OWA function, it is necessary to order all points from largest to smallest. This ordering process is straightforward in the univariate case, however the problem is more involved in the multivariate case [4]. Admittedly, the product order $\leq_m$ is a natural notion of order on $\mathbb{R}^m$. Based on this product order, an immediate extension of OWA functions to the multivariate setting could be thought of. More precisely, if $\mathbf{x}_1, \ldots, \mathbf{x}_n$ are such that they are linearly ordered according to $\leq_m$, then we could associate a weighting vector $\mathbf{w}$ to an OWA function similarly to the univariate case, as follows:

$$f_{\mathbf{w}}(\mathbf{x}_1, \ldots, \mathbf{x}_n) = \sum_{i=1}^{n} w_i \, \mathbf{x}_{(i)}, \tag{1}$$

where $\mathbf{x}_{(i)}$ denotes the i-th largest data point among $\mathbf{x}_1, \ldots, \mathbf{x}_n$ according to the product order $\leq_m$.

Example 6. Consider $\mathbf{x}_1 = (0.3, 0.3)^T$, $\mathbf{x}_2 = (0.4, 0.4)^T$, $\mathbf{x}_3 = (0.4, 0.5)^T$, $\mathbf{x}_4 = (0.6, 0.5)^T$ and $\mathbf{x}_5 = (0.7, 0.7)^T$. It holds that $\mathbf{x}_1 \leq_2 \mathbf{x}_2 \leq_2 \mathbf{x}_3 \leq_2 \mathbf{x}_4 \leq_2 \mathbf{x}_5$ and, therefore, we may define the multivariate minimum, maximum and median as follows:

$$f_{\min}(\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3, \mathbf{x}_4, \mathbf{x}_5) = \mathbf{x}_1 = (0.3, 0.3)^T,$$
$$f_{\max}(\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3, \mathbf{x}_4, \mathbf{x}_5) = \mathbf{x}_5 = (0.7, 0.7)^T,$$
$$f_{\mathrm{med}}(\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3, \mathbf{x}_4, \mathbf{x}_5) = \mathbf{x}_3 = (0.4, 0.5)^T. \quad \blacktriangleleft$$

Unfortunately, unlike in the case of the real line, the product order $\leq_m$ is not a linear order on $\mathbb{R}^m$ for $m \geq 2$. This means that we could find examples of data points that are not linearly ordered and, thus, the multivariate OWA function might not be defined unequivocally.

Example 7. Consider the list of data points in $\mathbb{R}^2$ presented in Table 1 and illustrated in Fig. 1. Note that the first five data points are those used in Example 6.

Although we have seen that $\mathbf{x}_1 \leq_2 \mathbf{x}_2 \leq_2 \mathbf{x}_3 \leq_2 \mathbf{x}_4 \leq_2 \mathbf{x}_5$, there exist data points that are incomparable with respect to the product order (such as $\mathbf{x}_1$ and $\mathbf{x}_7$). $\quad \blacktriangleleft$

It has been shown that it is not clear how a multivariate OWA function should be defined. Most proposals to extend OWA functions to the multivariate framework simply find a way of sorting the data points $\mathbf{x}_1, \ldots, \mathbf{x}_n$ and straightforwardly extend the definition of a univariate OWA function as in Eq. (1). A problem that typically arises in the multivariate framework is that instead of a linear order of the data points $\mathbf{x}_1, \ldots, \mathbf{x}_n$ in some cases we may only obtain a weak order. In such case, a natural approach is to simply distribute the weights for the data points tied at the same position equally, as follows:
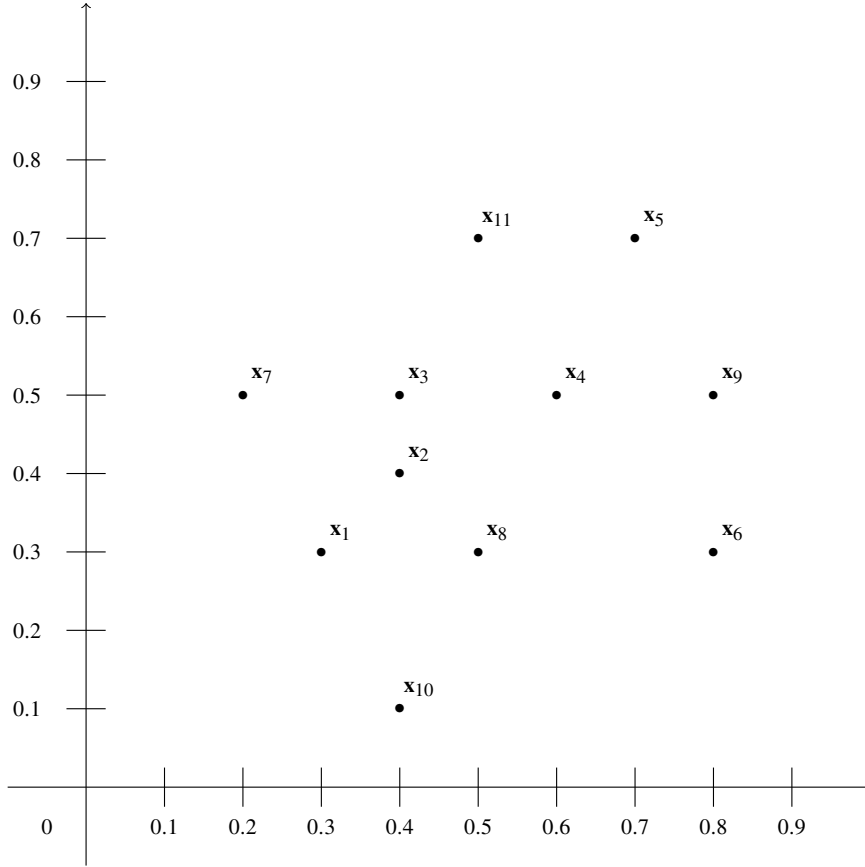
Fig. 1. Graphical representation of the list of data points of Table 1.

$$f_{\mathbf{w}}(\mathbf{x}_1, \ldots, \mathbf{x}_n) = \sum_{i=1}^{n} w_i \frac{1}{\# C_i} \sum_{\mathbf{x}_\ell \in C_i} \mathbf{x}_\ell \,, \tag{2}$$

where C_j represents the class associated with the data points at the j-th position in the weak order $\succsim$ of $\mathbf{x}_1, \ldots, \mathbf{x}_n$, i.e.,

$$C_j = \left\{ \mathbf{x}_i \in X \mid \#\{\mathbf{x}_\ell \in X \mid \mathbf{x}_\ell \succsim \mathbf{x}_i\} \geq j \geq n + 1 - \#\{\mathbf{x}_\ell \in X \mid \mathbf{x}_i \succsim \mathbf{x}_\ell\} \right\}.$$

4. Classes of multivariate OWA functions

This section is devoted to the introduction of different classes of multivariate OWA functions.

4.1. Multivariate OWA functions based on a componentwise extension of a univariate OWA function

The most natural extension of an OWA function from the univariate to the multivariate setting simply extends componentwisely a univariate OWA function, as follows:

$$f_{\mathbf{w}}(\mathbf{x}_1, \ldots, \mathbf{x}_n) = \left(\sum_{i=1}^{n} w_i \, \mathbf{x}(1)_{(i)}, \ldots, \sum_{i=1}^{n} w_i \, \mathbf{x}(m)_{(i)} \right),$$

where $\mathbf{x}(j)_{(i)}$ denotes the i-th greatest value at the j-th component among those of $\mathbf{x}_1(j), \ldots, \mathbf{x}_n(j)$.

