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Global Attractivity for Nonautonomous Delay-Differential Equations with Mixed Monotonicity and Two Delays

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Abstract
In this paper we study the scalar delay differential equation

$$x'(t) = \alpha(t)x(t - g_1(t))f(a(t), x(t - g_2(t))) - \beta(t)x(t)$$

where f is decreasing in both arguments and the coefficients are positive and bounded. Sufficient conditions for the permanence and global attractivity for a fixed positive solution are derived. We apply our results to nonautonomous variants of Nicholson's blowfly equation and the Beverton–Holt model.

Keywords Mixed Monotonicity · Nicholson's blowfly equation · Two delays · Beverton–Holt model

1 Introduction

Gurney et al. [14] proposed the delay differential equation

$$x'(t) = -dx(t) + px(t - \tau)e^{-ax(t-\tau)} \tag{1.1}$$

for analyzing the density of the Australian sheep blowfly *Lucilia cuprina*. With remarkable accuracy, this model was able to reproduce the population oscillations observed by Nicholson in [17]. The equation

$$x'(t) = -dx(t) + \frac{px(t - \tau)}{1 + x(t - \tau)^\gamma} \tag{1.2}$$

with $\gamma \geq 1$ is other marked model with a noteworthy ability to describe real patterns [2]. In this case, Mackey and Glass explained the oscillations in number of neutrophils detected in some cases of chronic myelogenous leukimia. From a mathematical point of view, Eqs. (1.1) and (1.2) share two common features:

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- A "humped" relationship between future states and current states.
- The presence of time delays.

It is broadly accepted that the interplay between these two factors normally produces oscillations in any system [4, 22]. However, this interplay is not well understood yet and many questions remain to be solved.

Apart from the biological situations mentioned above, delay differential equations have been employed as suitable models for many problems in ecology, physiology, engineering and epidemiology [5, 7, 12, 18–20, 22]. Many authors have focused on the analysis of the general class of equations

$$x'(t) = -dx(t) + x(t-\tau)f(x(t-\tau)) \tag{1.3}$$

that contains (1.1) and (1.2) as particular examples. See [2, 4, 8, 22] for results on permanence/persistence, nonlinear oscillations, chaotic dynamics and global attractivity for this equation. We stress that (1.3) can be derived from the McKendrick-von Forester equations

$$\begin{cases} \frac{\partial u}{\partial a}(a,t) + \frac{\partial u}{\partial t}(a,t) = -\mu(a)u(a,t) \\ u(0,t) = M(t)f(M(t)) \\ u(a,0) = u_0(a) \end{cases} \tag{1.4}$$

in studying the evolution of the population size of adult individuals $M(t) = \int_\tau^{+\infty} u(a,t)da$, see [15, 22]. Despite its undoubted utility, Eq. (1.3) has two limitations from a biological point of view. Seasonality is a critical environmental feature in any ecological system [16]. For example, the birth rate of any species depends on the temperature, light, humidity of the environment and these factors vary with seasons. However, (1.3) is based on the assumption that there are not temporal variations of the environmental conditions. On the other hand, when we employ equation (1.3) for analyzing the evolution of an insect population, we implicitly assume that the lag of the impact on the survival of a previous competition of individuals takes place at the reproduction stage. Nevertheless, in many applications, e.g. most tick populations, biological observations indicate that this lag occurs before reproduction [15, 23]. A possible modeling framework that solves the aforementioned limitations is

$$x'(t) = \alpha(t)x(t-g_1(t))f(a(t), x(t-g_2(t))) - \beta(t)x(t). \tag{1.5}$$

Compared with (1.3), the time dependence and the presence of two delays add many difficulties due to the complexity of the mathematical tools to deal with (1.5). We emphasize that model (1.5) shows dynamical patterns that do not appear in (1.3), see [1, 3]. The reader can consult [7] for other scalar delay differential equations with multiple delays coming from real problems.

The purpose of this paper is to extend for nonautonomous equations the results in [9, 10]. Specifically, we will provide criteria of global attractivity for a positive solution in (1.5). Our results enhance and extend some recent achievements [1, 13, 15] in the literature in the following directions:

- The time dependence in the model is not necessarily periodic or almost periodic.
- We do not impose that $h(x) = xf(x)$ is monotone or $f(x) = e^{-x}$.

The structure of the paper is as follows: In Sect. 2, we deduce some permanence/persistence results for the solutions of (1.5). In Sect. 3, we state the main theorem of this paper. Informally speaking, we construct a scalar difference equation that codes many dynamical behaviors of (1.5). In Sect. 4, we apply our approach in some classical models and compare our results with previous ones in the literature.

A key ingredient of this paper is the fluctuation lemma. We recall its statement for the reader's convenience.

Lemma 1.1 (Lemma A.1 page 154 in [22]) *Let $f : [a, +\infty) \longrightarrow \mathbb{R}$ be a map of class $\mathcal{C}^1$ and bounded. Then, there are two sequences $\{t_n\} \to +\infty$ and $\{s_n\} \to +\infty$ with the following properties:*

- $\lim_{n\to+\infty} f(s_n) = \limsup_{t\to+\infty} f(t)$ *with* $\lim_{n\to+\infty} f'(s_n) = 0$.
- $\lim_{n\to+\infty} f(t_n) = \liminf_{t\to+\infty} f(t)$ *with* $\lim_{n\to+\infty} f'(t_n) = 0$.

2 Basic Properties

We consider

$$x'(t) = \alpha(t)x(t - g_1(t))f(a(t), x(t - g_2(t))) - \beta(t)x(t) \tag{2.1}$$

with the following conditions:

(H1) $\alpha, a, \beta : [0, +\infty) \longrightarrow (0, +\infty)$ are continuous and bounded functions. Moreover, there is a constant $\theta > 0$ so that $\theta \leq \alpha(t), \beta(t)$ for all $t \in [0, +\infty)$.

(H2) $g_1, g_2 : [0, +\infty) \longrightarrow [0, +\infty)$ are continuous and bounded functions with $g_2(t) \leq g_1(t)$ for all $t \in [0, +\infty)$.

(H3) $f : [0, +\infty)^2 \longrightarrow (0, +\infty)$ is of class $\mathcal{C}^1$, decreasing in the first argument and strictly decreasing in the second argument. In addition, for each $b > 0$, $f(b, 0) = 1$ and $\lim_{x\longrightarrow+\infty} f(b, x) = 0$. We will denote by $f_\xi^{-1} : (0, 1] \longrightarrow [0, +\infty)$ the inverse of the function $f(\xi, \cdot)$.

(H4) There is a constant $\omega > 1$ so that $\frac{\alpha(t)}{\beta(t)} \geq \omega$ for all $t \in [0, +\infty)$.

Let $\tau = \sup\{g_1(t) : t \in [0, +\infty)\}$. We denote by $\mathcal{C}_0^+ = \mathcal{C}([-\tau, 0], [0, +\infty))$ and $\mathcal{C}^+ = \mathcal{C}([-\tau, 0], (0, +\infty))$ the space of the continuous functions defined on the interval $[-\tau, 0]$ and taking values on $[0, +\infty)$ and $(0, +\infty)$, respectively. Given a function $\phi \in \mathcal{C}_0^+$, there is a unique (local) solution $x(t) = x(t, \phi)$ of (2.1) that satisfies $x(t) = \phi(t)$ for all $t \in [-\tau, 0]$ and Eq. (2.1) for $t \geq 0$, see [4, 22] . From

$$\frac{d}{dt}\left(x(t)e^{\int_0^t \beta(s)ds}\right) = e^{\int_0^t \beta(s)ds}\alpha(t)x(t - g_1(t))f(a(t), x(t - g_2(t))), \tag{2.2}$$

we can obtain an useful representation of the solutions, namely,

$$x(t) = x(0)e^{-\int_0^t \beta(s)ds} + e^{-\int_0^t \beta(s)ds}\int_0^t e^{\int_0^s \beta(r)dr}\alpha(s)x(s - g_1(s))f(a(s), x(s - g_2(s)))ds. \tag{2.3}$$

Using that f, α and β are bounded (see **(H1)** and **(H3)**), we easily deduce that the solutions of (2.1) with initial function in $\mathcal{C}_0^+$ are defined for all $t \geq 0$ from the method of steps. We refer to the solutions with initial function in $\mathcal{C}^+$ as positive solutions. Notice that if $x(t)$ is a positive solution, then $x(t) > 0$ for all $t \geq 0$. To see this property, we observe that all elements within the integral in (2.3) are strictly positive, (see **(H1)** and **(H3)**). On the other hand, given $x(t)$ a positive solution, $z(t) = x(t)e^{\int_0^t \beta(s)ds}$ is strictly increasing, (see (2.2)). We repeatedly use this property along the paper.

