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Finite and Infinite Hypergeometric Sums Involving the Digamma Function

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Abstract: We calculate some finite and infinite sums containing the digamma function in closed form. For this purpose, we differentiate selected reduction formulas of the hypergeometric function with respect to the parameters applying some derivative formulas of the Pochhammer symbol. Additionally, we compare two different differentiation formulas of the generalized hypergeometric function with respect to the parameters. For some particular cases, we recover some results found in the literature. Finally, all the results have been numerically checked.

Keywords: digamma function; differentiation with respect to parameters; closed-form sums calculation

MSC: 33B15; 33C05

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1. Introduction

A large number of finite sums and series involving the digamma function have been compiled by Hansen [1] and more recently by Brychov [2]. Some authors have contributed to enhance this compilation, such as Doelder [3], who calculated series containing $\psi(x) - \psi(y)$ and $(\psi(x) - \psi(y))^2$ for certain values of x and y; and Coffey [4], who considered summations over digamma and polygamma functions. More recently, we found Miller [5], who used reduction formulas of the Kampé de Fériet function; and Cvijović [6], who used the derivative of the Pochhammer symbol. Sums involving the digamma function occur in the expressions of the derivatives of the Mittag–Leffler function and the Wright function with respect to parameters [7,8]. Additionally, they occur in the derivation of asymptotic expansions for Mellin–Barnes integrals [9] or in the evaluation of Feynman amplitudes in quantum field theory [4].

The aim of this paper is the derivation of several apparently new results by using also the derivative of the Pochhammer symbol to known reduction formulas of the hypergeometric function. Nevertheless, for the last result given in this paper, we use another approach. For this purpose, we compare the expression of the first derivative of the generalized hypergeometric function with respect to the parameters given in [10] to the one given in [11]. As a consitency test, for many particular values of the results obtained, we recover expressions given in the literature. In adittion, we have checked all the derived expressions with the aid of MATHEMATICA since we sometimes found some erratums in the literature.

This paper is organized as follows. In Section 2, we present some basic properties of the Pochhammer symbol and the beta and the digamma functions. In addition, we set the notation we use throughout the paper. In Sections 3 and 4, we derive some results for finite and infinite sums, respectively, involving the digamma function. Finally, we collect our conclusions in Section 5.

2. Preliminaries

The Pochhamer symbol is defined as [12] (Equation 18:12:1):

$$(x)_n = \frac{\Gamma(x+n)}{\Gamma(x)}, \tag{1}$$

where $\Gamma(x)$ denotes the gamma function. Additionally, the beta function, defined as [13] (Equation 1.5.3):

$$\begin{aligned} \mathrm{B}(x,y) &= \int_0^1 t^{x-1}(1-t)^{y-1}dt, \\ & \quad \operatorname{Re} x > 0,\ \operatorname{Re} y > 0, \end{aligned}$$

satisfies the property [13] (Equation 1.5.5):

$$\mathrm{B}(x,y) = \frac{\Gamma(x)\Gamma(y)}{\Gamma(x+y)}. \tag{2}$$

For $0 \le z \le 1$, the incomplete beta function is defined as [12] (Equation 58:3:1):

$$\mathrm{B}_z(x,y) = \int_0^z t^{x-1}(1-t)^{y-1}dt.$$

Next, we state some properties of the Pochhammer symbol, i.e., the reflection formula [12] (Equation 18:5:1),

$$(-x)_n = (-1)^n(x-n+1)_n, \tag{3}$$

the properties [12] (Equations 18:5:7 & 2:12:3),

$$(x)_{n+1} = x(x+1)_n, \tag{4}$$

$$\left(\frac{1}{2}\right)_n = \frac{(2n)!}{4^n n!}, \tag{5}$$

and the differentiation of the Pochhammer symbol [12] (Equation 18:10:1):

$$\frac{d}{dx}(x)_n = (x)_n[\psi(x+n) - \psi(x)], \tag{6}$$

thus:

$$\frac{d}{dx}\left[\frac{1}{(x)_n}\right] = \frac{1}{(x)_n}[\psi(x) - \psi(x+n)], \tag{7}$$

where $\psi(x)$ denotes the digamma function [12] (Chapter 44):

$$\psi(x) = \frac{\Gamma'(x)}{\Gamma(x)},$$

with the following properties [13] (Equations 1.3.3–9):