Example 8. Continue with Example 7. It follows that

$$\mathbf{x}_7(1) < \mathbf{x}_1(1) < \mathbf{x}_2(1) = \mathbf{x}_3(1) = \mathbf{x}_{10}(1) < \mathbf{x}_8(1) = \mathbf{x}_{11}(1) < \mathbf{x}_4(1) < \mathbf{x}_5(1) < \mathbf{x}_6(1) = \mathbf{x}_9(1),$$

and

$$\mathbf{x}_{10}(2) < \mathbf{x}_1(2) = \mathbf{x}_6(2) = \mathbf{x}_8(2) < \mathbf{x}_2(2) < \mathbf{x}_3(2) = \mathbf{x}_4(2) = \mathbf{x}_7(2) = \mathbf{x}_9(2) < \mathbf{x}_5(2) = \mathbf{x}_{11}(2).$$

Therefore, a possible definition of OWA function based on a componentwise extension of a univariate OWA function will simply apply the OWA function separately to each of the components. In this case, we obtain:

$$f_{\min}(\mathbf{x}_1, \ldots, \mathbf{x}_{11}) = (\mathbf{x}_7(1), \mathbf{x}_{10}(2))^T = (0.2, 0.1)^T,$$
$$f_{\max}(\mathbf{x}_1, \ldots, \mathbf{x}_{11}) = (\mathbf{x}_9(1), \mathbf{x}_{11}(2))^T = (0.8, 0.7)^T,$$
$$f_{\mathrm{med}}(\mathbf{x}_1, \ldots, \mathbf{x}_{11}) = (\mathbf{x}_8(1), \mathbf{x}_3(2))^T = (0.5, 0.5)^T. \quad \blacktriangleleft$$

This naive approach is very popular in statistics. For instance, the componentwise median (also referred to as the coordinatewise median) is the most classical extension of the univariate median to the multivariate framework [49]. Componentwise extensions of univariate functions are also among the preferred options in aggregation theory [24] and, actually, practitioners usually consider this approach to multivariate OWA functions without making it explicit (see, for instance, [12]). However, this type of functions does not embrace the multivariate nature of the data and, thus, fails to fulfill desirable properties such as affine equivariance and convex-hull internality (e.g., the component-wise median is not convex-hull internal if $m \geq 3$ [49]). Special attention deserves the centroid (i.e., the componentwise mean) that, even though it is characterizable as a componentwise function, succeeds in fulfilling many intuitive properties [25].

4.2. Multivariate OWA functions based on a linearization of the product order

In classical multiobjective optimization [31], the most basic approach to solve a vector maximization problem is based on utility functions, where the maximization/minimization of a multivariate objective function is transformed into the maximization/minimization of a utility function of the multiple objectives. Similarly, we could here obtain a weak order of the inputs by considering a utility function. A more advanced approach discussed in [15] is to linearly extend the product order $\leq_m$ on $\mathbb{R}^m$ by using several utility functions. A possible way to do so with a minimum number of utility functions is by considering m linearly independent weighted arithmetic means $M_1, \ldots, M_m : \mathbb{R}^m \to \mathbb{R}$. It is clarified that a weighted arithmetic mean is understood as a convex combination of the inputs. As discussed in [15], a linear extension $\preceq_{\mathbf{M}}$ of $\leq_m$ based on $\mathbf{M} = (M_1, \ldots, M_m)$ is defined as $\mathbf{x}_{i_1} \preceq_{\mathbf{M}} \mathbf{x}_{i_2}$ if $\mathbf{x}_{i_1} = \mathbf{x}_{i_2}$ or there exists $k \in \{1, \ldots, m\}$ such that $M_j(\mathbf{x}_{i_1}) = M_j(\mathbf{x}_{i_2})$ for any $j \in \{1, \ldots, k-1\}$ and $M_k(\mathbf{x}_{i_1}) < M_k(\mathbf{x}_{i_2})$. Typical examples of linear extensions of $\leq_m$ defined by means of m linearly independent weighted arithmetic means are the lexicographic orders [20], where the considered weighted arithmetic means are the projections, i.e., $M_j(\mathbf{x}_i) = \mathbf{x}_i(\sigma(j))$ with σ a permutation of $\{1, \ldots, n\}$. Another prominent example in case $m = 2$ is Xu and Yager's linear order on $\mathbb{R}^2$ [54], where $M_1(\mathbf{x}_i) = \frac{1}{2}\mathbf{x}_i(1) + \frac{1}{2}\mathbf{x}_i(2)$ and $M_2(\mathbf{x}_i) = \mathbf{x}_i(2)$.

Ultimately, after a linear extension of $\leq_m$ is agreed upon, a multivariate OWA function is introduced as in Eq. (1).

Example 9. Continue with Example 7 and the data points presented in Table 1. Consider $\preceq$ to be the lexicographic order induced by the identity permutation. It follows that:

$$\mathbf{x}_7 \preceq \mathbf{x}_1 \preceq \mathbf{x}_{10} \preceq \mathbf{x}_2 \preceq \mathbf{x}_3 \preceq \mathbf{x}_8 \preceq \mathbf{x}_{11} \preceq \mathbf{x}_4 \preceq \mathbf{x}_5 \preceq \mathbf{x}_6 \preceq \mathbf{x}_9.$$

Therefore, we obtain:

$$f_{\min}(\mathbf{x}_1, \ldots, \mathbf{x}_{11}) = \mathbf{x}_7 = (0.2, 0.5)^T,$$
$$f_{\max}(\mathbf{x}_1, \ldots, \mathbf{x}_{11}) = \mathbf{x}_9 = (0.8, 0.5)^T,$$
$$f_{\mathrm{med}}(\mathbf{x}_1, \ldots, \mathbf{x}_{11}) = \mathbf{x}_8 = (0.5, 0.3)^T. \quad \blacktriangleleft$$

Unfortunately, as mentioned in [46], the properties fulfilled by this type of multivariate OWA functions are not very appealing. Although they trivially are symmetric and fulfill most internality-related properties (they are CH-internal, BB-internal and idempotent, also being internal within the points if the weighting vector contains a 1), properties such as affine equivariance, orthogonal equivariance, most monotonicity properties (and in particular $\leq_m$-monotonicity) and continuity in each of the variables are assured to fail, regardless of the choice of linear extension of the product order. Interestingly, as mentioned in [47] for the case of the order-based multivariate median, uniform scale equivariance and translation equivariance (and thus MP-monotonicity and MC-monotonicity according to the terminology of [48]), which are not fulfilled in general, are assured to be fulfilled if the linear extension is based on m linearly independent weighted arithmetic means. If those m linearly independent weighted arithmetic means are the projections, then also scale equivariance is fulfilled.

Another interesting point raised in [47] concerns the lack of robustness of the multivariate median based on a linear extension of the product order. It is here recalled that one of several definitions of robustness for a function builds upon the notion of finite-sample breakdown point, which is the smallest proportion of 'arbitrarily bad' input data points that the function can handle before also giving an 'arbitrarily bad' output data point. It was proven in [47] that the finite-sample breakdown point of the multivariate median based on a linear extension of the product order is $\frac{1}{n}$, contrasting with the univariate case in which the median attains the maximum possible value of $\frac{1}{2}$ for translation equivariant functions.

4.3. Multivariate OWA functions based on an extension of the partially ordered set

A similar approach aims at (linearly) extending the partially ordered set $(X, \leq_m)$, where $X = \{\mathbf{x}_1, \dots, \mathbf{x}_n\}$. Note the subtle difference with respect to the previous case in which the aim was to linearly extend the partially ordered set $(\mathbb{R}^m, \leq_m)$. Different techniques for obtaining a weak order (complete order relation) from a partially ordered set have been developed. For instance, the cardinal method discussed in [21] assigns to each $\mathbf{x}_i$ the score

$$s(\mathbf{x}_i) = \#\{\ell \in \{1, \dots, n\} \mid \mathbf{x}_\ell \leq_m \mathbf{x}_i\} - \#\{\ell \in \{1, \dots, n\} \mid \mathbf{x}_i \leq_m \mathbf{x}_\ell\},$$

where $\#A$ denotes the cardinality of the set A, and ultimately orders the data points by increasing score. Another prominent method in this family, called the maximal method in [21], firstly identifies the set of maximal elements E_1 of $(X, \leq_m)$, secondly removes the elements in E_1 from X and identifies the set of maximal elements E_2 of $(X \backslash E_1, \leq_m)$, proceeding iteratively until all $\mathbf{x}_i$ have been assigned to a set E_ℓ. Finally, the elements $\mathbf{x}_{i_1} \in E_{\ell_1}$ and $\mathbf{x}_{i_2} \in E_{\ell_2}$ are ordered such that $\mathbf{x}_{i_1} \succsim \mathbf{x}_{i_2}$ if $\ell_1 \leq \ell_2$.

