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Research Article

Rakesh Arora and Sergey Shmarev*

Double-phase parabolic equations with variable growth and nonlinear sources

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Abstract: We study the homogeneous Dirichlet problem for the parabolic equations

$$u_t - \operatorname{div}(\mathcal{A}(z, |\nabla u|)\nabla u) = F(z, u, \nabla u), \quad z = (x, t) \in \Omega \times (0, T),$$

with the double phase flux $\mathcal{A}(z, |\nabla u|)\nabla u = (|\nabla u|^{p(z)-2} + a(z)|\nabla u|^{q(z)-2})\nabla u$ and the nonlinear source F. The initial function belongs to a Musielak-Orlicz space defined by the flux. The functions a, p, and q are Lipschitz-continuous, $a(z)$ is nonnegative, and may vanish on a set of nonzero measure. The exponents p, and q satisfy the balance conditions $\frac{2N}{N+2} < p^- \le p(z) \le q(z) < p(z) + \frac{r^*}{2}$ with $r^* = r^*(p^-, N)$, $p^- = \min_{\overline{Q}_T} p(z)$. It is shown that under suitable conditions on the growth of $F(z, u, \nabla u)$ with respect to the second and third arguments, the problem has a solution u with the following properties:

$$\begin{aligned} &u_t \in L^2(Q_T), \quad |\nabla u|^{p(z)+\delta} \in L^1(Q_T) \quad \text{for every } 0 \le \delta < r^*, \\ &|\nabla u|^{s(z)}, \ a(z)|\nabla u|^{q(z)} \in L^\infty(0, T; L^1(\Omega)) \quad \text{with } s(z) = \max\{2, p(z)\}. \end{aligned}$$

Uniqueness is proven under stronger assumptions on the source F. The same results are established for the equations with the regularized flux $\mathcal{A}(z, (\varepsilon^2 + |\nabla u|^2)^{1/2})\nabla u$, $\varepsilon > 0$.

Keywords: singular and degenerate parabolic equation, double phase problem, variable nonlinearity, nonlinear source

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1 Introduction

Let $\Omega \subset \mathbb{R}^N$ be a smooth bounded domain, $N \ge 2$, and $0 < T < \infty$. We consider the following parabolic problem with the homogeneous Dirichlet boundary conditions:

$$\begin{cases} u_t - \operatorname{div}(|\nabla u|^{p(z)-2}\nabla u + a(z)|\nabla u|^{q(z)-2}\nabla u) = F(z, u, \nabla u) & \text{in } Q_T, \\ u = 0 & \text{on } \Gamma_T, \\ u(x, 0) = u_0(x) & \text{in } \Omega, \end{cases} \tag{1.1}$$

where $z = (x, t)$ denotes the point in the cylinder $Q_T = \Omega \times (0, T]$ and $\Gamma_T = \partial\Omega \times (0, T)$ is the lateral boundary of the cylinder. The nonlinear source has the following form

$$F(z, v, \nabla v) = f_0(z) + \Psi(z, v) + \Phi(z, \nabla v). \tag{1.2}$$

Here, $a \ge 0$, p, q, f_0, Ψ, and Φ are given functions of their arguments.

*** Corresponding author: Sergey Shmarev,** Mathematics Department, University of Oviedo, c/Federico García Lorca 18, 33007, Oviedo, Spain, e-mail: shmarev@uniovi.es
Rakesh Arora: Department of Mathematics and Statistics, Masaryk University, Building 08, Kotlářská 2, Brno 611 37, Czech Republic, e-mail: arora@math.muni.cz; arora.npde@gmail.com

Equations given in (1.1) are often termed "the double-phase equations." This name, introduced in [17,18], reflects the fact that the flux function $(|\nabla u|^{p(z)-2} + a(z)|\nabla u|^{q(z)-2})\nabla u$ includes two terms with different properties. If $p(z) \le q(z)$ a.e. in the problem domain and $a(z)$ is allowed to vanish on a set of nonzero measure in Q_T, then the growth of the flux is determined by $p(z)$ on the set, where $a(z) = 0$ and by $q(z)$ wherever $a(z) > 0$.

1.1 Previous work

The study of the double-phase problems started in the late 80th by the works of Zhikov [47,48], where the models of strongly anisotropic materials were considered in the context of homogenization. Later on, the double-phase functionals

$$u \to \int_\Omega (|\nabla u|^p + a(x)|\nabla u|^q)\mathrm{d}x$$

attracted attention of many researchers. On the one hand, the study of these functionals is a challenging mathematical problem. On the other hand, the double-phase functionals appear in a variety of physical models. We refer here to [8,46] for applications in the elasticity theory, [6] for transonic flows, [9] for quantum physics, and [13] for reaction-diffusion systems.

Equations (1.1) with $p \ne q$ are also referred to as the equations with the (p, q)-growth because of the gap between the coercivity and growth conditions: If $p \le q$ and $0 \le a(x) \le L$, then for every $\xi \in \mathbb{R}^N$,

$$|\xi|^p \le (|\xi|^{p-2} + a(x)|\xi|^{q-2})|\xi|^2 \le C(1 + |\xi|^q), \quad C = \text{const} > 0.$$

These equations fall into the class of equations with nonstandard growth conditions, which have been actively studied during the last decades in the cases of constant or variable exponents p and q. We refer to the works [2,14,17,18,24,26–28,31,34,35,39,40,45] and references therein for the results on the existence and regularity of solutions, including optimal regularity results [24].

Results on the existence of solutions to the evolution double-phase equations can be found in papers [10,43,44]. These works deal with the Dirichlet problem for systems of parabolic equations of the form

$$u_t - \operatorname{div} a(x, t, \nabla u) = 0, \tag{1.3}$$

where the flux $a(x, t, \nabla u)$ is assumed to satisfy the (p, q)-growth conditions and certain regularity assumptions. As a partial case, the class of equations (1.3) includes equation (1.1) with constant exponents $p \le q$ and a nonnegative bounded coefficient $a(x, t)$. It is shown in [10, Th.1.6] that if

$$2 \le p \le q < p + \frac{4}{N+2},$$

then problem (1.1) with $F \equiv 0$ has a *very weak solution*:

$$u \in L^p(0, T; W_0^{1,p}(\Omega)) \cap L^q_{\mathrm{loc}}(0, T; W^{1,q}_{\mathrm{loc}}(\Omega)) \quad \text{with } u_t \in L^{\frac{p}{q-1}}\Big(0, T; W^{-1,\frac{p}{q-1}}(\Omega)\Big),$$

provided that $u_0 \in W_0^{1,r}(\Omega)$, $r = \frac{p(q-1)}{p-1}$. Moreover, $|\nabla u|$ is bounded on every strictly interior cylinder $Q'_T \Subset Q_T$ separated away from the parabolic boundary of Q_T. In [43], these results were extended to the case

$$\frac{2N}{N+2} < p < 2, \quad p \le q < p + \frac{4}{N+2}.$$

The paper [44] deals with *weak solutions* of systems of equations of the type (1.3) with (p, q) growth conditions. When applied to problem (1.1) with constant p and q, $F \equiv f_0(z)$, and $a(\cdot, t) \in C^\alpha(\Omega)$ with some $\alpha \in (0, 1)$ for a.e. $t \in (0, T)$, the result of [44] guarantees the existence of a weak solution:

$$u \in L^p(0, T; W_0^{1,p}(\Omega)) \cap L^q_{\mathrm{loc}}(0, T; W^{1,q}_{\mathrm{loc}}(\Omega)) \cap L^\infty(0, T; L^2(\Omega)),$$

provided that the exponents p and q obey the inequalities:

$$\frac{2N}{N+2} < p < q < p + \frac{\alpha \min\{2, p\}}{N+2}.$$

The proofs of the existence theorems in [10,43,44] rely on the property of local higher integrability of the gradient, $|\nabla u|^{p+\delta} \in L^1(Q_T')$ for every sub-cylinder $Q_T' \Subset Q_T$. The maximal possible value of $\delta > 0$ indicates the admissible gap between the exponents p and q and vary in dependence on the type of the solution.

Equation (1.1) with constant exponents p and q furnishes a prototype of the equations recently studied in papers [11,21,29,36] in the context of weak or variational solutions. The proofs of existence also use the local higher integrability of the gradient, but for the existence of variational solutions, a weaker assumption on the gap $q - p$ is required. For the solutions of the evolution $p(x,t)$-Laplace equation, global higher integrability of the gradient up to the lateral boundary of the cylinder was proven in [1] for very weak solutions and under mild restrictions on the regularity of the data. In [5], this property was established in the whole of the cylinder Q_T but under stronger assumptions on the data.

Nonhomogeneous parabolic equations of the form (1.3) with the flux $a(x,t,\nabla u)$ controlled by a generalized N-function were studied in [16] in the framework of Musielak-Orlicz spaces. The class of equations studied in [16] includes, as a partial case, equation (1.1) with $F = f_0(z)$ and variable exponents p, q. It is shown that problem (1.1) with bounded data u_0 and f_0 admits a unique solution $u \in L^\infty(0,T; L^2(\Omega)) \cap L^1(0,T; W_0^{1,1}(\Omega))$ with $\nabla u \in L_M(Q_T; \mathbb{R}^N)$, where L_M denotes the Musielak-Orlicz space defined by the flux $a(x,t,\xi)$. We refer here to [16,25], the survey article [14], and references therein for the issues of solvability of elliptic and parabolic equations in the Musielak-Orlicz spaces. A comprehensive review of the current state of the theory of double-phase problems is given in [38].

In the last years, the double-phase stationary problems with nonlinear convection terms were in the focus of attention of many researchers, see, e.g., papers [7,27,37], monograph [41], and references therein. However, to the best of our knowledge, there are no results on the counterpart evolution problems with nonlinear sources of the variable growth.

1.2 Description of results

In the present work, we prove the existence of *strong solutions* to problem (1.1). By the strong solution, we mean a solution whose time derivative is not a distribution but an element of a Lebesgue space, and the flux has better integrability properties than the properties prompted by the energy equality (the rigorous formulation is given in Definition 3.1).

- We consider first the case of the given source independent of the solution: $F = f_0(z)$. It is shown that if the exponents p, q and the data a, u_0, and f_0 are sufficiently smooth, and if the gap between the exponents satisfies the condition

$$\frac{2N}{N+2} < p(z) \le q(z) < p(z) + \frac{2p^-}{(N+2)p^- + 2N} \quad \text{in } Q_T, \tag{1.4}$$

where $p^- = \inf_{Q_T} p(z)$, then problem (1.1) has a strong solution u with

$$\begin{aligned} &u_t \in L^2(Q_T), \quad \operatorname*{ess\,sup}_{(0,T)} \mathcal{F}(u(\cdot,t),t) < \infty, \\ &\mathcal{F}(u(\cdot,t),t) \equiv \int_\Omega (|\nabla u|^{p(z)} + a(z)|\nabla u|^{q(z)})\mathrm{d}x. \end{aligned} \tag{1.5}$$

Moreover, the solution possesses the property of global higher integrability of the gradient:

$$\int_{Q_T} |\nabla u|^{p(z)+r}\mathrm{d}z \le C \quad \text{for every } 0 < r < \frac{4p^-}{(N+2)p^- + 2N}, \tag{1.6}$$

with a finite constant C depending only on $\mathcal{F}(u_0, 0)$, N, r, and the properties of $p(\cdot)$ and $q(\cdot)$. The same existence result is valid for problem (1.1) with the regularized nondegenerate flux:

$$\left((\varepsilon^2 + |\nabla u|^2)^{\frac{p(z)-2}{2}} + a(z)(\varepsilon^2 + |\nabla u|^2)^{\frac{q(z)-2}{2}}\right)\nabla u, \quad \varepsilon > 0.$$

- If the source F depends on u and ∇u, the existence of strong solutions of problem (1.1) with the property of global higher integrability (1.6) is proven for the nonlinear source function F of the form (1.2) subject to specific growth restrictions, which lead to an additional assumption on p^-. We assume that

$$\Psi \in C^0(\overline{Q}_T \times \mathbb{R}), \quad \Psi(z,0) = 0, \quad |\Psi(z,r)| \le C|r|^{\sigma(z)-1},$$
$$|\nabla \Psi(z,v)| \le C(|v|^{\sigma(z)-1}(1 + |\ln|v||) + |v|^{\sigma(z)-2}|\nabla v|),$$

with a constant C and a function $\sigma \in C^0(\overline{Q}_T)$ such that

$$2 \le \inf_{Q_T} \sigma(z) \le \sup_{Q_T} \sigma(z) < \min\left\{p^-, 1 + p^- \frac{N+2}{2N}\right\}.$$

The growth of F with respect to ∇u is subject to the following restrictions:

$$\Phi \in C^0(\overline{Q}_T \times \mathbb{R}^N), \quad |\Phi(z,s)| \le \begin{cases} L|s|^{p(z)-1} & \text{if } |s| < 1, \\ M^{\frac{1}{2}}|s|^{\frac{p(z)}{2}} & \text{if } |s| \ge 1 \end{cases}$$

with positive constants L, and M. The functions Ψ and Φ are not assumed to be sign-definite. An example of admissible Ψ and Φ is furnished by

$$\Phi(z,s) = \frac{c(z)|s|^{p(z)-1}}{1 + |s|^{\frac{p(z)}{2}-1}}, \quad \Psi(z,v) = b(z)|v|^{\sigma(z)-2}v,$$

with

$$b(z), c(z) \in C^0(\overline{Q}_T), \quad \|\nabla b\|_{\infty,Q_T} \le \text{const}, \quad \|\nabla \sigma\|_{\infty,Q_T} \le \text{const}.$$

If $\Psi \equiv 0$, the restriction on $p(z)$ transforms into $p^- \ge 2$. For the uniqueness of strong solutions, we assume that $\Phi \equiv 0$ and either $\Psi(z,v)$ is monotone decreasing with respect to v for a.e. $z \in Q_T$ or $\Psi(z,v) = b(z)v$.

In both cases, property (1.6) of global higher integrability of the gradient turns out to be crucial for the proof of the existence of a strong solution. A traditional approach to the double-phase equations involves regularization of the equation and derivation of a local estimate (1.6) with the use of Caccioppoli-type inequalities, which is sufficient for the proof of the existence of weak solutions. As distinguished from this method, we find a solution as the limit of a sequence of finite-dimensional Galerkin's approximations. These approximations do not solve the equation, but they are smooth up to the parabolic boundary of the cylinder Q_T. The higher regularity of the approximations allows us to prove global in Q_T uniform higher integrability of their gradients by means of the Gagliardo-Nirenberg inequality.

All results remain true for the multiphase equations:

$$u_t - \operatorname{div}\left(|\nabla u|^{p(z)-2}\nabla u + \sum_{i=1}^{K} a_i(z)|\nabla u|^{q_i(z)-2}\nabla u\right) = F(z,u,\nabla u),$$

provided the exponents $p(z)$ and $q_i(z)$ satisfy the balance condition (1.4) and the source F is subject to the above-described growth conditions.

2 The function spaces

We begin with a brief description of the Lebesgue and Sobolev spaces with variable exponents. A detailed insight into the theory of these spaces and a review of the bibliography can be found in [20,22,32,41]. Let $\Omega \subset \mathbb{R}^N$ be a bounded domain with the Lipschitz continuous boundary $\partial\Omega$. Define the set

$$\mathcal{P}(\Omega) := \{\text{measurable functions on } \Omega \text{ with values in } (1,\infty)\}.$$

2.1 Variable Lebesgue spaces

Throughout the rest of the paper, we assume that $\Omega \subset \mathbb{R}^N$, $N \geq 2$, is a bounded domain with the boundary $\partial\Omega \in C^2$. Given $r \in \mathcal{P}(\Omega)$, we introduce the modular

$$A_{r(\cdot)}(f) = \int_\Omega |f(x)|^{r(x)} \mathrm{d}x, \tag{2.1}$$

and the set

$$L^{r(\cdot)}(\Omega) = \{f : \Omega \to \mathbb{R} \quad | \text{ measurable on } \Omega, \quad A_{r(\cdot)}(f) < \infty\}.$$

The set $L^{r(\cdot)}(\Omega)$ equipped with the Luxemburg norm

$$\|f\|_{r(\cdot),\Omega} = \inf\left\{\lambda > 0 : A_{r(\cdot)}\left(\frac{f}{\lambda}\right) \leq 1\right\}$$

becomes a Banach space. By convention, from now on, we use the notation

$$r^- := \operatorname*{ess\,min}_{x\in\Omega} r(x), \quad r^+ := \operatorname*{ess\,max}_{x\in\Omega} r(x).$$

If $r \in \mathcal{P}(\Omega)$ and $1 < r^- \leq r(x) \leq r^+ < \infty$ in Ω, then the following properties hold.

(1) $L^{r(\cdot)}(\Omega)$ is a reflexive and separable Banach space.

(2) For every $f \in L^{r(\cdot)}(\Omega)$,

$$\min\left\{\|f\|^{r^-}_{r(\cdot),\Omega}, \|f\|^{r^+}_{r(\cdot),\Omega}\right\} \leq A_{r(\cdot)}(f) \leq \max\left\{\|f\|^{r^-}_{r(\cdot),\Omega}, \|f\|^{r^+}_{r(\cdot),\Omega}\right\}. \tag{2.2}$$

(3) For every $f \in L^{r(\cdot)}(\Omega)$ and $g \in L^{r'(\cdot)}(\Omega)$, the generalized Hölder inequality holds:

$$\int_\Omega |fg| \leq \left(\frac{1}{r^-} + \frac{1}{(r')^-}\right)\|f\|_{r(\cdot),\Omega}\|g\|_{r'(\cdot),\Omega} \leq 2\|f\|_{r(\cdot),\Omega}\|g\|_{r'(\cdot),\Omega}, \tag{2.3}$$

where $r' = \frac{r}{r-1}$ is the conjugate exponent of r.