$$\psi(z+1) = \frac{1}{z} + \psi(z), \tag{8}$$

$$\psi(1-z) - \psi(z) = \pi\cot(\pi z), \tag{9}$$

$$\psi(z) + \psi\left(z+\frac{1}{2}\right) + 2\log 2 = 2\psi(2z), \tag{10}$$

$$\psi(1) = -\gamma, \tag{11}$$

$$\psi\left(\frac{1}{2}\right) = -\gamma - \log 4, \tag{12}$$

$$\psi(n+1) = -\gamma + H_n, \tag{13}$$

$$\psi\left(n+\frac{1}{2}\right) = -\gamma - \log 4 + 2H_{2n} - H_n, \tag{14}$$

where

$$H_n = \sum_{k=1}^{n} \frac{1}{k}$$

is the n-th harmonic number.

Throughout the paper, we adopt the notation [14] (p. 797):

$$\beta(z) = \frac{1}{2}\left[\psi\left(\frac{z+1}{2}\right) - \psi\left(\frac{z}{2}\right)\right]. \tag{15}$$

Additionally, ${}_pF_q(z)$ denotes the generalized hypergeometric function, usually defined by means of the hypergeometric series [15] (Section 16.2):

$${}_pF_q\left(\begin{array}{c}(a_p)\\(b_q)\end{array}\middle| z\right) = {}_pF_q\left(\begin{array}{c}a_1,\ldots,a_p\\b_1,\ldots b_q\end{array}\middle| z\right) = \sum_{k=0}^{\infty} \frac{(a_1)_k \cdots (a_p)_k}{(b_1)_k \cdots (b_q)_k} \frac{z^k}{k!},$$

whenever this series converge and elsewhere by analytic continuation. Finally, we use the notation:

$$((a_p))_k = (a_1)_k \cdots (a_p)_k.$$

3. Finite Sums Involving Digamma Function

Theorem 1. *The following summation formula holds true:*

$$\begin{aligned} &\sum_{k=0}^{n} (-1)^k \binom{n}{k} \frac{(a)_k}{(c)_k} \psi(a+k) \\ = &\frac{(c-a)_n}{(c)_n}[\psi(a) - \psi(c-a+n) + \psi(c-a)]. \end{aligned} \tag{16}$$

Proof. The Chu–Vandermonde summation formula [16] (Corollary 2.2.3) is given by:

$${}_2F_1\left(\begin{array}{c}-n,a\\c\end{array}\middle| 1\right) = \frac{(c-a)_n}{(c)_n}, \quad n \in \mathbb{N}. \tag{17}$$

According to (1) and (3), we have:

$$\begin{aligned} {}_2F_1\left(\begin{array}{c}-n,a\\c\end{array}\middle| 1\right) &= \sum_{k=0}^{\infty} \frac{(-n)_k (a)_k}{k!(c)_k} \\ &= \sum_{k=0}^{\infty} \frac{(-1)^k (n-k+1)_k (a)_k}{k!(c)_k} \\ &= \sum_{k=0}^{\infty} \frac{(-1)^k \Gamma(n+1)(a)_k}{k!\,\Gamma(n-k+1)(c)_k} \\ &= \sum_{k=0}^{n} \binom{n}{k} \frac{(-1)^k (a)_k}{(c)_k}. \end{aligned}$$

We apply (6) to differentiate (17) with respect to the parameter a. On the one hand, we have:

$$\frac{\partial}{\partial a}\left[\frac{(c-a)_n}{(c)_n}\right] = -\frac{(c-a)_n}{(c)_n}[\psi(c-a+n) - \psi(c-a)], \tag{18}$$

and, on the other hand,

$$
\begin{aligned}
&\frac{\partial}{\partial a}\left[{}_2F_1\left(\begin{array}{c}-n,a\\c\end{array}\middle|1\right)\right]\\
&= \sum_{k=0}^{n}\binom{n}{k}\frac{(-1)^k}{(c)_k}\frac{d(a)_k}{da}\\
&= \sum_{k=0}^{n}(-1)^k\binom{n}{k}\frac{(a)_k}{(c)_k}\psi(a+k)-\psi(a)\sum_{k=0}^{n}\binom{n}{k}\frac{(-1)^k(a)_k}{(c)_k}\\
&= \sum_{k=0}^{n}(-1)^k\binom{n}{k}\frac{(a)_k}{(c)_k}\psi(a+k)-\psi(a)\frac{(c-a)_n}{(c)_n}.
\end{aligned}
\tag{19}
$$

