Rev. Real Acad. Cienc. Exactas Fis. Nat. Ser. A-Mat. (2022) 116:130
https://doi.org/10.1007/s13398-022-01273-x

ORIGINAL PAPER

Comonotone lower probabilities with robust marginal distributions functions

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Received: 5 May 2021 / Accepted: 14 May 2022 / Published online: 3 June 2022

Abstract
One of the usual dependence structures between random variables is comononicity, which refers to random variables that increase or decrease simultaneously. Besides the good mathematical properties, comonotonicity has been applied in choice theory under risk or in finance, among many other fields. The problem arises when the marginal distribution functions are only partially known, hence we only know bounds of their values. This can be mathematically modelled using p-boxes, allowing us to build a bridge with the theory of imprecise probabilities. This paper investigates the existence, construction and uniqueness of a joint (imprecise) comonotone model with the given marginal p-boxes. In particular, given that the joint comonotone model is not unique when it exists, we follow the philosophy of the imprecise probability theory and we characterise under which conditions there exists a least-committal comonotone model, called the comonotone natural extension.

Keywords Comonotonicity · Lower probabilities · p-boxes · Belief functions · Natural extension

Mathematics Subject Classification 60A05 · 60D05 · 62E17 · 62E15 · 62H05

1 Introduction

When we deal with random variables, it is necessary to take into account the dependence structure between them. The most common dependence structure between random variables is *independence*, which refers to the lack of dependence between them. Independent random variables are easy to handle because their joint cumulative distribution function (cdf, for short) can be expressed as the product of the marginals. Recently, many probabilistic properties have been studied without the assumption of independence, like for example strong laws of large numbers [1] or the convergence [37] or sum [9] of dependent random variables, among many others. When the random variables are not independent, Sklar's Theorem [32] assures that a somewhat similar decomposition of the joint cdf can be done using functions called *copulas*.

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Among the many different dependence structures between random variables, *comonotonicity* refers to the extreme case of positive dependence, meaning that two random variables are comonotone when there is an increasing connection between them. In that case, the joint cdf can be expressed as the minimum of the marginals and the support is an increasing set. Moreover, one of the main properties of comonotonicity is that given two marginal distributions, there is always a joint comonotone model with the given marginals. This joint comonotone model can be built using the representation of the joint cdf as the minimum of the marginals and such a comonotone model is unique.

Over the last years, comonotonicity has been applied in many different fields related to probability theory and its applications. For example, Yaari [38] proposed to replace the independence axiom in the expected utility representation theorem by a comonotonicity property; Dhaene et al. investigated the role of comonotonicity in risk measures [8]; De Meyer et al. [6] and Montes et al. [24, 25] investigated the role of comonotonicity in stochastic orderings; and Dhaene et al. [7] reviewed many applications of comonotonicity in finance.

The problem arises when the marginal probabilistic models are not precisely determined. This happens for example when the data are noisy or when the experts eliciting the probability distributions are unreliable. In those situations, instead of considering two marginal cdfs modelling the uncertainty about the marginal random variables, we can use robust distribution functions or *p-boxes* [11], which are pairs of ordered cdfs giving lower and upper bounds for the real but unknown cdf. This happens for example in robust statistics [13], when a cdf is elicited but a neighbourhood is built around it [21, 22].

The properties of p-boxes have been widely studied in the literature [12, 19, 27, 33], as well as its connection with other usual models within the Theory of Imprecise Probabilities [20, 33, 34]. The Theory of Imprecise Probabilities [36] encompasses all the models that can be used as an alternative to probability measures when the information is vague or imprecise. Among them, we mention here coherent lower and upper probabilities [35], which give lower and upper bounds for the values of a real but unknown probability measure, and belief and plausibility measures [30], which create a bridge between imprecise probabilities and evidence theory. These models are gaining popularity, and nowadays are applied in different fields such as finance [31], robust Bayesian analysis [28], game theory [14] or stochastic processes [4], among many others.

The notion of independence has been widely studied in the theory of imprecise probabilities (see [2] for a survey and [5, 15] for some interesting references). Also, many papers have investigated to which extent Sklar's theorem holds for imprecise probabilities [23, 26, 29]. In a previous paper, Montes and Destercke [18] analysed how to extend the notion of comonotonicity when dealing with coherent lower probabilities, and studied their properties. However, an important problem was not completely solved there: the existence, construction and uniqueness of a comonotone lower probability given marginal imprecise models. For example, when the marginal models are given in terms of possibility measures [10], it was proven [18, Prop.28] that there is always a joint comonotone model with the given marginals; when the marginal models are belief functions, only some sufficient [18, Sec.4.2] or necessary [18, Sec.4.3] conditions were given. In this paper, our aim is to dig into this problem: we consider two p-boxes modelling the available information about the random variables, and we look for a joint comonotone lower probability with the given (imprecise) marginals, that will be called *comonotone extension* of the given p-boxes. We will see that, when it exists, the comonotone extension is not unique, and for this reason we will study the existence of a most conservative comonotone extension, that will be called the *comonotone natural extension*.