(4) If $p_1, p_2 \in \mathcal{P}(\Omega)$ and satisfy the inequality $p_1(x) \leq p_2(x)$ a.e. in Ω, then $L^{p_1(\cdot)}(\Omega)$ is continuously embedded in $L^{p_2(\cdot)}(\Omega)$ and for all $u \in L^{p_2(\cdot)}(\Omega)$

$$\|u\|_{p_1(\cdot),\Omega} \leq C\|u\|_{p_2(\cdot),\Omega}, \quad C = C(|\Omega|, p_1^\pm, p_2^\pm). \tag{2.4}$$

(5) For every sequence $\{f_k\} \subset L^{r(\cdot)}(\Omega)$ and $f \in L^{r(\cdot)}(\Omega)$,

$$\|f_k - f\|_{r(\cdot),\Omega} \to 0 \quad \text{iff } A_{r(\cdot)}(f_k - f) \to 0 \text{ as } k \to \infty. \tag{2.5}$$

2.2 Variable Sobolev spaces

The variable Sobolev space $W_0^{1,r(\cdot)}(\Omega)$ is the set of functions:

$$W_0^{1,r(\cdot)}(\Omega) = \{u : \Omega \to \mathbb{R} | u \in L^{r(\cdot)}(\Omega) \cap W_0^{1,1}(\Omega), \ |\nabla u| \in L^{r(\cdot)}(\Omega)\}$$

equipped with the norm

$$\|u\|_{W_0^{1,r(\cdot)}(\Omega)} = \|u\|_{r(\cdot),\Omega} + \|\nabla u\|_{r(\cdot),\Omega}.$$

If $r \in C^0(\overline{\Omega})$, the Poincaré inequality holds: for every $u \in W_0^{1,r(\cdot)}(\Omega)$,

$$\|u\|_{r(\cdot),\Omega} \le C\|\nabla u\|_{r(\cdot),\Omega}. \tag{2.6}$$

Inequality (2.6) means that the equivalent norm of $W_0^{1,r(\cdot)}(\Omega)$ is given by

$$\|u\|_{W_0^{1,r(\cdot)}(\Omega)} = \|\nabla u\|_{r(\cdot),\Omega}. \tag{2.7}$$

Let us denote by $C_{\log}(\overline{\Omega})$ the subset of $\mathcal{P}(\Omega)$ composed of the functions continuous on $\overline{\Omega}$ with the logarithmic modulus of continuity:

$$p \in C_{\log}(\overline{\Omega}) \quad \Leftrightarrow \quad |p(x) - p(y)| \le \omega(|x-y|) \quad \forall x, y \in \overline{\Omega},\ |x-y| < \frac{1}{2},$$

where ω is a nonnegative function such that

$$\limsup_{s\to 0^+} \omega(s)\ln\frac{1}{s} = C, \qquad C = \text{const}.$$

If $r \in C_{\log}(\overline{\Omega})$, then the set $C_c^\infty(\Omega)$ of smooth functions with finite support is dense in $W_0^{1,r(\cdot)}(\Omega)$ (see Proposition 2.2). This property allows one to use the equivalent definition of the space $W_0^{1,r(\cdot)}(\Omega)$:

$$W_0^{1,r(\cdot)}(\Omega) = \{\text{the closure of } C_c^\infty(\Omega) \text{ with respect to the norm } \|\cdot\|_{W^{1,r(\cdot)}(\Omega)}\}.$$

2.3 Spaces of functions depending on x and t

For the study of parabolic problem (1.1), we need the spaces of functions depending on $z = (x, t) \in Q_T$. Given a function $q \in C_{\log}(\overline{Q}_T)$, we introduce the spaces:

$$\begin{aligned} &\mathcal{V}_{q(\cdot,\,t)}(\Omega) = \{u : \Omega \to \mathbb{R} | u \in L^2(\Omega) \cap W_0^{1,1}(\Omega),\ |\nabla u|^{q(x,t)} \in L^1(\Omega)\}, \quad t \in (0, T),\\ &\mathcal{W}_{q(\cdot)}(Q_T) = \{u : (0, T) \to \mathcal{V}_{q(\cdot,\,t)}(\Omega) | u \in L^2(Q_T),\ |\nabla u|^{q(z)} \in L^1(Q_T)\}. \end{aligned}$$

The norm of $\mathcal{W}_{q(\cdot)}(Q_T)$ is defined by

$$\|u\|_{\mathcal{W}_{q(\cdot)}(Q_T)} = \|u\|_{2,Q_T} + \|\nabla u\|_{q(\cdot),Q_T}.$$

Since $q \in C_{\log}(\overline{Q}_T)$, the space $\mathcal{W}_{q(\cdot)}(Q_T)$ is the closure of $C_c^\infty(Q_T)$ with respect to this norm.

2.4 Musielak-Sobolev spaces

Let $a_0 : \Omega \mapsto [0, \infty)$ be a given function, $a_0 \in C^{0,1}(\overline{\Omega})$. Assume that the exponents $p, q \in C^{0,1}(\overline{\Omega})$ take values in the intervals (p^-, p^+), (q^-, q^+), and $p(x) \le q(x)$ in Ω. Set

$$r(x) = \max\{2, p(x)\}, \quad s(x) = \max\{2, q(x)\}$$

and consider the function

$$\mathcal{H}(x, t) = t^{r(x)} + a_0(x)t^{s(x)}, \quad t \ge 0,\ x \in \Omega.$$

The set

$$L^{\mathcal{H}}(\Omega) = \left\{u : \Omega \mapsto \mathbb{R} |\ u \text{ is measurable},\ \rho_{\mathcal{H}}(u) = \int_\Omega \mathcal{H}(x, |u|)dx < \infty\right\}$$

equipped with the Luxemburg norm

$$\|u\|_{\mathcal{H}} = \inf\left\{\lambda > 0 : \rho_{\mathcal{H}}\left(\frac{u}{\lambda}\right) \le 1\right\}$$

becomes a Banach space. The space $L^{\mathcal{H}}(\Omega)$ is separable and reflexive [25]. By $W^{1,\mathcal{H}}(\Omega)$, we denote the Musielak-Sobolev space

$$W^{1,\mathcal{H}}(\Omega) = \{u \in L^{\mathcal{H}}(\Omega) : |\nabla u| \in L^{\mathcal{H}}(\Omega)\}$$

with the norm

$$\|u\|_{1,\mathcal{H}} = \|u\|_{\mathcal{H}} + \|\nabla u\|_{\mathcal{H}}.$$

2.5 Dense sets in $W_0^{1,p(\cdot)}(\Omega)$ and $W^{1,\mathcal{H}}(\Omega)$

Let $\{\phi_i\}$ and $\{\lambda_i\}$ be the eigenfunctions and the corresponding eigenvalues of the Dirichlet problem for the Laplacian:

$$(\nabla\phi_i, \nabla\psi)_{2,\Omega} = \lambda_i(\phi_i, \psi) \quad \forall\psi \in H_0^1(\Omega). \tag{2.8}$$

The functions ϕ_i form an orthonormal basis of $L^2(\Omega)$ and are mutually orthogonal in $H_0^1(\Omega)$. If $\partial\Omega \in C^k$, $k \ge 1$, then $\phi_i \in C^\infty(\Omega) \cap H^k(\Omega)$. Let us denote by $H_{\mathcal{D}}^k(\Omega)$ the subspace of the Hilbert space $H^k(\Omega)$ composed of the functions f for which

$$f = 0, \quad \Delta f = 0, \quad \dots, \Delta^{[\frac{k-1}{2}]} f = 0 \quad \text{on } \partial\Omega, \quad H_{\mathcal{D}}^0(\Omega) = L^2(\Omega).$$

The relations

$$[f, g]_k = \begin{cases} (\Delta^{\frac{k}{2}} f, \Delta^{\frac{k}{2}} g)_{2,\Omega} & \text{if } k \text{ is even}, \\ (\Delta^{\frac{k-1}{2}} f, \Delta^{\frac{k-1}{2}} g)_{H^1(\Omega)} & \text{if } k \text{ is odd} \end{cases}$$

define an equivalent inner product on $H_{\mathcal{D}}^k(\Omega)$: $[f, g]_k = \sum_{i=1}^\infty \lambda_i^k f_i g_i$, where f_i, and g_i are the Fourier coefficients of f, and g in the basis $\{\phi_i\}$ of $L^2(\Omega)$. The corresponding equivalent norm of $H_{\mathcal{D}}^k(\Omega)$ is defined by $\|f\|^2_{H_{\mathcal{D}}^k(\Omega)} = [f, f]_k$. Let $f^{(m)} = \sum_{i=1}^m f_i \phi_i$ be the partial sums of the Fourier series of $f \in L^2(\Omega)$. The following assertion is well known.

Proposition 2.1. *Let $\partial\Omega \in C^k$, $k \ge 1$. A function f can be represented by the Fourier series in the system $\{\phi_i\}$, convergent in the norm of $H^k(\Omega)$, if and only if $f \in H_{\mathcal{D}}^k(\Omega)$. If $f \in H_{\mathcal{D}}^k(\Omega)$, then the series $\sum_{i=1}^\infty \lambda_i^k f_i^2$ is convergent, its sum is bounded by $C\|f\|_{H^k(\Omega)}$ with an independent of f constant C, and $\|f^{(m)} - f\|_{H^k(\Omega)} \to 0$ as $m \to \infty$. If $k \ge [\frac{N}{2}] + 1$, then the Fourier series in the system $\{\phi_i\}$ of every function $f \in H_{\mathcal{D}}^k(\Omega)$ converges to f in $C^{k-[\frac{N}{2}]-1}(\overline{\Omega})$.*

Proposition 2.2. ([23], Th. 4.7, Proposition 4.10). *Let $\partial\Omega \in Lip$ and $p \in C_{\log}(\overline{\Omega})$. Then the set $C_c^\infty(\Omega)$ is dense in $W_0^{1,p(\cdot)}(\Omega)$.*

Given a function $u \in W_0^{1,p(\cdot)}(\Omega)$ with $p \in C_{\log}(\overline{\Omega})$, the smooth approximations of u in $W_0^{1,p(\cdot)}(\Omega)$ are obtained by means of mollification. The proof relies on the boundedness of the maximal Hardy-Littlewood operator in $L^{p(\cdot)}(\Omega)$ with $p \in C_{\log}(\overline{\Omega})$.

Let us denote $\mathcal{P}_m = \operatorname{span}\{\phi_1, \dots, \phi_m\}$, where ϕ_i are the solutions of problem (2.9).

Lemma 2.1. *If $\partial\Omega \in C^k$, $q \in C_{\log}(\overline{\Omega})$ and*

$$k \ge N\left(\frac{1}{2} + \frac{1}{N} - \frac{1}{q^+}\right), \quad q^+ = \max_{\overline{\Omega}} q(x), \tag{2.9}$$

then $\bigcup_{m=1}^\infty \mathcal{P}_m$ is dense in $W_0^{1,q(\cdot)}(\Omega)$.

Proof. Given $v \in W_0^{1,q(\cdot)}(\Omega)$ we have to show that for every $\varepsilon > 0$, there is $m \in \mathbb{N}$ and $v_m \in \mathcal{P}_m$ such that $\|v - v_m\|_{W_0^{1,q(\cdot)}(\Omega)} < \varepsilon$. Fix some $\varepsilon > 0$. By Proposition 2.2, there is $v_\varepsilon \in C_c^\infty(\Omega) \subset H_{\mathcal{D}}^k(\Omega)$ such that $\|v - v_\varepsilon\|_{W_0^{1,q(\cdot)}(\Omega)} < \varepsilon/2$. By Proposition 2.1 $v_\varepsilon^{(m)}(x) = \sum_{i=1}^m v_i\phi_i(x) \in H^k(\Omega)$ and $v_\varepsilon^{(m)} \to v_\varepsilon$ in $H^k(\Omega)$, therefore for every $\delta > 0$, there is $m \in \mathbb{N}$ such that $\|v_\varepsilon - v_\varepsilon^{(m)}\|_{H^k(\Omega)} < \delta$. Since k, N, and q satisfy condition (2.10), the embeddings $H_{\mathcal{D}}^k(\Omega) \subset W_0^{1,q^+}(\Omega) \subseteq W_0^{1,q(\cdot)}(\Omega)$ are continuous:

$$\|w\|_{W_0^{1,q(\cdot)}(\Omega)} \le C\|w\|_{W_0^{1,q^+}(\Omega)} \le C'\|w\|_{H^k(\Omega)} \quad \forall w \in H_{\mathcal{D}}^k(\Omega)$$

with independent of w constants C, C'. Set $C'\delta = \varepsilon/2$. Then

$$\|v_\varepsilon - v_\varepsilon^{(m)}\|_{W_0^{1,q(\cdot)}(\Omega)} \le C'\|v_\varepsilon - v_\varepsilon^{(m)}\|_{H^k(\Omega)} < C'\delta = \frac{\varepsilon}{2}.$$

It follows that

$$\|v - v_\varepsilon^{(m)}\|_{W_0^{1,q(\cdot)}(\Omega)} \le \|v - v_\varepsilon\|_{W_0^{1,q(\cdot)}(\Omega)} + \|v_\varepsilon - v_\varepsilon^{(m)}\|_{W_0^{1,q(\cdot)}(\Omega)} < \frac{\varepsilon}{2} + C'\delta = \varepsilon. \qquad \square$$

Corollary 2.1. *If $p \in C_{\log}(\overline{Q}_T)$ and condition (2.9) is fulfilled, then*

$$\left\{v(x,t):\ v = \sum_{i=1}^\infty v_i(t)\phi_i(x),\ v_i(t) \in C^{0,1}[0,T]\right\} \text{ is dense in } \mathcal{W}_{p(\cdot)}(Q_T).$$

Since $\partial\Omega \in C^k$ with $k \ge 1$ and $r(x), s(x) \ge 2$, then $W^{1,\mathcal{H}}(\Omega) \subset W^{1,2}(\Omega)$ and the elements of $W^{1,\mathcal{H}}(\Omega)$ have traces on $\partial\Omega$. Let us denote by

$$W_0^{1,\mathcal{H}}(\Omega) := \{u \in W^{1,\mathcal{H}}(\Omega):\ u = 0 \text{ on } \partial\Omega \text{ in the sense of traces}\}$$

the closed subspace of $W^{1,\mathcal{H}}(\Omega)$.

Proposition 2.3. *Let $\partial\Omega \in C^1$ and $a_0, p, q \in C^{0,1}(\overline{\Omega})$. If $p(x) \le q(x)$ in Ω and*

$$\frac{s^+}{r^-} \le 1 + \frac{1}{N}, \quad s^+ = \max_{\overline{\Omega}}\{\max\{2, q(x)\}\}, \quad r^- = \min_{\overline{\Omega}}\{\max\{2, p(x)\}\},$$

then $W_0^{1,\mathcal{H}}(\Omega) = D^{1,\mathcal{H}}(\Omega)$, where $D^{1,\mathcal{H}}(\Omega)$ is the closure of $C_c^\infty(\Omega) \cap W^{1,\mathcal{H}}(\Omega)$ with respect to the norm of $W^{1,\mathcal{H}}(\Omega)$.

The assertion of Proposition 2.3 can be derived from [15, Th. 3.1] or [30, Ch. 6]. For a straightforward construction of the approximating sequence, one may adapt the proof of Proposition 2.2. The properties of the maximal operator and mollifiers in the space $L^{\mathcal{H}}(\Omega)$ needed for the proof are established in [30, Th. 4.3.4, Th. 4.3.7]. It is proven in [19, Theorem 2.23] that the function $\mathcal{H}$ satisfies all conditions of these theorems.

Lemma 2.2. *If $a_0, p, q \in C^{0,1}(\overline{\Omega})$ and $\partial\Omega \in C^k$ with*

$$k \ge N\left(\frac{1}{2} + \frac{1}{N} - \frac{1}{s^+}\right), \tag{2.10}$$

then $\bigcup_{m=1}^\infty \mathcal{P}_m$ is dense in $W_0^{1,\mathcal{H}}(\Omega)$.

We omit the details of the proof which imitates the proof of Lemma 2.1: by Proposition 2.3 $C_c^\infty(\Omega)$ is dense in $W_0^{1,\mathcal{H}}(\Omega)$, and since $C_c^\infty(\Omega) \subset H_{\mathcal{D}}^{(k)}(\Omega)$, every $v_\varepsilon \in H_{\mathcal{D}}^k(\Omega)$ can be approximated by $v_\varepsilon^{(m)} \in \mathcal{P}_m$.

3 Assumptions and main results

Let $p, q : Q_T \mapsto \mathbb{R}$ be measurable functions satisfying the conditions

$$\begin{aligned} &\frac{2N}{N+2} < p_- \le p(z) \le p_+ \text{ in } \overline{Q}_T, \\ &\frac{2N}{N+2} < q_- \le q(z) \le q_+ \text{ in } \overline{Q}_T, \quad p^\pm,\ q^\pm = \text{const}. \end{aligned} \tag{3.1}$$

Assume that $p,\ q \in W^{1,\infty}(Q_T)$ as functions of variables $z = (x, t)$: there exist positive constants C^*, C^{**}, C_*, C_{**} such that

$$\begin{aligned} &\operatorname*{ess\,sup}_{Q_T} |\nabla p| \le C_* < \infty, \quad \operatorname*{ess\,sup}_{Q_T} |p_t| \le C^*, \\ &\operatorname*{ess\,sup}_{Q_T} |\nabla q| \le C_{**} < \infty, \quad \operatorname*{ess\,sup}_{Q_T} |q_t| \le C^{**}. \end{aligned} \tag{3.2}$$

The modulating coefficient $a(\cdot)$ is assumed to satisfy the following conditions:

$$a(z) \ge 0 \text{ in } \overline{Q}_T, \quad a \in C([0, T]; C^{0,1}(\overline{\Omega})), \quad \operatorname*{ess\,sup}_{Q_T} |a_t| \le C_a, \tag{3.3}$$

$C_a = \text{const}$. We do not impose any condition on the null set of the function a in $\overline{Q}_T$. If $F = f_0(z)$, we do not distinguish between the cases of degenerate and singular equations. It is possible that $p(z) < 2$ and $q(z) > 2$ at the same point $z \in Q_T$. To study the equation with the nonlinear source, we assume $p^- \ge 2$.

Definition 3.1. A function $u : Q_T \mapsto \mathbb{R}$ is called *strong solution* of problem (1.1) if
(1) $u \in \mathcal{W}_{q(\cdot)}(Q_T)$, $u_t \in L^2(Q_T)$, $|\nabla u| \in L^\infty(0, T; L^{s(\cdot)}(\Omega))$ with $s(z) = \max\{2, p(z)\}$,
(2) for every $\psi \in \mathcal{W}_{q(\cdot)}(Q_T)$,

$$\int_{Q_T} u_t \psi \mathrm{d}z + \int_{Q_T} (|\nabla u|^{p(z)-2} + a(z)|\nabla u|^{q(z)-2}) \nabla u \cdot \nabla \psi \mathrm{d}z = \int_{Q_T} F(z, u, \nabla u) \psi \mathrm{d}z, \tag{3.4}$$

(3) for every $\phi \in C_0^1(\Omega)$

$$\int_\Omega (u(x, t) - u_0(x)) \phi \mathrm{d}x \to 0 \quad \text{as } t \to 0.$$

The main results are given in the following theorems.

Theorem 3.1. *Let $\Omega \subset \mathbb{R}^N$, $N \ge 2$, be a bounded domain with the boundary $\partial\Omega \in C^k$, $k \ge 2 + \left[\frac{N}{2}\right]$. Assume that $p(\cdot)$, and $q(\cdot)$ satisfy conditions* (3.1), and (3.2), *and there exists a constant,*

$$r \in (0, r^*), \quad r^* = \frac{4p^-}{p^-(N+2) + 2N},$$

such that

$$p(z) \le q(z) \le p(z) + \frac{r}{2} \text{ in } \overline{Q}_T. \tag{3.5}$$

If $a(\cdot)$ satisfies conditions (3.3) *and $F \equiv f_0(z)$, then for every $f_0 \in L^2(0, T; W_0^{1,2}(\Omega))$ and $u_0 \in W_0^{1,\mathcal{H}}(\Omega)$ with*

$$\int_\Omega (|\nabla u_0|^2 + |\nabla u_0|^{p(x,0)} + a(x, 0)|\nabla u_0|^{q(x,0)}) \mathrm{d}x = K < \infty, \tag{3.6}$$

problem (1.1) *has a unique strong solution u. This solution satisfies the estimate*

$$\|u_t\|^2_{2,Q_T} + \operatorname*{ess\,sup}_{(0,T)} \int_\Omega (|\nabla u|^{s(z)} + a(z)|\nabla u|^{q(z)})\mathrm{d}x + \int_{Q_T} |\nabla u|^{p(z)+r}\,\mathrm{d}z \le C \tag{3.7}$$

with the exponent $s(z) = \max\{2, p(z)\}$ *and a constant* C, *which depends on* N, $\partial\Omega$, T, $p^\pm$, $q^\pm$, r, *the constants in conditions* (3.2), (3.3), $\|f_0\|_{L^2(0,T;\,W_0^{1,2}(\Omega))}$, *and* K.