Equating (18) to (19), we obtain (16), as we wanted to prove. □

Corollary 1. *For $a=1$, taking into account (11), we obtain:*

$$
\begin{aligned}
&\sum_{k=0}^{n}\frac{(-1)^k\psi(k+1)}{(n-k)!(c)_k}\\
&= \frac{(c-1)_n}{n!(c)_n}[-\gamma-\psi(c-1+n)+\psi(c-1)].
\end{aligned}
$$

Theorem 2. *Similarly to (16), if we perform the derivative with respect to the c parameter and apply (7), we will obtain:*

$$
\begin{aligned}
&\sum_{k=0}^{n}(-1)^k\binom{n}{k}\frac{(a)_k}{(c)_k}\psi(c+k)\\
&= \frac{(c-a)_n}{(c)_n}[\psi(c+n)+\psi(c-a)-\psi(c-a+n)].
\end{aligned}
$$

Corollary 2. *For $a=1$, we get:*

$$
\begin{aligned}
&\sum_{k=0}^{n}\frac{(-1)^k\psi(c+k)}{(n-k)!(c)_k}\\
&= \frac{c-1}{n!(c-1+n)}\left[\frac{1}{c-1+n}+\psi(c-1)\right].
\end{aligned}
\tag{20}
$$

Corollary 3. *Taking the limit $c\to 1$ in (20), and applying (8), we obtain:*

$$
\sum_{k=0}^{n}(-1)^{k+1}\binom{n}{k}\psi(k+1)=\frac{1}{n}.
$$

Theorem 3. *For $n\in\mathbb{N}$, the following finite sum holds true:*

$$
\begin{aligned}
&\sum_{k=0}^{n}\frac{2^k(a)_k(2n-k-1)!}{k!(n-k)!}\psi(a+k)\\
&= \frac{2^{2(n-1)}}{n}\left\{\left(\frac{1+a}{2}\right)_n\left[\psi\left(\frac{1+a}{2}+n\right)+\psi\left(\frac{a}{2}\right)+\log 4\right]\right.\\
&\left.+\left(\frac{a}{2}\right)_n\left[\psi\left(\frac{a}{2}+n\right)+\psi\left(\frac{1+a}{2}\right)+\log 4\right]\right\}.
\end{aligned}
\tag{21}
$$

Proof. We apply the reflection formula (3) and the property (5) to the reduction formula [17]:

$$ {}_2F_1\left(\begin{array}{c} -n, a \\ -2n+1 \end{array} \middle| 2 \right) = \frac{1}{\left(\frac{1}{2}\right)_n} \left[\left(\frac{1+a}{2}\right)_n + \left(\frac{a}{2}\right)_n \right], \quad n \in \mathbb{N}, $$

to arrive at:

$$ \sum_{k=0}^{n} \frac{2^k (a)_k (2n-k-1)!}{k!(n-k)!} = \frac{2^{2n-1}}{n} \left[\left(\frac{1+a}{2}\right)_n + \left(\frac{a}{2}\right)_n \right]. \tag{22} $$

We differentiate (22) with respect to parameter a, taking into account (10) to obtain (21), as we wanted to prove. □

Corollary 4. *For $a = 1$, taking into account (5), (13), and (14), Equation (21) is reduced to:*

$$ \begin{aligned} & \sum_{k=0}^{n} \frac{2^k (2n-k-1)!}{(n-k)!} \psi(k+1) \\ = \; & \frac{(n-1)!}{4} \left[4^n (H_n - 2\gamma) + \binom{2n}{n} (2\, H_{2n} - H_n - 2\gamma) \right], \end{aligned} $$

or equivalently, reversing the sum order,

$$ \begin{aligned} & \sum_{k=0}^{n} \frac{(n+k-1)!}{2^k k!} \psi(n+1-k) \\ = \; & \frac{(n-1)!}{2^{n+2}} \left[4^n (H_n - 2\gamma) + \binom{2n}{n} (2\, H_{2n} - H_n - 2\gamma) \right]. \end{aligned} $$