Proposition 2.1 *Assume that* **(H1), (H2), (H3)** *and* **(H4)** *hold. Then, for any positive solution $x(t)$ of (2.1), $x(t)$ is bounded and* $\liminf_{t\to+\infty} x(t) > 0$.

Proof We split the proof into two steps:
Step 1: The positive solutions of (2.1) are bounded.
Assume, by contradiction, that there is an unbounded positive solution $x(t)$. Then, we can find a sequence $\{s_n\} \nearrow +\infty$ with the following properties:

(Q1) $x(s_n) = \max\{x(t) : t \in [0, s_n]\}$ for all $n \in \mathbb{N}$.
(Q2) $x'(s_n) \geq 0$ for all $n \in \mathbb{N}$.
(Q3) $\lim_{n\to+\infty} x(s_n) = +\infty$.

The construction of this sequence is as follows: Take $m = \max\{x(t) : t \in [-\tau, 0]\}$. Define $s_n = \min\{t \in [0, +\infty) : x(t) = n \cdot (m + 1)\}$ with $n \in \mathbb{N}$.

Using the expression of Eq. (2.1) together with **(Q2)**, we deduce that

$$\frac{\beta(s_n)}{\alpha(s_n)} x(s_n) \leq x(s_n - g_1(s_n)) f(a(s_n), x(s_n - g_2(s_n))) \tag{2.4}$$

for all $n \in \mathbb{N}$. Using **(Q1)**, we get that

$$\frac{\beta(s_n)}{\alpha(s_n)} \leq f(a(s_n), x(s_n - g_2(s_n))) \tag{2.5}$$

for all $n \in \mathbb{N}$. Let

$$\Delta = \inf\left\{\frac{\beta(t)}{\alpha(t)} : t \in [0, +\infty)\right\} \in (0, 1), \tag{2.6}$$

(see **(H4)**) and

$$\delta = \inf\{a(t) : t \in [0 + \infty]\} \geq 0.$$

We observe that by **(H3)**,

$$f(a(s_n), x(s_n - g_2(s_n))) \leq f(\delta, x(s_n - g_2(s_n))) \tag{2.7}$$

for all $n \in \mathbb{N}$. Combining (2.5), (2.6) and (2.7), we conclude that

$$\Delta \leq f(\delta, x(s_n - g_2(s_n))),$$

or, equivalently, by **(H3)**,

$$x(s_n - g_2(s_n)) \leq f_\delta^{-1}(\Delta) \tag{2.8}$$

for all $n \in \mathbb{N}$. Notice that we can define $f_\delta^{-1}(\Delta)$ because $f(\delta, 0) = 1$ and $\lim_{x\longrightarrow+\infty} f(\delta, x) = 0$. On the other hand, using that $x(t)e^{\int_0^t \beta(s)ds}$ is strictly increasing and **(H2)**, we have that

$$x(s_n - g_1(s_n))e^{\int_0^{s_n - g_1(s_n)} \beta(s)ds} \leq x(s_n - g_2(s_n))e^{\int_0^{s_n - g_2(s_n)} \beta(s)ds}$$

for all $n \in \mathbb{N}$. Thus,

$$x(s_n - g_1(s_n)) \leq x(s_n - g_2(s_n))e^{\int_{s_n - g_1(s_n)}^{s_n - g_2(s_n)} \beta(s)ds}$$

for all $n \in \mathbb{N}$. By **(H1), (H2)** and (2.8), it is clear that $x(s_n - g_1(s_n))$ is bounded. Since $\{x(s_n - g_1(s_n))\}$ and $\{x(s_n - g_2(s_n))\}$ are bounded, we conclude easily from (2.4) that $x(s_n)$ is bounded as well. This is a contradiction with **(Q3)**.

Step 2: The positive solutions of (2.1) are bounded apart from 0.
Assume, by contradiction, that there is a positive solution $x(t)$ so that

$$\liminf_{t\to+\infty} x(t) = 0.$$

Then, we can find a sequence $\{t_n\} \nearrow +\infty$ with the following properties:

(P1) $x(t_n) = \min\{x(t) : t \in [0, t_n]\}$ for all $n \in \mathbb{N}$.
(P2) $x'(t_n) \leq 0$ for all $n \in \mathbb{N}$.
(P3) $\lim_{n\to+\infty} x(t_n) = 0$.

The construction of this sequence is analogous to $\{s_n\}$ of the previous step.

By expression of (2.1) together with **(P2)**, we deduce that

$$\frac{\beta(t_n)}{\alpha(t_n)} x(t_n) \geq x(t_n - g_1(t_n)) f(a(t_n), x(t_n - g_2(t_n))) \tag{2.9}$$

for all $n \in \mathbb{N}$. Using **(P1)**, we get that

$$\frac{\beta(t_n)}{\alpha(t_n)} \geq f(a(t_n), x(t_n - g_2(t_n))) \geq f(\tilde{a}, x(t_n - g_2(t_n))) \tag{2.10}$$

for all $n \in \mathbb{N}$ with $\tilde{a} = \sup\{a(t) : t \geq 0\}$, (use **(H3)** in the second inequality). By **(H4)**, we know that

$$\frac{1}{\omega} \geq \frac{\beta(t)}{\alpha(t)} \tag{2.11}$$

for all $t \in [0, +\infty)$ with $\omega > 1$. Inserting (2.11) in (2.10), we arrive at

$$\frac{1}{\omega} \geq f(\widetilde{a}, x(t_n - g_2(t_n)))$$

or equivalently, by **(H3)**,

$$x(t_n - g_2(t_n)) \geq f_{\tilde{a}}^{-1}\left(\frac{1}{\omega}\right) \tag{2.12}$$

for all $n \in \mathbb{N}$. Notice that we can define $f_{\tilde{a}}^{-1}(\frac{1}{\omega})$ because $f(\tilde{a}, 0) = 1$ and $\lim_{x\longrightarrow+\infty} f(\tilde{a}, x) = 0$. On the other hand, using that $x(t)e^{\int_0^t \beta(s)ds}$ is strictly increasing and $g_2(t) \geq 0$, we have that

$$x(t_n)e^{\int_0^{t_n} \beta(s)ds} \geq x(t_n - g_2(t_n))e^{\int_0^{t_n-g_2(t_n)} \beta(s)ds}$$

for all $n \in \mathbb{N}$. Thus,

$$x(t_n) \geq x(t_n - g_2(t_n))e^{-\int_{t_n-g_2(t_n)}^{t_n} \beta(s)ds}$$

for all $n \in \mathbb{N}$. Using (2.12), **(H1)** and **(H2)** in the previous inequality, we observe that $x(t_n)$ does not converge to zero as $n \to +\infty$. This is a contradiction with **(P3)**. □

Now we provide uniform bounds for the upper and lower limits of the positive solutions of (2.1). We write these bounds using the next notation:

$$\varphi = \limsup_{t\to+\infty} \int_{t-g_2(t)}^{t} \beta(s)ds, \tag{2.13}$$

$$\widetilde{\varphi} = \limsup_{t\to+\infty} \int_{t-g_1(t)}^{t-g_2(t)} \beta(s)ds, \tag{2.14}$$

$$a^* = \limsup_{t\to+\infty} a(t), \tag{2.15}$$

$$a_* = \liminf_{t\to+\infty} a(t), \tag{2.16}$$

and

$$\Omega = \liminf_{t\to+\infty} \frac{\beta(t)}{\alpha(t)}. \tag{2.17}$$

These quantities are meaningful by **(H1)**, **(H2)** and **(H4)** and satisfy $\varphi, \widetilde{\varphi} \geq 0$; $a^* \geq a_* \geq 0$; and $1 > \Omega > 0$.