Assume that the source F may depend on the solution and its gradient. Let F be defined by (1.2) with Ψ and Φ satisfying the following conditions:

$$\begin{aligned}&\Psi \in C^0(\overline{Q}_T \times \mathbb{R}), \quad \Psi(z,0) = 0, \quad |\Psi(z,r)| \le C|r|^{\sigma(z)-1},\\ &|\nabla\Psi(z,v)| \le C(|v|^{\sigma(z)-1}(1 + |\ln|v||) + |v|^{\sigma(z)-2}|\nabla v|)\\ &\text{with a constant C and a continuous function } \sigma \text{ in } \overline{Q}_T \text{ such that}\\ &2 \le \sigma^- \le \sigma^+ < \infty, \quad \sigma^- = \inf_{Q_T}\sigma(z), \quad \sigma^+ = \sup_{Q_T}\sigma(z),\end{aligned} \tag{3.8}$$

and

$$\begin{aligned}&\Phi \in C^0(\overline{Q}_T \times \mathbb{R}^N),\\ &\text{there exist constants L, M such that for all } z \in \overline{Q}_T\\ &|\Phi(z,s)| \le \begin{cases} L|s|^{p(z)-1} & \text{if } |s| < 1,\\ M|s|^{\frac{p(z)}{2}} & \text{if } |s| \ge 1.\end{cases}\end{aligned} \tag{3.9}$$

Theorem 3.2. *Let in the conditions of* Theorem 3.1, $\Psi \not\equiv 0$ *and/or* $\Phi \not\equiv 0$.

(1) *Assume that* Ψ *satisfies conditions* (3.8) *and* Φ *satisfies conditions* (3.9). *If*

$$2 < p^-, \quad \sigma^+ < \min\left\{p^-, 1 + p^- \frac{N+2}{2N}\right\}, \tag{3.10}$$

then for every $f_0 \in L^2(0,T;W_0^{1,2}(\Omega))$ *and* $u_0 \in W_0^{1,\mathcal{H}}(\Omega)$, *problem* (1.1) *has at least one strong solution* u. *The solution* u *satisfies estimate* (3.7) *with the constant depending on the same quantities as in the case* $F = f_0(z)$, *the constants in conditions* (3.8), (3.9), *and* $\sigma^\pm$. *If* $\Psi \equiv 0$, *the assertion is true if* (3.10) *is substituted by the inequality* $p^- \ge 2$.

(2) *The strong solution is unique if* $p(\cdot), q(\cdot)$ *satisfy the conditions of Theorem* 3.1, $\Phi \equiv 0$, *and either* $\Psi(z,u) = b(z)u$ *with a bounded coefficient* $b(z)$ *or* $\Psi(z,s)$ *is monotone decreasing with respect to* s *for a.e.* $z \in Q_T$.

An outline of the work. In Section 4, we collect several auxiliary assertions. We present estimates on the gradient trace on $\partial\Omega$ for the functions from variable Sobolev spaces and formulate the interpolation inequality, which enables us to prove global higher integrability of the gradient. This property turns out to be the key element in the proof of the existence theorems for problem (1.1) and the counterpart regularized problems (problem (5.1)).

A solution of problem (1.1) is obtained as the limit of the family of solutions of the nondegenerate problems (5.1) with the regularized fluxes:

$$\left((\varepsilon^2 + |\nabla u|^2)^{\frac{p(z)-2}{2}} + a(z)(\varepsilon^2 + |\nabla u|^2)^{\frac{q(z)-2}{2}}\right)\nabla u, \quad \varepsilon > 0.$$

For every $\varepsilon \in (0,1)$, problem (5.1) is solved with the method of Galerkin. In Section 5, we formulate the problems for the approximations. Section 6 is devoted to derive a priori estimates on the approximate solutions and their derivatives. For the convenience of presentation, we separate the cases $F = f_0(z)$ when the source is independent of the solution and its gradient, and the general case (1.2). In the latter case, the derivation of the a priori estimates requires additional restrictions on the range of the exponent p and the rate of growth of $F(z,u,\nabla u)$ with respect to the second and third arguments. The a priori estimates of Section 6 involve higher-order derivatives of the approximate solutions. This is where we make use of the interpolation inequalities of Section 4 to obtain the global higher integrability of the gradient which, in turn, yields uniform boundedness of the $L^{q(\cdot)}(Q_T)$-norms of the gradients of the approximate solutions. Despite the fact that the initial datum belongs to a Musielak-Orlicz space, it turns out that the approximate solutions form a uniformly bounded sequence in a variable Sobolev space. This property is crucial in the study of convergence of the sequence of approximations and the regularity of its limit.

Theorems 3.1 and 3.2 are proven in Section 7. We show first that for every $\varepsilon > 0$, the constructed sequence of Galerkin's approximations contains a subsequence, which converges to a strong solution u_ε of the regularized problem (5.1). The proof relies on the compactness and monotonicity of the fluxes. To pass to the limit in the nonlinear source term, we need the pointwise convergence of the gradients, which turns out to be a byproduct of the uniform higher integrability. Existence of a solution to problem (1.1) is established in a similar way.

Remark 3.1. The condition on the regularity of $\partial\Omega$ allows us to use Galerkin's finite-dimensional approximations in the proof of better regularity of the solution. However, this assumption is not necessary. It can be relaxed by means of approximation of the domain with the boundary $\partial\Omega \in C^2$ by an expanding family of smooth domains. The family of the corresponding solutions converges to the solution of the problem in the approximated domain.

Remark 3.2. The Lipschitz continuity of the modulating coefficient $a(x,t)$ and the exponents $p(x,t)$, $q(x,t)$ at $t = 0$ is essential for approximation of the initial datum u_0 via density of smooth functions in the Musielak-Orlicz-Sobolev space. The proof relies on results of the recent work [19] where this regularity of the data was assumed. When $t > 0$, the same regularity of the data is assumed for the derivation of the interpolation inequalities, which involve second-order derivatives.

Notation: Throughout the rest of the text, the symbol C will be used to denote the constants that can be calculated or estimated through the data but whose exact values are unimportant. The value of C may vary from line to line even inside the same formula. Whenever it does not cause a confusion, we omit the arguments of the variable exponents of nonlinearity and the coefficients. We will use the shorthand notation $|v_{xx}|^2 = \sum_{i,j=1}^{N}|v_{x_ix_j}|^2$.

4 Auxiliary propositions

Until the end of this section, the notation $p(\cdot)$, $q(\cdot)$, $a(\cdot)$ is used for functions not related to the exponents and coefficient in (1.1) and (5.1).

Lemma 4.1. (Lemma 1.32, [4]) *Let $\partial\Omega \in \mathrm{Lip}$ and $p \in C^0(\overline{Q}_T)$. Assume that $u \in L^\infty(0,T; L^2(\Omega)) \cap W_0^{1,p(\cdot)}(Q_T)$ and*

$$\operatorname*{ess\,sup}_{(0,T)} \|u(\cdot,t)\|_{2,\Omega}^2 + \int_{Q_T} |\nabla u|^{p(z)}\,\mathrm{d}z = M < \infty.$$

Then

$$\|u\|_{p(\cdot),Q_T} \le C, \qquad C = C(M, p^\pm, N, \omega),$$

where ω is the modulus of continuity of the exponent $p(\cdot)$.

The proof in [4] is given for the case $\Omega = B_R(x_0)$. To adapt it to the general case, it is sufficient to consider the zero continuation of u to a circular cylinder containing Q_T.

Let us accept the notation

$$\begin{aligned}&\beta_\varepsilon(\mathbf{s}) = \varepsilon^2 + |\mathbf{s}|^2,\\&\gamma_\varepsilon(z,\mathbf{s}) = (\varepsilon^2 + |\mathbf{s}|^2)^{\frac{p(z)-2}{2}} + a(z)(\varepsilon^2 + |\mathbf{s}|^2)^{\frac{q(z)-2}{2}}, \quad \mathbf{s} \in \mathbb{R}^N, \ z \in Q_T, \ \varepsilon > 0.\end{aligned} \tag{4.1}$$

With certain abuse of notation, we will denote by $\gamma_\varepsilon(x,\mathbf{s})$ the same function but with the exponents p, and q and the coefficient a depending on the variable $x \in \Omega$.

Lemma 4.2. (Lemma 4.1, [5]) *Let $\partial\Omega \in C^1$, $u \in C^2(\overline{\Omega})$ and $u = 0$ on $\partial\Omega$. Assume that*

$$\begin{aligned}&p : \Omega \mapsto [p^-, p^+], \quad p^\pm = \text{const},\\&\frac{2N}{N+2} < p^-, \quad p(\cdot) \in C^0(\overline{\Omega}), \quad \operatorname*{ess\,sup}_{\Omega} |\nabla p| = L,\\&\int_\Omega \beta_\varepsilon^{\frac{p(x)-2}{2}}(\nabla u)|u_{xx}|^2\,dx < \infty, \quad \int_\Omega u^2\,dx = M_0, \quad \int_\Omega |\nabla u|^{p(x)}\,dx = M_1.\end{aligned} \tag{4.2}$$

Then for every

$$\frac{2}{N+2} =: r_* < r < r^* := \frac{4p^-}{p^-(N+2) + 2N} \tag{4.3}$$

and every $\delta \in (0,1)$

$$\int_\Omega \beta_\varepsilon^{\frac{p(x)+r-2}{2}}(\nabla u)|\nabla u|^2\,dx \le \delta \int_\Omega \beta_\varepsilon^{\frac{p(x)-2}{2}}(\nabla u)|u_{xx}|^2\,dx + C\left(1 + \int_\Omega |\nabla u|^{p(x)}\,dx\right) \tag{4.4}$$

with an independent of u constant $C = C(\partial\Omega, \delta, p^\pm, N, r, M_0, M_1)$.

Theorem 4.1. (Theorem 4.1, [5]) *Let $\partial\Omega \in C^1$, $u \in C^0([0,T]; C^2(\overline{\Omega}))$ and $u = 0$ on $\partial\Omega \times [0,T]$. Assume that*

$$\begin{aligned}&p : Q_T \mapsto [p^-, p^+], \ p^\pm = \text{const},\\&p \in C^0(\overline{Q}_T) \text{ with the modulus of continuity } \omega\\&\frac{2N}{N+2} < p^-, \quad \operatorname*{ess\,sup}_{Q_T} |\nabla p| = L,\\&\int_{Q_T} \beta_\varepsilon^{\frac{p(z)-2}{2}}(\nabla u)|u_{xx}|^2\,dz < \infty, \quad \sup_{(0,T)} \|u(t)\|_{2,\Omega}^2 = M_0, \quad \int_{Q_T} |\nabla u|^{p(z)}\,dz = M_1.\end{aligned} \tag{4.5}$$

Then for every

$$\frac{2}{N+2} = r_* < r < r^* = \frac{4p^-}{p^-(N+2) + 2N}$$

and every $\delta \in (0,1)$, the function u satisfies the inequality

$$\int_{Q_T} \beta_\varepsilon^{\frac{p(z)+r-2}{2}}(\nabla u)|\nabla u|^2\,dz \le \delta \int_{Q_T} \beta_\varepsilon^{\frac{p(z)-2}{2}}(\nabla u)|u_{xx}|^2\,dz + C\left(1 + \int_{Q_T} |\nabla u|^{p(z)}\,dz\right) \tag{4.6}$$

with an independent of u constant $C = C(N, \partial\Omega, T, \delta, p^\pm, \omega, r, M_0, M_1)$.

Lemma 4.3. *Let $\Omega \subset \mathbb{R}^N$, $N \ge 2$ be a bounded domain with the boundary $\partial\Omega \in C^2$, and $a \in W^{1,\infty}(\Omega)$ be a given nonnegative function. Assume that $v \in W^{3,2}(\Omega) \cap W_0^{1,2}(\Omega)$ and denote*

$$K = \int_{\partial\Omega} a(x)(\varepsilon^2 + |\nabla v|^2)^{\frac{p(x)-2}{2}}(\Delta v(\nabla v \cdot \mathbf{n}) - \nabla(\nabla v \cdot \mathbf{n})\cdot\nabla v)\,dS, \tag{4.7}$$

where $\mathbf{n}$ stands for the exterior normal to $\partial\Omega$. There exists a constant $L = L(\partial\Omega)$ such that

$$K \leq L \int_{\partial\Omega} a(x)(\varepsilon^2 + |\nabla v|^2)^{\frac{p(x)-2}{2}} |\nabla v|^2 \, dS.$$

Lemma 4.3 follows from the well-known assertions, see, e.g., [33, Ch. 1, Sec. 1.5] for the case $a \equiv 1$, $N = 2, 3$, or [3, Lemma A.1] for the case of an arbitrary dimension.

Lemma 4.4. *Let $\partial\Omega$ be a Lipschitz-continuous surface and $a(\cdot)$ be a nonnegative function on $\overline{\Omega}$. Assume that $a, q \in W^{1,\infty}(\Omega)$, with*

$$\|\nabla q\|_{\infty,\Omega} \leq L < \infty, \quad \|\nabla a\|_{\infty,\Omega} \leq L_0 < \infty.$$

There exists a constant $\delta = \delta(\partial\Omega)$ such that for every $u \in W^{1,q(\cdot)}(\Omega)$

$$\delta \int_{\partial\Omega} a(x)(\varepsilon^2 + |u|^2)^{\frac{q(x)-2}{2}} |u|^2 \, dS \leq C \int_{\Omega} (a(x)|u|^{q(x)-1}|\nabla u| + a(z)|u|^{q(x)} |\ln|u|| + |u|^{q(x)} + 1) dx \tag{4.8}$$

with a constant $C = C(q^+, L, L_0, N, \Omega)$.

The assertion immediately follows from [5, Lemma 4.4].

Corollary 4.1. *Under the conditions of Lemma 4.4, for every $\lambda \in (0, 1)$ and $\varepsilon \in (0, 1)$*

$$\begin{aligned}\int_{\partial\Omega} a(x)(\varepsilon^2 + |u|^2)^{\frac{q(x)-2}{2}} |u|^2 dS \leq \lambda \int_{\Omega} a(x)(\varepsilon^2 + |u|^2)^{\frac{q(x)-2}{2}} |\nabla u|^2 dx + L_0 \int_{\Omega} |u|^{q(x)} dx \\ + L \int_{\Omega} a(z)|u|^{q(x)} |\ln|u|| dx + K\end{aligned} \tag{4.9}$$

with independent of u constants K, L, L_0.

Theorem 4.2. *Let $\partial\Omega \in C^2$, $u \in C^2(\overline{\Omega})$ and $u = 0$ on $\partial\Omega$. Assume that $a(\cdot)$ satisfies the conditions of Lemma 4.4, $p(\cdot)$ satisfies the conditions of Lemma 4.2, and*

$$q : \Omega \mapsto [q^-, q^+] \subset \left(\frac{2N}{N+2}, \infty\right), \quad q \in W^{1,\infty}(\Omega), \quad \operatorname*{ess\,sup}_{\Omega} |\nabla q| = L.$$

If for a.e. $x \in \Omega$

$$q(x) < p(x) + r \quad \text{with } \frac{2}{N+2} < r < \frac{4p^-}{p^-(N+2) + 2N},$$

then for every $\lambda \in (0, 1)$

$$\int_{\partial\Omega} \gamma_\varepsilon(x, \nabla u)|\nabla u|^2 \, dS \leq \lambda \int_{\Omega} \gamma_\varepsilon(x, \nabla u)|u_{xx}|^2 \, dx + C\left(1 + \int_{\Omega} |\nabla u|^{p(x)} dx\right) \tag{4.10}$$

with a constant C depending on λ and the constants $p^\pm$, N, L, L_0, but independent of u.

We omit the proof that follows from Lemma 4.4 and Corollary 4.1 – see the proofs of Theorem 4.2 and Corollary 4.1 in [5].

Since $\mathcal{P}_m \subset C^1(\overline{\Omega}) \cap H_0^3(\Omega)$, the interpolation inequalities of this section remain true for every function $w \in \mathcal{P}_m$, $m \in \mathbb{N}$.

5 Regularized problem

Given $\varepsilon > 0$, let us consider the following family of regularized double-phase parabolic equations:

$$\begin{cases} \partial_t u - \operatorname{div}(\gamma_\varepsilon(z, \nabla u)\nabla u) = F(z, u, \nabla u) & \text{in } Q_T, \\ u = 0 & \text{on } \Gamma_T, \\ u(0, .) = u_0 & \text{in } \Omega, \ \ \varepsilon \in (0, 1), \end{cases} \tag{5.1}$$

where $F(z, u)$ is defined in (1.2), $\gamma_\varepsilon(z, \mathbf{s})$ is introduced in (4.1), and $\gamma_\varepsilon(z, \nabla u)\nabla u$ is the regularized flux function. The solution of problem (5.1) is understood in the sense of Definition 3.1.

5.1 Galerkin's method

Let $\varepsilon > 0$ be a fixed parameter. The sequence $\{u_\varepsilon^{(m)}\}$ of finite-dimensional Galerkin's approximations for the solutions of the regularized problem (5.1) is sought in the form

$$u_\varepsilon^{(m)}(x, t) = \sum_{j=1}^{m} u_j^{(m)}(t)\phi_j(x), \tag{5.2}$$

where $\phi_j \in W_0^{1,2}(\Omega)$ and $\lambda_j > 0$ are the eigenfunctions and the corresponding eigenvalues of problem (2.8). The coefficients $u_j^{(m)}(t)$ are defined as the solutions of the Cauchy problem for the system of m ordinary differential equations:

$$\begin{cases} (u_j^{(m)})'(t) = -\displaystyle\int_\Omega \gamma_\varepsilon(z, \nabla u_\varepsilon^{(m)})\nabla u_\varepsilon^{(m)} \cdot \nabla\phi_j \mathrm{d}x + \int_\Omega F(z, u_\varepsilon^{(m)}, \nabla u_\varepsilon^{(m)})\phi_j \mathrm{d}x, \\ u_j^{(m)}(0) = (u_0^{(m)}, \phi_j)_{2,\Omega}, \quad j = 1, 2, \dots, m, \end{cases} \tag{5.3}$$

where γ_ε is defined in (4.1). By Lemma 2.2, the functions $u_0^{(m)} \in \mathcal{P}_m$ can be chosen so that

$$\begin{aligned} & u_0^{(m)} = \sum_{j=1}^{m} (u_0, \phi_j)_{2,\Omega}\phi_j \in \operatorname{span}\{\phi_1, \phi_2, \dots, \phi_m\}, \\ & u_0^{(m)} \to u_0 \quad \text{in } W_0^{1,\mathcal{H}}(\Omega). \end{aligned} \tag{5.4}$$

By the Carathéodory existence theorem, for every finite m, system (5.3) has a solution $(u_1^{(m)}, u_2^{(m)}, \dots, u_m^{(m)})$ in the extended sense on an interval $(0, T_m)$, and the functions $u_i^{(m)}(t)$ are absolutely continuous and differentiable a.e. in $(0, T_m)$. The a priori estimates derived in Section 6 show that for every m the function $u_\varepsilon^{(m)}(x, T_m)$ belongs to $\operatorname{span}\{\phi_1, \dots, \phi_m\}$ and satisfies the estimate

$$\|\nabla u_\varepsilon^{(m)}(\cdot, T_m)\|_{2,\Omega}^2 + \mathcal{F}(u_\varepsilon(\cdot, T_m), T_m) \le C + \|f_0\|_{2,Q_T}^2 + \|\nabla u_0^{(m)}\|_{2,\Omega}^2 + \mathcal{F}(u_0^{(m)}, 0)$$

with the function $\mathcal{F}$ defined in (1.5) and a constant C independent of m and ε. This estimate allows one to continue each of $u_\varepsilon^{(m)}$ to the maximal existence interval $(0, T)$.