4. Infinite Sums Involving Digamma Function

Theorem 4. *For $\mathrm{Re}(c-a-b) > 0$, the following infinite series holds true:*

$$ \begin{aligned} & \sum_{k=0}^{\infty} \frac{(a)_k (b)_k}{k!(c)_k} \psi(a+k) \\ = \; & \frac{\Gamma(c)\Gamma(c-a-b)}{\Gamma(c-a)\Gamma(c-b)} [\psi(c-a) - \psi(c-a-b) + \psi(a)]. \end{aligned} \tag{23} $$

Proof. We differentiate the Gauss summation formula [16] (Theorem 2.2.2):

$$ \begin{aligned} & {}_2F_1\left(\begin{array}{c} a, b \\ c \end{array} \middle| 1 \right) = \sum_{k=0}^{\infty} \frac{(a)_k (b)_k}{k!(c)_k} = \frac{\Gamma(c)\Gamma(c-a-b)}{\Gamma(c-a)\Gamma(c-b)}, \\ & \mathrm{Re}(c-a-b) > 0, \end{aligned} \tag{24} $$

with respect to the parameter a. □

Remark 1. *In [1] (Addendum. Equation 55.4.5.2), we found an equivalent form, but with an erratum:*

$$ \begin{aligned} & \sum_{k=0}^{\infty} \frac{(b)_k (c)_k}{k!(a)_k} [\psi(c-k) - \psi(c)] \\ \neq \; & \frac{\Gamma(a)\Gamma(a-b-c)}{\Gamma(a-b)\Gamma(a-c)} [\psi(a-c) - \psi(a-b-c)], \\ & \mathrm{Re}(c+b-a) < 1, \end{aligned} $$

where we have to change in the sum $\psi(c-k)$ by $\psi(c+k)$. Additionally, the condition seems to be wrong.

Corollary 5. *For the particular case $c = 2$ and $b = 1/2$, taking into account (8), (9), we recover the formula given in [2] (Equation 6.2.1(67)),*

$$\begin{aligned} &\sum_{k=0}^{\infty} \frac{(a)_k \left(\frac{1}{2}\right)_k}{k!(k+1)!} \psi(a+k) \\ = \; &\frac{2\,\Gamma\left(\frac{3}{2}-a\right)}{\sqrt{\pi}\Gamma(2-a)} \left[\frac{1}{1-a} + \pi \cot(\pi a) + 2\psi(a) - \psi\left(\frac{3}{2}-a\right)\right], \\ &\operatorname{Re} a < 1. \end{aligned} \tag{25}$$

However, from (23), we can extend the validity of (25) to $\operatorname{Re} a < 3/2$.

Corollary 6. *For $b > 0$, the following expansion of the beta function holds true:*

$$\mathrm{B}(a,b) = -\sum_{k=0}^{\infty} \frac{(-b)_k}{k!} \psi(a+k).$$

Proof. We calculate the following limit, taking into account (8) and (11):

$$\lim_{x\to 0} \frac{\psi(x)}{\Gamma(x)} = \lim_{x\to 0} \frac{1}{\Gamma(x)} \left[\psi(x+1) - \frac{1}{x}\right] = -\lim_{x\to 0} \frac{1}{\Gamma(x+1)} = -1. \tag{26}$$

Take $c = a$ in (23) and apply (26) and (2) to obtain:

$$\begin{aligned} &\sum_{k=0}^{\infty} \frac{(-b)_k}{k!} \psi(a+k) \\ = \; &\lim_{c\to a} \frac{\Gamma(c)\Gamma(c-a+b)}{\Gamma(c-a)\Gamma(c+b)} [\psi(c-a) - \psi(c-a+b) + \psi(a)] \\ = \; &-\frac{\Gamma(a)\Gamma(b)}{\Gamma(a+b)} = -\mathrm{B}(a,b), \end{aligned}$$