The paper is organised as follows: in Sect. 2 we introduce the models within the imprecise probability theory that will be used in this paper: coherent lower probabilities, belief functions and p-boxes. In Sect. 3 we review the definition and main properties of comonotone probability measures, as well as the definition of comonotonicity for lower probabilities given in [18]. In Sect. 4 we tackle the main problem of this paper: given two marginal p-boxes, we analyse the existence (Sect. 4.1), construction (Sect. 4.2) and uniqueness (Sect. 4.3) of a comonotone extension of the given p-boxes. Later, in Sect. 5.1 and 5.2 we explore some properties of the comonotone extensions and in Sect. 5.3 we characterise the conditions under which the comonotone natural extension exists. Section 6 ends the paper discussing the main results and posing some open problems. Note that this work is an extended version of the preliminary conference paper presented at the ISIPTA conference 2021 [17].

2 Imprecise probability models

We devote this section to the introduction of the main models of the imprecise probability theory we will need throughout the paper.

Just to fix the notation, from now on $(\Omega, \mathcal{A}, P)$ denotes a finite probability space and $\mathcal{X} = \{x_1, \ldots, x_n\}$ and $\mathcal{Y} = \{y_1, \ldots, y_m\}$ denote two ordered possibility spaces, where we assume that the elements in $\mathcal{X}$ and $\mathcal{Y}$ are ordered according to their index, i.e., $x_1 < x_2 < \cdots < x_n$ and $y_1 < y_2 < \cdots < y_m$. Also, we denote by $\mathbb{P}(\Omega)$ the set of (finitely additive) probability measures on Ω, and similarly for $\mathbb{P}(\mathcal{X})$, $\mathbb{P}(\mathcal{Y})$ and $\mathbb{P}(\mathcal{X} \times \mathcal{Y})$. Finally, we use the notation A_{x_i}, A_{y_j} and $A_{(x_i,y_j)}$ for the cumulative events in $\mathcal{X}$, $\mathcal{Y}$ and $\mathcal{X} \times \mathcal{Y}$ given by:

$$A_{x_i} = \{x \in \mathcal{X} \mid x \leq x_i\} = \{x_1, x_2, \ldots, x_i\}. \tag{1}$$

$$A_{y_j} = \{y \in \mathcal{Y} \mid y \leq y_j\} = \{y_1, y_2, \ldots, y_j\}. \tag{2}$$

$$A_{(x_i,y_j)} = A_{x_i} \times A_{y_j} = \{x_1, x_2, \ldots, x_i\} \times \{y_1, y_2, \ldots, y_j\}. \tag{3}$$

2.1 Lower probabilities

When the probability measure P_0 is partially known, we can make use of the machinery of imprecise probabilities [36] to model the available information.

A *lower probability* is a function $\underline{P} : \mathcal{P}(\Omega) \to [0, 1]$ satisfying $\underline{P}(\emptyset) = 0$, $\underline{P}(\Omega) = 1$ and monotonicity that gives lower bounds for the real, but unknown, probability of the events $A \in \mathcal{P}(\Omega)$. Hence, the available information about P_0 is that $P_0(A) \geq \underline{P}(A)$ for every $A \in \mathcal{P}(\Omega)$.

Using this *epistemic interpretation* of $\underline{P}$ we consider the set of probability measures compatible with $\underline{P}$, called *credal set* and given by:

$$\mathcal{M}(\underline{P}) = \{P \in \mathbb{P}(\Omega) \mid P(A) \geq \underline{P}(A) \ \ \forall A \in \mathcal{P}(\Omega)\}.$$

$\mathcal{M}(\underline{P})$ is closed and convex. Furthermore, following the epistemic interpretation of $\underline{P}$, $\mathcal{M}(\underline{P})$ contains the candidate probability measures for P_0. Also, given that the credal set $\mathcal{M}(\underline{P})$ is closed and convex, it is characterised by its extreme points, which are those probability measures $P \in \mathcal{M}(\underline{P})$ such that if $P = \alpha P_1 + (1-\alpha)P_2$ for some $P_1, P_2 \in \mathcal{M}(\underline{P})$ and $\alpha \in (0, 1)$, then $P_1 = P_2 = P$.

To guarantee some rationality in the model, the property of coherence is usually imposed. $\underline{P}$ is *coherent* when $\underline{P}(A) = \min_{P \in \mathcal{M}(\underline{P})} P(A)$ for every $A \in \mathcal{P}(\Omega)$. This means that the bounds given by $\underline{P}$ are tight, in the sense that for every $A \in \mathcal{P}(\Omega)$, there exists $P \in \mathcal{M}(\underline{P})$ such that $P(A) = \underline{P}(A)$.

Using $\underline{P}$, we define its *conjugate upper probability*, a function $\overline{P} : \mathcal{P}(\Omega) \to [0, 1]$ defined through the conjugacy relation $\overline{P}(A) = 1 - \underline{P}(A^c)$ for every $A \in \mathcal{P}(\Omega)$. In particular, $\underline{P}$ and $\overline{P}$ satisfy the following property:

$$\forall A \in \mathcal{P}(\Omega)\colon \ \big(P_0(A) \geq \underline{P}(A) \Leftrightarrow P_0(A^c) \leq \overline{P}(A^c)\big)\,,$$

hence $\underline{P}$ and its conjugate $\overline{P}$ contain the same probabilistic information. Also, $\underline{P}$ is coherent if and only if $\overline{P}(A) = \max_{P \in \mathcal{M}(\underline{P})} P(A)$ for every $A \in \mathcal{P}(\Omega)$.