For the definition of a multivariate OWA function it suffices to consider the obtained weak order and resort to Eq. (2).

Example 10. Continue with Example 7 and the data points presented in Table 1. The Hasse diagram of the poset $(X, \leq_m)$ associated with these data points is shown in Fig. 2.

The cardinal method yields the following scores $s(\mathbf{x}_1) = -8$, $s(\mathbf{x}_2) = -3$, $s(\mathbf{x}_3) = 0$, $s(\mathbf{x}_4) = 4$, $s(\mathbf{x}_5) = 8$, $s(\mathbf{x}_6) = 2$, $s(\mathbf{x}_7) = -5$, $s(\mathbf{x}_8) = -3$, $s(\mathbf{x}_9) = 8$, $s(\mathbf{x}_{10}) = -8$ and $s(\mathbf{x}_{11}) = 5$. Therefore, we obtain the weak order:

$$\mathbf{x}_5 \sim \mathbf{x}_9 \succ \mathbf{x}_{11} \succ \mathbf{x}_4 \succ \mathbf{x}_6 \succ \mathbf{x}_3 \succ \mathbf{x}_2 \sim \mathbf{x}_8 \succ \mathbf{x}_7 \succ \mathbf{x}_1 \sim \mathbf{x}_{10},$$

and the following multivariate OWA functions:

$$f_{\min}(\mathbf{x}_1, \dots, \mathbf{x}_{11}) = \frac{\mathbf{x}_1 + \mathbf{x}_{10}}{2} = (0.35, 0.2)^T,$$

$$f_{\max}(\mathbf{x}_1, \dots, \mathbf{x}_{11}) = \frac{\mathbf{x}_5 + \mathbf{x}_9}{2} = (0.75, 0.6)^T,$$

$$f_{\mathrm{med}}(\mathbf{x}_1, \dots, \mathbf{x}_{11}) = \mathbf{x}_3 = (0.4, 0.5)^T.$$

Similarly, the maximal method yields the following sets $E_1 = \{\mathbf{x}_5, \mathbf{x}_9\}$, $E_2 = \{\mathbf{x}_4, \mathbf{x}_6, \mathbf{x}_{11}\}$, $E_3 = \{\mathbf{x}_3, \mathbf{x}_8\}$, $E_4 = \{\mathbf{x}_2, \mathbf{x}_7\}$ and $E_5 = \{\mathbf{x}_1, \mathbf{x}_{10}\}$. Therefore, we obtain the weak order:

$$\mathbf{x}_5 \sim \mathbf{x}_9 \succ \mathbf{x}_4 \sim \mathbf{x}_6 \sim \mathbf{x}_{11} \succ \mathbf{x}_3 \sim \mathbf{x}_8 \succ \mathbf{x}_2 \sim \mathbf{x}_7 \succ \mathbf{x}_1 \sim \mathbf{x}_{10},$$
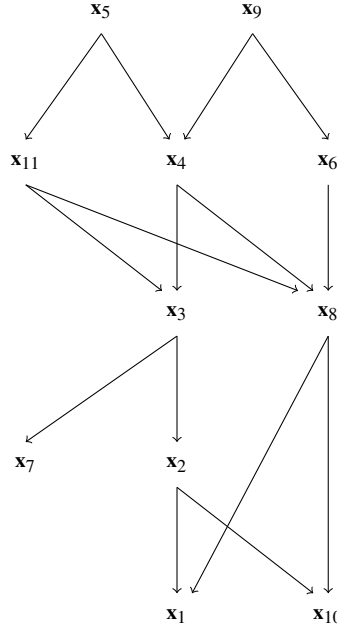
and the following multivariate OWA functions:

Fig. 2. Hasse diagram of the partially ordered set $(X, \leq_m)$ for the data points presented in Table 1.

$$f_{\min}(\mathbf{x}_1, \ldots, \mathbf{x}_{11}) = \frac{\mathbf{x}_1 + \mathbf{x}_{10}}{2} = (0.35, 0.2)^T \, ,$$

$$f_{\max}(\mathbf{x}_1, \ldots, \mathbf{x}_{11}) = \frac{\mathbf{x}_5 + \mathbf{x}_9}{2} = (0.75, 0.6)^T \, ,$$

$$f_{\mathrm{med}}(\mathbf{x}_1, \ldots, \mathbf{x}_{11}) = \frac{\mathbf{x}_3 + \mathbf{x}_8}{2} = (0.45, 0.4)^T \, . \quad \blacktriangleleft$$

These functions are symmetric and fulfill most internality properties, except internality within the points (which is not fulfilled even if the weighting vector contains a 1). Continuity in each variable, $\leq_2$-monotonicity and orthogonal equivariance (and thus affine equivariance) are not fulfilled. Interestingly, since the partial order $(X, \leq_2)$ is not affected by translations or changes of scale, translation equivariance, scale equivariance and uniform-scale equivariance are fulfilled.

The most important drawback of this approach concerns the behavior of the method for different arities. In particular, the addition and removal of a data point may greatly affect the final result. Another drawback is that it is less likely to obtain comparable data points as the number of dimensions increases.

4.4. Multivariate OWA functions based on an enumeration of all linear extensions of the partially ordered set

Another alternative initiated in [45] (see Definition 9) in the context of interval-valued aggregation starts with an enumeration of all possible linear extensions of $(X, \leq_m)$. Since the data points $\mathbf{x}_1, \ldots, \mathbf{x}_n$ may be ordered with respect to each linear extension, we can compute the corresponding multivariate OWA function as in Eq. (1) for each linear extension and ultimately average the results:

$$f_{\mathbf{w}}(\mathbf{x}_1, \ldots, \mathbf{x}_n) = \frac{1}{p} \sum_{i=1}^{n} \sum_{\ell=1}^{p} w_i \, \mathbf{x}_{(i), \prec_\ell} \, ,$$

where $\prec_1, \ldots, \prec_p$ denotes all linear extensions of $(X, \leq_m)$ and $\mathbf{x}_{(i), \prec_\ell}$ denotes the i-th largest data point among $\mathbf{x}_1, \ldots, \mathbf{x}_n$ according to the linear extension $\prec_\ell$ of $\leq_m$.

Example 11. Continue with Example 7 and the data points presented in Table 1. There exist 628 linear extensions of the poset $(X, \leq_m)$ illustrated in Fig. 2. The average results for the associated multivariate minimum, maximum and median are the following ones:

$$f_{\min}(\mathbf{x}_1, \ldots, \mathbf{x}_{11}) = (0.3199045, 0.2601911)^T ,$$
$$f_{\max}(\mathbf{x}_1, \ldots, \mathbf{x}_{11}) = (0.7471338, 0.6057325)^T ,$$
$$f_{\mathrm{med}}(\mathbf{x}_1, \ldots, \mathbf{x}_{11}) = (0.4745223, 0.4194268)^T . \quad \blacktriangleleft$$

Multivariate OWA functions of this type fulfill similar properties as those presented in Subsection 4.3. However, an important problem of these multivariate OWA functions is that they require to obtain all possible linear extensions of the partially ordered set $(X, \leq_m)$, which is in fact a difficult and computationally expensive problem [11] whenever n is moderately large (since the number of linear orders on a set of n elements is $n!$).