□

Remark 2. *If we differentiate the Gauss summation Formula (24) with respect to parameter c, and if we apply (7), we will obtain for* $\operatorname{Re}(c-a-b) > 0$,

$$\begin{aligned} &\sum_{k=0}^{\infty} \frac{(a)_k (b)_k}{k!(c)_k} \psi(c+k) \\ = \; &\frac{\Gamma(c)\Gamma(c-a-b)}{\Gamma(c-a)\Gamma(c-b)} [\psi(c-a) + \psi(c-b) - \psi(c-a-b)], \end{aligned}$$

which is equivalent to [1] (Addendum. Equation 55.4.5.1), but the condition $\operatorname{Re}(a+b-c) < 1$ *seems to be wrong.*

Theorem 5. *The following series holds true:*

$$\begin{aligned} &\sum_{k=0}^{\infty} \frac{(a)_k (1-a)_k}{2^k k! (b)_k} \psi(b+k) \\ = \; &\frac{\sqrt{\pi}\,\Gamma(b)}{2^b \Gamma\left(\frac{a+b}{2}\right) \Gamma\left(\frac{b-a+1}{2}\right)} \left[\psi\left(\frac{a+b}{2}\right) + \psi\left(\frac{b-a+1}{2}\right) + \log 4\right]. \end{aligned}$$

Proof. We differentiate the summation formula [14] (Equation 7.3.7(8))

$$ {}_2F_1\left(\begin{array}{c} a, 1-a \\ b \end{array} \middle| \frac{1}{2} \right) = \sum_{k=0}^{\infty} \frac{(a)_k (1-a)_k}{2^k k! (b)_k} = \frac{2^{1-b} \sqrt{\pi}\, \Gamma(b)}{\Gamma\left(\frac{a+b}{2}\right) \Gamma\left(\frac{b-a+1}{2}\right)}, $$

with respect to parameter b. □

Corollary 7. *We take $a = c$ and apply (12) to obtain:*

$$ \sum_{k=0}^{\infty} \frac{(1-a)_k}{2^k k!} \psi(a+k) = \frac{\psi(a) - \gamma}{2^a}. $$

Theorem 6. *For $|z| < 1$, the following series holds true:*

$$ \begin{aligned} & \sum_{k=0}^{\infty} \frac{(b)_k\, z^k}{(k+1)!} \psi(k+b) \\ = & \frac{(1-z)^{1-b}}{z(1-b)^2} \left\{ (1-b) \log(1-z) - \left[1 - (1-z)^{b-1} \right] [1 + (1-b)\psi(b)] \right\}. \end{aligned} $$

Proof. We differentiate the reduction formula [14] (Equation 7.3.1(125)),

$$ {}_2F_1\left(\begin{array}{c} 1, b \\ 2 \end{array} \middle| z \right) = \sum_{k=0}^{\infty} \frac{(b)_k}{(k+1)!} z^k = \frac{(1-z)^{1-b} - 1}{z(b-1)}, $$

with respect to the parameter b. □

Remark 3. *Taking the limit $b \to 1$, we arrive at:*

$$ \begin{aligned} & \sum_{k=0}^{\infty} \frac{z^k}{k+1} \psi(k+1) = \frac{\log(1-z)}{2z} [2\gamma + \log(1-z)], \\ & |z| < 1, \end{aligned} $$

which is equivalent to [2] (Equation 6.2.1(2)).

Remark 4. *For $b = 2$, and taking into account (13) for $n = 1$, i.e., $\psi(2) = 1 - \gamma$, we arrive at:*

$$ \begin{aligned} & \sum_{k=1}^{\infty} z^k \psi(k+1) = \frac{\gamma z + \log(1-z)}{z-1}, \\ & |z| < 1, \end{aligned} $$

which is equivalent to [2] (Equation 6.2.1(1)).

Theorem 7. *For $|z| < 1$, the following infinite sum holds true:*

$$ \begin{aligned} & \sum_{k=0}^{\infty} \frac{z^k (a+1)_k (b)_k}{k! (a)_k} \psi(k+b) \\ = & \frac{[\psi(b) - \log(1-z)] \left[1 - \left(1 - \frac{b}{a} \right) z \right] + z/a}{(1-z)^{1+b}}. \end{aligned} $$

Proof. We differentiate the following reduction formula [15] (Equation 15.4.19):

$$ {}_2F_1\left(\begin{array}{c} a+1, b \\ a \end{array} \middle| z \right) = \left[1 - \left(1 - \frac{b}{a} \right) z \right] (1-z)^{-1-b}, \tag{27} $$

with respect to parameter b. □

Corollary 8. *For the particular case $b = a$, we obtain:*

$$\begin{aligned}
&\sum_{k=0}^{\infty} \frac{z^k (a+1)_k}{k!} \psi(k+a) \\
= & \frac{\psi(a) - \log(1-z) + z/a}{(1-z)^{1+a}}.
\end{aligned}$$