Proposition 2.2 *Assume that* **(H1), (H2), (H3)** *and* **(H4)** *hold. Then,*

$$\liminf_{t\to+\infty} x(t) \geq e^{-\varphi} f_{a^*}^{-1}\left(\frac{1}{\omega}\right)$$

for any positive solution $x(t)$ of (2.1).

Proof Take $x(t)$ a positive solution of (2.1). By Proposition 2.1 and applying the fluctuation lemma (see Lemma 1.1), we can find a sequence $\{t_n\} \nearrow +\infty$ so that

$$\liminf_{t\to+\infty} x(t) = \lim_{n\to+\infty} x(t_n) = L > 0$$

with $\lim_{n\to+\infty} x'(t_n) = 0$. Evaluating (2.1) at t_n, we obtain

$$x'(t_n) = \alpha(t_n)x(t_n - g_1(t_n))f(a(t_n), x(t_n - g_2(t_n))) - \beta(t_n)x(t_n)$$

or, equivalently,

$$\frac{x'(t_n)}{\alpha(t_n)} + \frac{\beta(t_n)}{\alpha(t_n)}x(t_n) = x(t_n - g_1(t_n))f(a(t_n), x(t_n - g_2(t_n))). \tag{2.18}$$

By **(H1)**, **(H2)** and **(H4)**, it is not restrictive, after taking subsequences, to assume that $\alpha(t_n) \longrightarrow \alpha_1 > 0$; $a(t_n) \longrightarrow a_1$ with $a_1 \leq a^*$; $\frac{\beta(t_n)}{\alpha(t_n)} \longrightarrow \eta$ with $\eta \leq \frac{1}{\omega}$; $x(t_n - g_1(t_n)) \longrightarrow L_1$, $x(t_n - g_2(t_n)) \longrightarrow L_2$ with $L_1, L_2 \geq L$. Recall that $x(t)$ is bounded by Proposition 2.1. Making $n \longrightarrow +\infty$ in (2.18), we arrive at

$$\eta L = L_1 f(a_1, L_2).$$

As mentioned previously, $\eta \leq \frac{1}{\omega}$ and $a_1 \leq a^*$. Thus, by **(H3)**, we deduce

$$\frac{1}{\omega}L \geq L_1 f(a^*, L_2).$$

Now, using $L_1 \geq L$, we have that

$$\frac{1}{\omega} \geq f(a^*, L_2).$$

Applying the inverse of $f(a^*, \cdot)$ above (see **(H3)**),

$$L_2 \geq f_{a^*}^{-1}\left(\frac{1}{\omega}\right). \tag{2.19}$$

On the other hand, using that $x(t)e^{\int_0^t \beta(s)ds}$ is increasing and $g_2(t) \geq 0$, we obtain that

$$x(t_n - g_2(t_n))e^{\int_0^{t_n - g_2(t_n)} \beta(s)ds} \leq x(t_n)e^{\int_0^{t_n} \beta(s)ds},$$

what implies

$$x(t_n - g_2(t_n)) \leq x(t_n)e^{\int_{t_n - g_2(t_n)}^{t_n} \beta(s)ds}. \tag{2.20}$$

Since $g_2(t) \geq 0$ and $\beta(t)$ are bounded (see **(H1)** and **(H2)**), it is not restrictive to suppose that $\lim_{n\to+\infty} \int_{t_n - g_2(t_n)}^{t_n} \beta(s)ds$ exists and $\lim_{n\to+\infty} \int_{t_n - g_2(t_n)}^{t_n} \beta(s)ds \leq \varphi$, (see (2.13)). Making $n \longrightarrow +\infty$ in (2.20), we conclude that

$$L_2 \leq Le^{\varphi}.$$

Finally, by (2.19), it is clear that

$$e^{-\varphi} f_{a^*}^{-1}\left(\frac{1}{\omega}\right) \leq L.$$

□

Proposition 2.3 *Assume that* **(H1), (H2), (H3)** *and* **(H4)** *hold. Then,*

$$\limsup_{t\to+\infty} x(t) \leq \frac{e^{\widetilde{\varphi}}}{\Omega} f_{a_*}^{-1}(\Omega)$$

for any positive solution $x(t)$ *of* (2.1).

Proof Take $x(t)$ a positive solution of (2.1). By Proposition 2.1 and applying the fluctuation lemma (see Lemma 1.1), we can take a sequence $\{s_n\} \nearrow +\infty$ so that

$$\limsup_{t\to+\infty} x(t) = \lim_{n\to+\infty} x(s_n) = S$$

with $\lim_{n\to+\infty} x'(s_n) = 0$. Evaluating (2.1) at $\{s_n\}$, we obtain

$$x'(s_n) = \alpha(s_n)x(s_n - g_1(s_n)) f(a(s_n), x(s_n - g_2(s_n))) - \beta(s_n)x(s_n)$$

or, equivalently,

$$\frac{x'(s_n)}{\alpha(s_n)} + \frac{\beta(s_n)}{\alpha(s_n)} x(s_n) = x(s_n - g_1(s_n)) f(a(s_n), x(s_n - g_2(s_n))). \tag{2.21}$$

It is not restrictive, after passing to subsequences, to assume that $\alpha(s_n) \longrightarrow \alpha_2 > 0$; $a(s_n) \longrightarrow a_2$ with $a_2 \geq a_*$; $\frac{\beta(s_n)}{\alpha(s_n)} \longrightarrow \mu$ with $1 > \mu \geq \Omega$; $x(s_n - g_1(s_n)) \longrightarrow S_1$, $x(s_n - g_2(s_n)) \longrightarrow S_2$ with $0 < S_1, S_2 \leq S$. Recall that $\liminf_{t\to+\infty} x(t) > 0$ by Proposition 2.1 and $\alpha(t) > 0$, (see also **(H1)**, **(H2)** and **(H4)**). Making $n \longrightarrow +\infty$ in (2.21),

$$\mu S = S_1 f(a_2, S_2).$$

Using that $\mu \geq \Omega$ and $a_2 \geq a_*$ together with **(H3)**, we obtain that

$$\Omega S \leq S_1 f(a_*, S_2). \tag{2.22}$$

From this inequality, we get

$$\Omega \leq f(a_*, S_2)$$

because $S_1 \leq S$. Applying the inverse of $f(a_*, \cdot)$ above (see **(H3)**), we have that

$$S_2 \leq f_{a_*}^{-1}(\Omega). \tag{2.23}$$

On the other hand, using that $x(t)e^{\int_0^t \beta(s)ds}$ is increasing and $g_1(s_n) \geq g_2(s_n)$ for all $n \in \mathbb{N}$ (see **(H2)**), we deduce that

$$x(s_n - g_1(s_n)) \leq x(s_n - g_2(s_n)) e^{\int_{s_n - g_1(s_n)}^{s_n - g_2(s_n)} \beta(t)dt}. \tag{2.24}$$

Since $g_1(t)$, $g_2(t)$ and $\beta(t)$ are bounded (see **(H1)** and **(H2)**), it is not restrictive to assume that $\lim_{n\to+\infty} \int_{s_n - g_1(s_n)}^{s_n - g_2(s_n)} \beta(t)dt$ exists and $\lim_{n\to+\infty} \int_{s_n - g_1(s_n)}^{s_n - g_2(s_n)} \beta(t)dt \leq \widetilde{\varphi}$, see (2.14). Making $n \longrightarrow +\infty$ in (2.24) and using (2.23), we obtain that

$$S_1 \leq f_{a_*}^{-1}(\Omega)e^{\widetilde{\varphi}}. \tag{2.25}$$

Finally, the conclusion follows directly by (2.22), (2.25) and **(H3)** because $S_2 \geq 0$ and $f(a_*, 0) = 1$. □

3 Main Results

In this section we fix a positive solution $x_*(t)$ of (2.1). Our aim is to establish sufficient conditions for the global attractivity of $x_*(t)$. Specifically, we want to prove that, for any $x(t)$ positive solution of (2.1),

$$\lim_{t\to+\infty}[x(t)-x_*(t)]=0.$$

To this task, we define

$$y(t)=\frac{x(t)}{x_*(t)}.$$

By Proposition 2.1, for any positive solution $x(t)$, $0<\liminf_{t\to+\infty} y(t)$ and $\limsup_{t\to+\infty} y(t)\in(0,+\infty)$. After simple computations, $y(t)$ satisfies the equation

$$y'(t)=b(t)\Big(y(t-g_1(t))f(a(t),x_*(t-g_2(t))y(t-g_2(t)))-y(t)f(a(t),x_*(t-g_2(t)))\Big) \tag{3.1}$$

with

$$b(t)=\frac{\alpha(t)x_*(t-g_1(t))}{x_*(t)}. \tag{3.2}$$

Lemma 3.1 *Assume that* **(H1), (H2), (H3), (H4)** *hold. If there exists* $y(t)$ *a solution of* (3.1) *so that* $\lim_{t\to+\infty} y(t)=\zeta$*, then* $\zeta=1$.