From now on, we assume that the lower probability $\underline{P}$ is coherent, and that $\overline{P}$ denotes always its conjugate upper probability.

There are a number of interesting properties of coherent conjugate lower and upper probabilities that will be useful later on (see [36, Sec.2.7.4]):

Consistency: $0 \leq \underline{P}(A) \leq \overline{P}(A) \leq 1$ for every $A \subseteq \Omega$.
Super-additivity of $\underline{P}$: $\underline{P}(A \cup B) \geq \underline{P}(A) + \underline{P}(B)$ whenever $A \cap B = \emptyset$.
Sub-additivity of $\overline{P}$: $\overline{P}(A \cup B) \leq \overline{P}(A) + \overline{P}(B)$ for every $A, B \subseteq \Omega$.
Inequalities for $\underline{P}$, $\overline{P}$: $\underline{P}(A \cup B) \leq \underline{P}(A) + \overline{P}(B)$ for every $A, B \subseteq \Omega$.

2.2 Belief functions

Belief functions are the key concept of the Theory of Evidence developed by Dempster and Shafer [30] and they can be embedded into the theory of coherent lower probabilities. A function $m : \mathcal{P}(\Omega) \to [0, 1]$ satisfying $\sum_{A \subseteq \Omega} m(A) = 1$ is called *basic probability assignment*, and $m(A)$, the mass of A, represents the amount of evidence supporting the occurrence of the event A. Using m, we define the *belief* and *plausibility functions*, $\underline{P}$ and $\overline{P}$ respectively, by:

$$\underline{P}(A) = \sum_{B \subseteq A} m(B), \quad \overline{P}(A) = \sum_{B | A \cap B \neq \emptyset} m(B) \quad \forall A \subseteq \Omega. \tag{4}$$

$\underline{P}(A)$ and $\overline{P}(A)$ represent the minimum and maximum probability of A deduced from the basic probability assignment. Also, the belief function $\underline{P}$ is a coherent lower probability, while the plausibility function $\overline{P}$ is its conjugate upper probability, hence $\underline{P}(A) = 1 - \overline{P}(A^c)$ for every $A \subseteq \Omega$.

Conversely, given a belief function $\underline{P}$, we can retrieve the initial basic probability assignment using the formula:

$$m(A) = \sum_{B \subseteq A} (-1)^{|A \setminus B|} \underline{P}(B) \quad \forall A \subseteq \Omega. \tag{5}$$

In fact, a coherent lower probability is a belief function if and only if the function m in the previous equation is non-negative.

In this framework, the events with strictly positive mass, usually called *focal events*, are of particular interest. Using the notation $\mathcal{F} = \{A \subseteq \Omega \mid m(A) > 0\}$, the belief and plausibility functions in Eq. (4) can be computed as:

$$\underline{P}(A) = \sum_{B \in \mathcal{F} | B \subseteq A} m(B), \quad \overline{P}(A) = \sum_{B \in \mathcal{F} | A \cap B \neq \emptyset} m(B) \quad \forall A \subseteq \Omega.$$

Hence, the focal events and their masses characterise the belief function.

2.3 Univariate random variables

Assume now that in the initial probability space $(\Omega, \mathcal{P}(\Omega), P_0)$ we replace P_0 by a coherent lower probability $\underline{P}$ modelling the available information about the real but unknown probability measure P_0. Given the random variable $X : \Omega \to \mathcal{X}$, we consider the coherent lower probability $\underline{P}_X : \mathcal{P}(\mathcal{X}) \to [0, 1]$ given by $\underline{P}_X(A) = \underline{P}\big(X^{-1}(A)\big)$ for every $A \subseteq \mathcal{X}$, and its conjugate upper probability $\overline{P}_X(A) = 1 - \underline{P}_X(A^c)$ for every $A \subseteq \mathcal{X}$.

In the precise framework, a probability measure $P_X \in \mathbb{P}(\mathcal{X})$ can be equivalently represented using its associated cdf $F_X : \mathcal{X} \to [0, 1]$ given by $F_X(x_i) = P(X \leq x_i)$. In our imprecise framework, we can make use of robust distribution functions, also known as *p-boxes* [11].

Definition 1 A (univariate) p-box $(\underline{F}, \overline{F})$ is a pair of cdfs $\underline{F}, \overline{F} : \mathcal{X} \to [0, 1]$ such that $\underline{F} \leq \overline{F}$.