Corollary 9. *For the particular case $b = 1$, we obtain:*

$$\begin{aligned}
&\sum_{k=0}^{\infty} z^k (a+k) \psi(k+1) \\
= & \frac{z - [\gamma + \log(1-z)][a + (1-a)z]}{(1-z)^2}.
\end{aligned}$$

Theorem 8. *The following series holds true:*

$$\begin{aligned}
&\sum_{k=0}^{\infty} \frac{(k+1)!}{(b)_k} 2^{-k} \psi(k+b) \\
= & 2[(b-1)\psi(b) - 1] + 4[2b - 3 - (b-1)(b-2)\psi(b)]\beta(b-1) \\
& +4(b-1)(b-2)\beta'(b-1).
\end{aligned} \tag{28}$$

Proof. We differentiate the reduction formula [14] (Equation 7.3.7(18)):

$${}_2F_1\left(\begin{matrix} 1,2 \\ b \end{matrix} \middle| \frac{1}{2} \right) = \sum_{k=0}^{\infty} \frac{(k+1)!}{(b)_k} 2^{-k} = 2(b-1)[1 - 2(b-2)\beta(b-1)],$$

with respect to parameter b. □

Remark 5. *If we differentiate the reduction formula [14] (Equation 7.3.7(17)):*

$${}_2F_1\left(\begin{matrix} 1,1 \\ b \end{matrix} \middle| \frac{1}{2} \right) = \sum_{k=0}^{\infty} \frac{k!}{(b)_k} 2^{-k} = 2(b-1)\beta(b-1),$$

with respect to parameter b, we will obtain [2] (Equation 6.2.1(64)):

$$\begin{aligned}
&\sum_{k=0}^{\infty} \frac{2^{-k} k!}{(b)_k} \psi(k+b) \\
= & 2[(b-1)\psi(b) - 1]\beta(b-1) - 2(b-1)\beta'(b-1).
\end{aligned} \tag{29}$$

Corollary 10. *Substracting (29) from (28), we arrive at:*

$$\begin{aligned}
&\sum_{k=0}^{\infty} \frac{k\, k!}{(b)_k} 2^{-k-1} \psi(k+b) \\
= & (b-1)\left[\psi(b) + (2b-3)\beta'(b-1)\right] \\
& +[4b - 5 - (b-1)(2b-3)\psi(b)]\beta(b-1) - 1.
\end{aligned}$$

Theorem 9. *For $|z| < 1$, the following infinite sum holds true:*

$$\begin{aligned}
&\sum_{k=0}^{\infty} \frac{z^k (a+1)_k (b)_k}{(a)_k (c)_k} \psi(c+k) \qquad (30)\\
&= \psi(c-1)\, {}_3F_2\left(\begin{array}{c} 1, a+1, b \\ a, c \end{array} \middle| z \right) \\
&+ \frac{1}{a(1-z)^{1+b}} \left\{ \frac{a+(b-a)z}{c-1}\, {}_3F_2\left(\begin{array}{c} b, c-1, c-1 \\ c, c \end{array} \middle| \frac{z}{z-1} \right) \right. \\
&\left. + \frac{b(c-1)z}{c^2(z-1)}\, {}_3F_2\left(\begin{array}{c} b+1, c, c \\ c+1, c+1 \end{array} \middle| \frac{z}{z-1} \right) \right\}.
\end{aligned}$$