Proof First we notice that by **(H1)** and Propositions 2.2 and 2.3, there are two strictly positive constants ϖ_1 and ϖ_2 so that

$$\varpi_1\le b(t)\le\varpi_2 \tag{3.3}$$

for all $t\in(0,+\infty)$. Next we take $y(t)$ a solution of (3.1) so that $\lim_{t\to+\infty} y(t)=\zeta$. Since $y(t)$ is bounded apart from zero (in an uniform manner), we deduce that $\zeta>0$. Then, there exists a sequence $\{t_n\}\to+\infty$ so that $y'(t_n)\to 0$. It is not restrictive, after passing to subsequences if necessary, to assume that $a(t_n)\to a_0\ge 0$ and $x_*(t_n-g_2(t_n))\to \digamma_0>0$ as $n\to+\infty$. Evaluating Eq. (3.1) at t_n we arrive at

$$y'(t_n)=b(t_n)\Big(y(t_n-g_1(t_n))f(a(t_n),x_*(t_n-g_2(t_n))y(t_n-g_2(t_n)))$$
$$-y(t_n)f(a(t_n),x_*(t_n-g_2(t_n)))\Big).$$

Making $n\to+\infty$ and using (3.3), we conclude that

$$\zeta f(a_0,\digamma_0\zeta)=\zeta f(a_0,\digamma_0).$$

Finally, by **(H3)**, we conclude that $\zeta=1$. □

Since $x(t)$, $x_*(t)$ are bounded and bounded apart from zero; and

$$x(t)-x_*(t)=x_*(t)\left(\frac{x(t)}{x_*(t)}-1\right),$$

we note that

$$\lim_{t\longrightarrow+\infty} y(t)=1 \Longleftrightarrow \lim_{t\longrightarrow+\infty}[x(t)-x_*(t)]=0.$$

From now on we focus on the analysis of $\lim_{t\longrightarrow+\infty} y(t)$.

In the rest of the section, the map

$$F : (0, +\infty)^3 \longrightarrow (0, \infty)$$
$$F(\lambda_1, \lambda_2, x) = \frac{f(\lambda_1, \lambda_2 x)}{f(\lambda_1, \lambda_2)}$$

plays a critical role. We assume the following conditions:

(F1) $\frac{\partial F}{\partial \lambda_i}(\lambda_1, \lambda_2, x) \geq 0$ for all $\lambda_i > 0$, $x \in (0, 1)$, and $i = 1, 2$.
(F2) $\frac{\partial F}{\partial \lambda_i}(\lambda_1, \lambda_2, x) \leq 0$ for all $\lambda_i > 0$, $x > 1$, and $i = 1, 2$.

We stress that F is strictly decreasing in the third variable by **(H3)**.

Proposition 3.1 *Assume that* **(H1), (H2), (H3), (H4)** *and* **(F1), (F2)** *hold. Consider a positive solution $x(t)$ with $\mathcal{L} = \liminf_{t\to+\infty} y(t)$ and $\mathcal{S} = \limsup_{t\to+\infty} y(t)$. Then, there are four positive constants $\mathcal{L}_1, \mathcal{L}_2, \mathcal{S}_1, \mathcal{S}_2$ with the following properties:*

*(**C1**)* $\mathcal{L}_1, \mathcal{L}_2, \mathcal{S}_1, \mathcal{S}_2 \in [\mathcal{L}, \mathcal{S}]$.
*(**C2**)* $\mathcal{S}_2 \leq 1 \leq \mathcal{L}_2$.
*(**C3**)* $\mathcal{L} \geq \mathcal{L}_1 F(a^*, \lambda_*, \mathcal{L}_2)$ *and* $\mathcal{S} \leq \mathcal{S}_1 F(a^*, \lambda_*, \mathcal{S}_2)$ *with λ_* a constant so that* $\lambda_* \geq \limsup_{t\to+\infty} x_*(t)$, *(see (2.15) for the definition of a^*).*

Proof By the fluctuation lemma (Lemma 1.1), we can take $\{t_n\}$ and $\{s_n\}$ tending to $+\infty$ so that

$$\lim_{n\to+\infty} y(t_n) = \mathcal{L} \quad \text{and} \quad \lim_{n\to+\infty} y'(t_n) = 0$$
$$\lim_{n\to+\infty} y(s_n) = \mathcal{S} \quad \text{and} \quad \lim_{n\to+\infty} y'(s_n) = 0.$$

It is not restrictive (after passing to sub-sequences if necessary) to assume that there are four positive constants $\mathcal{L}_1, \mathcal{L}_2, \mathcal{S}_1, \mathcal{S}_2 \in [\mathcal{L}, \mathcal{S}]$ so that $\lim_{n\to+\infty} y(t_n - g_1(t_n)) = \mathcal{L}_1$, $\lim_{n\to+\infty} y(t_n - g_2(t_n)) = \mathcal{L}_2$, $\lim_{n\to+\infty} y(s_n - g_1(s_n)) = \mathcal{S}_1$ and $\lim_{n\to+\infty} y(s_n - g_2(s_n)) = \mathcal{S}_2$. We can also suppose that there are other four positive constants $\mathcal{A}_1, \mathcal{A}_2, \nu_1, \nu_2$ with $\nu_1, \nu_2 \leq \lambda_*$ and $\mathcal{A}_1, \mathcal{A}_2 \leq a^*$ so that $\lim_{n\to+\infty} a(t_n) = \mathcal{A}_1$, $\lim_{n\to+\infty} a(s_n) = \mathcal{A}_2$, $\lim_{n\to+\infty} x_*(t_n - g_2(t_n)) = \nu_1$ and $\lim_{n\to+\infty} x_*(s_n - g_2(s_n)) = \nu_2$. Evaluating (3.1) at $\{t_n\}$ and $\{s_n\}$ respectively, we obtain that

$$y'(t_n) = b(t_n)\Big(y(t_n - g_1(t_n)) f(a(t_n), x_*(t_n - g_2(t_n)) y(t_n - g_2(t_n))) - y(t_n) f(a(t_n), x_*(t_n - g_2(t_n)))\Big)$$

and

$$y'(s_n) = b(s_n)\Big(y(s_n - g_1(s_n)) f(a(s_n), x_*(s_n - g_2(s_n)) y(s_n - g_2(s_n))) - y(s_n) f(a(s_n), x_*(s_n - g_2(s_n)))\Big).$$

Observe that, by Proposition 2.1 and **(H1)**, $b(t) \geq \xi > 0$ for all $t > 0$ with ξ a suitable positive constant (see (3.2) for the precise definition of $b(t)$). Making $n \longrightarrow +\infty$ in the previous expressions, we deduce that

$$\begin{cases} \mathcal{L}_1 f(\mathcal{A}_1, \nu_1 \mathcal{L}_2) - \mathcal{L} f(\mathcal{A}_1, \nu_1) = 0 \\ \mathcal{S}_1 f(\mathcal{A}_2, \nu_2 \mathcal{S}_2) - \mathcal{S} f(\mathcal{A}_2, \nu_2) = 0, \end{cases}$$

what implies

$$\begin{cases} \mathcal{L} = \mathcal{L}_1 F(\mathcal{A}_1, \nu_1, \mathcal{L}_2) \\ \mathcal{S} = \mathcal{S}_1 F(\mathcal{A}_2, \nu_2, \mathcal{S}_2). \end{cases} \tag{3.4}$$