The coherent lower probability $\underline{P}_X$ defines a (univariate) p-box $(\underline{F}_X, \overline{F}_X)$ by:

$$\begin{aligned} \underline{F}_X(x_i) &= \underline{P}(X \leq x_i) = \underline{P}_X\big(\{x_1, \ldots, x_i\}\big) = \underline{P}_X\big(A_{x_i}\big), \\ \overline{F}_X(x_i) &= \overline{P}(X \leq x_i) = \overline{P}_X\big(\{x_1, \ldots, x_i\}\big) = \overline{P}_X\big(A_{x_i}\big) \quad \forall i = 1, \ldots, n. \end{aligned} \tag{6}$$

The p-box $(\underline{F}_X, \overline{F}_X)$ also defines a credal set formed by the probability measures whose associated cdfs are bounded by $\underline{F}_X$ and $\overline{F}_X$:

$$\mathcal{M}\big(\underline{F}_X, \overline{F}_X\big) = \big\{P \in \mathbb{P}(\mathcal{X}) \mid \underline{F}_X \leq F_P \leq \overline{F}_X\big\}.$$

Taking lower and upper envelopes in $\mathcal{M}\big(\underline{F}_X, \overline{F}_X\big)$ we obtain the coherent conjugate lower and upper probabilities given, for every $A \subseteq \mathcal{X}$, by:

$$\underline{P}_{(\underline{F}_X, \overline{F}_X)}(A) = \min_{P \in \mathcal{M}(\underline{F}_X, \overline{F}_X)} P(A), \quad \overline{P}_{(\underline{F}_X, \overline{F}_X)}(A) = \max_{P \in \mathcal{M}(\underline{F}_X, \overline{F}_X)} P(A). \tag{7}$$

These conjugate lower and upper probabilities are not only coherent but also belief and plausibility functions [33, Thm.17], respectively. Moreover:

$$\underline{P}_{(\underline{F}_X, \overline{F}_X)}(\{x_1, \ldots, x_i\}) = \underline{F}_X(x_i), \quad \overline{P}_{(\underline{F}_X, \overline{F}_X)}(\{x_1, \ldots, x_i\}) = \overline{F}_X(x_i)$$

for every $i = 1, \ldots, n$ and there are simple formulas for computing the values in Eq. (7) (see [33, Prop.4]). In particular, for any $i = 2, \ldots, n$ it holds that:

$$\overline{P}_{(\underline{F}_X, \overline{F}_X)}(\{x_i\}) = \overline{F}_X(x_i) - \underline{F}_X(x_{i-1}). \tag{8}$$

Also, $\mathcal{M}\big(\underline{F}_X, \overline{F}_X\big) = \mathcal{M}\Big(\underline{P}_{(\underline{F}, \overline{F})}\Big)$, meaning that both $(\underline{F}_X, \overline{F}_X)$ and $\underline{P}_{(\underline{F}_X, \overline{F}_X)}$ contain the same probabilistic information, and their credal sets are generally larger than that of the initial coherent lower probability $\underline{P}_X$:

$$\mathcal{M}\big(\underline{F}_X, \overline{F}_X\big) = \mathcal{M}\Big(\underline{P}_{(\underline{F}, \overline{F})}\Big) \supseteq \mathcal{M}(\underline{P}_X),$$

meaning that $\underline{P}_{(\underline{F}_X, \overline{F}_X)} \leq \underline{P}_X$. This means that different coherent lower probabilities may induce the same p-box using Eq. (6), $\underline{P}_{(\underline{F}_X, \overline{F}_X)}$ being the only one with the same credal set as the p-box.

2.4 Random vectors

Now, instead of considering a random variable X, we consider a random vector (X, Y) defined from $(\Omega, \mathcal{P}(\Omega), \underline{P})$ to $\mathcal{X} \times \mathcal{Y}$. The random vector (X, Y) induces the coherent lower probability $\underline{P}_{X,Y} : \mathcal{P}(\mathcal{X} \times \mathcal{Y}) \to [0, 1]$ by

$$\underline{P}_{X,Y}(A) = \underline{P}\big((X, Y)^{-1}(A)\big) \quad \forall A \subseteq \mathcal{X} \times \mathcal{Y}.$$

As we did in the univariate framework, we consider the notion of p-box for bivariate spaces.

Definition 2 [27] A bivariate p-box $(\underline{F}, \overline{F})$ is a pair of ordered $(\underline{F} \leq \overline{F})$ functions $\underline{F}, \overline{F} : \mathcal{X} \times \mathcal{Y} \to [0, 1]$ that are component-wise increasing and satisfy $\underline{F}(x_n, y_m) = \overline{F}(x_n, y_m) = 1$.

In contrast with the univariate framework, $\underline{F}$ and $\overline{F}$ need not be bivariate cdfs, because they may not satisfy the rectangle inequality (see [27, Ex.1]).

The coherent lower probability $\underline{P}_{X,Y}$ defines a bivariate p-box by:

$$\begin{aligned} \underline{F}_{X,Y}(x_i, y_j) &= \underline{P}_{X,Y}\big(\{x_1, \dots, x_i\} \times \{y_1, \dots, y_j\}\big) = \underline{P}_{X,Y}\big(A_{(x_i,y_j)}\big), \\ \overline{F}_{X,Y}(x_i, y_j) &= \overline{P}_{X,Y}\big(\{x_1, \dots, x_i\} \times \{y_1, \dots, y_j\}\big) = \overline{P}_{X,Y}\big(A_{(x_i,y_j)}\big), \end{aligned} \tag{9}$$

for every $i = 1, \dots, n$ and $j = 1, \dots, m$. The bivariate p-box $(\underline{F}_{X,Y}, \overline{F}_{X,Y})$ defines a credal set:

$$\mathcal{M}\big(\underline{F}_{X,Y}, \overline{F}_{X,Y}\big) = \big\{P \in \mathbb{P}(\mathcal{X} \times \mathcal{Y}) \mid \underline{F}_{X,Y} \leq F_P \leq \overline{F}_{X,Y}\big\}. \tag{10}$$