6 *A priori* estimates

6.1 *A priori* estimates I: the case $\Phi \equiv 0$ and $\Psi \equiv 0$

Lemma 6.1. *Let Ω be a bounded domain with the boundary $\partial\Omega \in$ Lip, $p(\cdot)$, $q(\cdot)$ satisfy (3.1), $a(\cdot)$ satisfies (3.3), $u_0 \in L^2(\Omega)$, and $f_0 \in L^2(Q_T)$. The function $u_\varepsilon^{(m)}$ satisfies the estimates*

$$\sup_{t\in(0,T)} \|u_\varepsilon^{(m)}(\cdot, t)\|_{2,\Omega}^2 + \int_{Q_T} \gamma_\varepsilon(z, \nabla u_\varepsilon^{(m)})|\nabla u_\varepsilon^{(m)}|^2 \,\mathrm{d}z \le C_1 \mathrm{e}^T\Big(\|f_0\|_{2,Q_T}^2 + \|u_0\|_{2,\Omega}^2\Big) \tag{6.1}$$

and

$$\int_{Q_T} (|\nabla u_\varepsilon^{(m)}|^{p(z)} + a(z)|\nabla u_\varepsilon^{(m)}|^{q(z)})\mathrm{d}z \le C_2 \int_{Q_T} \gamma_\varepsilon(z, \nabla u_\varepsilon^{(m)})|\nabla u_\varepsilon^{(m)}|^2\,\mathrm{d}z + C_3, \tag{6.2}$$

where the constants C_i are independent of ε and m.

Proof. By multiplying jth equation of (5.3) by $u_j^{(m)}(t)$ and then by summing up the results for $j = 1, 2, \dots, m$, we obtain

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|u_\varepsilon^{(m)}(\cdot, t)\|_{2,\Omega}^2 = \sum_{j=1}^m u_j^{(m)}(t)(u_j^{(m)})'(t) = -\int_\Omega \gamma_\varepsilon(z, \nabla u_\varepsilon^{(m)})|\nabla u_\varepsilon^{(m)}|^2\,\mathrm{d}x + \int_\Omega f_0(x,t)u_\varepsilon^{(m)}\mathrm{d}x. \tag{6.3}$$

By the Cauchy inequality,

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|u_\varepsilon^{(m)}(\cdot, t)\|_{2,\Omega}^2 + \int_\Omega \gamma_\varepsilon(z, \nabla u_\varepsilon^{(m)})|\nabla u_\varepsilon^{(m)}|^2\,\mathrm{d}x \le \frac{1}{2}\|f_0(\cdot, t)\|_{2,\Omega}^2 + \frac{1}{2}\|u_\varepsilon^{(m)}(\cdot, t)\|_{2,\Omega}^2. \tag{6.4}$$

Now, rewriting the last inequality in the equivalent form,

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}(\mathrm{e}^{-t}\|u_\varepsilon^{(m)}(\cdot, t)\|_{L^2(\Omega)}^2) + \mathrm{e}^{-t}\int_\Omega \gamma_\varepsilon(z, \nabla u_\varepsilon^{(m)})|\nabla u_\varepsilon^{(m)}|^2\,\mathrm{d}x \le \frac{\mathrm{e}^{-t}}{2}\|f_0(\cdot, t)\|_{2,\Omega}^2,$$

and integrating with respect to t, we arrive at the inequality

$$\sup_{t\in(0,T)} \|u_\varepsilon^{(m)}(\cdot, t)\|_{L^2(\Omega)}^2 + \int_{Q_T} \gamma_\varepsilon(z, \nabla u_\varepsilon^{(m)})|\nabla u_\varepsilon^{(m)}|^2\,\mathrm{d}z \le C\mathrm{e}^T(\|f_0\|_{2,Q_T}^2 + \|u_0\|_{2,\Omega}^2), \tag{6.5}$$

where the constant C is independent of ε and m. Since $a(z)$ is a nonnegative bounded function, the second assertion follows from (6.5) and the inequality

$$a(z)|\nabla u_\varepsilon^{(m)}|^{q(z)} \le a(z)(\varepsilon^2 + |\nabla u_\varepsilon^{(m)}|^2)^{\frac{q(z)}{2}} \le \begin{cases} 2a(z)(\varepsilon^2 + |\nabla u_\varepsilon^{(m)}|^2)^{\frac{q(z)-2}{2}}|\nabla u_\varepsilon^{(m)}|^2 & \text{if } |\nabla u_\varepsilon^{(m)}| \ge \varepsilon, \\ (2\varepsilon^2)^{\frac{q(z)}{2}}a(z) \le 2^{\frac{q^+}{2}}a(z) & \text{otherwise}. \end{cases} \tag{6.6}$$

□

Lemma 6.2. *Let Ω be a bounded domain with $\partial\Omega \in C^k$, $k \ge 2 + \left[\frac{N}{2}\right]$. Assume that $p(\cdot)$, $q(\cdot)$ satisfies* (3.1), (3.2), *and* (3.5) *and $a(\cdot)$ satisfy* (3.3). *If $u_0 \in W_0^{1,2}(\Omega)$ and $f_0 \in L^2((0,T); W_0^{1,2}(\Omega))$, then for a.e. $t \in (0,T)$, the following inequality holds*:

$$\begin{aligned} &\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|\nabla u_\varepsilon^{(m)}(\cdot, t)\|_{2,\Omega}^2 + C_0\int_\Omega \gamma_\varepsilon(z, \nabla u_\varepsilon^{(m)})|(u_\varepsilon^{(m)})_{xx}|^2\,\mathrm{d}x \\ &\quad \le C_1\left(1 + \int_\Omega |\nabla u_\varepsilon^{(m)}|^{p(z)}\,\mathrm{d}x + \|\nabla u_\varepsilon^{(m)}(\cdot, t)\|_{2,\Omega}^2 + \|f_0(\cdot, t)\|_{W_0^{1,2}(\Omega)}^2\right) \end{aligned} \tag{6.7}$$

with independent of m and ε constants $0 < C_0 < \min\{p^- - 1, 1\}$ *and* $C_1 > 0$.

Proof. Let us multiply each of equations in (5.3) by $\lambda_j u_j^{(m)}$ and sum up the results for $j = 1, 2, \dots, m$:

$$\begin{aligned} \frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|\nabla u_\varepsilon^{(m)}(\cdot, t)\|_{2,\Omega}^2 &= \sum_{j=1}^m \lambda_j (u_j^{(m)})'(t)u_j^{(m)}(t) \\ &= \sum_{j=1}^m \lambda_j u_j^{(m)} \int_\Omega \operatorname{div}(\gamma_\varepsilon(z, \nabla u_\varepsilon^{(m)})\nabla u_\varepsilon^{(m)})\phi_j\mathrm{d}x + \sum_{j=1}^m \lambda_j u_j^{(m)}\int_\Omega f_0(x,t)\phi_j\mathrm{d}x \\ &= -\int_\Omega \operatorname{div}(\gamma_\varepsilon(z, \nabla u_\varepsilon^{(m)})\nabla u_\varepsilon^{(m)})\Delta u_\varepsilon^{(m)}\mathrm{d}x - \int_\Omega f_0(x,t)\Delta u_\varepsilon^{(m)}\mathrm{d}x. \end{aligned} \tag{6.8}$$

Since $\partial\Omega \in C^k$ with $k \geq 2 + [\frac{N}{2}]$, then $u_\varepsilon^{(m)}(\cdot, t) \in \mathcal{P}_m \subset H_0^3(\Omega) \cap C^1(\overline{\Omega})$. Therefore, the first term on the right-hand of (6.8) can be transformed by means of the Green formula:

$$
\begin{aligned}
-\int_\Omega \operatorname{div}(\gamma_\varepsilon(z, \nabla u_\varepsilon^{(m)})\nabla u_\varepsilon^{(m)})\Delta u_\varepsilon^{(m)} \mathrm{d}x &= -\int_\Omega \left(\sum_{k=1}^N (u_\varepsilon^{(m)})_{x_k x_k}\right)\left(\sum_{i=1}^N (\gamma_\varepsilon(z, \nabla u_\varepsilon^{(m)})(u_\varepsilon^{(m)})_{x_i})_{x_i}\right)\mathrm{d}x \\
&= -\int_{\partial\Omega} \Delta u_\varepsilon^{(m)} \gamma_\varepsilon(z, \nabla u_\varepsilon^{(m)})(\nabla u_\varepsilon^{(m)} \cdot \mathbf{n})\mathrm{d}S \\
&\quad + \int_\Omega \sum_{k,i=1}^N (u_\varepsilon^{(m)})_{x_k x_k x_i} \gamma_\varepsilon(z, \nabla u_\varepsilon^{(m)})(u_\varepsilon^{(m)})_{x_i}\mathrm{d}x \\
&= -\int_{\partial\Omega} \gamma_\varepsilon(z, \nabla u_\varepsilon^{(m)}) \sum_{k,i=1}^N ((u_\varepsilon^{(m)})_{x_k x_k}(u_\varepsilon^{(m)})_{x_i} n_i - (u_\varepsilon^{(m)})_{x_k x_i}(u_\varepsilon^{(m)})_{x_i} n_k)\mathrm{d}S \\
&\quad - \int_\Omega \sum_{k,i=1}^N (u_\varepsilon^{(m)})_{x_k x_i}(\gamma_\varepsilon(z, \nabla u_\varepsilon^{(m)})(u_\varepsilon^{(m)})_{x_i})_{x_k}\mathrm{d}x \\
&= -\int_\Omega \gamma_\varepsilon(z, \nabla u_\varepsilon^{(m)})|(u_\varepsilon^{(m)})_{xx}|^2\mathrm{d}x + J_1 + J_2 + J_{\partial\Omega} + J_a,
\end{aligned}
$$

where $\mathbf{n} = (n_1, \ldots, n_N)$ is the outer normal vector to $\partial\Omega$,

$$
\begin{aligned}
J_1 &:= \int_\Omega (2 - p(z))(\varepsilon^2 + |\nabla u_\varepsilon^{(m)}|^2)^{\frac{p(z)-2}{2}-1}\left(\sum_{k=1}^N (\nabla u_\varepsilon^{(m)} \cdot \nabla(u_\varepsilon^{(m)})_{x_k})^2\right)\mathrm{d}x \\
&\quad + \int_\Omega (2 - q(z))a(z)(\varepsilon^2 + |\nabla u_\varepsilon^{(m)}|^2)^{\frac{q(z)-2}{2}-1}\left(\sum_{k=1}^N (\nabla u_\varepsilon^{(m)} \cdot \nabla(u_\varepsilon^{(m)})_{x_k})^2\right)\mathrm{d}x, \\
J_2 &= -\int_\Omega \sum_{k,i=1}^N (u_\varepsilon^{(m)})_{x_k x_i}(u_\varepsilon^{(m)})_{x_i}(\varepsilon^2 + |\nabla u_\varepsilon^{(m)}|^2)^{\frac{p(z)-2}{2}} \frac{p_{x_k}}{2} \ln(\varepsilon^2 + |\nabla u_\varepsilon^{(m)}|^2)\mathrm{d}x \\
&\quad - \int_\Omega \sum_{k,i=1}^N (u_\varepsilon^{(m)})_{x_k x_i}(u_\varepsilon^{(m)})_{x_i} a(z)(\varepsilon^2 + |\nabla u_\varepsilon^{(m)}|^2)^{\frac{q(z)-2}{2}} \frac{q_{x_k}}{2} \ln(\varepsilon^2 + |\nabla u_\varepsilon^{(m)}|^2)\mathrm{d}x, \\
J_{\partial\Omega} &= -\int_{\partial\Omega} \gamma_\varepsilon(z, \nabla u_\varepsilon^{(m)})(\Delta u_\varepsilon^{(m)}(\nabla u_\varepsilon^{(m)} \cdot \mathbf{n}) - \nabla u_\varepsilon^{(m)} \cdot \nabla(\nabla u_\varepsilon^{(m)} \cdot \mathbf{n}))\mathrm{d}S, \\
J_a &= -\int_\Omega \sum_{i,k=1}^N a_{x_k}(u_\varepsilon^{(m)})_{x_i}(\varepsilon^2 + |\nabla u_\varepsilon^{(m)}|^2)^{\frac{q(z)-2}{2}}(u_\varepsilon^{(m)})_{x_k x_i}.
\end{aligned}
$$

Substitution into (6.8) leads to the inequality

$$
\begin{aligned}
&\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|\nabla u_\varepsilon^{(m)}(\cdot, t)\|_{2,\Omega}^2 + \int_\Omega \gamma_\varepsilon(z, \nabla u_\varepsilon^{(m)})|(u_\varepsilon^{(m)})_{xx}|^2\mathrm{d}x \\
&\quad = J_1 + J_2 + J_{\partial\Omega} + J_a - \int_\Omega \nabla f_0 \cdot \nabla u_\varepsilon^{(m)}\mathrm{d}x \\
&\quad \leq J_1 + J_2 + J_{\partial\Omega} + J_a + \frac{1}{2}\|\nabla u_\varepsilon^{(m)}(\cdot, t)\|_{2,\Omega}^2 + \frac{1}{2}\|f_0(\cdot, t)\|_{W_0^{1,2}(\Omega)}^2.
\end{aligned}
\tag{6.9}
$$

The terms on the right-hand side of (6.9) are estimated in three steps.

Step 1: Estimate on J_1. Since $a(z) \geq 0$ and $p(z) < q(z)$ in Q_T, the term J_1 is merged on the left-hand side.

Indeed:

$$
\begin{aligned}
J_1 &= \int\limits_{\{x\in\Omega:p(z)\geq 2\}} (2-p(z))\cdots + \int\limits_{\{x\in\Omega:p(z)<2\}} (2-p(z))\cdots + \int\limits_{\{x\in\Omega:q(z)\geq 2\}} (2-q(z))\cdots + \int\limits_{\{x\in\Omega:q(z)<2\}} (2-q(z))\cdots \\
&\leq \int\limits_{\{p(z)<2\}} (2-p(z))(\varepsilon^2+|\nabla u_\varepsilon^{(m)}|^2)^{\frac{p(z)-2}{2}-1}\left(\sum_{k=1}^{N}(\nabla u_\varepsilon^{(m)}\cdot\nabla(u_\varepsilon^{(m)})_{x_k})^2\right)\mathrm{d}x \\
&\quad + \int\limits_{\{q(z)<2\}} (2-q(z))a(z)(\varepsilon^2+|\nabla u_\varepsilon^{(m)}|^2)^{\frac{q(z)-2}{2}-1}\left(\sum_{k=1}^{N}(\nabla u_\varepsilon^{(m)}\cdot\nabla(u_\varepsilon^{(m)})_{x_k})^2\right)\mathrm{d}x,
\end{aligned}
$$

whence

$$
|J_1| \leq \max\{0, 2-p^-\}\int\limits_\Omega (\varepsilon^2+|\nabla u_\varepsilon^{(m)}|^2)^{\frac{p(z)-2}{2}}|(u_\varepsilon^{(m)})_{xx}|^2\,\mathrm{d}x + \max\{0,2-q^-\}\int\limits_\Omega a(z)(\varepsilon^2+|\nabla u_\varepsilon^{(m)}|^2)^{\frac{q(z)-2}{2}}|(u_\varepsilon^{(m)})_{xx}|^2\,\mathrm{d}x.
$$

Step 2: Estimate on J_2. By the Cauchy inequality, for every $\delta_0 > 0$,

$$
\begin{aligned}
|J_2| &\leq \frac{1}{2}\|\nabla p\|_{\infty,\Omega}\int\limits_\Omega\left((\varepsilon^2+|\nabla u_\varepsilon^{(m)}|^2)^{\frac{p(z)-2}{4}}\sum_{k,i=1}^{N}|(u_\varepsilon^{(m)})_{x_kx_i}|\right)\left(|(u_\varepsilon^{(m)})_{x_i}||\ln(\varepsilon^2+|\nabla u_\varepsilon^{(m)}|^2)|(\varepsilon^2+|\nabla u_\varepsilon^{(m)}|^2)^{\frac{p(z)-2}{4}}\right)\mathrm{d}x \\
&\quad + \frac{1}{2}\|\nabla q\|_{\infty,\Omega}\int\limits_\Omega\left((a(z))^{\frac{1}{2}}(\varepsilon^2+|\nabla u_\varepsilon^{(m)}|^2)^{\frac{q(z)-2}{4}}\sum_{k,i=1}^{N}|(u_\varepsilon^{(m)})_{x_kx_i}|\right)\left((a(z))^{\frac{1}{2}}|(u_\varepsilon^{(m)})_{x_i}||\ln(\varepsilon^2\right. \\
&\quad \left. + |\nabla u_\varepsilon^{(m)}|^2)|(\varepsilon^2+|\nabla u_\varepsilon^{(m)}|^2)^{\frac{q(z)-2}{4}}\right)\mathrm{d}x \\
&\leq \delta_0\int\limits_\Omega \gamma_\varepsilon(z,\nabla u_\varepsilon^{(m)})\sum_{k,i=1}^{N}|(u_\varepsilon^{(m)})_{x_kx_i}|^2\,\mathrm{d}x + C_1\int\limits_\Omega \ln^2(\varepsilon^2+|\nabla u_\varepsilon^{(m)}|^2)\gamma_\varepsilon(z,\nabla u_\varepsilon^{(m)})|\nabla u_\varepsilon^{(m)}|^2\,\mathrm{d}x
\end{aligned}
\tag{6.10}
$$

with a constant $C_1 = C_1(C^*, C^{**}, N, \delta_0)$. Let us denote

$$
\mathcal{M} = C_1\int\limits_\Omega \ln^2(\varepsilon^2+|\nabla u_\varepsilon^{(m)}|^2)\gamma_\varepsilon(z,\nabla u_\varepsilon^{(m)})|\nabla u_\varepsilon^{(m)}|^2\,\mathrm{d}x.
$$

For $\mu_1 \in (0,1)$ and $y > 0$, the following inequality holds:

$$
y^{\frac{p}{2}}\ln^2 y \leq \begin{cases} y^{\frac{p+\mu_1}{2}}\left(y^{\frac{-\mu_1}{2}}\ln^2(y)\right) \leq C(\mu_1,p^+)\left(y^{\frac{p+\mu_1}{2}}\right) & \text{if } y \geq 1, \\ y^{\frac{p^-}{2}}\ln^2(y) \leq C(p^-) & \text{if } y \in (0,1). \end{cases}
\tag{6.11}
$$

Let

$$
r_* = \frac{2}{N+2} \quad \text{and} \quad r^* = \frac{4p^-}{p^-(N+2)+2N}.
\tag{6.12}
$$