4.5. Multivariate OWA functions based on aggregation of rankings

A different approach is to exploit the ordinal information within each of the attributes separately, thus yielding m rankings (possibly with ties) of the data points. By means of a technique for the aggregation of rankings, a global ranking of the data points may thus be obtained and finally used for defining the multivariate OWA function. The literature on the aggregation of rankings is quite rich, having been addressed in many disciplines such as multi-objective optimization [26], medicine [36] and computer science [18]. However, there is little doubt that the discipline that has been the most interested in the aggregation of rankings is the field of social choice theory [1], where different methods have been proposed, always being assumed that all methods are symmetric with respect to the indices of the rankings (anonimity). Prominent examples of such methods for the aggregation of rankings studied in this field are the Borda count [10], the method of Kemeny [32] and the method of Schulze [51]. Here, we restrict our attention to one of the simplest and most natural methods – the Borda count – adapted to the setting in which the rankings may contain ties [9,19]. Firstly, for each component $j \in \{1, \ldots, m\}$, a ranking (weak order) $\succsim_j$ of the n data points is obtained by sorting from largest to smallest the values of the data points at the j-th component. Secondly, each data point $\mathbf{x}_i$ ($i \in \{1, \ldots, n\}$) is awarded a mark every time that a different data point $\mathbf{x}_\ell$ is ranked at a worse position in one of the rankings $\succsim_1, \ldots, \succsim_m$. In the event of several data points being placed at the same equivalence class in a certain $\succsim_j$, we follow the approach of Method I in [9] where each data point is awarded the mean of the marks they would receive if they appeared separately on $\succsim_j$. The sum of awarded marks for a data point $\mathbf{x}_i$, denoted by $B(\mathbf{x}_i)$, is used for obtaining a weak order on $\{\mathbf{x}_1, \ldots, \mathbf{x}_n\}$ (where the first ranked data point is the one with the highest sum of awarded marks).

Finally, for the definition of a multivariate OWA function, it suffices to consider the obtained weak order and apply Eq. (2).

Example 12. Continue with Example 7 and the data points presented in Table 1. The first component yields the ranking:

$$\mathbf{x}_7 \prec_1 \mathbf{x}_1 \prec_1 \mathbf{x}_2 \sim_1 \mathbf{x}_3 \sim_1 \mathbf{x}_{10} \prec_1 \mathbf{x}_8 \sim_1 \mathbf{x}_{11} \prec_1 \mathbf{x}_4 \prec_1 \mathbf{x}_5 \prec_1 \mathbf{x}_6 \sim_1 \mathbf{x}_9 ,$$

whereas the second component yields the ranking:

$$\mathbf{x}_{10} \prec_2 \mathbf{x}_1 \sim_2 \mathbf{x}_6 \sim_2 \mathbf{x}_8 \prec_2 \mathbf{x}_2 \prec_2 \mathbf{x}_3 \sim_2 \mathbf{x}_4 \sim_2 \mathbf{x}_7 \sim_2 \mathbf{x}_9 \prec_2 \mathbf{x}_5 \sim_2 \mathbf{x}_{11} .$$

The Borda count results in the following marks $B(\mathbf{x}_1) = 3$, $B(\mathbf{x}_2) = 7$, $B(\mathbf{x}_3) = 9.5$, $B(\mathbf{x}_4) = 13.5$, $B(\mathbf{x}_5) = 17.5$, $B(\mathbf{x}_6) = 11.5$, $B(\mathbf{x}_7) = 6.5$, $B(\mathbf{x}_8) = 7.5$, $B(\mathbf{x}_9) = 16$, $B(\mathbf{x}_{10}) = 3$ and $B(\mathbf{x}_{11}) = 15$. Therefore, the Borda ranking is the following:

$$\mathbf{x}_1 \sim \mathbf{x}_{10} \prec \mathbf{x}_7 \prec \mathbf{x}_2 \prec \mathbf{x}_8 \prec \mathbf{x}_3 \prec \mathbf{x}_6 \prec \mathbf{x}_4 \prec \mathbf{x}_{11} \prec \mathbf{x}_9 \prec \mathbf{x}_5 .$$

Thus, we obtain the following multivariate OWA functions:

Table 2

Data points $\mathbf{b}_1, \ldots, \mathbf{b}_{11}$ constructed from $\mathbf{x}_1, \ldots, \mathbf{x}_{11}$ by means of lattice operations.

i	1	2	3	4	5	6	7	8	9	10	11
$\mathbf{b}_i(1)$	0.8	0.8	0.7	0.6	0.5	0.5	0.4	0.4	0.4	0.3	0.2
$\mathbf{b}_i(2)$	0.7	0.7	0.5	0.5	0.5	0.5	0.4	0.3	0.3	0.3	0.1

$$f_{\min}(\mathbf{x}_1, \ldots, \mathbf{x}_{11}) = \frac{\mathbf{x}_1 + \mathbf{x}_{10}}{2} = (0.35, 0.2)^T \,,$$

$$f_{\max}(\mathbf{x}_1, \ldots, \mathbf{x}_{11}) = \mathbf{x}_5 = (0.7, 0.7)^T \,,$$

$$f_{\mathrm{med}}(\mathbf{x}_1, \ldots, \mathbf{x}_{11}) = \mathbf{x}_3 = (0.4, 0.5)^T \,. \quad \blacktriangleleft$$

The properties fulfilled by this type of multivariate OWA functions coincide with those fulfilled by the multivariate OWA functions presented in Subsection 4.3. Similar drawbacks concerning the behavior under varying arity (i.e., addition or substraction of data points) are noted. It should also be remarked that some of the methods for the aggregation of rankings become intractable as the number of data points increases [5]. Finally, some of the existing methods for the aggregation of rankings (e.g., the Kemeny method [32]) are not assured to result into a unique aggregated ranking, being unclear how to proceed in such case.

4.6. Multivariate OWA functions based on lattice operations

Lizasoain and Moreno [39] presented a natural extension of OWA functions to complete lattices [14]. Therefore, since the poset $(\mathbb{R}^m, \leq_m)$ actually is a complete lattice where the meet operation $\wedge$ is given by the componentwise minimum and the joint operation $\vee$ is given by the componentwise maximum, we may be interested in defining multivariate OWA functions based on lattice operations. For this purpose, as presented in Lemma 3.1 in [39], instead of the data point $\mathbf{x}_1, \ldots, \mathbf{x}_n$ we may consider the data points $\mathbf{b}_1, \ldots, \mathbf{b}_n$, defined as follows:

$$\mathbf{b}_1 = \vee \{\mathbf{x}_1, \ldots, \mathbf{x}_n\} \,,$$

$$\vdots$$

$$\mathbf{b}_k = \vee \left\{ \wedge \{\mathbf{x}_{i_1}, \ldots, \mathbf{x}_{i_k}\} \mid \{i_1, \ldots, i_k\} \subseteq \{1, \ldots, n\} \right\} \,,$$

$$\vdots$$

$$\mathbf{b}_n = \wedge \{\mathbf{x}_1, \ldots, \mathbf{x}_n\} \,.$$

From construction, it follows that $\mathbf{b}_n \leq_m \ldots \leq_m \mathbf{b}_1$. Based on these data points $\mathbf{b}_1, \ldots, \mathbf{b}_n$, a multivariate OWA function associated with the weighting vector $\mathbf{w}$ may be defined as follows:

$$f_{\mathbf{w}}(\mathbf{x}_1, \ldots, \mathbf{x}_n) = \sum_{i=1}^{n} w_i \, \mathbf{b}_i \,.$$

Example 13. Continue with Example 7 and the data points presented in Table 1. We obtain the data points $\mathbf{b}_1, \ldots, \mathbf{b}_{11}$ shown in Table 2.