There is an important difference with respect to the univariate framework, because the credal set in Eq. (10) could be empty (see [27, Ex.2]). When it is non-empty, it defines a coherent lower and upper probability by taking lower and upper envelopes:

$$\underline{P}_{(\underline{F}_{X,Y}, \overline{F}_{X,Y})}(A) = \min_{P \in \mathcal{M}(\underline{F}_{X,Y}, \overline{F}_{X,Y})} P(A).$$

$$\overline{P}_{(\underline{F}_{X,Y}, \overline{F}_{X,Y})}(A) = \max_{P \in \mathcal{M}(\underline{F}_{X,Y}, \overline{F}_{X,Y})} P(A) \quad \forall A \subseteq \mathcal{X} \times \mathcal{Y}.$$

For every $(x_i, y_j) \in \mathcal{X} \times \mathcal{Y}$, $\underline{P}_{(\underline{F}_{X,Y}, \overline{F}_{X,Y})}$ and $\overline{P}_{(\underline{F}_{X,Y}, \overline{F}_{X,Y})}$ satisfy

$$\underline{P}_{(\underline{F}_{X,Y}, \overline{F}_{X,Y})}(A_{(x_i,y_j)}) = \underline{F}_{X,Y}(x_i, y_j), \; \overline{P}_{(\underline{F}_{X,Y}, \overline{F}_{X,Y})}(A_{(x_i,y_j)}) = \overline{F}_{X,Y}(x_i, y_j).$$

$(\underline{F}_X, \overline{F}_X)$ and its associated coherent lower probability contain the same probabilistic information, and they are more imprecise than the initial $\underline{P}_{X,Y}$:

$$\mathcal{M}\big(\underline{F}_{X,Y}, \overline{F}_{X,Y}\big) = \mathcal{M}\Big(\underline{P}_{(\underline{F}_{X,Y}, \overline{F}_{X,Y})}\Big) \supseteq \mathcal{M}\big(\underline{P}_{X,Y}\big).$$

The coherent lower probability $\underline{P}_{X,Y} : \mathcal{P}(\mathcal{X} \times \mathcal{Y}) \to [0, 1]$ defines the *marginal* coherent lower probabilities $\underline{P}_X : \mathcal{P}(\mathcal{X}) \to [0, 1]$ and $\underline{P}_Y : \mathcal{P}(\mathcal{Y}) \to [0, 1]$ by:

$$\underline{P}_X(A) = \underline{P}_{X,Y}(A \times \mathcal{Y}), \quad \underline{P}_Y(B) = \underline{P}_{X,Y}(\mathcal{X} \times B) \tag{11}$$

for every $A \subseteq \mathcal{X}$ and $B \subseteq \mathcal{Y}$. Similarly, if $(\underline{F}_{X,Y}, \overline{F}_{X,Y})$ is the bivariate p-box associated with $\underline{P}_{X,Y}$ by means of Eq. (9), the *marginal p-boxes* $(\underline{F}_X, \overline{F}_X)$ and $(\underline{F}_Y, \overline{F}_Y)$ are given by:

$$\begin{aligned} \underline{F}_X(x_i) &= \underline{F}_{X,Y}(x_i, y_m), \quad \overline{F}_X(x_i) = \overline{F}_{X,Y}(x_i, y_m), \quad \forall i = 1, \dots, n. \\ \underline{F}_Y(y_j) &= \underline{F}_{X,Y}(x_n, y_j), \quad \overline{F}_Y(y_j) = \overline{F}_{X,Y}(x_n, y_j), \quad \forall j = 1, \dots, m. \end{aligned} \tag{12}$$

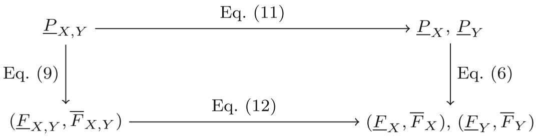

Fig. 1 Relationship between the bivariate and univariate imprecise models

In fact, $(\underline{F}_X, \overline{F}_X)$ and $(\underline{F}_Y, \overline{F}_Y)$ are the univariate p-boxes associated with the marginal coherent lower probabilities $\underline{P}_X$ and $\underline{P}_Y$ given in Eq. (11). These relationships are graphically summarised in Fig. 1.

It is worth noting that if $\underline{P}_{X,Y}$ is a belief function, the marginals $\underline{P}_X$ and $\underline{P}_Y$ are belief functions too, with basic probability assignments:

$$m_X(A) = \sum_{C^{\downarrow\mathcal{X}}=A} m(C), \quad \text{and} \quad m_Y(B) = \sum_{C^{\downarrow\mathcal{Y}}=B} m(C),$$

for every $A \subseteq \mathcal{X}$ and $B \subseteq \mathcal{Y}$, where $C^{\downarrow\mathcal{X}}$ and $C^{\downarrow\mathcal{Y}}$ denote the $\mathcal{X}$- and $\mathcal{Y}$-projections of C over $\mathcal{X}$ and $\mathcal{Y}$, respectively, given by:

$$C^{\downarrow\mathcal{X}} = \{x \in \mathcal{X} \mid \exists y \in \mathcal{Y} \text{ such that } (x, y) \in C\}.$$
$$C^{\downarrow\mathcal{Y}} = \{y \in \mathcal{Y} \mid \exists x \in \mathcal{X} \text{ such that } (x, y) \in C\}.$$

3 Comonotonicity

In this section, we review the concept and main properties of comonotonicity for both probability measures and coherent lower probabilities.