Take the numbers r_1 and r_2 such that

$$
r_1 \in (r_*, r^*), \quad r_2 \in (0,1), \quad q(z)+r_2 \leq p(z)+r_1 < p(z)+r^*
$$

and estimate $\mathcal{M}$ applying (6.11):

$$
\mathcal{M} \leq C\left(1 + \int\limits_\Omega(\varepsilon^2+|\nabla u_\varepsilon^{(m)}|^2)^{\frac{p(z)+r_1-2}{2}}|\nabla u_\varepsilon^{(m)}|^2\,\mathrm{d}x + \int\limits_\Omega a(z)(\varepsilon^2+|\nabla u_\varepsilon^{(m)}|^2)^{\frac{q(z)+r_2-2}{2}}|\nabla u_\varepsilon^{(m)}|^2\,\mathrm{d}x\right)
$$

with a constant $C = C(C_1, r_1, r_2)$. Let us transform the integrand of the second integral using the following inequality:

$$
\begin{aligned}
(\varepsilon^2+|\nabla u_\varepsilon^{(m)}|^2)^{\frac{q(z)+r_2-2}{2}}|\nabla u_\varepsilon^{(m)}|^2 &\le (\varepsilon^2+|\nabla u_\varepsilon^{(m)}|^2)^{\frac{q(z)+r_2}{2}}\\
&\le 1+(\varepsilon^2+|\nabla u_\varepsilon^{(m)}|^2)^{\frac{p(z)+r_1}{2}}\\
&\le 1+\begin{cases}(2\varepsilon^2)^{\frac{p(z)+r_1}{2}} & \text{if } |\nabla u_\varepsilon^{(m)}|<\varepsilon,\\ 2(\varepsilon^2+|\nabla u_\varepsilon^{(m)}|^2)^{\frac{p(z)+r_1-2}{2}}|\nabla u_\varepsilon^{(m)}|^2 & \text{if } |\nabla u_\varepsilon^{(m)}|\ge\varepsilon.\end{cases}
\end{aligned}
\tag{6.13}
$$

By using (6.13) and the interpolation inequality of Lemma 4.2, we finally obtain

$$
\begin{aligned}
\mathcal{M} &\le C\left(1+\int_\Omega(\varepsilon^2+|\nabla u_\varepsilon^{(m)}|^2)^{\frac{p(z)+r_1-2}{2}}|\nabla u_\varepsilon^{(m)}|^2\,\mathrm{d}x\right)\\
&\le \delta_1\int_\Omega(\varepsilon^2+|\nabla u_\varepsilon^{(m)}|^2)^{\frac{p(z)-2}{2}}|(u_\varepsilon^{(m)})_{xx}|^2\mathrm{d}x + C\left(1+\int_\Omega|\nabla u_\varepsilon^{(m)}|^{p(z)}\,\mathrm{d}x\right)
\end{aligned}
\tag{6.14}
$$

with any $\delta_1\in(0,1)$ and $C=C(\delta_1)$. Gathering (6.10) and (6.14), we finally obtain:

$$
|J_2|\le(\delta_0+\delta_1)\int_\Omega(\varepsilon^2+|\nabla u_\varepsilon^{(m)}|^2)^{\frac{p(z)-2}{2}}|(u_\varepsilon^{(m)})_{xx}|^2\mathrm{d}x + C\left(1+\int_\Omega|\nabla u_\varepsilon^{(m)}|^{p(z)}\,\mathrm{d}x\right)
$$

with a constant C depending on δ_i and $\|a(\cdot,t)\|_{\infty,\Omega}$, but independent of ε and m.

Step 3: Estimates on J_a and $J_{\partial\Omega}$. Let $\rho\in(r_*,r^*)$ be such that $2q(z)-p(z)<p(z)+\rho<p(z)+r^*$. Applying Young's inequality and (6.13), we obtain the estimate

$$
\begin{aligned}
|J_a| &\le \int_\Omega\sum_{i,k=1}^N|a_{x_k}||(u_\varepsilon^{(m)})_{x_i}|(\varepsilon^2+|\nabla u_\varepsilon^{(m)}|^2)^{\frac{q(z)-2}{2}}|(u_\varepsilon^{(m)})_{x_kx_i}|\mathrm{d}x\\
&\le \|\nabla a\|_{\infty,\Omega}\int_\Omega(\varepsilon^2+|\nabla u_\varepsilon^{(m)}|^2)^{\frac{2q(z)-p(z)}{4}}\left((\varepsilon^2+|\nabla u_\varepsilon^{(m)}|^2)^{\frac{p(z)-2}{4}}|(u_\varepsilon^{(m)})_{xx}|\right)\mathrm{d}x\\
&\le \tilde\delta\int_\Omega(\varepsilon^2+|\nabla u_\varepsilon^{(m)}|^2)^{\frac{p(z)-2}{2}}|(u_\varepsilon^{(m)})_{xx}|^2\mathrm{d}x + C(\tilde\delta)\int_\Omega(\varepsilon^2+|\nabla u_\varepsilon^{(m)}|^2)^{\frac{2q(z)-p(z)}{2}}\mathrm{d}x\\
&\le \tilde\delta\int_\Omega(\varepsilon^2+|\nabla u_\varepsilon^{(m)}|^2)^{\frac{p(z)-2}{2}}|(u_\varepsilon^{(m)})_{xx}|^2\mathrm{d}x + C'\left(1+\int_\Omega(\varepsilon^2+|\nabla u_\varepsilon^{(m)}|^2)^{\frac{p(z)+\rho}{2}}\mathrm{d}x\right)\\
&\le \tilde\delta\int_\Omega(\varepsilon^2+|\nabla u_\varepsilon^{(m)}|^2)^{\frac{p(z)-2}{2}}|(u_\varepsilon^{(m)})_{xx}|^2\mathrm{d}x + C''\left(1+\int_\Omega(\varepsilon^2+|\nabla u_\varepsilon^{(m)}|^2)^{\frac{p(z)+\rho-2}{2}}|\nabla u_\varepsilon^{(m)}|^2\,\mathrm{d}x\right),
\end{aligned}
$$

where $C''=C''(\|\nabla a\|_{\infty,\Omega},N,q)$ is independent of ε and m. By Lemma 4.2, we obtain

$$
|J_a|\le\delta_2\int_\Omega(\varepsilon^2+|\nabla u_\varepsilon^{(m)}|^2)^{\frac{p(z)-2}{2}}|(u_\varepsilon^{(m)})_{xx}|^2\mathrm{d}x + C\left(1+\int_\Omega|\nabla u_\varepsilon^{(m)}|^{p(z)}\,\mathrm{d}x\right)
$$

for any $\delta_2\in(0,1)$ and a constant C independent of ε and m.

To estimate $J_{\partial\Omega}$, we use Lemma 4.3 and Theorem 4.2:

$$
\begin{aligned}
|J_{\partial\Omega}| &\le \left|\int_{\partial\Omega}\gamma_\varepsilon(z,\nabla u_\varepsilon^{(m)})(\Delta u_\varepsilon^{(m)}(\nabla u_\varepsilon^{(m)}\cdot\mathbf{n})-\nabla u_\varepsilon^{(m)}\cdot\nabla(\nabla u_\varepsilon^{(m)}\cdot\mathbf{n}))\mathrm{d}S\right|\\
&\le C\int_{\partial\Omega}\gamma_\varepsilon(z,\nabla u_\varepsilon^{(m)})|\nabla u_\varepsilon^{(m)}|^2\,\mathrm{d}S\\
&\le \delta_3\int_\Omega\gamma_\varepsilon(z,\nabla u_\varepsilon^{(m)})|(u_\varepsilon^{(m)})_{xx}|^2\mathrm{d}x + C\left(1+\int_\Omega|\nabla u_\varepsilon^{(m)}|^{p(z)}\,\mathrm{d}x\right)
\end{aligned}
$$

with an arbitrary $\delta_3 \in (0,1)$ and C depending on δ_3, p, q, a, $\partial\Omega$, and their differential properties, but not on ε and m. To complete the proof and obtain (6.7), we gather the estimates of J_1, J_2, J_a, and $J_{\partial\Omega}$ and choose δ_i so small that

$$\min\{1, p^- - 1\} - \sum_{i=0}^{3} \delta_i = \eta > 0. \qquad \square$$

Lemma 6.3. *Under the conditions of* Lemma 6.2

$$\sup_{(0,T)} \|\nabla u_\varepsilon^{(m)}(\cdot, t)\|_{2,\Omega}^2 + \int_{Q_T} \gamma_\varepsilon(z, \nabla u_\varepsilon^{(m)})|(u_\varepsilon^{(m)})_{xx}|^2 \mathrm{d}z \le C\mathrm{e}^{C'T}(1 + \|\nabla u_0\|_{2,\Omega}^2 + \|f_0\|^2_{L^2(0,T;\, W_0^{1,2}(\Omega))}) \tag{6.15}$$

and for every $0 < r < \frac{4p^-}{p^-(N+2)+2N}$,

$$\int_{Q_T} |\nabla u_\varepsilon^{(m)}|^{q(z)} \mathrm{d}z + \int_{Q_T} |\nabla u_\varepsilon^{(m)}|^{p(z)+r} \mathrm{d}z \le C'' \tag{6.16}$$

with constants C, C', and C'' *independent of* m *and* ε.

Proof. Multiplying (6.7) by $\mathrm{e}^{-2C_1 t}$ and simplifying, we obtain the following differential inequality:

$$\frac{\mathrm{d}}{\mathrm{d}t}(\mathrm{e}^{-2C_1 t}\|\nabla u_\varepsilon^{(m)}(\cdot, t)\|_{2,\Omega}^2) \le C\mathrm{e}^{-2C_1 t}\left(1 + \int_\Omega |\nabla u_\varepsilon^{(m)}|^{p(z)} \mathrm{d}x + \|f_0(\cdot, t)\|^2_{W_0^{1,2}(\Omega)}\right).$$

Integrating it with respect to t and taking into account (6.1) and (6.2), we arrive at the following estimate: for every $t \in [0, T]$,

$$\begin{aligned}\|\nabla u_\varepsilon^{(m)}(\cdot, t)\|_{2,\Omega}^2 &\le C\mathrm{e}^{2C_1 T}(\|\nabla u_0\|_{2,\Omega}^2 + \mathrm{e}^T(1 + \|u_0\|_{2,\Omega}^2 + \|f_0\|_{2,Q_T}^2) + \|\nabla f\|_{2,Q_T}^2) \\ &\le C\mathrm{e}^{C'T}(1 + \|u_0\|^2_{W_0^{1,2}(\Omega)} + \|f_0\|^2_{L^2(0,T;\, W_0^{1,2}(\Omega))}).\end{aligned}$$

Substitution of the aforementioned estimate into (6.7) gives

$$\begin{aligned}&\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|\nabla u_\varepsilon^{(m)}(\cdot, t)\|_{2,\Omega}^2 + C_0 \int_\Omega \gamma_\varepsilon(z, \nabla u_\varepsilon^{(m)})|(u_\varepsilon^{(m)})_{xx}|^2 \mathrm{d}x \\ &\quad \le C_1\left(1 + \int_\Omega |\nabla u_\varepsilon^{(m)}|^{p(z)} \mathrm{d}x + \|\nabla u_0\|_{2,\Omega}^2 + \|f_0(\cdot, t)\|^2_{W_0^{1,2}(\Omega)}\right).\end{aligned}$$

Integrating it with respect to t and using (6.2) to estimate the integral of $|\nabla u_\varepsilon^{(m)}|^{p(z)}$ on the right-hand side, we obtain

$$\int_{Q_T} \gamma_\varepsilon(z, \nabla u_\varepsilon^{(m)})|(u_\varepsilon^{(m)})_{xx}|^2 \mathrm{d}z \le C\mathrm{e}^{C'T}(1 + \|\nabla u_0\|_{2,\Omega}^2 + \|f_0\|^2_{L^2(0,T;\, W_0^{1,2}(\Omega))}).$$

To prove estimate (6.16), we make use of Theorem 4.1. Let us fix a number $r \in (r_*, r^*)$ with r_*, r^* defined in (6.12). Split the cylinder Q_T into the two parts $Q_T^+ = Q_T \cap \{p(z) + r \ge 2\}$ and $Q_T^- = Q_T \cap \{p(z) + r < 2\}$ and represent

$$\int_{Q_T} |\nabla u_\varepsilon^{(m)}|^{p(z)+r} \mathrm{d}z = \int_{Q_T^+} |\nabla u_\varepsilon^{(m)}|^{p(z)+r} \mathrm{d}z + \int_{Q_T^-} |\nabla u_\varepsilon^{(m)}|^{p(z)+r} \mathrm{d}z \equiv I_+ + I_-.$$

Since

$$I_+ \le \int_{Q_T^+} (\varepsilon^2 + |\nabla u_\varepsilon^{(m)}|^2)^{\frac{p(z)+r-2}{2}} |\nabla u_\varepsilon^{(m)}|^2 \mathrm{d}z \le \int_{Q_T} (\varepsilon^2 + |\nabla u_\varepsilon^{(m)}|^2)^{\frac{p(z)+r-2}{2}} |\nabla u_\varepsilon^{(m)}|^2 \mathrm{d}z,$$

the estimate on I_+ follows immediately from Theorem 4.1 and (6.15). To estimate I_-, we set $B_+ = Q_T^- \cap \{z : |\nabla u_\varepsilon^{(m)}| \geq \varepsilon\}$, $B_- = Q_T^- \cap \{z : |\nabla u_\varepsilon^{(m)}| < \varepsilon\}$. The estimate on I_- follows from Theorem 4.1 and (6.15) because

$$\begin{aligned} I_- &= \int_{B_+\cup B_-} |\nabla u_\varepsilon^{(m)}|^{p(z)+r}\,\mathrm{d}z = \int_{B_+} (|\nabla u_\varepsilon^{(m)}|^2)^{\frac{p(z)+r-2}{2}} |\nabla u_\varepsilon^{(m)}|^2\,\mathrm{d}z + \int_{B_-} \varepsilon^{p(z)+r}\mathrm{d}z \\ &\leq 2^{\frac{2-r-p^-}{2}} \int_{B_+} (\varepsilon^2 + |\nabla u_\varepsilon^{(m)}|^2)^{\frac{p(z)+r-2}{2}} |\nabla u_\varepsilon^{(m)}|^2\,\mathrm{d}z + \varepsilon^{p^-+r} T|\Omega| \\ &\leq C\left(1 + \int_{Q_T} (\varepsilon^2 + |\nabla u_\varepsilon^{(m)}|^2)^{\frac{p(z)+r-2}{2}} |\nabla u_\varepsilon^{(m)}|^2\,\mathrm{d}z\right). \end{aligned}$$

By combining the aforementioned estimates, using the Young inequality, and applying (6.15), (6.2), and Theorem 4.1, we obtain (6.16) with $r \in (r_*, r^*)$:

$$\begin{aligned} \int_{Q_T} |\nabla u_\varepsilon^{(m)}|^{q(z)}\,\mathrm{d}z + \int_{Q_T} |\nabla u_\varepsilon^{(m)}|^{p(z)+r}\,\mathrm{d}z &\leq 1 + \int_{Q_T} |\nabla u_\varepsilon^{(m)}|^{p(z)+r}\,\mathrm{d}z \\ &\leq C\left(1 + \int_{Q_T} \gamma_\varepsilon(z, \nabla u_\varepsilon^{(m)}) |(u_\varepsilon^{(m)})_{xx}|^2 \mathrm{d}z + \int_{Q_T} |\nabla u_\varepsilon^{(m)}|^{p(z)}\,\mathrm{d}z\right) \leq C. \end{aligned}$$

If $r \in (0, r_*]$, the required inequality follows from Young's inequality. ☐

Corollary 6.1. *Under the conditions of* Lemma 6.3

$$\int_{Q_T} (\varepsilon^2 + |\nabla u_\varepsilon^{(m)}|^2)^{\frac{p(z)+r}{2}} \mathrm{d}z \leq C, \quad \varepsilon \in (0, 1), \tag{6.17}$$

with an independent of ε and m constant C.

Corollary 6.2. *Let condition* (3.5) *be fulfilled. Under the conditions of Lemma* 6.3

$$\|(\varepsilon^2 + |\nabla u_\varepsilon^{(m)}|^2)^{\frac{p(z)-2}{2}} \nabla u_\varepsilon^{(m)}\|_{q'(\cdot),Q_T} \leq C, \tag{6.18}$$

with a constant C independent of m and ε.

Proof. Condition (3.5) entails the inequality

$$\frac{q(z)(p(z)-1)}{q(z)-1} \leq q(z) \leq p(z) + r.$$

By Young's inequality, the assertion follows then from (6.17):

$$\int_{Q_T} (\varepsilon^2 + |\nabla u_\varepsilon^{(m)}|^2)^{\frac{q(z)(p(z)-1)}{2(q(z)-1)}} \mathrm{d}z \leq C\left(1 + \int_{Q_T} |\nabla u_\varepsilon^{(m)}|^{p(z)+r}\,\mathrm{d}z\right) \leq C. \qquad ☐$$

Lemma 6.4. *Assume that in the conditions of* Lemma 6.2, $u_0 \in W_0^{1,2}(\Omega) \cap W_0^{1,\mathcal{H}}(\Omega)$. *Then*

$$\|(u_\varepsilon^{(m)})_t\|_{2,Q_T}^2 + \sup_{(0,T)} \int_\Omega \gamma_\varepsilon(z, \nabla u_\varepsilon^{(m)}) |\nabla u_\varepsilon^{(m)}|^2\,\mathrm{d}x \leq C\left(1 + \int_\Omega (|\nabla u_0|^{p(x,0)} + a(x,0)|\nabla u_0|^{q(x,0)})\mathrm{d}x\right) + \|f_0\|_{2,Q_T}^2, \tag{6.19}$$

with an independent of m and ε constant C, which depends on the constants in conditions (3.2).