Therefore, we obtain:

$$f_{\min}(\mathbf{x}_1, \ldots, \mathbf{x}_{11}) = (0.2, 0.1)^T \,,$$

$$f_{\max}(\mathbf{x}_1, \ldots, \mathbf{x}_{11}) = (0.8, 0.7)^T \,,$$

$$f_{\mathrm{med}}(\mathbf{x}_1, \ldots, \mathbf{x}_{11}) = (0.5, 0.5)^T \,. \quad \blacktriangleleft$$

One may note that the multivariate OWA functions in Examples 7 and 13 return the same results. This is not a coincidence since, as a result of $(\mathbb{R}^m, \leq_m)$ being a product lattice [14], the meet and joint operations of the lattice are componentwise extensions of the meet and joint operations on $(\mathbb{R}, \leq)$ (see Section 2.15 in [14]). Therefore, unless a

Table 3
Majority vectors of the data points in Table 1.

i	1	2	3	4	5	6	7	8	9	10	11
$\mathbf{r}_i(1)$	0.3	0.4	0.4	0.5	0.7	0.3	0.2	0.3	0.5	0.1	0.5
$\mathbf{r}_i(2)$	0.3	0.4	0.5	0.6	0.7	0.8	0.5	0.5	0.8	0.4	0.7

different lattice structure is considered (for instance, considering a linear extension of $\leq_m$), this type of multivariate OWA functions based on lattice operations essentially are multivariate OWA functions based on a componentwise extension of a univariate OWA function, which have been already discussed in Subsection 4.1.

4.7. Multivariate OWA functions based on orders on distributions

Under the assumption that there exists a common scale on which all components are measured, it is reasonable to think of each of the data points $\mathbf{x}_i$ as a distribution (or multiset) of m real values rather than as a data point. In this case, one could resort to techniques used for ordering distributions such as the method of majority judgment [2,3] arising in the field of social choice theory or to stochastic orders [8] arising in the field of statistics.

On the one hand, majority judgment [2,3] proceeds as follows. For each $\mathbf{x}_i$, compute the majority vector $\mathbf{r}_i$, which is the m-dimensional vector defined as $\mathbf{r}_i = (\mathbf{x}_{i(\frac{m+1}{2})}, \mathbf{x}_{i(\frac{m-1}{2})}, \mathbf{x}_{i(\frac{m+3}{2})}, \ldots, \mathbf{x}_{i(1)}, \mathbf{x}_{i(m)})$ if m is odd and as $\mathbf{r}_i = (\mathbf{x}_{i(\frac{m}{2})}, \mathbf{x}_{i(\frac{m+2}{2})}, \mathbf{x}_{i(\frac{m-2}{2})}, \ldots, \mathbf{x}_{i(1)}, \mathbf{x}_{i(m)})$ if m is even, where $\mathbf{x}_{i(\ell)}$ represents the ℓ-th largest value among the components of $\mathbf{x}_i$. Finally, the data points $\mathbf{x}_1, \ldots, \mathbf{x}_n$ are weakly ordered considering the first lexicographical order (from the first to the last component) according to their majority vectors. Finally, Eq. (2) may be used once a weak order of the data points is obtained.

On the other hand, we discuss some prominent cases of stochastic orders used for comparing probability distributions. Technically, in order to use stochastic orders in our setting, we will associate with each $\mathbf{x}_i$ a random variable X_i such that $P(X_i = \mathbf{x}_i(j)) = 1/m$ for any $j \in \{1, \ldots, m\}$ and the goal becomes to order all random variables $X_1, \ldots, X_n$ by means of a certain stochastic order. The easiest example of stochastic order is that of expected utility [43], where we compute for each X_i its expected utility $E(u(X_i))$ for a certain increasing function $u : \mathbb{R} \to \mathbb{R}$ and sort the random variables according to this expected utility. This stochastic order yields a weak order on the set of random variables, thus allowing to define a multivariate OWA function. Another prominent stochastic order is that of stochastic dominance [34,35], yet it results in a partial order rather than a weak order. More recent stochastic orders include statistical preference [16] and probabilistic preference [42]. Again, Eq. (2) may be used once a weak order of the data points is obtained.

Example 14. Continue with Example 7 and the data points presented in Table 1. On the one hand, the majority vectors $\mathbf{r}_1, \ldots, \mathbf{r}_{11}$ of $\mathbf{x}_1, \ldots, \mathbf{x}_{11}$ are shown in Table 3.

According to the first lexicographical order, we obtain:

$$\mathbf{r}_{10} \prec_{\text{Lex}_1} \mathbf{r}_7 \prec_{\text{Lex}_1} \mathbf{r}_1 \prec_{\text{Lex}_1} \mathbf{r}_8 \prec_{\text{Lex}_1} \mathbf{r}_6 \prec_{\text{Lex}_1} \mathbf{r}_2 \prec_{\text{Lex}_1} \mathbf{r}_3 \prec_{\text{Lex}_1} \mathbf{r}_4 \prec_{\text{Lex}_1} \mathbf{r}_{11} \prec_{\text{Lex}_1} \mathbf{r}_9 \prec_{\text{Lex}_1} \mathbf{r}_5 \,.$$

Therefore, multivariate OWA functions based on majority judgment would result in:

$$f_{\min}(\mathbf{x}_1, \ldots, \mathbf{x}_{11}) = \mathbf{x}_{10} = (0.4, 0.1)^T \,,$$
$$f_{\max}(\mathbf{x}_1, \ldots, \mathbf{x}_{11}) = \mathbf{x}_5 = (0.7, 0.7)^T \,,$$
$$f_{\text{med}}(\mathbf{x}_1, \ldots, \mathbf{x}_{11}) = \mathbf{x}_2 = (0.4, 0.4)^T \,.$$

On the other hand, for expected utility considering the identity as the considered utility function u, we obtain $E(X_1) = 0.3$, $E(X_2) = 0.4$, $E(X_3) = 0.45$, $E(X_4) = 0.55$, $E(X_5) = 0.7$, $E(X_6) = 0.55$, $E(X_7) = 0.35$, $E(X_8) = 0.4$, $E(X_9) = 0.65$, $E(X_{10}) = 0.25$ and $E(X_{11}) = 0.6$. We can thus order those values and obtain:

$$\mathbf{x}_{10} \prec_{\text{EU}} \mathbf{x}_1 \prec_{\text{EU}} \mathbf{x}_7 \prec_{\text{EU}} \mathbf{x}_2 \sim_{\text{EU}} \mathbf{x}_8 \prec_{\text{EU}} \mathbf{x}_3 \prec_{\text{EU}} \mathbf{x}_4 \sim_{\text{EU}} \mathbf{x}_6 \prec_{\text{EU}} \mathbf{x}_{11} \prec_{\text{EU}} \mathbf{x}_9 \prec_{\text{EU}} \mathbf{x}_5 \,.$$

Therefore, multivariate OWA functions based on the comparison of expected (identity) utility would result in:

$$f_{\min}(\mathbf{x}_1, \ldots, \mathbf{x}_{11}) = \mathbf{x}_{10} = (0.4, 0.1)^T \,,$$
$$f_{\max}(\mathbf{x}_1, \ldots, \mathbf{x}_{11}) = \mathbf{x}_5 = (0.7, 0.7)^T \,,$$
$$f_{\mathrm{med}}(\mathbf{x}_1, \ldots, \mathbf{x}_{11}) = \mathbf{x}_3 = (0.4, 0.5)^T \,. \quad \blacktriangleleft$$

It is important to note that these methods will never produce a linear order on $\mathbb{R}^m$ since permutations of the components of any data point will all be in the same equivalence class. Therefore, internality within the points is not guaranteed for associated multivariate OWA functions, even if the weighting vector contains a 1. In fact, no natural property among $\leq_m$-monotonicity, continuity in each variable or equivariance under different types of transformations is fulfilled. Nevertheless, the reader should not totally discard multivariate OWA functions based on orders on distributions since their use is strongly encouraged in case all values $\mathbf{x}_i(j)$ are measured on the same scale and are thus perfectly comparable with each other.