Since $\mathcal{L}_1, \mathcal{S}_1 \in [\mathcal{L}, \mathcal{S}]$, we have that

$$\begin{cases} 1 \geq F(\mathcal{A}_1, \nu_1, \mathcal{L}_2) \\ 1 \leq F(\mathcal{A}_2, \nu_2, \mathcal{S}_2). \end{cases} \tag{3.5}$$

We know that $F(\lambda_1, \lambda_2, 1) = 1$ for all $(\lambda_1, \lambda_2) \in (0, +\infty)^2$ and that $F(\lambda_1, \lambda_2, x)$ is strictly decreasing in the third component for all $(\lambda_1, \lambda_2, x) \in (0, +\infty)^3$, (see **(H3)** and the definition of F). Thus, $\mathcal{S}_2 \leq 1 \leq \mathcal{L}_2$. Using these inequalities together with $\mathcal{A}_1, \mathcal{A}_2 \leq a^*$; $\nu_1, \nu_2 \leq \lambda_*$ and **(F1)**, **(F2)**, we deduce from expression (3.4) that

$$\begin{cases} \mathcal{L} \geq \mathcal{L}_1 F(a^*, \lambda_*, \mathcal{L}_2) \\ \mathcal{S} \leq \mathcal{S}_1 F(a^*, \lambda_*, \mathcal{S}_2). \end{cases} \tag{3.6}$$

□

The next result is a refinement of Proposition 3.1. Before its statement, we fix a positive constant κ so that

$$\limsup_{t \to +\infty} b(t) f(a(t), x_*(t - g_2(t))) \leq \kappa. \tag{3.7}$$

Let

$$\sigma = \limsup_{t \to +\infty} g_2(t). \tag{3.8}$$

Remark 3.1 Assume that **(H1), (H2), (H3), (H4)** and **(F1), (F2)** hold. If $\sigma = 0$, we directly have that $\mathcal{L}_2 = \mathcal{L}$ and $\mathcal{S}_2 = \mathcal{S}$, (see the proof of Proposition 3.1). Therefore, from **(C2)**, we can deduce that $\mathcal{L} = \mathcal{S} = 1$ and so, for any positive solution $x(t)$, $\lim_{t \to +\infty} [x(t) - x_*(t)] = 0$. Informally speaking, we have proved that in the absence of the delay g_2, the delay g_1 is harmless on the convergence of $x(t) - x_*(t)$ to zero.

By the previous remark, it is enough to analyze the case $\sigma > 0$.

Proposition 3.2 *Assume that* **(H1), (H2), (H3), (H4)** *and* **(F1), (F2)** *hold. Suppose that there exists a positive solution $x(t)$ so that $\mathcal{L} < \mathcal{S}$ with $\mathcal{L} = \liminf_{t \to +\infty} y(t)$ and $\mathcal{S} = \limsup_{t \to +\infty} y(t)$. Then, there are four positive real constants $\widetilde{\mathcal{A}}_1, \widetilde{\mathcal{A}}_2, \widetilde{\mathcal{S}}_1, \widetilde{\mathcal{S}}_2$ with the following properties:*

- *(**R1**)* $\widetilde{\mathcal{A}}_1, \widetilde{\mathcal{A}}_2, \widetilde{\mathcal{S}}_1, \widetilde{\mathcal{S}}_2 \in [\mathcal{L}, \mathcal{S}]$.
- *(**R2**)* $\widetilde{\mathcal{S}}_2 \leq 1 \leq \widetilde{\mathcal{A}}_2$.
- *(**R3**)* $\mathcal{L} \geq e^{-\kappa\sigma} + (1 - e^{-\kappa\sigma}) \widetilde{\mathcal{A}}_1 F(a^*, \lambda_*, \widetilde{\mathcal{A}}_2)$ *and* $\mathcal{S} \leq e^{-\kappa\sigma} + (1 - e^{-\kappa\sigma}) \widetilde{\mathcal{S}}_1 F(a^*, \lambda_*, \widetilde{\mathcal{S}}_2)$ *with λ_* a constant so that $\lambda_* \geq \limsup_{t \to +\infty} x_*(t)$, (see (2.15) for the definition of a^*).*

Proof Let

$$c(t) = b(t) f(a(t), x_*(t - g_2(t))). \tag{3.9}$$

From the definition of $b(t)$, the results in Sect. 2 and **(H1)**, **(H3)**, $c(t)$ is uniformly bounded and bounded apart from zero. We can write Eq. (3.1) as

$$y'(t) = b(t) y(t - g_1(t)) f(a(t), x_*(t - g_2(t)) y(t - g_2(t))) - c(t) y(t), \tag{3.10}$$

(see (3.2) for the definition of $b(t)$). Using the variation of the constant formula, we deduce that

$$y(t) = y(t - g_2(t))e^{-\int_{t-g_2(t)}^{t} c(s)ds} + e^{-\int_0^t c(s)ds} A(t) \tag{3.11}$$

with

$$A(t) = \int_{t-g_2(t)}^{t} e^{\int_0^s c(r)dr} b(s)y(s - g_1(s)) f(a(s), x_*(s - g_2(s))y(s - g_2(s)))ds.$$

After multiplying and dividing by $f(a(s), x_*(s - g_2(s)))$ within the integral, we realize that

$$A(t) = \int_{t-g_2(t)}^{t} \frac{d}{ds}(e^{\int_0^s c(r)dr})y(s - g_1(s))F(a(s), x_*(s - g_2(s)), y(s - g_2(s)))ds. \tag{3.12}$$

Evaluating (3.11) at the sequence $\{s_n\}$ given in Proposition 3.1, we have that

$$y(s_n) = y(s_n - g_2(s_n))e^{-\int_{s_n-g_2(s_n)}^{s_n} c(s)ds} + e^{-\int_0^{s_n} c(s)ds} A(s_n).$$

Notice that

$$A(s_n) \leq M_n \left(\int_{s_n-g_2(s_n)}^{s_n} \frac{d}{ds}(e^{\int_0^s c(r)dr})ds \right) = M_n \left(e^{\int_0^{s_n} c(r)dr} - e^{\int_0^{s_n-g_2(s_n)} c(r)dr} \right)$$

with

$$M_n = \max\{y(s - g_1(s))F(a(s), x_*(s - g_2(s)), y(s - g_2(s))) : s \in [s_n - g_2(s_n), s_n]\}.$$

Inserting this inequality above, we obtain

$$y(s_n) \leq y(s_n - g_2(s_n))e^{-\int_{s_n-g_2(s_n)}^{s_n} c(s)ds} + \left(1 - e^{-\int_{s_n-g_2(s_n)}^{s_n} c(s)ds}\right) M_n. \tag{3.13}$$