3.1 Comonotone probability measures

Consider the random vector (X, Y) with joint and marginal probability measures $P_{X,Y}$, P_X and P_Y, respectively, and denote by $F_{X,Y}$, F_X and F_Y their joint and marginal cdfs.

The joint probability measure $P_{X,Y}$ captures the dependence between the random variables X and Y. We consider in this paper a particular type of dependence structure called comonotonicity. Recall that:

- Two elements $(x_i, y_j), (x_k, y_l) \in \mathcal{X} \times \mathcal{Y}$ are *comonotone* if $x_i < x_k$ implies $y_j \leq y_l$ and $y_j < y_l$ implies $x_i \leq x_k$.
- An event $A \subseteq \mathcal{X} \times \mathcal{Y}$ is *increasing* if all the pairs of elements in A are comonotone.

Also, recall that the support of the probability measure $P_{X,Y}$ is formed by those elements with strictly positive probability:

$$\text{Supp}(P_{X,Y}) = \left\{(x_i, y_j) \in \mathcal{X} \times \mathcal{Y} \,\middle|\, P_{X,Y}\big(\{(x_i, y_j)\}\big) > 0\right\}.$$

Using these preliminary definitions, we introduce the main notion of this paper.

Definition 3 Given a random vector (X, Y), $P_{X,Y}$ is *comonotone* if its support $\text{Supp}(P_{X,Y})$ is an increasing set in $\mathcal{X} \times \mathcal{Y}$.

According to this definition, comonotonicity refers to random variables with increasing support, or in other words, to random variables that increase or decrease simultaneously.

Comonotonicity can be equivalently expressed in different ways. The next theorem summarises the main characterisations that can be found in the literature, such that for example in [7, Thm.2] or [3, Prop.2.1], among others.

Theorem 1 *Given a random vector (X, Y), $P_{X,Y}$ is comonotone if and only if any, hence all, of the following conditions holds:*

1. *$Supp(P_{X,Y})$ is an increasing set in $\mathcal{X} \times \mathcal{Y}$.*
2. *For every (x, y), $P\big(D_1^{(x,y)}\big) = 0$ or $P\big(D_2^{(x,y)}\big) = 0$, where*

$$D_1^{(x,y)} = \{(x_i, y_j) \in \mathcal{X} \times \mathcal{Y} \mid x_i > x,\ y_j \leq y\},\ \textit{and}$$
$$D_2^{(x,y)} = \{(x_i, y_j) \in \mathcal{X} \times \mathcal{Y} \mid x_i \leq x,\ y_j > y\}.$$

3. *For every $(x, y) \in \mathcal{X} \times \mathcal{Y}$, $F_{X,Y}(x, y) = \min\{F_X(x), F_Y(y)\}$.*

One very important property is related to the construction of a comonotone model with given marginals.

Proposition 1 *Given two probability measures $P_X \in \mathbb{P}(\mathcal{X})$ and $P_Y \in \mathbb{P}(\mathcal{Y})$, there is always a comonotone probability measure $P_{X,Y} \in \mathbb{P}(\mathcal{X} \times \mathcal{Y})$ whose marginals are P_X and P_Y. Furthermore, such $P_{X,Y}$ is unique.*

In fact, from Theorem 1, the joint cdf of the comonotone probability measure is given by $F_{X,Y}(x, y) = \min\{F_X(x), F_Y(y)\}$ for every $(x, y) \in \mathcal{X} \times \mathcal{Y}$. This gives a simple constructive approach for building the comonotone model.

3.2 Comonotone lower probabilities

Consider a coherent lower probability $\underline{P}_{X,Y}$ representing the uncertainty about (X, Y). We start recalling the definition of comonotonicity.

Definition 4 [18] The coherent lower probability $\underline{P}_{X,Y} : \mathcal{P}(\mathcal{X} \times \mathcal{Y}) \to [0, 1]$ associated with the random vector (X, Y) is *comonotone* when every $P \in \mathcal{M}(\underline{P}_{X,Y})$ is a comonotone probability measure.

The interpretation of this definition is the following: from the epistemic point of view adopted at the beginning of the paper, $\underline{P}_{X,Y}$ models the imprecise information about the real but unknown probability measure associated with the random vector (X, Y). Hence, all the probability measures in $\mathcal{M}\big(\underline{P}_{X,Y}\big)$ are candidates for being such real but unknown probability measure. Consequently, for $\underline{P}_{X,Y}$ to be comonotone it seems reasonable to require that all the candidate probability measures are comonotone as well.

When $\underline{P}_{X,Y}$ is comonotone, it is worth noting that every $P_{X,Y} \in \mathcal{M}(\underline{P}_{X,Y})$ is comonotone (by Def. 4) and its marginals P_X and P_Y belong to the credal sets of $\underline{P}_X$ and $\underline{P}_Y$, respectively.