4.8. Multivariate OWA functions based on data depth

There exists a quite prominent school of thought in multivariate statistics encouraging us to abandon order-based thinking for multivariate data, as one could see, e.g., in Kendall [33]: "order properties [. . .] exist only in one dimension". It is under this school of thought that Tukey [53] initiated a field of research on statistical depth functions [58] that has become very popular since. Intuitively, statistical depth functions associate with each data point a measure of how 'deep' (i.e., central) this data point is with respect to a set of data points (or with respect to a certain probability distribution). A statistical depth function thus induces a weak order of the data points from deepest to least deep, thus allowing to introduce a multivariate OWA function. The reader is referred to [38] for an early reference to this type of functions, therein called DL-statistics. It is to be noted that, unlike all other presented multivariate OWA functions, depth-based multivariate OWA functions do not allow to distinguish a 'minimum' from a 'maximum' and treat them both equally as 'extremes'. For this reason, one may think of multivariate OWA functions based on data depth as a generalization of centered OWA functions (i.e., OWA functions with a symmetric weighting vector) to the multivariate setting.

There exist many different proposals of statistical depth functions. Some of the most popular ones are due to Tukey [53] (also referred to as 'halfspace depth'), to Oja [44] (also referred to as 'simplex depth' or 'Oja depth'), to Liu [37] (also referred to as 'simplicial depth') and to Rousseeuw and Hubert [50] (also referred to as 'regression depth'). Here, we only present the first and probably most prominent one, Tukey's halfspace depth function, defined for any $\mathbf{x} \in \mathbb{R}^m$ as:

$$HD(\mathbf{x}) = \inf\{P(H) \mid H \text{ is a closed halfspace such that } \mathbf{x} \in H\}$$

where P is a probability measure on $\mathbb{R}^m$, in this case given by $P(H) = \frac{1}{n}\#\{i \in \{1, \ldots, n\} \mid \mathbf{x}_i \in H\}$.

According to any depth function, all data points $\mathbf{x}_1, \ldots, \mathbf{x}_n$ can be ordered from deepest to least deep and obtain a multivariate OWA function, as in Eq. (2).

In this case, the weighting vector $\mathbf{w}$ should change its meaning. More specifically, $\mathbf{w}_1$ should now represent the weight for the deepest data point and $\mathbf{w}_n$ should represent the weight for the least deep data point. Typically in this case $\mathbf{w}$ will always be decreasing and the associated multivariate OWA function will be used for representing centers/medians rather than extremes. This approach was already considered in [38] for the definition of DL-statistics (see also [41] for the definition of multivariate trimmed means based on the halfspace depth). It should also be noted that in the context of multivariate medians [52] it is actually more usual to consider as median the center of gravity of the set of deepest data points rather than the data point (or average of the data points) maximizing depth within the given n data points $\mathbf{x}_1, \ldots, \mathbf{x}_n$.

Example 15. Continue with Example 7 and the data points presented in Table 1. According to the halfspace depth, $\mathbb{R}^m$ can be divided into six zones depending on whether the associated halfspace depth equals 0, $\frac{1}{11}$, $\frac{2}{11}$, $\frac{3}{11}$, $\frac{4}{11}$ or $\frac{5}{11}$ (which is the maximum possible depth given the data points presented in Table 1). All these zones are illustrated in Fig. 3.

It is concluded that $HD(\mathbf{x}_1) = \frac{2}{11}$, $HD(\mathbf{x}_2) = \frac{4}{11}$, $HD(\mathbf{x}_3) = \frac{3}{11}$, $HD(\mathbf{x}_4) = \frac{4}{11}$, $HD(\mathbf{x}_5) = \frac{1}{11}$, $HD(\mathbf{x}_6) = \frac{1}{11}$, $HD(\mathbf{x}_7) = \frac{1}{11}$, $HD(\mathbf{x}_8) = \frac{3}{11}$, $HD(\mathbf{x}_9) = \frac{1}{11}$, $HD(\mathbf{x}_{10}) = \frac{1}{11}$ and $HD(\mathbf{x}_{11}) = \frac{1}{11}$, resulting in the following order of the data points from deepest to least deep:
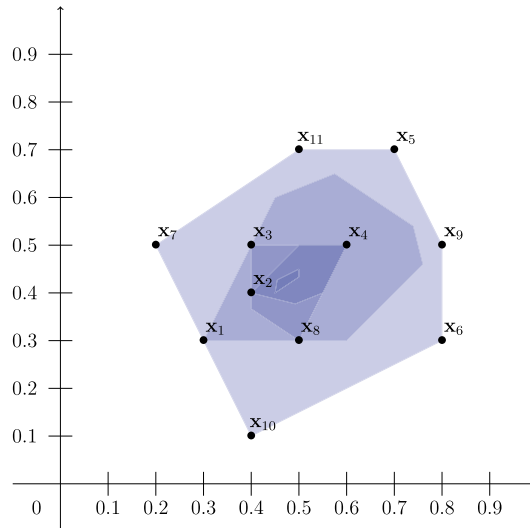
Fig. 3. Illustration of five contour levels of the halfspace depth for the list of data points of Table 1.

$$\mathbf{x}_2 \sim \mathbf{x}_4 \succ \mathbf{x}_3 \sim \mathbf{x}_8 \succ \mathbf{x}_1 \succ \mathbf{x}_5 \sim \mathbf{x}_6 \sim \mathbf{x}_7 \sim \mathbf{x}_9 \sim \mathbf{x}_{10} \sim \mathbf{x}_{11} .$$

The order above leads to the following multivariate OWA function:

$$f_{\mathrm{med}}(\mathbf{x}_1, \ldots, \mathbf{x}_{11}) = \frac{\mathbf{x}_2 + \mathbf{x}_4}{2} = (0.5, 0.45)^T .$$

As mentioned above, there is not much interest in the notion of minimum or maximum when dealing with depth-based multivariate OWA functions. ◄

Depth-based multivariate OWA functions are symmetric and satisfy most internality properties (except internality within the points). Unsurprisingly, since the product order does not interact well with affine transformations [25] and most statistical depth functions are affine equivariant [58], depth-based multivariate OWA functions are not $\leq_m$-monotone. Interestingly, the lack of $\leq_m$-monotonicity in case the associated statistical depth function is affine equivariant is compensated by the fulfillment of affine equivariance by the depth-based multivariate OWA function. This property is regarded as a golden standard in the field of multivariate statistics and is not fulfilled by any other class of multivariate OWA functions presented in this section.

5. Discussion on properties

In this section, we provide a discussion on the properties fulfilled by the different multivariate OWA functions presented in this work.

Internality within the points (IWP) is not fulfilled in general by any of the multivariate OWA functions since, for instance, the centroid (associated with the weighting vector $\mathbf{w} = (\frac{1}{n}, \ldots, \frac{1}{n})$) is within each of the classes and is not internal within the points. However, if the weighting vector is formed by a one and $n - 1$ zeros, multivariate OWA functions based on a linearization of the product order are internal within the points since they actually resort to symmetrization of a projection to one of the components. It is easy to find counterexamples for all other classes in the two-dimensional case by considering, for instance, $n = 2$, the multivariate OWA function associated with the weighting vector $\mathbf{w} = (1, 0)^T$ ('the maximum'), and $\mathbf{x}_1 = (1, 0)^T$ and $\mathbf{x}_2 = (0, 1)^T$. This very same example illustrates why multivariate OWA functions based on a componentwise extension of a univariate OWA function and multivariate OWA functions based on lattice operations are not internal within the convex hull (ICH), yet they still are internal within the bounding box (IBB) and idempotent (ID) (see, e.g., [24]). All other presented classes of multivariate OWA functions already are internal within the convex hull (ICH) – and therefore internal within the bounding box (IBB) and idempotent (ID) – since they are the result of a convex combination of the inputs.