Proof. On the one hand, consider the reduction formula [14] (Equation 7.4.4(94)):

$$\begin{aligned}
&{}_3F_2\left(\begin{array}{c} -k, a, b \\ a+\ell, b+n \end{array} \middle| 1 \right) \\
&= k!(a)_\ell (b)_n \left[\frac{1}{(\ell-1)!(a)_{k+1}(b-a)_n}\, {}_3F_2\left(\begin{array}{c} 1-\ell, a, 1+a-b-n \\ 1+a+k, 1+a-b \end{array} \middle| 1 \right) \right. \\
&\left. + \frac{1}{(n-1)!(b)_{k+1}(a-b)_\ell}\, {}_3F_2\left(\begin{array}{c} 1-n, b, 1+b-a-\ell \\ 1+b+k, 1+b-a \end{array} \middle| 1 \right) \right],
\end{aligned}$$

for the particular case $\ell = 1$, $n = 1$, to obtain:

$$ {}_3F_2\left(\begin{array}{c} -k, a, b \\ a+1, b+1 \end{array} \middle| 1 \right) = \frac{k!a\,b}{(a)_{k+1}(b)_{k+1}} \left[\frac{(a)_{k+1} - (b)_{k+1}}{a-b} \right].$$

We take the limit $a \to b$ and apply (6) as well as the property (4) to obtain:

$$\begin{aligned}
{}_3F_2\left(\begin{array}{c} -k, b, b \\ b+1, b+1 \end{array} \middle| 1 \right) &= \frac{k!\,b^2}{\left[(b)_{k+1}\right]^2} \lim_{a\to b} \left[\frac{(a)_{k+1} - (b)_{k+1}}{a-b} \right] \\
&= \frac{k!\,b^2}{\left[(b)_{k+1}\right]^2} \frac{d}{dx}\left[(x)_{k+1}\right]_{x=b} \\
&= \frac{k!\,b}{(b+1)_k} [\psi(b+1+k) - \psi(b)]. \qquad (31)
\end{aligned}$$

On the other hand, from (27), we have:

$$ {}_2F_1\left(\begin{array}{c} \alpha+k+2, \beta+k+1 \\ \alpha+k+1 \end{array} \middle| z \right) = \left(1 + \frac{\beta-\alpha}{\alpha+k+1} z \right) (1-z)^{-2-\beta-k}. \qquad (32)$$

Now, equate the results given in [10,11],

$$\begin{aligned}
&\frac{\partial^m}{\partial b^m} \left[{}_pF_{q+1}\left(\begin{array}{c} (a_p) \\ b, (b_q) \end{array} \middle| z \right) \right] \qquad (33)\\
&= \frac{m!(-1)^m ((a_p))_1}{b^{m+1} ((b_q))_1} \\
&\quad \sum_{k=0}^{\infty} \frac{z^{k+1} ((a_p+1))_k}{k!(k+1)!((b_q+1))_k}\, {}_{m+2}F_{m+1}\left(\begin{array}{c} -k, b, \ldots, b \\ b+1, \ldots, b+1 \end{array} \middle| 1 \right) \\
&= m!(-1)^m z \\
&\quad \sum_{k=0}^{\infty} \frac{(-z)^k ((a_p))_{k+1}}{k!(k+1)!((b_q))_{k+1} (k+b)^{m+1}}\, {}_pF_{q+1}\left(\begin{array}{c} (a_p)+k+1 \\ (b_q)+k+1, k+2 \end{array} \middle| z \right),
\end{aligned}$$

for the particular case $m = 1$, $(a_p) = (\alpha+1, \beta, 1)$ and $(b_q) = (\alpha)$, to obtain:

$$
\begin{aligned}
&= \frac{(\alpha+1)\beta}{b^2\alpha} \sum_{k=0}^{\infty} \frac{z^k(\alpha+2)_k(\beta+1)_k}{k!(\alpha+1)_k} \, {}_3F_2\left(\left. \begin{array}{c} -k, b, b \\ b+1, b+1 \end{array} \right| 1 \right) \\
&= \sum_{k=0}^{\infty} \frac{(-z)^k(\alpha+1)_{k+1}(\beta)_{k+1}}{k!(\alpha)_{k+1}(k+b)^2} \, {}_2F_1\left(\left. \begin{array}{c} \alpha+k+2, \beta+k+1 \\ \alpha+k+1 \end{array} \right| z \right).
\end{aligned}
$$

Next, insert (31) and (32) and simplify the result using (4) to arrive at:

$$
\begin{aligned}
& \sum_{k=0}^{\infty} \frac{z^k(\alpha+2)_k(\beta+1)_k}{(\alpha+1)_k(b+1)_k} [\psi(b+1+k) - \psi(b)] \\
= \; & \frac{1}{b(\alpha+1)(1-z)^{2+\beta}} \\
& \sum_{k=0}^{\infty} \frac{(\beta+1)_k[(b)_k]^2}{k!\,[(b+1)_k]^2} [\alpha+k+1+(\beta-\alpha)z] \left(\frac{z}{z-1}\right)^k.
\end{aligned}
$$