Proof. By multiplying (5.3) with $(u_j^{(m)})_t$ and summing over $j = 1, 2, \dots, m$, we obtain the equality

$$\int_\Omega (u_\varepsilon^{(m)})_t^2 \mathrm{d}x + \int_\Omega \gamma_\varepsilon(z, \nabla u_\varepsilon^{(m)}) \nabla u_\varepsilon^{(m)} \cdot \nabla (u_\varepsilon^{(m)})_t \mathrm{d}x = \int_\Omega f_0 (u_\varepsilon^{(m)})_t \mathrm{d}x. \tag{6.20}$$

By using the identity

$$\begin{aligned} a(z)(\varepsilon^2 + |\nabla u_\varepsilon^{(m)}|^2)^{\frac{q(z)-2}{2}} \nabla u_\varepsilon^{(m)} \cdot \nabla (u_\varepsilon^{(m)})_t &= \frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{a(z)(\varepsilon^2 + |\nabla u_\varepsilon^{(m)}|^2)^{\frac{q(z)}{2}}}{q(z)}\right) + \frac{a(z)q_t(z)(\varepsilon^2 + |\nabla u_\varepsilon^{(m)}|^2)^{\frac{q(z)}{2}}}{q^2(z)} \\ &\quad \times \left(1 - \frac{q(z)}{2} \ln((\varepsilon^2 + |\nabla u_\varepsilon^{(m)}|^2))\right) - \frac{a_t(\varepsilon^2 + |\nabla u_\varepsilon^{(m)}|^2)^{\frac{q(z)}{2}}}{q(z)}, \end{aligned}$$

we rewrite (6.20) as:

$$\begin{aligned} &\|(u_\varepsilon^{(m)})_t(\cdot, t)\|_{2,\Omega}^2 + \frac{\mathrm{d}}{\mathrm{d}t} \int_\Omega \left(\frac{(\varepsilon^2 + |\nabla u_\varepsilon^{(m)}|^2)^{\frac{p(z)}{2}}}{p(z)} + \frac{a(z)(\varepsilon^2 + |\nabla u_\varepsilon^{(m)}|^2)^{\frac{q(z)}{2}}}{q(z)} \right) \mathrm{d}x \\ &= \int_\Omega f_0 (u_\varepsilon^{(m)})_t \mathrm{d}x - \int_\Omega \frac{p_t(\varepsilon^2 + |\nabla u_\varepsilon^{(m)}|^2)^{\frac{p(z)}{2}}}{p^2(z)} \left(1 - \frac{p(z)}{2} \ln(\varepsilon^2 + |\nabla u_\varepsilon^{(m)}|^2)\right) \mathrm{d}x \\ &\quad - \int_\Omega \frac{a(z)q_t(z)(\varepsilon^2 + |\nabla u_\varepsilon^{(m)}|^2)^{\frac{q(z)}{2}}}{q^2(z)} \left(1 - \frac{q(z)}{2} \ln((\varepsilon^2 + |\nabla u_\varepsilon^{(m)}|^2))\right) \mathrm{d}x \\ &\quad + \int_\Omega \frac{a_t(\varepsilon^2 + |\nabla u_\varepsilon^{(m)}|^2)^{\frac{q(z)}{2}}}{q(z)} \mathrm{d}x \equiv \int_\Omega f_0 (u_\varepsilon^{(m)})_t \mathrm{d}x + \mathcal{J}_1 + \mathcal{J}_2 + \mathcal{J}_3. \end{aligned} \tag{6.21}$$

The first term on the right-hand side of (6.21) is estimated by the Cauchy inequality:

$$\left| \int_\Omega f_0 (u_\varepsilon^{(m)})_t \mathrm{d}x \right| \le \frac{1}{2} \|(u_\varepsilon^{(m)})_t(\cdot, t)\|_{2,\Omega}^2 + \frac{1}{2} \|f_0(\cdot, t)\|_{2,\Omega}^2. \tag{6.22}$$

To estimate $\mathcal{J}_i$, we use (6.1), (6.2), (6.11), (6.14), and (6.16). Fix two numbers $r_1 \in (r_*, r^*)$, $r_2 \in (0, 1)$ such that

$$q(z) + r_2 < p(z) + r_1 < p(z) + r^*.$$

Then

$$\begin{aligned} \sum_{i=1}^3 |\mathcal{J}_i| &\le C_1 \left(1 + \int_\Omega |\nabla u_\varepsilon^{(m)}|^{p(z)} \mathrm{d}x + \int_\Omega |\nabla u_\varepsilon^{(m)}|^{q(z)} \mathrm{d}x \right) + C_2 \int_\Omega (\varepsilon^2 + |\nabla u_\varepsilon^{(m)}|^2)^{\frac{p(z)}{2}} \ln(\varepsilon^2 + |\nabla u_\varepsilon^{(m)}|^2) \mathrm{d}x \\ &\quad + C_3 \int_\Omega (\varepsilon^2 + |\nabla u_\varepsilon^{(m)}|^2)^{\frac{q(z)}{2}} \ln(\varepsilon^2 + |\nabla u_\varepsilon^{(m)}|^2) \mathrm{d}x \\ &\le C_4 \left(1 + \int_\Omega (\varepsilon^2 + |\nabla u_\varepsilon^{(m)}|^2)^{\frac{p(z)+r_1}{2}} \mathrm{d}x \right). \end{aligned}$$

The required inequality (6.19) follows after gathering the aforementioned estimates, integrating the result in t and applying (5.4). □

6.2 *A priori* estimates II: the case $\Phi \not\equiv 0$ and/or $\Psi \not\equiv 0$

We proceed to derive a priori estimates in the case when the equation contains the nonlinear source that depends on the solution and its gradient. The difference in the arguments consists in the necessity to estimate the integrals of the terms that appear after multiplication of $F(z, u_\varepsilon^{(m)}, \nabla u_\varepsilon^{(m)})$ by $u_\varepsilon^{(m)}$, $\Delta u_\varepsilon^{(m)}$, and $u_{\varepsilon t}^{(m)}$.

(1) Let us multiply jth equation of (5.3) by $u_j^{(m)}$ and sum up. In the result we arrive at equality (6.3) with the right-hand side containing the additional terms,

$$\mathcal{I}_0 \equiv \int_\Omega \Psi(z, u_\varepsilon^{(m)}) u_\varepsilon^{(m)} \mathrm{d}x + \int_\Omega \Phi(z, \nabla u_\varepsilon^{(m)}) u_\varepsilon^{(m)} \mathrm{d}x.$$

Assume that Ψ satisfies (3.8) and Φ satisfies (3.9) with $1 < \sigma^+ < p^-$. Using the Cauchy, Young, and Poincaré inequalities we find that for every $t \in (0, T)$

$$\begin{aligned}
|\mathcal{I}_0| &\le C_\delta + \delta \int_\Omega |u_\varepsilon^{(m)}|^{p^-} \mathrm{d}x + C_\delta \int_\Omega |u_\varepsilon^{(m)}|^2 \mathrm{d}x + \delta \int_\Omega \Phi^2(z, \nabla u_\varepsilon^{(m)}) \mathrm{d}x \\
&\le C_\delta(1 + \|u_\varepsilon^{(m)}\|_{2,\Omega}^2) + \delta \widehat{C} \int_\Omega |\nabla u_\varepsilon^{(m)}|^{p^-} \mathrm{d}x \\
&\quad + \delta M^2 \int_{\Omega \cap \{|\nabla u_\varepsilon^{(m)}| \ge 1\}} |\nabla u_\varepsilon^{(m)}|^{p(z)} \mathrm{d}x + \delta L^2 \int_{\Omega \cap \{|\nabla u_\varepsilon^{(m)}| < 1\}} \mathrm{d}x \\
&\le C_\delta(1 + \|u_\varepsilon^{(m)}\|_{2,\Omega}^2) + \delta(\widehat{C} + M^2) \int_\Omega |\nabla u_\varepsilon^{(m)}|^{p(z)} \mathrm{d}x,
\end{aligned}$$

where $\delta \in (0, 1)$ is an arbitrary constant and $\widehat{C}$ is the constant from inequality (2.6) with $r = p^-$. We plug this estimate into (6.4) and use (6.6) with $a \equiv 1$ and q substituted by p. Choosing δ sufficiently small, we transform (6.4) to the form

$$\frac{1}{2} \frac{\mathrm{d}}{\mathrm{d}t} \|u_\varepsilon^{(m)}\|_{2,\Omega}^2 + C \int_\Omega \gamma_\varepsilon(z, \nabla u_\varepsilon^{(m)}) |\nabla u_\varepsilon^{(m)}|^2 \mathrm{d}x \le C'(1 + \|f_0\|_{2,\Omega}^2 + \|u_\varepsilon^{(m)}\|_{2,\Omega}^2).$$

Integrating this inequality in t, we obtain the following counterpart of Lemma 6.1.

Lemma 6.5. *Assume that $a(\cdot)$, $p(\cdot)$, $q(\cdot)$, u_0, and f_0 satisfy the conditions of Lemma* 6.1. *If Ψ and Φ satisfy conditions* (3.8) *and* (3.9), *respectively*, *then*

$$\sup_{t \in (0,T)} \|u_\varepsilon^{(m)}(\cdot, t)\|_{2,\Omega}^2 + \int_{Q_T} \gamma_\varepsilon(z, \nabla u_\varepsilon^{(m)}) |\nabla u_\varepsilon^{(m)}|^2 \mathrm{d}z \le C_1 \mathrm{e}^{C_2 T} \Big(\|f_0\|_{2,Q_T}^2 + \|u_0\|_{2,\Omega}^2\Big) + C_0 \tag{6.23}$$

and

$$\int_{Q_T} (|\nabla u_\varepsilon^{(m)}|^{p(z)} + a(z) |\nabla u_\varepsilon^{(m)}|^{q(z)}) \mathrm{d}x \mathrm{d}t \le C_2 \int_{Q_T} \gamma_\varepsilon(z, \nabla u_\varepsilon^{(m)}) |\nabla u_\varepsilon^{(m)}|^2 \mathrm{d}z + C_3 \tag{6.24}$$

with independent of ε and m constants C_i.

(2) Estimate on $\|\nabla u_\varepsilon^{(m)}(t)\|_{2,\Omega}$. We follow the proof of Lemma 6.2: by multiplying each of equations in (5.3) by $\lambda_j u_j^{(m)}$ and summing the results, we arrive at equality (6.8) with the additional terms on the right-hand side:

$$\mathcal{I}_1^{(1)} + \mathcal{I}_1^{(2)} := \int_\Omega \Psi(z, u_\varepsilon^{(m)}) u_\varepsilon^{(m)} \Delta u_\varepsilon^{(m)} \mathrm{d}x + \int_\Omega \Phi(z, \nabla u_\varepsilon^{(m)}) \Delta u_\varepsilon^{(m)} \mathrm{d}x.$$

Estimate of $\mathcal{I}_1^{(1)}$. By the Green formula and by virtue of (3.8),

$$
\begin{aligned}
|\mathcal{I}_1^{(1)}| &\le \int_\Omega |\nabla\Psi(z, u_\varepsilon^{(m)})\cdot\nabla u_\varepsilon^{(m)}|\mathrm{d}x \le C\int_\Omega |u_\varepsilon^{(m)}|^{\sigma(z)-2}|\nabla u_\varepsilon^{(m)}|^2\,\mathrm{d}x\\
&\quad + C\int_\Omega |u_\varepsilon^{(m)}|^{\sigma(z)-1}|\nabla u_\varepsilon^{(m)}|\mathrm{d}x + C\int_\Omega |u_\varepsilon^{(m)}|^{\sigma(z)-1}|\ln|(u_\varepsilon^{(m)})||\nabla u_\varepsilon^{(m)}|\mathrm{d}x\\
&\equiv \mathcal{K}_1 + \mathcal{K}_2 + \mathcal{K}_3.
\end{aligned}
$$

Since $\sigma^+ < p^-$ by assumption, we may apply (6.11) with a constant μ chosen so small that $\sigma^+ - 1 + 2\mu \le p^- - 1$. Then

$$
\begin{aligned}
\mathcal{K}_3 &\le \delta\int_\Omega |\nabla u_\varepsilon^{(m)}|^{p^-}\,\mathrm{d}x + C\left(1 + \int_\Omega |u_\varepsilon^{(m)}|^{(\sigma(z)-1+\mu)\frac{p^-}{p^- -1}}\,\mathrm{d}x\right)\\
&\le C' + \delta\int_\Omega |\nabla u_\varepsilon^{(m)}|^{p^-}\,\mathrm{d}x + \delta\int_\Omega |u_\varepsilon^{(m)}|^{p^-}\,\mathrm{d}x\\
&\le C'' + \delta(1 + C''')\int_\Omega |\nabla u_\varepsilon^{(m)}|^{p(z)}\,\mathrm{d}x
\end{aligned}
$$

with arbitrary $\delta > 0$ and independent of ε and m constants. $\mathcal{K}_2$ is estimated likewise:

$$
\begin{aligned}
\mathcal{K}_2 &\le C\int_\Omega |u_\varepsilon^{(m)}|^{\sigma(z)-1}|\nabla u_\varepsilon^{(m)}|\mathrm{d}x \le \delta\int_\Omega |\nabla u_\varepsilon^{(m)}|^{p^-}\,\mathrm{d}x + C_\delta\int_\Omega |u_\varepsilon^{(m)}|^{(\sigma(z)-1)\frac{p^-}{p^- -1}}\,\mathrm{d}x\\
&\le C + \delta(1 + C')\int_\Omega |\nabla u_\varepsilon^{(m)}|^{p(z)}\,\mathrm{d}x.
\end{aligned}
$$

To estimate $\mathcal{K}_1$, we assume $2 \le \sigma^- \le \sigma^+ < p^-$. By the inequalities of Young and Poincaré,

$$
\mathcal{K}_1 \le \delta\int_\Omega |\nabla u_\varepsilon^{(m)}|^{p^-}\,\mathrm{d}x + C_\delta\int_\Omega |u_\varepsilon^{(m)}|^{p^-\frac{\sigma(z)-2}{p^- -2}}\,\mathrm{d}x \le C' + \delta(1 + C)\int_\Omega |\nabla u_\varepsilon^{(m)}|^{p(z)}\,\mathrm{d}x.
$$

Observe that in the case $\Psi \equiv 0$, we have $\mathcal{I}_1^{(1)} = 0$, and the assumption $2 < p^-$ becomes superfluous.

Estimate on $\mathcal{I}_2^{(2)}$. Assume that $p(z) \ge 2$ in Q_T, fix $t \in (0, T)$ and denote $\Omega^+ = \Omega \cap \{|\nabla u_\varepsilon^{(m)}| \ge 1\}$ and $\Omega^- = \Omega \cap \{|\nabla u_\varepsilon^{(m)}| < 1\}$. By the Cauchy inequality,

$$
\begin{aligned}
\int_{\Omega^-} \Phi(z, \nabla u_\varepsilon^{(m)})\Delta u_\varepsilon^{(m)}\mathrm{d}x &\le L\int_{\Omega^-} |\nabla u_\varepsilon^{(m)}|^{\frac{p}{2}}\left(|\nabla u_\varepsilon^{(m)}|^{\frac{p-2}{2}}|\Delta_\varepsilon^{(m)}u|\right)\mathrm{d}x\\
&\le \delta\int_\Omega (\varepsilon^2 + |\nabla u_\varepsilon^{(m)}|^2)^{\frac{p-2}{2}}|\Delta u_\varepsilon^{(m)}|^2\,\mathrm{d}x + C_{\delta,L}|\Omega|,
\end{aligned}
$$

$$
\begin{aligned}
\int_{\Omega^+} \Phi(z, \nabla u_\varepsilon^{(m)})\Delta u_\varepsilon^{(m)}\mathrm{d}x &\le \int_{\Omega^+} |\Phi(z, \nabla u_\varepsilon^{(m)})|\left(|\nabla u_\varepsilon^{(m)}|^{\frac{p-2}{2}}|\Delta u_\varepsilon^{(m)}|\right)\mathrm{d}x\\
&\le M\int_{\Omega^+} |\nabla u_\varepsilon^{(m)}|^{\frac{p}{2}}\left(|\nabla u_\varepsilon^{(m)}|^{\frac{p-2}{2}}|\Delta u_\varepsilon^{(m)}|\right)\mathrm{d}x\\
&\le \delta\int_\Omega \left((\varepsilon^2 + |\nabla u_\varepsilon^{(m)}|^2)^{\frac{p-2}{2}}|\Delta u_\varepsilon^{(m)}|^2\right)\mathrm{d}x + C_{\delta,M}\int_\Omega |\nabla u_\varepsilon^{(m)}|^p\,\mathrm{d}x.
\end{aligned}
$$

Following the proof of Lemma 6.3, we arrive at the inequality

$$
\begin{aligned}
&\sup_{(0,T)} \|\nabla u_\varepsilon^{(m)}(\cdot, t)\|_{2,\Omega}^2 + \int_{Q_T} \gamma_\varepsilon(z, \nabla u_\varepsilon^{(m)})|(u_\varepsilon^{(m)})_{xx}|^2\mathrm{d}z\\
&\le Ce^{C'T}\left(\|\nabla u_0\|_{2,\Omega}^2 + \|f_0\|_{L^2(0,T;\, W_0^{1,2}(\Omega))}^2 + \int_{Q_T} |\nabla u_\varepsilon^{(m)}|^{p(z)}\,\mathrm{d}z + \int_{Q_T} |u_\varepsilon^{(m)}|^{p(z)}\,\mathrm{d}z + 1\right)
\end{aligned}
\tag{6.25}
$$

with new constants C and C', which do not depend on ε and m. The last term on the right-hand side of this inequality is estimated by virtue of Lemma 4.1 and estimates (6.23) and (6.24).

Lemma 6.6. *Under the conditions of* Lemma 6.5,

$$\sup_{(0,T)} \|\nabla u_\varepsilon^{(m)}(\cdot, t)\|_{2,\Omega}^2 + \int_{Q_T} \gamma_\varepsilon(z, \nabla u_\varepsilon^{(m)})|(u_\varepsilon^{(m)})_{xx}|^2 dz \leq C e^{C'T}(\widehat{C} + \|u_0\|^2_{W_0^{1,2}(\Omega)} + \|f_0\|^2_{L^2(0,T;\, W_0^{1,2}(\Omega))}) \tag{6.26}$$

with an independent of ε and m constants C, C', and a constant $\widehat{C}$ depending only on T, and the quantities on the right-hand sides of (6.1), (6.2). *If* $\Psi \equiv 0$, *estimate* (6.26) *holds for* $p^- \geq 2$.

(3) Estimate on $\|u_{\varepsilon t}^{(m)}\|_{2,Q_T}$. We follow the proof of Lemma 6.4. Multiplying (5.3) by $(u_\varepsilon^{(m)})_t$ and summing the results, we obtain equality (6.21) with the additional terms on the right-hand side:

$$\mathcal{M}_0 \equiv \int_\Omega \Psi(z, u_\varepsilon^{(m)}) u_\varepsilon^{(m)} u_{\varepsilon t}^{(m)} dx + \int_\Omega \Phi(z, \nabla u_\varepsilon^{(m)}) u_{\varepsilon t}^{(m)} dx.$$

By (3.8), (3.9), and Young's inequality,

$$\mathcal{M}_0 \leq C_0 + C_1 \int_\Omega |u_\varepsilon^{(m)}|^{2(\sigma^+-1)} dx + C_2 \int_\Omega |\nabla u_\varepsilon^{(m)}|^{p(z)} dx + \delta \int_\Omega (u_\varepsilon^{(m)})_t^2 dx.$$

Let us make use of the Gagliardo-Nirenberg inequality. If $\theta = \frac{\frac{1}{2}\frac{\sigma^+-2}{\sigma^+-1}}{\frac{1}{2}+\frac{1}{N}-\frac{1}{p^-}} \in [0, 1]$, then for every $v \in W_0^{1,p^-}(\Omega)$,

$$\|v\|_{2(\sigma^+-1),\Omega} \leq C\|\nabla v\|^\theta_{p^-,\Omega} \|v\|^{1-\theta}_{2,\Omega}$$

with an independent of v constant C. The inclusion $\theta \in [0, 1]$ holds if

$$2 \leq \sigma^+ < \sigma^* = \begin{cases} 1 + \dfrac{Np^-}{2(N-p^-)} & \text{if } p^- < N, \\ \infty & \text{if } p^- \geq N. \end{cases}$$

Let us claim that

$$2\theta(\sigma^+ - 1) \leq p^- \Leftrightarrow \sigma^+ \leq 1 + p^- \frac{N+2}{2N} < \sigma^*.$$

Then by Young's inequality

$$\int_\Omega |u_\varepsilon^{(m)}|^{2(\sigma^+-1)} dx \leq C\|\nabla u_\varepsilon^{(m)}\|^{2\theta(\sigma^+-1)}_{p^-,\Omega} \|u_\varepsilon^{(m)}\|^{2(1-\theta)(\sigma^+-1)}_{2,\Omega} \leq C\left(1 + \int_\Omega |\nabla u_\varepsilon^{(m)}|^{p^-} dx\right) \tag{6.27}$$

with a constant C depending on $\|u_\varepsilon^{(m)}\|_{2,\Omega}$, already estimated in Lemma 6.5 for $\sigma^+ < p^-$. Combining these inequalities with (6.19), we obtain

$$\begin{aligned} &\frac{1}{2}\|(u_\varepsilon^{(m)})_t\|^2_{2,Q_T} + \sup_{(0,T)} \int_\Omega \gamma_\varepsilon(z, \nabla u_\varepsilon^{(m)})|\nabla u_\varepsilon^{(m)}|^2 dx \\ &\quad \leq \|f_0\|^2_{2,Q_T} + C\left(1 + \int_\Omega (|\nabla u_0|^{p(x,0)} + a(x,0)|\nabla u_0|^{q(x,0)}) dx + \int_{Q_T} |u_\varepsilon^{(m)}|^{p(z)}\right). \end{aligned} \tag{6.28}$$

The last integral on the right-hand side is estimated by virtue of Lemma 4.1 and the estimates of Lemma 6.5.