Multivariate OWA functions based on a componentwise extension of a univariate OWA function (and, thus, multivariate OWA functions based on lattice operations) are proven to be $\leq_m$-monotone (M) in [24]. A proof showing that the order-based multivariate median is not $\leq_m$-monotone if $n > 2$ and, therefore, multivariate OWA functions based on a linearization of the product order are not $\leq_m$-monotone is given in [47]. This proof serves as a starting point for the construction of counterexamples showing that all other multivariate OWA functions are not $\leq_m$-monotone. Consider, for instance, $n = 3$ and $m = 2$, the multivariate OWA function associated with the weighting vector $\mathbf{w} = (0, 0, 1)^T$ ('the minimum'), and $\mathbf{x}_1 = (1, 0)^T$, $\mathbf{x}_2 = (0, 1)^T$ and $\mathbf{x}_3 = (1, 1)^T$. It follows that for the multivariate OWA functions in Subsections 4.3, 4.4, 4.5 and 4.7, the associated multivariate OWA function will result in the value $A_{\mathbf{w}}(\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3) = (0.5, 0.5)^T$. If one substitutes $\mathbf{x}_2$ by $\mathbf{x}_3$, it holds that $A_{\mathbf{w}}(\mathbf{x}_1, \mathbf{x}_3, \mathbf{x}_3) = (1, 0)^T$, thus $A_{\mathbf{w}}$ cannot be $\leq_m$-monotone since $\mathbf{x}_2 \leq_2 \mathbf{x}_3$ yet it is not true that $A_{\mathbf{w}}(\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3) \leq_2 A_{\mathbf{w}}(\mathbf{x}_1, \mathbf{x}_3, \mathbf{x}_3)$. Finally, for multivariate OWA functions based on data depth, the result follows from [25].

Symmetry (S) follows for all presented multivariate OWA functions since all presented methods actually are the result of a symmetrization process in which data points are initially reordered according to different criteria.

Continuity in each variable (C) follows naturally for multivariate OWA functions based on a componentwise extension of a univariate OWA function (and, thus, multivariate OWA functions based on lattice operations). A proof showing that the order-based multivariate median is not continuous in each variable and, therefore, multivariate OWA functions based on a linearization of the product order are not necessarily continuous in each variable is given in [47]. Similarly, it is easy to find counterexamples for all other classes by considering, for instance, $n = 2$, the multivariate OWA function associated with the weighting vector $\mathbf{w} = (1, 0)^T$ ('the maximum'), and $\mathbf{x}_1 = (0, 0)^T$ and $\mathbf{x}_2 = (0, 1)^T$. It follows that $\mathbf{x}_1 \leq_2 \mathbf{x}_2$ and, actually, $A_{\mathbf{w}}(\mathbf{x}_1, \mathbf{x}_2) = \mathbf{x}_2$ for all multivariate OWA functions presented in Subsections 4.3, 4.4, 4.5 and 4.7, however different results are obtained when substituting $\mathbf{x}_2$ by $\mathbf{x}_2' = (-\varepsilon, 1)^T$ with $\varepsilon > 0$. For multivariate OWA functions based on data depth, one can just substitute $\mathbf{x}_1$ in Example 15 by $\mathbf{x}_1' = (0.3 - \varepsilon, 0.3 - \varepsilon)^T$ with $\varepsilon > 0$, thus changing its depth from $HD(\mathbf{x}_1) = \frac{2}{11}$ to $HD(\mathbf{x}_1') = \frac{1}{11}$.

Affine equivariance (EA) is known not to be satisfied by multivariate OWA functions based on a componentwise extension of a univariate OWA function [49] (and, thus, multivariate OWA functions based on lattice operations). The fact that the order-based multivariate median is not affine equivariant and, therefore, multivariate OWA functions based on a linearization of the product order are not necessarily affine equivariant is proven in [47]. Similarly, it is easy to find counterexamples for all other classes by considering, for instance, $n = 2$, the multivariate OWA function associated with the weighting vector $\mathbf{w} = (1, 0)^T$ ('the maximum'), and $\mathbf{x}_1 = (0, 0)^T$ and $\mathbf{x}_2 = (1, 0)^T$. It follows that $A_{\mathbf{w}}(\mathbf{x}_1, \mathbf{x}_2) = \mathbf{x}_2$ for all multivariate OWA functions presented in Subsections 4.3, 4.4, 4.5 and 4.7. Consider $\mathbf{O}$ the rotation matrix associated with a 90° clockwise rotation. It follows that $A_{\mathbf{w}}(\mathbf{O}\mathbf{x}_1, \mathbf{O}\mathbf{x}_2) = \mathbf{O}\mathbf{x}_1$ for all multivariate OWA functions presented in Subsections 4.3, 4.4, 4.5 and 4.7, which differs from $\mathbf{O}\mathbf{x}_2$. On the contrary, multivariate OWA functions based on data depth inherit the affine equivariance from the affine equivariance of its associated statistical depth function [58]. Note that the same applies to orthogonal equivariance (EO) for all discussed multivariate OWA functions. However, since a translation or a rescaling does not affect the order between the components, the translation equivariance (ET), scale equivariance (ES) and uniform-scale equivariance (EUS) actually follow for all multivariate OWA functions presented in this paper with the exception of those based on orders on distributions. Important attention should be devoted to multivariate OWA functions based on a linearization of the product order since the chosen linear extension of $(\mathbb{R}^m, \leq_m)$ should not be affected by translation or rescalings in order to guarantee the corresponding equivariance property (see [47]).

A summary of the main properties fulfilled by the different multivariate OWA functions presented in this work is shown in Table 4. It should be remarked that the centroid is recovered by all discussed multivariate OWA functions when considering the weighting vector $\mathbf{w} = (\frac{1}{n}, \ldots, \frac{1}{n})$. Interestingly, the centroid fulfills all properties listed in Table 4 with the exception of internality within the points (IWP).

As a final discussion on properties of multivariate OWA functions, it is recalled that several alternatives to $\leq_m$-monotonicity for multivariate functions were explored in [48]. The property of (SP)-monotonicity closely relates to weighted centroids and is not fulfilled by any multivariate OWA function different than the centroid. The properties of (MP)-monotonicity and (MC)-monotonicity are weaker than translation equivariance and, therefore, are fulfilled by all multivariate OWA functions discussed in this paper with the exception of those based on an order on distributions. As a counterexample showing that multivariate OWA functions based on an order on distributions are not necessarily (MC)-monotone (and therefore (MP)-monotone or translation equivariant), consider $n = 2$, $\mathbf{w} = (1, 0)^T$ ('the maximum'), and $\mathbf{x}_1 = (0, 1)^T$ and $\mathbf{x}_2 = (1, 0)^T$. For instance, for the multivariate OWA function based on majority judgment,

Table 4

Properties fulfilled by the different multivariate OWA functions presented in this work. For the abbreviations at the header of the columns, the reader is referred to the abbreviations of the properties presented in Definition 4. $*_1$: If the weighting vector contains a 1; $*_2$: if the linear extension is generated by m linearly independent weighted arithmetic means; $*_3$: if the linear extension is generated by the m projections; $*_4$ if the order for distributions is invariant under uniform changes of scale (as it is the case with majority judgment, stochastic dominance and expected utility for an increasing utility). For multivariate OWA functions based on lattice operations the product lattice on $\mathbb{R}^m$ is considered, whereas for depth-based multivariate OWA functions it is assumed that the statistical depth function is affine equivariant.