Grouping the terms

$$
\begin{aligned}
& \sum_{k=0}^{\infty} \frac{z^k(\alpha+2)_k(\beta+1)_k}{(\alpha+1)_k(b+1)_k} \psi(b+1+k) \\
= \; & \psi(b) \sum_{k=0}^{\infty} \frac{z^k(\alpha+2)_k(\beta+1)_k(1)_k}{k!(\alpha+1)_k(b+1)_k} \\
& + \frac{1}{b(\alpha+1)(1-z)^{2+\beta}} \\
& \left\{ (\alpha+1+(\beta-\alpha)z) \sum_{k=0}^{\infty} \frac{(\beta+1)_k[(b)_k]^2}{k!\,[(b+1)_k]^2} \left(\frac{z}{z-1}\right)^k \right. \\
& \left. + \frac{(\beta+1)b^2}{(b+1)^2} \left(\frac{z}{z-1}\right) \sum_{k=0}^{\infty} \frac{(\beta+2)_k[(b+1)_k]^2}{k!\,[(b+2)_k]^2} \left(\frac{z}{z-1}\right)^k \right\},
\end{aligned}
$$

and recasting the sums with hypergeometric functions (renaming the parameters), we finally arrive at (30), as we wanted to prove. □

Remark 6. *For the particular case $a = b$, and taking into account the reduction formula [14] (Equation 7.3.1(119)):*

$$
{}_2F_1\left(\left. \begin{array}{c} 1, a \\ c \end{array} \right| z \right) = z^{1-c}(1-z)^{c-a-1}(c-1)\mathrm{B}_z(c-1, a-c+1),
$$

we obtain for $|z| < 1$:

$$
\begin{aligned}
& \sum_{k=0}^{\infty} \frac{z^k(a)_k}{(c)_k} \psi(c+k) \\
= \; & \psi(c-1)\, z^{1-c}(1-z)^{c-a-1}(c-1)\mathrm{B}_z(c-1, a-c+1) \\
& + \frac{1}{(1-z)^a} \left\{ \frac{1}{c-1} \, {}_3F_2\left(\left. \begin{array}{c} a-1, c-1, c-1 \\ c, c \end{array} \right| \frac{z}{z-1} \right) \right. \\
& \left. + \frac{(c-1)z}{c^2(z-1)} \, {}_3F_2\left(\left. \begin{array}{c} a, c, c \\ c+1, c+1 \end{array} \right| \frac{z}{z-1} \right) \right\},
\end{aligned} \tag{34}
$$

which is a non-trivial alternative form of the result given in [6]:

$$\begin{aligned} & \sum_{k=0}^{\infty} \frac{z^k (a)_k}{(c)_k} [\psi(c+k) - \psi(c)] \\ = \; & \frac{a\, z}{c^2 (1-z)^{a+1}} \, {}_3F_2\left(\left. \begin{array}{c} a+1, c, c \\ c+1, c+1 \end{array} \right| \frac{z}{z-1} \right), \\ & a \in \mathbb{C}, |z| < 1. \end{aligned} \tag{35}$$

5. Conclusions

We have calculated some finite and infinite sums involving the digamma function differentiating some reduction formulas of the hypergeometric function with respect to the parameters and applying the differentiation formulas of the Pochhammer symbol given in (6) and (7). It is worth noting that this method can be applied to many other reduction formulas of hypergeometric and generalized hypergeometric functions. Here, we have only selected some interesting new cases, some of which have allowed us to detect errors in the literature. Additionally, as a consistency test, we have recovered some formulas found in the literature from some particular cases of the results obtained.

Nevertheless, in (30), we have applied another approach, wherein we have compared the differentiation formulas given in (33) for a particular case of the parameters. This approach is not as straightforward as the other one. However, note that the particular case given in (34) applying this method provides a non-trivial alternative form of the result (35) found in the literature.

Finally, we point out that all the sums presented in this paper have been numerically checked with MATHEMATICA, and they are available at https://bit.ly/3dCZFCJ (accessed on 11 August 2022).

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