Lemma 6.7. *Let in the conditions of Lemma 6.6*

$$2 < p^-, \quad 2 \le \sigma^- \le \sigma^+ < \min\left\{p^-, 1 + p^- \frac{N+2}{2N}\right\}.$$

Then

$$\begin{aligned} &\frac{1}{2}\|(u_\varepsilon^{(m)})_t\|^2_{2,Q_T} + \sup_{(0,T)} \int_\Omega \gamma_\varepsilon(z, \nabla u_\varepsilon^{(m)})|\nabla u_\varepsilon^{(m)}|^2\,dx \\ &\le C\left(1 + \int_\Omega (|\nabla u_0|^{p(x,0)} + a(x,0)|\nabla u_0|^{q(x,0)})dx\right) + \|f_0\|^2_{2,Q_T}, \end{aligned} \tag{6.29}$$

with a constant C independent of ε and m. If $\Psi \equiv 0$, estimate (6.29) *is true for $p^- \ge 2$.*

7 Existence and uniqueness of strong solutions

7.1 Regularized problem

Theorem 7.1. *Let u_0, f, p, q, a, and $\partial\Omega$ satisfy the conditions of* Theorem 3.1. *Then for every $\varepsilon \in (0,1)$, problem* (5.1) *has a unique solution u_ε, which satisfies the estimates*

$$\begin{aligned} &\|u_\varepsilon\|_{\mathcal{W}_{q(\cdot)}(Q_T)} \le C_0, \\ &\operatorname*{ess\,sup}_{(0,T)} \|u_\varepsilon(\cdot,t)\|^2_{2,\Omega} + \|u_{\varepsilon t}\|^2_{2,Q_T} + \operatorname*{ess\,sup}_{(0,T)} \|\nabla u_\varepsilon(\cdot,t)\|^2_{2,\Omega} \\ &+ \operatorname*{ess\,sup}_{(0,T)} \int_\Omega \left((\varepsilon^2 + |\nabla u_\varepsilon^{(m)}|^2)^{\frac{p(z)}{2}} + a(z)(\varepsilon^2 + |\nabla u_\varepsilon^{(m)}|^2)^{\frac{q(z)}{2}}\right)dx \le C_0, \end{aligned} \tag{7.1}$$

with a constant C_0 depending on the data but not on ε. Moreover, u_ε possesses the property of global higher integrability of the gradient: for every,

$$\delta \in (0, r^*), \quad r^* = \frac{4p^-}{p^-(N+2) + 2N},$$

there exists a constant $C = C(\partial\Omega, N, p^\pm, \delta, \|u_0\|_{W_0^{1,2}(\Omega)}, \|f\|_{L^2(0,T;\, W_0^{1,2}(\Omega))})$ such that

$$\int_{Q_T} |\nabla u_\varepsilon|^{p(z)+\delta}\,dz \le C. \tag{7.2}$$

Proof. Let $\varepsilon \in (0,1)$ be a fixed parameter. Under the assumptions of Theorem 3.1, there exists a sequence of Galerkin approximations $u_\varepsilon^{(m)}$ defined by formulas (5.2). The functions $u_\varepsilon^{(m)}$ satisfy estimates (6.1), (6.2), (6.15), (6.16), (6.18), and (6.19). These uniform in m and ε estimates enable one to extract a subsequence $u_\varepsilon^{(m)}$ (for which we keep the same name), and functions u_ε, η_ε, χ_ε such that

$$\begin{aligned} &u_\varepsilon^{(m)} \to u_\varepsilon \quad \star\text{-weakly in } L^\infty(0,T; L^2(\Omega)), \quad (u_\varepsilon^{(m)})_t \rightharpoonup (u_\varepsilon)_t \text{ in } L^2(Q_T), \\ &\nabla u_\varepsilon^{(m)} \rightharpoonup \nabla u_\varepsilon \text{ in } (L^{p(\cdot)}(Q_T))^N, \quad \nabla u_\varepsilon^{(m)} \rightharpoonup \nabla u_\varepsilon \text{ in } (L^{q(\cdot)}(Q_T))^N, \\ &(\varepsilon^2 + |\nabla u_\varepsilon^{(m)}|^2)^{\frac{p(z)-2}{2}} \nabla u_\varepsilon^{(m)} \rightharpoonup \eta_\varepsilon \text{ in } (L^{q'(\cdot)}(Q_T))^N, \\ &(\varepsilon^2 + |\nabla u_\varepsilon^{(m)}|^2)^{\frac{q(z)-2}{2}} \nabla u_\varepsilon^{(m)} \rightharpoonup \chi_\varepsilon \text{ in } (L^{q'(\cdot)}(Q_T))^N. \end{aligned} \tag{7.3}$$

In the third line, we made use of the uniform estimate

$$\int_{Q_T} (\varepsilon^2 + |\nabla u_\varepsilon^{(m)}|^2)^{\frac{q(z)(p(z)-1)}{2(q(z)-1)}}dz \le C\left(1 + \int_{Q_T} |\nabla u_\varepsilon^{(m)}|^{p(z)+r}\,dz\right) \le C,$$

which follows from (3.5) and (6.16). The functions $u_\varepsilon^{(m)}$ and $(u_\varepsilon^{(m)})_t$ are uniformly bounded in $L^\infty(0, T; W_0^{1,p^-}(\Omega))$ and $L^2(0, T; L^2(\Omega))$, respectively, and $W_0^{1,q(\cdot,t)}(\Omega) \subseteq W_0^{1,q^-}(\Omega) \hookrightarrow L^2(\Omega)$. By [42, Sec. 8, Corollary 4], the sequence $\{u_\varepsilon^{(m)}\}$ is relatively compact in $C([0, T]; L^2(\Omega))$, *i.e.*, there exists a subsequence $\{u_\varepsilon^{(m_k)}\}$, which we assume coinciding with $\{u_\varepsilon^{(m)}\}$, such that $u_\varepsilon^{(m)} \to u_\varepsilon$ in $C([0, T]; L^2(\Omega))$ and a.e. in Q_T. Let us define

$$\mathcal{P}_m = \left\{\phi : \ \phi = \sum_{i=1}^m \psi_i(t)\phi_i(x), \quad \psi_i \text{ are absolutely continuous in } [0, T]\right\}.$$

Fix some $m \in \mathbb{N}$. By the method of construction $u_\varepsilon^{(m)} \in \mathcal{P}_m$. Since $\mathcal{P}_k \subset \mathcal{P}_m$ for $k < m$, then for every $\xi_k \in \mathcal{P}_k$ with $k \le m$,

$$\int_{Q_T} u_{\varepsilon t}^{(m)} \xi_k \mathrm{d}z + \int_{Q_T} \gamma_\varepsilon(z, \nabla u_\varepsilon^{(m)}) \nabla u_\varepsilon^{(m)} \cdot \nabla \xi_k \mathrm{d}z = \int_{Q_T} f_0 \xi_k \mathrm{d}z. \tag{7.4}$$

Let $\xi \in \mathcal{W}_{q(\cdot)}(Q_T)$. The space $C^\infty([0, T]; C_0^\infty(\Omega))$ is dense in $\mathcal{W}_{q(\cdot)}(Q_T)$; therefore, there exists a sequence $\{\xi_k\}$ such that $\xi_k \in \mathcal{P}_k$ and $\xi_k \to \xi \in \mathcal{W}_{q(\cdot)}(Q_T)$. If $U_m \rightharpoonup U$ in $L^{q'(\cdot)}(Q_T)$, then for every $V \in L^{q(\cdot)}(Q_T)$, we have

$$a(z)V \in L^{q(\cdot)}(Q_T) \quad \text{and} \quad \int_{Q_T} a U_m V \mathrm{d}z \to \int_{Q_T} a U V \mathrm{d}z.$$

Using this fact, we pass to the limit as $m \to \infty$ in (7.4) with a fixed k, and then, letting $k \to \infty$, we conclude that

$$\int_{Q_T} u_{\varepsilon t} \xi \mathrm{d}z + \int_{Q_T} \eta_\varepsilon \cdot \nabla \xi \mathrm{d}z + \int_{Q_T} a(z) \quad \chi_\varepsilon \cdot \nabla \xi \mathrm{d}z = \int_{Q_T} f_0 \xi \mathrm{d}z \tag{7.5}$$

for all $\xi \in \mathcal{W}_{q(\cdot)}(Q_T)$. To identify the limit vectors η_ε and χ_ε, we use the classical argument based on monotonicity. The flux function $\gamma_\varepsilon(z, \nabla u_\varepsilon^{(m)}) \nabla u_\varepsilon^{(m)}$ is monotone:

$$(\gamma_\varepsilon(z, \xi)\xi - \gamma_\varepsilon(z, \zeta)\zeta, \xi - \zeta) \ge 0 \quad \text{for all } \xi, \zeta \in \mathbb{R}^N, \ z \in Q_T, \ \varepsilon \ge 0, \tag{7.6}$$

see, e.g., [5, Lemma 6.1] for the proof. By virtue of (7.6), for every $\psi \in \mathcal{P}_m$,

$$\begin{aligned} \gamma_\varepsilon(z, \nabla u_\varepsilon^{(m)}) |\nabla u_\varepsilon^{(m)}|^2 &= \gamma_\varepsilon(z, \nabla u_\varepsilon^{(m)}) \nabla u_\varepsilon^{(m)} \cdot (\nabla u_\varepsilon^{(m)} - \nabla \psi) + \gamma_\varepsilon(z, \nabla u_\varepsilon^{(m)}) \nabla u_\varepsilon^{(m)} \cdot \nabla \psi \\ &\ge \gamma_\varepsilon(z, \nabla \psi) \nabla \psi \cdot (\nabla u_\varepsilon^{(m)} - \nabla \psi) + \gamma_\varepsilon(z, \nabla u_\varepsilon^{(m)}) \nabla u_\varepsilon^{(m)} \cdot \nabla \psi. \end{aligned} \tag{7.7}$$

By taking $\xi_k = u_\varepsilon^{(m)}$ in (7.4), we obtain: for every $\psi \in \mathcal{P}_k$ with $k \le m$,

$$\begin{aligned} 0 &= \int_{Q_T} (u_\varepsilon^{(m)})_t u_\varepsilon^{(m)} \mathrm{d}z + \int_{Q_T} \gamma_\varepsilon(z, \nabla u_\varepsilon^{(m)}) |\nabla u_\varepsilon^{(m)}|^2 \mathrm{d}z - \int_{Q_T} f_0 u_\varepsilon^{(m)} \mathrm{d}z \\ &\ge \int_{Q_T} (u_\varepsilon^{(m)})_t u_\varepsilon^{(m)} \mathrm{d}z + \int_{Q_T} \gamma_\varepsilon(z, \nabla \psi) \nabla \psi \cdot \nabla (u_\varepsilon^{(m)} - \psi) \mathrm{d}z \\ &\quad + \int_{Q_T} \gamma_\varepsilon(z, \nabla u_\varepsilon^{(m)}) \nabla u_\varepsilon^{(m)} \cdot \nabla \psi \mathrm{d}z - \int_{Q_T} f_0 u_\varepsilon^{(m)} \mathrm{d}z. \end{aligned}$$

Notice that $(u_\varepsilon^{(m)}, (u_\varepsilon^{(m)})_t)_{2,Q_T} \to (u_{\varepsilon t}, u_\varepsilon)_{2,Q_T}$ as $m \to \infty$ as the product of weakly and strongly convergent sequences. This fact together with (7.3) means that each term of the last inequality has a limit as $m \to \infty$. Letting $m \to \infty$ and using (7.5), we find that for every $\psi \in \mathcal{P}_k$

$$\begin{aligned} 0 &\ge \int_{Q_T} u_\varepsilon u_{\varepsilon t} \mathrm{d}z + \int_{Q_T} \gamma_\varepsilon(z, \nabla \psi) \nabla \psi \cdot \nabla (u_\varepsilon - \psi) \mathrm{d}z + \int_{Q_T} (\eta_\varepsilon + a(z)\chi_\varepsilon) \cdot \nabla \psi \mathrm{d}z - \int_{Q_T} f_0 u_\varepsilon \mathrm{d}z \\ &= \int_{Q_T} \left((\varepsilon^2 + |\nabla \psi|^2)^{\frac{p(z)-2}{2}} \nabla \psi - \eta_\varepsilon \right) \cdot \nabla (u_\varepsilon - \psi) \mathrm{d}z + \int_{Q_T} a(z) \left((\varepsilon^2 + |\nabla \psi|^2)^{\frac{q(z)-2}{2}} \nabla \psi - \chi_\varepsilon \right) \cdot \nabla (u_\varepsilon - \psi) \mathrm{d}z. \end{aligned}$$

By the density of $\bigcup_{k=1}^{\infty}\mathcal{P}_k$ in $\mathcal{W}_{q(\cdot)}(Q_T)$, the last inequality also holds for every $\psi \in \mathcal{W}_{q(\cdot)}(Q_T)$. Take $\psi = u_\varepsilon + \lambda\xi$ with a constant $\lambda > 0$ and an arbitrary $\xi \in \mathcal{W}_{q(\cdot)}(Q_T)$. Then

$$\lambda\left[\int\limits_{Q_T}\left((\varepsilon^2 + |\nabla(u_\varepsilon + \lambda\xi)|^2)^{\frac{p(z)-2}{2}}\nabla(u_\varepsilon + \lambda\xi) - \eta_\varepsilon\right)\cdot\nabla\xi\,dz + \int\limits_{Q_T} a(z)\left((\varepsilon^2 + |\nabla(u_\varepsilon + \lambda\xi)|^2)^{\frac{q(z)-2}{2}}\nabla(u_\varepsilon + \lambda\xi) - \chi_\varepsilon\right)\cdot\nabla\xi\,dz\right] \le 0. \tag{7.8}$$

Simplifying and letting $\lambda \to 0$ we find that

$$\int\limits_{Q_T}(\gamma_\varepsilon(z, \nabla u_\varepsilon)\nabla u_\varepsilon - (\eta_\varepsilon + a(z)\chi_\varepsilon))\cdot\nabla\xi\,dz \le 0 \quad \forall\xi \in \mathcal{W}_{q(\cdot)}(Q_T),$$

which is possible only if

$$\int\limits_{Q_T}(\gamma_\varepsilon(z, \nabla u_\varepsilon)\nabla u_\varepsilon - (\eta_\varepsilon + a(z)\chi_\varepsilon))\cdot\nabla\xi\,dz = 0 \quad \forall\xi \in \mathcal{W}_{q(\cdot)}(Q_T).$$

The initial condition for u_ε is fulfilled by continuity because $u_\varepsilon \in C([0, T]; L^2(\Omega))$.

Uniqueness of the weak solution is an immediate byproduct of monotonicity. Let u and v are two solutions of problem (5.1). Take an arbitrary $\tau \in (0, T]$. Choosing $u - v$ for the test function in equalities (3.4) for u and v in the cylinder $Q_\tau = \Omega \times (0, \tau)$ and subtracting the results and applying (7.6), we arrive at the inequality

$$\frac{1}{2}\|u - v\|_{2,\Omega}^2(\tau) = \int\limits_{Q_\tau}(u - v)(u - v)_t\,dz \le 0.$$

It follows that $u(x, \tau) = v(x, \tau)$ a.e. in Ω for every $\tau \in [0, T]$.

Estimates (7.1) follow from the uniform in m estimates on the functions $u_\varepsilon^{(m)}$ and their derivatives, the properties of weak convergence (7.3), and lower semicontinuity of the modular. Inequality (6.17) yields that for every $\delta \in (0, r^*)$ the sequence $\{\nabla u_\varepsilon^{(m)}\}$ contains a subsequence, which converges to ∇u_ε weakly in $(L^{p(\cdot)+\delta}(Q_T))^N$, whence (7.2). □

Theorem 7.2. *Let in the conditions of* Theorem 7.1, $\Psi \not\equiv 0$ *and/or* $\Phi \not\equiv 0$.

(i) *If* a, p, q, Ψ, *and* Φ *satisfy the conditions of Theorem* 3.2, *then for every* $\varepsilon \in (0, 1)$, *problem* (5.1) *has at least one strong solution* u_ε, *which satisfies estimates* (7.1) and (7.2) *and*

$$\||u_\varepsilon|^{\sigma(z)-2}u_\varepsilon\|_{2,Q_T} \le C_1, \quad \|\Phi(z, \nabla u_\varepsilon)\|_{2,Q_T} \le C_2$$

with independent of ε *constants* C_i.

(ii) *The solution is unique if* $\Phi \equiv 0$ *and either* $\Psi(z, v) \equiv b(z)v$ *with a bounded* $b(z)$, *or* $\Psi(z, v)$ *is monotone decreasing with respect to* v *for a.e.* $z \in Q_T$.

Proof. The estimates of Lemmas 6.5–6.7 allow one to extract a subsequence $\{u_\varepsilon^{(m)}\}$ with the convergence properties (7.3). The sequences $v_m = \Psi(z, u_\varepsilon^{(m)})$ and $\mathbf{w}_m = \Phi(z, \nabla u_\varepsilon^{(m)})$ are uniformly bounded in $L^2(Q_T)$ because of (6.27) and (6.24). It follows that v_m and $\mathbf{w}_m$ converge weakly in $L^2(Q_T)$ to some functions v and $\mathbf{w}$ (up to a subsequence). Due to the pointwise convergence $u_\varepsilon^{(m)} \to u_\varepsilon$, it is necessary that $v = \Psi(z, u_\varepsilon)$ a.e. in Q_T. As in the proof of Theorem 7.1, we conclude that for every test-function $\xi_k \in \mathcal{P}_m$, $k \le m$, $u_\varepsilon^{(m)}$ satisfies (7.4) with f_0 substituted by $F(z, u_\varepsilon^{(m)}, \nabla u_\varepsilon^{(m)})$, and for every $\xi \in \mathcal{W}_{q(\cdot)}(Q_T)$, the limit u_ε satisfies (7.5) with f_0 substituted by $f_0 + \Psi(z, u) + \mathbf{w}$. By using (7.7), we find that

$$
\begin{aligned}
0 &= \int_{Q_T} (u_\varepsilon^{(m)})_t u_\varepsilon^{(m)} \mathrm{d}z + \int_{Q_T} \gamma_\varepsilon(z, \nabla u_\varepsilon^{(m)})|\nabla u_\varepsilon^{(m)}|^2 \,\mathrm{d}z - \int_{Q_T} F(z, u_\varepsilon^{(m)}, \nabla u_\varepsilon^{(m)}) u_\varepsilon^{(m)} \mathrm{d}z \\
&\geq \int_{Q_T} (u_\varepsilon^{(m)})_t u_\varepsilon^{(m)} \mathrm{d}z + \int_{Q_T} \gamma_\varepsilon(z, \nabla\psi)\nabla\psi \cdot \nabla(u_\varepsilon^{(m)} - \psi)\mathrm{d}z \\
&\quad + \int_{Q_T} \gamma_\varepsilon(z, \nabla u_\varepsilon^{(m)})\nabla u_\varepsilon^{(m)} \cdot \nabla\psi \mathrm{d}z - \int_{Q_T} F(z, u_\varepsilon^{(m)}, \nabla u_\varepsilon^{(m)}) u_\varepsilon^{(m)} \mathrm{d}z.
\end{aligned}
$$

Letting $m \to \infty$, by using the convergence properties of the sequences $\{u_\varepsilon^{(m)}\}$, $\{v_m\}$, $\{\mathbf{w}_m\}$, and then comparing the result with (7.5) with $f_0 + \Psi(z, u_\varepsilon) + \mathbf{w}$ on the right-hand side, we conclude that the limit function satisfies (7.8) for every $\xi \in \mathcal{W}_{q(\cdot)}(Q_T)$. The limits η_ε and χ_ε are identified then by monotonicity.