Next we take a sequence $\xi_n \in [s_n - g_2(s_n), s_n]$ so that

$$M_n = y(\xi_n - g_1(\xi_n))F(a(\xi_n), x_*(\xi_n - g_2(\xi_n)), y(\xi_n - g_2(\xi_n)))$$

for all $n \in \mathbb{N}$. Since $y(t)$, $c(t)$, $x_*(t)$ are bounded and bounded apart from zero, (see Proposition 2.1, **(H1), (H2), (H4)**), we can suppose that

$$y(\xi_n - g_1(\xi_n)) \longrightarrow \widetilde{\mathcal{S}}_1$$
$$y(\xi_n - g_2(\xi_n)) \longrightarrow \widetilde{\mathcal{S}}_2$$

with $\widetilde{\mathcal{S}}_1, \widetilde{\mathcal{S}}_2 \in [\mathcal{L}, \mathcal{S}]$;

$$a(\xi_n) \longrightarrow \widetilde{a}_2$$

with $\widetilde{a}_2 \leq a^* = \limsup_{t\to+\infty} a(t)$; and

$$x_*(\xi_n - g_2(\xi_n)) \longrightarrow \gamma_2$$

with $0 < \gamma_2 \leq \limsup_{t\to+\infty} x_*(t) \leq \lambda_*$. Regarding the sequence $\{s_n\}$, we know by Proposition 3.1 that

$$y(s_n - g_2(s_n)) \longrightarrow \mathcal{S}_2$$

with $\mathcal{S}_2 \leq 1$. Moreover, since $c(s)$ is bounded and bounded apart from zero and $g_2(t) \geq 0$ is bounded (see **(H2)**), it is not restrictive to assume that

$$\int_{s_n-g_2(s_n)}^{s_n} c(s)ds \longrightarrow \kappa_2$$

with $0 \leq \kappa_2 \leq \kappa\sigma$, (see (3.7) and (3.8)). We stress that by **(C1)** and **(C2)** in Proposition 3.1, $\mathcal{S} \geq 1$. Collecting the above information and making $n \to +\infty$ in (3.13), we arrive at

$$\mathcal{S} \leq \mathcal{S}_2 e^{-\kappa_2} + (1 - e^{-\kappa_2})\widetilde{\mathcal{S}}_1 F(\widetilde{a}_2, \gamma_2, \widetilde{\mathcal{S}}_2). \tag{3.14}$$

We remark that $\mathcal{S}_2 \leq 1$ and $\widetilde{\mathcal{S}}_1 \leq \mathcal{S}$. Arguing in a similar manner with the sequence $\{t_n\}$ of Proposition 3.1 instead of $\{s_n\}$, we obtain

$$\mathcal{L} \geq \mathcal{L}_2 e^{-\kappa_1} + (1 - e^{-\kappa_1})\widetilde{\mathcal{A}}_1 F(\widetilde{a}_1, \gamma_1, \widetilde{\mathcal{A}}_2) \tag{3.15}$$

with $0 \leq \kappa_1 \leq \kappa\sigma$; $\mathcal{L} \leq 1$; $\mathcal{L}_2 \geq 1$; $\widetilde{\mathcal{A}}_1, \widetilde{\mathcal{A}}_2 \in [\mathcal{L}, \mathcal{S}]$; $\widetilde{a}_1 \leq \limsup_{t \to +\infty} a(t) = a^*$; and $\gamma_1 \leq \limsup_{t \to +\infty} x_*(t) \leq \lambda_*$. Now we distinguish between two cases:
Case 1: Assume that $\kappa_1 > 0$ and $\kappa_2 > 0$.
Using that $\mathcal{S} \geq 1$, $\mathcal{S}_2 \leq 1$ and $\widetilde{\mathcal{S}}_1 \leq \mathcal{S}$, we deduce from (3.14) that

$$F(\widetilde{a}_2, \gamma_2, \widetilde{\mathcal{S}}_2) \geq 1$$

and

$$\widetilde{\mathcal{S}}_1 F(\widetilde{a}_2, \gamma_2, \widetilde{\mathcal{S}}_2) \geq 1. \tag{3.16}$$

We know that $F(\widetilde{a}_2, \gamma_2, 1) = 1$ and F is strictly decreasing in the third variable (see **(H3)**). Hence, $\widetilde{\mathcal{S}}_2 \leq 1$. On the other hand, by **(F1)** and $\gamma_2 \leq \lambda_*$, $\widetilde{a}_2 \leq a^*$, we obtain that

$$F(\widetilde{a}_2, \gamma_2, \widetilde{\mathcal{S}}_2) \leq F(a^*, \lambda_*, \widetilde{\mathcal{S}}_2). \tag{3.17}$$

Inserting (3.17) in (3.14), we arrive at

$$\mathcal{S} \leq \mathcal{S}_2 e^{-\kappa_2} + (1 - e^{-\kappa_2})\widetilde{\mathcal{S}}_1 F(a^*, \lambda_*, \widetilde{\mathcal{S}}_2). \tag{3.18}$$

A simple computation of the derivative of $P(x) = e^{-x} + (1 - e^{-x})\widetilde{\mathcal{S}}_1 F(a^*, \lambda_*, \widetilde{\mathcal{S}}_2)$ together with (3.16) show that P is increasing. Thus, since $\kappa_2 \leq \kappa\sigma$ and $\mathcal{S}_2 \leq 1$, we have that

$$\mathcal{S} \leq e^{-\kappa\sigma} + (1 - e^{-\kappa\sigma})\widetilde{\mathcal{S}}_1 F(a^*, \lambda_*, \widetilde{\mathcal{S}}_2).$$

The inequality $\mathcal{L} \geq e^{-\kappa\sigma} + (1 - e^{-\kappa\sigma})\widetilde{\mathcal{A}}_1 F(a^*, \lambda_*, \widetilde{\mathcal{A}}_2)$ can be proved analogously using (3.15).
Case 2: Assume that $\kappa_1 = 0$ or $\kappa_2 = 0$.
Let us prove that this case can not occur. Suppose, for instance, that $\kappa_2 = 0$. Then, by (3.14), we have that $\mathcal{S} \leq \mathcal{S}_2$. We can conclude that $\mathcal{S} = 1$ because we knew by Proposition 3.1 that $\mathcal{S}_2 \leq 1$, $\mathcal{S}_2 \leq \mathcal{S}$ and $\mathcal{S} \geq 1$. On the other hand, by Proposition 3.1 and $\mathcal{S} = 1$, we deduce that $\mathcal{L}_2 = 1$. Now, (3.15) writes as

$$\mathcal{L} \geq e^{-\kappa_1} + (1 - e^{-\kappa_1})\widetilde{\mathcal{A}}_1 F(\widetilde{a}_1, \gamma_1, \widetilde{\mathcal{A}}_2) \tag{3.19}$$

Since $1 \geq \mathcal{L}$ and $\widetilde{\mathcal{A}}_1 \geq \mathcal{L}$, we obtain that $F(\widetilde{a}_1, \gamma_1, \widetilde{\mathcal{A}}_2) \leq 1$. Using that $F(\widetilde{a}_1, \gamma_1, 1) = 1$ and F is strictly decreasing in the third component, we get that $\widetilde{\mathcal{A}}_2 \geq 1$. As $1 = \mathcal{S} \geq \widetilde{\mathcal{A}}_2$, we also obtain that $\widetilde{\mathcal{A}}_2 = 1$. Therefore, (3.19) becomes

$$\mathcal{L} \geq e^{-\kappa_1} + (1 - e^{-\kappa_1})\widetilde{\mathcal{A}}_1.$$

Note that this expression implies that

$$\mathcal{L} \geq e^{-\kappa_1} + (1 - e^{-\kappa_1})\mathcal{L}$$

because $\widetilde{\mathcal{A}}_1 \geq \mathcal{L}$. Now, it is clear that $\mathcal{L} = 1$ because $\mathcal{L} \leq 1$. At this moment we have proved that $\mathcal{L} = \mathcal{S} = 1$. This is a contradiction because $\mathcal{L} < \mathcal{S}$ by assumptions. □

Remark 3.2 Assume that **(H1), (H2), (H3), (H4)** and **(F1), (F2)** hold. Given $x(t)$ a positive solution,

$$\liminf_{t\to+\infty} y(t) = \liminf_{t\to+\infty} \frac{x(t)}{x_*(t)} > e^{-\kappa\sigma}.$$

To see this inequality we argue as follows: If $\limsup_{t\to+\infty} y(t) > \liminf_{t\to+\infty} y(t)$, then the conclusion is clear by Proposition 3.2. If $\limsup_{t\to+\infty} y(t) = \liminf_{t\to+\infty} y(t)$, then $\lim_{t\to+\infty} y(t) = 1$ by Lemma 3.1.

The following result is stated in terms of the global attraction of a suitable difference equation. We recall that a fixed point $\bar{x}$ of $H : (\Gamma, +\infty) \longrightarrow (\Gamma, +\infty)$ with $\Gamma \in [-\infty, +\infty)$ is globally attracting in $(\Gamma, +\infty)$ for the difference equation

$$x_{n+1} = H(x_n)$$

if, for all $x_0 \in (\Gamma, +\infty)$, $H^n(x_0) \to \bar{x}$ as $n \to +\infty$ with $H^n = H \circ ..^{n)}.. \circ H$.