After introducing the notion of comonotone lower probabilities, one of the main contributions of [18] was the analysis of the properties in Theorem 1 in this framework. Properties (1)–(3) in Theorem 1 were rewritten in terms of lower and upper probabilities, giving rise to the next result (see [18, Thms.14,16,17,18]).

Theorem 2 [18] *Let $\underline{P}_{X,Y} : \mathcal{P}(\mathcal{X} \times \mathcal{Y}) \to [0, 1]$ be a coherent lower probability with conjugate $\overline{P}_{X,Y}$. The following statements are equivalent:*

1. $\underline{P}_{X,Y}$ *is a comonotone lower probability.*
2. *Supp*$\big(\underline{P}_{X,Y}\big)$ *is an increasing set in* $\mathcal{X} \times \mathcal{Y}$ *where:*

$$Supp\big(\underline{P}_{X,Y}\big) = \bigcup_{P \in \mathcal{M}(\underline{P}_{X,Y})} Supp(P) = \Big\{(x, y) \in \mathcal{X} \times \mathcal{Y} \mid \overline{P}_{X,Y}\big(\{(x, y)\}\big) > 0\Big\}.$$

3. *For every* $(x, y) \in \mathcal{X} \times \mathcal{Y}$, *either* $\overline{P}_{X,Y}\big(D_1^{(x,y)}\big)=0$ *or* $\overline{P}_{X,Y}\big(D_2^{(x,y)}\big)=0.$

Also, if (1)–(3) *hold, then* $\underline{P}_{X,Y}$ *satisfies the following properties:*

(4) *For every* $(x, y) \in \mathcal{X} \times \mathcal{Y}$, *the bivariate p-box* $(\underline{F}_{X,Y}, \overline{F}_{X,Y})$ *associated with* $\underline{P}_{X,Y}$ *can be expressed as:*

$$\underline{F}_{X,Y}(x, y) = \min\big\{\underline{F}_X(x), \underline{F}_Y(y)\big\},\ \overline{F}_{X,Y}(x, y) = \min\big\{\overline{F}_X(x), \overline{F}_Y(y)\big\}. \tag{13}$$

(5) *Every extreme point* P *of* $\mathcal{M}\big(\underline{P}_{X,Y}\big)$ *is comonotone.*

However, these two necessary conditions are not sufficient.

The connections in Theorem 2 are graphically summarised in Fig. 2.

Besides defining comonotonicity for coherent lower probabilities and analysing their main properties, reference [18] briefly investigated the problem of building a comonotone belief function with given marginal belief functions. However, the problem was not completely solved and only sufficient and necessary conditions for the existence of the comonotone model were found.

In the next section we dig into the existence, construction and uniqueness of a comonotone model with given marginal p-boxes, analysing to which extent Proposition 1 still holds in this framework.

4 Comonotone extension with given marginal p-boxes

Proposition 1 shows one of the main features of comonotonicity: any pair of marginal distributions F_X and F_Y defines a *unique* joint comonotone probability $P_{X,Y}$ whose associated cdf is the minimum of the marginals. Now, we analyse to which extent this property is preserved when dealing with lower probabilities.

We introduce the following notation:

Definition 5 Consider two marginal p-boxes $(\underline{F}_X, \overline{F}_X)$ and $(\underline{F}_Y, \overline{F}_Y)$. A coherent lower probability $\underline{P}_{X,Y} : \mathcal{P}(\mathcal{X}\times\mathcal{Y}) \to [0, 1]$ is a *comonotone extension* of $(\underline{F}_X, \overline{F}_X)$ and $(\underline{F}_Y, \overline{F}_Y)$ if it is comonotone and its marginal p-boxes are $(\underline{F}_X, \overline{F}_X)$ and $(\underline{F}_Y, \overline{F}_Y)$.

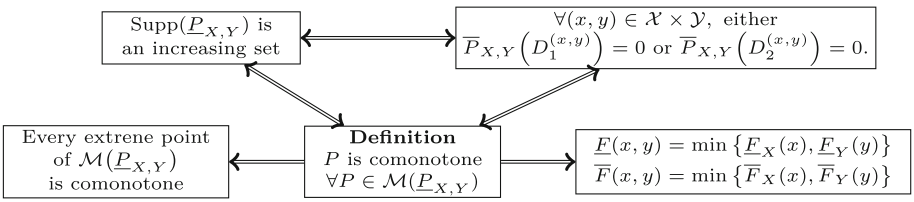

Fig. 2 Equivalences and implications concerning the comonotonicity of $\underline{P}_{X,Y}$

Our aim in this section is to analyse the following natural questions:

Existence: Given marginal p-boxes, does a comonotone extension exist?
Construction: In case it exists, how can we construct it?
Uniqueness: In case it exists, is this comonotone extension unique?

For the sake of technical simplicity, we assume throughout the paper that the upper probabilities $\overline{P}_{(\underline{F}_X,\overline{F}_X)}$ and $\overline{P}_{(\underline{F}_Y,\overline{F}_Y)}$ are strictly positive in the singletons. From Eq. (8), this assumption implies that:

- $\overline{F}_X(x_1) > 0$ and $\overline{F}_Y(y_1) > 0$.
- $\underline{F}_X(x_i) < \overline{F}_X(x_{i+1})$ and $\underline{F}_Y(y_j) < \overline{F}_Y(y_{j+1})$ for every $i = 1, \ldots, n-1$, $j = 1, \ldots, m-1$.