Type	IWP	ICH	IBB	ID	M	S	C	EA	EO	ET	ES	EUS
Componentwise (Subsection 4.1)	×	×	✓	✓	✓	✓	✓	×	×	✓	✓	✓
Linear extension of $(\mathbb{R}^m, \leq_m)$ (Subsection 4.2)	$*_1$	✓	✓	✓	×	✓	×	×	×	$*_2$	$*_3$	$*_2$
Extension of the poset $(X, \leq_m)$ (Subsection 4.3)	×	✓	✓	✓	×	✓	×	×	×	✓	✓	✓
Enumeration of linear extensions (Subsection 4.4)	×	✓	✓	✓	×	✓	×	×	×	✓	✓	✓
Aggregation of rankings (Subsection 4.5)	×	✓	✓	✓	×	✓	×	×	×	✓	✓	✓
Lattice operations (Subsection 4.6)	×	×	✓	✓	✓	✓	✓	×	×	✓	✓	✓
Order on distributions (Subsection 4.7)	×	✓	✓	✓	×	✓	×	×	×	×	×	$*_4$
Depth-based (Subsection 4.8)	×	✓	✓	✓	×	✓	×	✓	✓	✓	✓	✓

Table 5

Monotonicity properties fulfilled by the different multivariate OWA functions presented in this work. For the abbreviations at the header of the columns, the reader is referred to the taxonomy introduced in [48]. $*$: if the linear extension is generated by m linearly independent weighted arithmetic means. For multivariate OWA functions based on lattice operations the product lattice on $\mathbb{R}^m$ is considered, whereas for depth-based multivariate OWA functions it is assumed that the statistical depth function is affine equivariant.

Type	M	SP	SC	MP	MC
Componentwise (Subsection 4.1)	✓	×	✓	✓	✓
Linear extension of $(\mathbb{R}^m, \leq_m)$ (Subsection 4.2)	×	×	×	$*$	$*$
Extension of the poset $(X, \leq_m)$ (Subsection 4.3)	×	×	×	✓	✓
Enumeration of linear extensions (Subsection 4.4)	×	×	×	✓	✓
Aggregation of rankings (Subsection 4.5)	×	×	×	✓	✓
Lattice operations (Subsection 4.6)	✓	×	✓	✓	✓
Order on distributions (Subsection 4.7)	×	×	×	×	×
Depth-based (Subsection 4.8)	×	×	×	✓	✓

it follows that $A_{\mathbf{w}}(\mathbf{x}_1, \mathbf{x}_2) = (0.5, 0.5)^T$, but for $\mathbf{t} = (0.1, 0)^T$ it holds that $A_{\mathbf{w}}(\mathbf{x}_1 + \mathbf{t}, \mathbf{x}_2 + \mathbf{t}) = (0.1, 1)^T$. Finally, the property of (SC)-monotonicity is fulfilled by multivariate OWA functions based on a componentwise extension of a univariate OWA function (and, thus, multivariate OWA functions based on lattice operations) since they are componentwise extensions of a monotone univariate function (see [48]). It is already known that multivariate OWA functions based on a linear extension of $(\mathbb{R}^m, \leq_m)$ are not necessarily (SC)-monotone (see [47]) and it follows from the already-proven fact that multivariate OWA functions based on an order on distributions are not necessarily (MC)-monotone that multivariate OWA functions based on an order on distributions are not necessarily (SC)-monotone. For multivariate OWA functions based on an extension of the partially ordered set (e.g., cardinal method), consider $n = 3$, $\mathbf{w} = (0, 0, 1)^T$ ('the minimum'), and $\mathbf{x}_1 = (0, 1)^T$, $\mathbf{x}_2 = (0.2, 0.5)^T$ and $\mathbf{x}_3 = (1, 0)^T$. It follows that $A_{\mathbf{w}}(\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3) = (0.4, 0.5)^T$, but for $\mathbf{t} = (0.4, 0)^T$ it holds that $A_{\mathbf{w}}(\mathbf{x}_1 + \mathbf{t}, \mathbf{x}_2, \mathbf{x}_3) = (0.2, 0.5)^T$. For multivariate OWA functions based on an enumeration of all linear extensions of the partially ordered set, consider $n = 3$, $\mathbf{w} = (0, 0, 1)^T$ ('the minimum'), and $\mathbf{x}_1 = (2.4, 1.6)^T$, $\mathbf{x}_2 = (0, 1.9)^T$ and $\mathbf{x}_3 = (2.8, 2.1)^T$. It follows that $A_{\mathbf{w}}(\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3) = (1.2, 1.75)^T$, but for $\mathbf{t} = (0.6, 0)^T$ it holds that $A_{\mathbf{w}}(\mathbf{x}_1 + \mathbf{t}, \mathbf{x}_2, \mathbf{x}_3) = (1, 1.8)^T$. The previous counterexample is also valid for multivariate OWA functions based on aggregation of rankings (e.g., Borda count), now obtaining $A_{\mathbf{w}}(\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3) = (1.2, 1.75)^T$ and $A_{\mathbf{w}}(\mathbf{x}_1 + \mathbf{t}, \mathbf{x}_2, \mathbf{x}_3) = (0, 1.9)^T$. Finally, for multivariate OWA functions based on data depth, consider $n = 4$, $\mathbf{w} = (1, 0, 0, 0)^T$ ('the median'), and $\mathbf{x}_1 = (1, 3.1)^T$, $\mathbf{x}_2 = (0, 3)^T$, $\mathbf{x}_3 = (0.8, 0.2)^T$ and $\mathbf{x}_4 = (1.1, 3.9)^T$. It follows that $A_{\mathbf{w}}(\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3, \mathbf{x}_4) = (1, 3.1)^T$, but for $\mathbf{t} = (0.5, 0)^T$ it holds that $A_{\mathbf{w}}(\mathbf{x}_1 + \mathbf{t}, \mathbf{x}_2, \mathbf{x}_3, \mathbf{x}_4) = (0.85, 2.55)^T$. A summary of the monotonicity properties fulfilled by the different multivariate OWA functions presented in this work is shown in Table 5.

6. Conclusions

In this paper, several classes of multivariate OWA functions have been presented and the properties fulfilled by the multivariate OWA functions in each of these classes have been compared. As a concluding remark, it is noted that there does not exist an absolutely best multivariate OWA function and one could always find a reason to use one class of multivariate OWA function over another one. In the following, some reasons for selecting each of the different classes are presented:

- Multivariate OWA functions based on a componentwise extension of a univariate OWA function (discussed in Subsection 4.1) are a simple alternative fulfilling $\leq_m$-monotonicity and make the most sense if all components are actually independent.
- Multivariate OWA functions based on a linearization of the product order (discussed in Subsection 4.2) should be used if there exist an underlying univariate notion allowing to order individuals from smallest to largest (see, e.g., the conclusions of [47] for one such case).
- Multivariate OWA functions based on an extension of the partially ordered set (discussed in Subsection 4.3) are a similar alternative, where the order is searched for at the level of the data points rather than at the level of $(\mathbb{R}^m, \leq_m)$, thus taking into account the frequency with which the data points appear.
- Multivariate OWA functions based on an enumeration of all linear extensions of the partially ordered set (discussed in Subsection 4.4) are a compromise solution in which, instead of selecting a single linear extension, all potential cases are averaged.
- Multivariate OWA functions based on aggregation of rankings (discussed in Subsection 4.5) are another similar approach more in line with the rank-based point of view adopted by the field of non-parametric statistics.
- If one is not actually dealing with $\mathbb{R}^m$ but a different complete lattice, the user may opt for a multivariate OWA function based on lattice operations (discussed in Subsection 4.6).
- Under the assumption that there exists a common scale in which all components are measured, multivariate OWA functions based on orders on distributions (discussed in Subsection 4.7) seem to be among the most natural choices.
- If the idea of order is to be abandoned and a good behavior with respect to geometric transformations is pursued, the user should resort to multivariate OWA functions based on data depth (discussed in Subsection 4.8).

In summary, at least eight different classes of multivariate OWA functions are available in the literature. The choice of one or another class is left as a personal decision for the user, who should typically pay attention to the characteristics of the problem of interest.

Declaration of competing interest

The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

Data availability

No data was used for the research described in the article.

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