Theorem 3.1 *Assume that* **(H1), (H2), (H3), (H4)** *and* **(F1), (F2)** *hold. With the notation of Proposition 3.2, suppose that*

$$F(a^*, \lambda_*, e^{-\kappa\sigma}) < \frac{1}{1 - e^{-\kappa\sigma}}. \tag{3.20}$$

If 1 is globally attracting in $(e^{-\kappa\sigma}, +\infty)$ for the difference equation

$$x_{n+1} = H(x_n) \tag{3.21}$$

with

$$H(x) = \frac{e^{-\kappa\sigma}}{1 - (1 - e^{-\kappa\sigma})F(a^*, \lambda_*, x)},$$

then, $x_*(t)$ is globally attracting, that is, for any positive solution $x(t)$ of (2.1),

$$\lim_{t\longrightarrow+\infty} [x(t) - x_*(t)] = 0.$$

Proof First we observe that by (3.20) and **(H3)**, H is strictly decreasing and $H((e^{-\kappa\sigma}, +\infty)) \subset (e^{-\kappa\sigma}, +\infty)$. Since 1 is globally attracting in $(e^{-\kappa\sigma}, +\infty)$ for the difference equation (3.21), then there is no a non-trivial compact interval $I \subset (e^{-\kappa\sigma}, +\infty)$ so that $I \subset H(I)$, see Lemma 4.1 in [6]. After this preliminary fact, we assume, by contradiction, that there is a positive solution $x(t)$ so that $y(t) = \frac{x(t)}{x_*(t)} \nrightarrow 1$ as $t \to +\infty$. Set $\mathcal{L} = \liminf_{t\to+\infty} y(t)$ and $\mathcal{S} = \limsup_{t\to+\infty} y(t)$. By Proposition 2.1, we know that $0 < \mathcal{L}$ and $\mathcal{S} \in (0, +\infty)$. By Lemma 3.1, we deduce that $\mathcal{L} < \mathcal{S}$. By the previous proposition, we conclude that $\sigma > 0$ and that there are four positive constants $\widetilde{\mathcal{A}}_1, \widetilde{\mathcal{A}}_2, \widetilde{\mathcal{S}}_1, \widetilde{\mathcal{S}}_2$ so that

$$\mathcal{L} \geq e^{-\kappa\sigma} + (1 - e^{-\kappa\sigma})\mathcal{L}F(a^*, \lambda_*, \widetilde{\mathcal{A}}_2)$$

and

$$\mathcal{S} \leq e^{-\kappa\sigma} + (1 - e^{-\kappa\sigma})\mathcal{S}F(a^*, \lambda_*, \widetilde{\mathcal{S}}_2).$$

After simple manipulations, we obtain that

$$\mathcal{L} \geq \frac{e^{-\kappa\sigma}}{1 - (1 - e^{-\kappa\sigma})F(a^*, \lambda_*, \widetilde{\mathcal{A}}_2)}$$

and

$$\mathcal{S} \le \frac{e^{-\kappa\sigma}}{1-(1-e^{-\kappa\sigma})F(a^*, \lambda_*, \widetilde{\mathcal{S}}_2)},$$

what implies $H(\widetilde{\mathcal{S}}_2) \ge \mathcal{S}$ and $H(\widetilde{\mathcal{A}}_2) \le \mathcal{L}$ with $\widetilde{\mathcal{S}}_2, \widetilde{\mathcal{A}}_2 \in [\mathcal{L}, \mathcal{S}]$. We can deduce that $[\mathcal{L}, \mathcal{S}] \subset H([\mathcal{L}, \mathcal{S}])$. We have obtained a contradiction with the comments at the beginning of the proof. We note that $\mathcal{L} > e^{-\kappa\sigma}$ by Remark 3.2. □

If the coefficients of (2.1) are T-periodic and we know in advance the existence of a positive T-periodic solution $x_*(t)$ for Eq. (2.1), Theorem 3.1 guarantees that $x_*(t)$ is globally attracting. The reader can consult [11] for nice results on the existence of positive T-periodic solutions of (2.1) when the coefficients are T-periodic.

4 Examples

In this section we apply Theorem 3.1 in two classical models: A nonautonomous variant of Nicholson's blowfly equation and a nonautonomous model of Beverton–Holt type. We will show that the assumptions introduced previously are satisfied for the usual nonlinearities $f(a, x) = e^{-ax}$ and $f(a, x) = \frac{1}{1+ax}$.

Next, for the reader's convenience, we recall a well-known criterion for global attractivity of an equilibrium in scalar difference equations, (see [6]).

Proposition 4.1 *Assume that $\varphi : (c, +\infty) \longrightarrow (c, +\infty)$ is a decreasing function of class $\mathcal{C}^3$ with negative Schwarzian derivative, that is,*

$$(S\varphi)(x) = \frac{\varphi'''(x)}{\varphi'(x)} - \frac{3}{2}\left(\frac{\varphi''(x)}{\varphi'(x)}\right)^2 < 0, \quad \textit{for all } x > 0$$

provided $\varphi'(x) \ne 0$. If $\bar{x} \in (c, +\infty)$ is the equilibrium of

$$x_{n+1} = \varphi(x_n) \tag{4.1}$$

and $|\varphi'(\bar{x})| \le 1$, then $\bar{x}$ is globally attracting in $(c, +\infty)$ for the difference equation (4.1).

We stress that if $G(x) = F(a^*, \lambda_*, x)$ is a map of class $\mathcal{C}^3$ with negative Schwartzian derivative in $(e^{-\kappa\sigma}, +\infty)$ and (3.20) is satisfied, then H has negative Schwartzian derivative in $(e^{-\kappa\sigma}, +\infty)$ as well. This is a consequence of the results in [21] because $H(x) = \Phi \circ G$ with $\Phi(x) = \frac{e^{-\kappa\sigma}}{1-(1-e^{-\kappa\sigma})x}$ and the Schwartzian derivative is negative for both functions.

4.1 A Nonautonomous Nicholson's Blowfly Equation with Two Different Delays

Consider

$$x'(t) = \alpha(t)x(t-\tau_1)e^{-ax(t-\tau_2)} - \beta(t)x(t) \tag{4.2}$$

with the following conditions:

(A1) $a > 0, \alpha, \beta : [0, +\infty) \to (0, +\infty)$ are continuous and bounded. Moreover, there is a constant $\theta > 0$ so that $\theta \le \alpha(t), \beta(t)$ for all $t \in [0, +\infty)$.
(A2) $0 \le \tau_2 \le \tau_1$.
(A3) There is a constant $\omega > 1$ so that $\frac{\alpha(t)}{\beta(t)} \ge \omega$ for all $t \in [0, +\infty)$.

In (4.2), τ_1 represents the lag of the impact on the survival and/or reproduction of a previous competition of individuals and τ_2 is the maturation time. When $\tau_1 = \tau_2$, the impact takes place at the reproduction stage. However, in many applications, e.g. most tick populations, experimental results suggest that $\tau_2 \leq \tau_1$, see [23].

Under **(A1)**, **(A2)** and **(A3)**, conditions **(H1)-(H4)** of Sect. 3 are satisfied for Eq. (4.2). Notice that $g_1(t) = \tau_1$, $g_2(t) = \sigma = \tau_2$, $f(a, x) = e^{-ax}$ and $F(\lambda_1, \lambda_2, x) = e^{\lambda_1\lambda_2(1-x)}$. Thus, it is clear that **(F1)** and **(F2)** hold. It is worth mentioning that for each λ_1, λ_2; $F(\lambda_1, \lambda_2, x)$ has negative Schwartzian derivative. As mentioned above, this implies that

$$H(x) = \frac{e^{-\kappa\tau_2}}{1 - (1 - e^{-\kappa\tau_2})F(a, \lambda_*, x)}$$

has negative Schwartzian derivative as well.