To prove that $\mathbf{w} = \Phi(z, \nabla u_\varepsilon)$, it is enough to show that $\nabla u_\varepsilon^{(m)} \to \nabla u_\varepsilon$ a.e. in Q_T. It follows from the weak convergence $\nabla u_\varepsilon^{(m)} \rightharpoonup \nabla u_\varepsilon$ in $L^{q(\cdot)}(Q_T)$, the strong convergence $u_\varepsilon^{(m)} \to u_\varepsilon$ in $L^2(Q_T)$, and the Mazur Lemma, see [12, Ch.3, Cor.3.8], that there exists a sequence $\{v_\varepsilon^{(m)}\}$ such that $v_\varepsilon^{(m)} \in \mathcal{P}_m$, each $v_\varepsilon^{(m)}$ is a convex combination of $\{u_\varepsilon^{(1)}, u_\varepsilon^{(2)}, \ldots, u_\varepsilon^{(m)}\}$, and

$$
v_\varepsilon^{(m)} \to u_\varepsilon \text{ in } \mathcal{W}_{q(\cdot)}(Q_T). \tag{7.9}
$$

Let us define $w_m \in \mathcal{P}_m$ as follows:

$$
\|u_\varepsilon - w_m\|_{\mathcal{W}_{q(\cdot)}(Q_T)} = \operatorname{dist}(u_\varepsilon, \mathcal{P}_m) \equiv \min\left\{\|u_\varepsilon - w\|_{\mathcal{W}_{q(\cdot)}(Q_T)} : w \in \mathcal{P}_m\right\}.
$$

Because of (7.9) such w_m exists and

$$
\operatorname{dist}(u_\varepsilon, \mathcal{P}_m) = \|u_\varepsilon - w_m\|_{\mathcal{W}_{q(\cdot)}(Q_T)} \leq \|u_\varepsilon - v_\varepsilon^{(m)}\|_{\mathcal{W}_{q(\cdot)}(Q_T)} \to 0, \ \ m \to \infty. \tag{7.10}
$$

By the properties of the modular and the strong convergence $u_\varepsilon^{(m)} \to u_\varepsilon$ in $L^2(Q_T)$, we also have

$$
\begin{aligned}
&\|u_\varepsilon^{(m)} - w_m\|_{2,Q_T}^2 + \int_{Q_T} |\nabla(u_\varepsilon - w_m)|^{q(z)} \mathrm{d}z \\
&\leq \ 2\|u_\varepsilon^{(m)} - u_\varepsilon\|_{2,Q_T}^2 + 2\|u_\varepsilon - w_m\|_{2,Q_T}^2 + \int_{Q_T} |\nabla(u_\varepsilon - w_m)|^{q(z)} \mathrm{d}z \to 0
\end{aligned} \tag{7.11}
$$

as $m \to \infty$. Since $u_\varepsilon^{(m)}$ satisfy identities (7.4) with f_0 substituted by $F(z, u_\varepsilon^{(m)}, \nabla u_\varepsilon^{(m)})$, and u_ε satisfy (3.4) with the regularized fluxes $\gamma_\varepsilon(z, \nabla u_\varepsilon)\nabla u_\varepsilon$ and the source $f_0 + \Psi(z, u_\varepsilon) + \mathbf{w}$, the functions $u_\varepsilon^{(m)}, w_m \in \mathcal{N}_m$ are admissible test functions in the integral identities for $u_\varepsilon^{(m)}$ and u_ε. By subtracting these identities and rearranging the result, we obtain

$$
\begin{aligned}
&\int_{Q_T} (\gamma_\varepsilon(z, \nabla u_\varepsilon^{(m)})\nabla u_\varepsilon^{(m)} - \gamma_\varepsilon(z, \nabla u_\varepsilon)\nabla u_\varepsilon)\nabla(u_\varepsilon^{(m)} - u_\varepsilon)\mathrm{d}z \\
&\quad = \ - \int_{Q_T} ((u_\varepsilon^{(m)})_t - u_{\varepsilon t})(u_\varepsilon^{(m)} - w_m)\mathrm{d}z + \int_{Q_T} (\mathbf{w}_m - \mathbf{w})(u_\varepsilon^{(m)} - w_m)\mathrm{d}z \\
&\qquad + \int_{Q_T} (\Psi(z, u_\varepsilon^{(m)}) - \Psi(z, u_\varepsilon))(u_\varepsilon^{(m)} - w_m)\mathrm{d}z \\
&\qquad - \int_{Q_T} (\gamma_\varepsilon(z, \nabla u_\varepsilon^{(m)})\nabla u_\varepsilon^{(m)} - \gamma_\varepsilon(z, \nabla u_\varepsilon)\nabla u_\varepsilon)\nabla(u_\varepsilon - w_m)\mathrm{d}z.
\end{aligned} \tag{7.12}
$$

By the choice of w_m and due to the convergence properties (7.3), all terms on the right-hand side of (7.12) tend to zero as $m \to \infty$. The left-hand side is bounded from below because of the strict monotonicity of the function $(\varepsilon^2 + |\xi|^2)^{\frac{p-2}{2}}\xi$ with $\varepsilon \in [0, 1)$, $p \geq 2$. By using (7.6) with $p(z) \geq 2$ and (7.12) we find that

$|\nabla(u_\varepsilon^{(m)} - u_\varepsilon)| \to 0$ in $L^{p(\cdot)}(Q_T)$. Hence, $\nabla u_\varepsilon^{(m)} \to \nabla u_\varepsilon$ a.e. in Q_T up to a subsequence, and $\mathbf{w} = \Phi(z, \nabla u_\varepsilon)$ as required.

To prove the uniqueness we assume, for contradiction, that problem (5.1) with $\Phi \equiv 0$ has two strong solutions $u_1, u_2 \in \mathcal{W}_{q(\cdot)}(Q_T)$. The function $u_1 - u_2$ is an admissible test function in the integral identities (3.4) for u_i. Combining these identities and using (7.6), we arrive at the inequality:

$$\frac{1}{2}\|u_1 - u_2\|_{2,\Omega}^2(t) \le \int_0^t \int_\Omega (\Psi(z, u_1) - \Psi(z, u_2))(u_1 - u_2)\mathrm{d}z.$$

If $\Psi(z, s) = b(z)s$, this inequality takes the form

$$\frac{1}{2}\|u_1 - u_2\|_{2,\Omega}^2(t) \le B\int_0^t \|u_1 - u_2\|_{2,\Omega}^2(\tau)\mathrm{d}\tau, \quad t \in (0, T), \quad B = \operatorname*{ess\,sup}_{Q_T} b(z),$$

whence $\|u_1 - u_2\|_{2,\Omega}(t) = 0$ in $(0, T)$ by Grönwall's inequality. Let $\Psi(z, s)$ be monotone decreasing in s for a.e. $z \in Q_T$. Then $(\Psi(z, u_1) - \Psi(z, u_2))(u_1 - u_2) \le 0$ a.e. in Q_T and

$$\frac{1}{2}\|u_1 - u_2\|_{2,\Omega}^2(t) \le 0 \quad \text{in } (0, T). \qquad \square$$

7.2 Degenerate problem. Proofs of Theorems 3.1 and 3.2

We begin with the proof of Theorem 3.1 and assume that $F \equiv f_0(z)$. Let $\{u_\varepsilon\}$ be the family of strong solutions of the regularized problems (5.1) satisfying estimates (7.1). These uniform in ε estimates enable one to extract a sequence $\{u_{\varepsilon_k}\}$ and find functions $u \in \mathcal{W}_{q(\cdot)}(Q_T)$, $\eta, \chi \in (L^{q'(\cdot)}(Q_T))^N$ with the following properties:

$$\begin{aligned}
&u_{\varepsilon_k} \to u \quad \text{in } L^2(Q_T) \text{ and a.e. in } Q_T,\ u_{\varepsilon_k t} \rightharpoonup u_t \text{ in } L^2(Q_T),\\
&\nabla u_{\varepsilon_k} \rightharpoonup \nabla u \text{ in } (L^{q(\cdot)}(Q_T))^N,\\
&\left(\varepsilon_k^2 + |\nabla u_{\varepsilon_k}|^2\right)^{\frac{p(z)-2}{2}}\nabla u_{\varepsilon_k} \rightharpoonup \eta \text{ in } (L^{q'(\cdot)}(Q_T))^N,\\
&\left(\varepsilon_k^2 + |\nabla u_{\varepsilon_k}|^2\right)^{\frac{q(z)-2}{2}}\nabla u_{\varepsilon_k} \rightharpoonup \chi \text{ in } (L^{q'(\cdot)}(Q_T))^N.
\end{aligned} \tag{7.13}$$

In the third line, we make use of the uniform estimate

$$\int_{Q_T} (\varepsilon^2 + |\nabla u_\varepsilon|^2)^{\frac{q(z)(p(z)-1)}{2(q(z)-1)}}\mathrm{d}z \le C\left(1 + \int_{Q_T} |\nabla u_\varepsilon|^{p(z)+r}\,\mathrm{d}z\right) \le C,$$

which follows from (3.5) and (7.2). Moreover, $u \in C([0, T]; L^2(\Omega))$. Each of u_{ε_k} satisfies the identity

$$\int_{Q_T} u_{\varepsilon_k t}\xi\,\mathrm{d}z + \int_{Q_T} \gamma_{\varepsilon_k}\left(z, \nabla u_{\varepsilon_k}\right)\nabla u_{\varepsilon_k} \cdot \nabla\xi\,\mathrm{d}z = \int_{Q_T} f_0\xi\,\mathrm{d}z \tag{7.14}$$

for all $\xi \in \mathcal{W}_{q(\cdot)}(Q_T)$, which yields

$$\int_{Q_T} u_t\xi\,\mathrm{d}z + \int_{Q_T} (\eta + a(z)\chi)\cdot\nabla\xi\,\mathrm{d}z = \int_{Q_T} f_0\xi\,\mathrm{d}z \quad \forall \xi \in \mathcal{W}_{q(\cdot)}(Q_T). \tag{7.15}$$

To identify η and χ, we use the monotonicity argument. Take $\xi = u_{\varepsilon_k}$ in (7.14):

$$\int_{Q_T} u_{\varepsilon_k t}u_{\varepsilon_k}\mathrm{d}z + \int_{Q_T} \gamma_{\varepsilon_k}\left(z, \nabla u_{\varepsilon_k}\right)\nabla u_{\varepsilon_k} \cdot \nabla u_{\varepsilon_k}\mathrm{d}z = \int_{Q_T} f_0 u_{\varepsilon_k}\mathrm{d}z. \tag{7.16}$$

According to (7.7), for every $\phi \in \mathcal{W}_{q(\cdot)}(Q_T)$,

$$
\begin{aligned}
\int_{Q_T} \gamma_{\varepsilon_k}\big(z, \nabla u_{\varepsilon_k}\big)\nabla u_{\varepsilon_k} \cdot \nabla u_{\varepsilon_k} \mathrm{d}z \geq & \int_{Q_T} \gamma_{\varepsilon_k}(z, \nabla\phi) - (|\nabla\phi|^{p-2} + a|\nabla\phi|^{q-2})\nabla\phi \cdot \nabla\big(u_{\varepsilon_k} - \phi\big)\mathrm{d}z \\
& + \int_{Q_T} \gamma_{\varepsilon_k}\big(z, \nabla u_{\varepsilon_k}\big)\nabla u_{\varepsilon_k} \cdot \nabla\phi \mathrm{d}z + \int_{Q_T} (|\nabla\phi|^{p-2} + a|\nabla\phi|^{q-2})\nabla\phi \cdot \nabla\big(u_{\varepsilon_k} - \phi\big)\mathrm{d}z \\
& \equiv J_{1,k} + J_{2,k} + J_{3,k},
\end{aligned}
$$

where

$$
\begin{aligned}
& J_{2,k} \to \int_{Q_T} (\eta + a(z)\chi)\cdot\nabla\phi \mathrm{d}z, \\
& J_{3,k} \to \int_{Q_T} (|\nabla\phi|^{p-2} + a(z)|\nabla\phi|^{q-2})\nabla\phi \cdot \nabla(u - \phi)\mathrm{d}z \quad \text{as } k \to \infty.
\end{aligned}
$$

Since $|(\gamma_{\varepsilon_k}(z, \nabla\phi)\nabla\phi - (|\nabla\phi|^{p-2} + a^{\frac{q-1}{q}}(z)|\nabla\phi|^{q-2})\nabla\phi| \to 0$ a.e. in Q_T as $k \to \infty$, and because the integrand of $J_{1,k}$ has the majorant,

$$
\begin{aligned}
& \left| ((\varepsilon_k^2 + |\nabla\phi|^2)^{\frac{p-2}{2}} - |\nabla\phi|^{p-2})\nabla\phi \right|^{p'} + \left| a^{\frac{q-1}{q}}(z)\Big((\varepsilon_k^2 + |\nabla\phi|^2)^{\frac{q-2}{2}} - |\nabla\phi|^{q-2}\Big)\nabla\phi\Big) \right|^{q'} \\
& \leq C\Big((1 + |\nabla\phi|^2)^{\frac{p(z)}{2}} + a(z)(1 + |\nabla\phi|^2)^{\frac{q(z)}{2}}\Big) \\
& \leq C(1 + |\nabla\phi|^{p(z)} + a(z)|\nabla\phi|^{q(z)}),
\end{aligned}
$$

then $J_{1,k} \to 0$ by the dominated convergence theorem. Combining (7.15) with (7.16) and letting $k \to \infty$, we find that for every $\phi \in \mathcal{W}_{q(\cdot)}(Q_T)$,

$$
\int_{Q_T} ((|\nabla\phi|^{p(z)-2} + a(z)|\nabla\phi|^{q(z)-2})\nabla\phi - (\eta + a(z)\chi))\cdot\nabla(u - \phi)\mathrm{d}z \geq 0.
$$

Choosing $\phi = u + \lambda\zeta$ with $\lambda > 0$ and $\zeta \in \mathcal{W}_{q(\cdot)}(Q_T)$, simplifying, and then letting $\lambda \to 0^+$, we conclude that for every $\zeta \in \mathcal{W}_{q(\cdot)}(Q_T)$

$$
\int_{Q_T} ((|\nabla u|^{p(z)-2}\nabla u + a(z)|\nabla u|^{q(z)-2}\nabla u) - (\eta + a(z)\chi))\cdot\nabla\zeta \mathrm{d}z \geq 0.
$$

Since the sign of ζ is arbitrary, the last relation must be the equality. It follows that in (7.15) $\eta + a(z)\chi$ can be substituted by $|\nabla u|^{p(z)-2}\nabla u + a(z)|\nabla u|^{q(z)-2}\nabla u$. Since $u \in C([0, T]; L^2(\Omega))$, the initial condition is fulfilled by continuity. Estimates (3.7) follow from the uniform in ε estimates of Theorem 7.1 and the lower semicontinuity of the modular exactly as in the proof of Theorem 7.1. Uniqueness of a strong solution is an immediate consequence of the monotonicity. Theorem 3.1 is proven.

Proof of Theorem 3.2. The sequence of solutions of the regularized problems (5.1) contains a subsequence $\{u_{\varepsilon_k}\}$ that has the convergence properties (7.13) and satisfies the uniform estimates of Theorem 7.2 (i). It follows that there exist $v, \mathbf{w} \in L^2(Q_T)$ such that $v_k = \Psi(z, u_{\varepsilon_k}) \rightharpoonup v$ and $\mathbf{w}_k \equiv \Phi(z, \nabla u_{\varepsilon_k}) \rightharpoonup \mathbf{w}$ in $L^2(Q_T)$. The pointwise convergence $u_\varepsilon \to u$ entails the equality $v = \Psi(z, u)$. Following the proof of Theorem 7.2, we conclude that the limit function u solves problem (1.1) with the source $f_0 + \Psi(z, u) + \mathbf{w}$. The equality $\mathbf{w} = \Phi(z, \nabla u)$ will follow if we show that $\nabla u_{\varepsilon_k} \to \nabla u$ a.e. in Q_T. Take $u_{\varepsilon_k} - u \in \mathcal{W}_{q(\cdot)}(Q_T)$ for the test function in identities (3.4) for u_{ε_k} and u. Subtract these identities and rewrite the result in the following form:

$$
\begin{aligned}
& \int_{Q_T} (\gamma_{\varepsilon_k}\big(z, \nabla u_{\varepsilon_k}\big)\nabla u_{\varepsilon_k} - \gamma_{\varepsilon_k}(z, \nabla u)\nabla u)\cdot\nabla\big(u_{\varepsilon_k} - u\big)\mathrm{d}z \\
& \quad = -\int_{Q_T} \big(u_{\varepsilon_k} - u\big)_t\big(u_{\varepsilon_k} - u\big)\mathrm{d}z + \int_{Q_T} (\mathbf{w}_k - \mathbf{w})\big(u_{\varepsilon_k} - u\big)\mathrm{d}z + \int_{Q_T} (\Psi\big(z, u_{\varepsilon_k}\big) - \Psi(z, u))\big(u_{\varepsilon_k} - u\big)\mathrm{d}z \\
& \qquad + \int_{Q_T} (\gamma_0(z, \nabla u)\nabla u - \gamma_{\varepsilon_k}(z, \nabla u)\nabla u)\cdot\nabla\big(u_{\varepsilon_k} - u\big)\mathrm{d}z.
\end{aligned}
$$

The first two terms on the right-hand side of these inequality tend to zero when $\varepsilon_k \to 0$ as the products of weakly and strongly convergent sequences. The third term tends to zero because $\|u_{\varepsilon_k} - u\|_{2,Q_T} \to 0$, while $\Psi(z, u_{\varepsilon_k})$ and $\Psi(z, u)$ are uniformly bounded in $L^2(Q_T)$. To prove that the last term tends to zero, we use the same arguments as in the case of the integral $J_{1,k}$ in the proof of Theorem 3.1. By virtue of (7.6) and (2.5) $\nabla u_{\varepsilon_k} \to \nabla u$ in $(L^{p(\cdot)}(Q_T))^N$ and a.e. in Q_T.

The uniqueness follows by the same arguments as in the case of the nondegenerate problem.

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