Fix $x_*(t) > 0$ a positive solutions of (4.2). By Propositions 2.2 and 2.3,

$$\limsup_{t\to+\infty} x_*(t) \leq \frac{e^{\widetilde{\varphi}}}{\Omega}\left(\frac{-\ln\Omega}{a}\right)$$

$$\liminf_{t\to+\infty} x_*(t) \geq e^{-\varphi}\left(\frac{\ln\omega}{a}\right)$$

with $\Omega = \liminf_{t\to+\infty} \frac{\beta(t)}{\alpha(t)}$, $\varphi = \limsup_{t\to+\infty} \int_{t-\tau_2}^{t} \beta(s)ds$, $\widetilde{\varphi} = \limsup_{t\to+\infty} \int_{t-\tau_1}^{t-\tau_2} \beta(s)ds$. Next we identify a constant κ so that $\kappa \geq \limsup_{t\to+\infty} b(t) f(a, x_*(t-\tau_2))$, (recall that $b(t) = \frac{\alpha(t)x_*(t-\tau_1)}{x_*(t)}$). Notice that

$$\limsup_{t\to+\infty} \alpha(t)\frac{\limsup_{t\to+\infty} x_*(t)}{\liminf_{t\to+\infty} x_*(t)} \geq \limsup_{t\to+\infty} b(t) f(a, x_*(t-\tau_2)).$$

By the previous estimates, if $\alpha_* = \limsup_{t\to+\infty} \alpha(t)$, we can take

$$\kappa = \alpha_* \frac{e^{\widetilde{\varphi}+\varphi}}{\Omega}\left(\frac{-\ln\Omega}{\ln\omega}\right)$$

and

$$\lambda_* = \frac{e^{\widetilde{\varphi}}(-\ln\Omega)}{\Omega a}.$$

Once we have an estimate of the elements involved in Theorem 3.1, we analyze the conditions

$$F(a, \lambda_*, e^{-\kappa\tau_2}) < \frac{1}{1 - e^{-\kappa\tau_2}} \tag{4.3}$$

and the global attractivity of 1 in $(e^{-\kappa\tau_2}, +\infty)$ for the difference equation

$$x_{n+1} = H(x_n)$$

with

$$H(x) = \frac{e^{-\kappa\tau_2}}{1 - (1 - e^{-\kappa\tau_2})F(a, \lambda_*, x)}.$$

Using Proposition 4.1 and assuming (4.3), the global attractivity for 1 is guaranteed if $H'(1) \geq -1$. On the other hand, we observe that

$$H'(1) = -(e^{\kappa\tau_2} - 1)\frac{e^{\widetilde{\varphi}}(\ln\Omega)}{\Omega} = -(e^{\kappa\tau_2} - 1)a\lambda_* \geq -1 \tag{4.4}$$

implies (4.3). Indeed, observe that (4.4) leads to

$$(1 - e^{-\kappa\tau_2})a\lambda_* \le e^{-\kappa\tau_2}.$$

On the other hand, after taking logarithms, (4.3) writes as

$$(1 - e^{-\kappa\tau_2})a\lambda_* < -\ln(1 - e^{-\kappa\tau_2}).$$

Since $e^{-\kappa\tau_2} < -\ln(1 - e^{-\kappa\tau_2})$, it is clear that (4.3) implies (4.4). Collecting the above discussion, we can deduce the following result:

Theorem 4.1 *Assume that* **(A1)***,* **(A2)** *and* **(A3)** *hold. If*

$$(e^{\kappa\tau_2} - 1)\frac{e^{\widetilde{\varphi}}(\ln\Omega)}{\Omega} \le 1$$

then, for any pair of positive solutions $x(t), x_*(t)$ *of* (4.2)*,*

$$\lim_{t\to+\infty}[x(t) - x_*(t)] = 0.$$

Nonautonomous Nicholson's blowfly models with a single delay has been extensively analyzed in the literature [4, 8, 22]. However, the global analysis for (4.2) has been recognized to be challenging and there are few available results in the literature. The reader can consult [13, 15] for nice results on global attraction for the equation

$$x'(t) = \beta(t)(Px(t - g_1(t))e^{-ax(t-g_2(t))} - \delta x(t)). \tag{4.5}$$

If we applied Theorem 3.1 for studying the global attractivity of the equilibrium $\frac{\ln\frac{P}{\delta}}{a}$ for Eq. (4.5), we should impose $g_2(t) \le g_1(t)$ for all $t \in [0, +\infty)$ to recover Theorem 3.1 in [13]. In other words, we need a condition not required in [13]. However, our approach has two advantages in comparison with [13]:

- It is not restricted to Nicholson's blowfly equations.
- We can derive global attractivity criteria for non-constant solutions.

4.2 A Nonautonomous Beverton–Holt Equation with Two Different Delays

Consider

$$x'(t) = \alpha(t)\frac{x(t - \tau_1)}{1 + x(t - \tau_2)} - \beta(t)x(t) \tag{4.6}$$

with the following conditions:

(B1) $\alpha, \beta : [0, +\infty) \to (0, +\infty)$ are continuous and bounded. Moreover, there is a constant $\theta > 0$ so that $\theta \le \alpha(t), \beta(t)$ for all $t \in [0, +\infty)$.

(B2) $0 \le \tau_2 \le \tau_1$.

(B3) There is a constant $\omega > 1$ so that $\frac{\alpha(t)}{\beta(t)} \ge \omega$ for all $t \in [0, +\infty)$.

Under **(B1)**, **(B2)** and **(B3)**, conditions **(H1)-(H4)** of Sect. 3 are satisfied for Eq. (4.6). Notice that $g_1(t) = \tau_1$, $g_2(t) = \sigma = \tau_2$, $f(a, x) = \frac{1}{1+x}$ and $f_a^{-1}(x) = \frac{1}{x} - 1$. It is clear that **(F1)** and **(F2)** are satisfied because $F(\lambda_1, \lambda_2, x) = \frac{1+\lambda_2}{1+\lambda_2 x}$. Fix $x_*(t) > 0$ a positive solutions of (4.6). By Propositions 2.2 and 2.3, we deduce that

$$\liminf_{t\to+\infty} x_*(t) \ge e^{-\varphi}(\omega - 1)$$

and

$$\limsup_{t\to+\infty} x_*(t) \le e^{\widetilde{\varphi}} \frac{1-\Omega}{\Omega^2}$$

with $\Omega = \liminf_{t\to+\infty} \frac{\beta(t)}{\alpha(t)}, \varphi = \limsup_{t\to+\infty} \int_{t-\tau_2}^{t} \beta(s)ds, \widetilde{\varphi} = \limsup_{t\to+\infty} \int_{t-\tau_1}^{t-\tau_2} \beta(s)ds.$

Arguing as in (4.2), we take

$$\kappa = \alpha_* e^{\widetilde{\varphi}+\varphi} \left(\frac{1-\Omega}{\Omega^2(\omega-1)} \right)$$

and

$$\lambda_* = e^{\widetilde{\varphi}} \frac{1-\Omega}{\Omega^2}$$

with $\alpha_* = \limsup_{t\to+\infty} \alpha(t)$.

Now we define

$$H(x) = \frac{e^{-\kappa\sigma}}{1-(1-e^{-\kappa\sigma})F(a^*, \lambda_*, x)}.$$

This function has negative Schwartzian derivative and $H'(1) = -(e^{\kappa\sigma}-1)\frac{\lambda_*}{1+\lambda_*}$.

After straightforward computations, we can deduce that

$$F(a^*, \lambda_*, e^{-\kappa\sigma}) = \frac{1+\lambda_*}{1+\lambda_* e^{-\kappa\sigma}} < \frac{1}{1-e^{-\kappa\sigma}} \iff (e^{\kappa\sigma}-1)\frac{\lambda_*}{1+\lambda_*} < 1.$$

Thus, as a direct consequence of Theorem 3.1 and Proposition 4.1, we arrive at the following result:

Theorem 4.2 *Assume that* **(B1)***,* **(B2)** *and* **(B3)** *hold. If*

$$(e^{\kappa\sigma}-1)\frac{\lambda_*}{1+\lambda_*} < 1$$

then, for any pair $x(t)$, $x_*(t)$ of positive solutions of (4.6)*,*

$$\lim_{t\to+\infty} [x(t) - x_*(t)] = 0.$$

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