As we will explain in Sect. 6, these conditions are not restrictive at all.

4.1 Existence of a comonotone extension with given marginal p-boxes

We start responding to the first question related to the existence of a comonotone extension given marginal p-boxes. Unfortunately, the answer is negative.

Example 1 Consider the possibility spaces $\mathcal{X} = \{x_1, x_2\}$ and $\mathcal{Y} = \{y_1, y_2\}$ and the marginal p-boxes $(\underline{F}_X, \overline{F}_X)$ and $(\underline{F}_Y, \overline{F}_Y)$ given by:

$\mathcal{X}$	x_1	x_2
$\underline{F}_X(x_i)$	0	1
$\overline{F}_X(x_i)$	1	1

$\mathcal{Y}$	y_1	y_2
$\underline{F}_Y(y_i)$	0.5	1
$\overline{F}_Y(y_i)$	0.5	1

If there exists a comonotone extension $\underline{P}_{X,Y}$ of $(\underline{F}_X, \overline{F}_X)$ and $(\underline{F}_Y, \overline{F}_Y)$, from Theorem 2 its associated bivariate p-box $(\underline{F}_{X,Y}, \overline{F}_{X,Y})$ satisfies Eq. (13). Hence, $(\underline{F}_{X,Y}, \overline{F}_{X,Y})$ would be given by:

$\left[\underline{F}_X(x_i), \overline{F}_X(x_i)\right]$	[0, 1]	[1, 1]	
y_2	[0, 1]	[1, 1]	[1, 1]
y_1	[0, 0.5]	[0.5, 1]	[0.5, 0.5]
$\left[\underline{F}_{X,Y}(x_i, y_j), \overline{F}_{X,Y}(x_i, y_j)\right]$	x_1	x_2	$\left[\underline{F}_Y(y_j), \overline{F}_Y(y_j)\right]$

Applying the sub-additivity of $\overline{P}_{X,Y}$ to $A = \{(x_1, y_2)\}$ and $B = \{(x_1, y_1)\}$:

$$\begin{aligned}\overline{P}_{X,Y}\big(\{(x_1, y_2)\}\big) &\geq \overline{P}_{X,Y}\big(\{(x_1, y_1), (x_1, y_2)\}\big) - \overline{P}_{X,Y}\big(\{(x_1, y_1)\}\big)\\ &= \overline{F}_{X,Y}(x_1, y_2) - \overline{F}_{X,Y}(x_1, y_1) = 1 - 0.5 = 0.5 > 0.\end{aligned}$$

Also, taking $A = \{(x_1, y_1)\}$ and $B = \{(x_2, y_1)\}$, and applying the inequality with $\underline{P}_{X,Y}$ and $\overline{P}_{X,Y}$, we obtain:

$$\begin{aligned}\overline{P}_{X,Y}\big(\{(x_2, y_1)\}\big) &\geq \underline{P}_{X,Y}\big(\{(x_1, y_1), (x_2, y_1)\}\big) - \underline{P}_{X,Y}\big(\{(x_1, y_1)\}\big)\\ &= \underline{F}_{X,Y}(x_2, y_1) - \underline{F}_{X,Y}(x_1, y_1) = 0.5 - 0 = 0.5 > 0.\end{aligned}$$

Hence, $\overline{P}_{X,Y}\big(\{(x_2, y_1)\}\big) > 0$ and $\overline{P}_{X,Y}\big(\{(x_1, y_2)\}\big) > 0$, so both (x_2, y_1) and (x_1, y_2) belong to $\mathrm{Supp}\big(\underline{P}_{X,Y}\big)$. Since these two elements are not comonotone, $\mathrm{Supp}\big(\underline{P}_{X,Y}\big)$ is not increasing, so from Theorem 2, $\underline{P}_{X,Y}$ is not comonotone. □

We conclude that not all the marginal p-boxes have a comonotone extension, even when one of the p-boxes is a precise cdf.

In the next subsection, we characterise the conditions that the marginal p-boxes $(\underline{F}_X, \overline{F}_X)$ and $(\underline{F}_Y, \overline{F}_Y)$ must satisfy in order to guarantee the existence of a comonotone extension and we give a constructive method for building it.

4.2 Construction of a comonotone extension with given marginal p-boxes

Example 1 shows that there are marginal p-boxes for which there is not a comonotone extension. Next, we characterise the conditions $(\underline{F}_X, \overline{F}_X)$ and $(\underline{F}_Y, \overline{F}_Y)$ must satisfy to guarantee the existence of a comonotone extension $\underline{P}_{X,Y}$.

In order to alleviate the technical complexity of the forthcoming results, we introduce the following notation and terminology:

- Given $(x_i, y_j), (x_k, y_l) \in \mathcal{X} \times \mathcal{Y}$, $(x_i, y_j) \prec (x_k, y_l)$ if $x_i \leq x_k$ and $y_j \leq y_l$, where one of the inequalities is strict.
- We use the short notation $\overline{\underline{a}}$ for denoting the interval[1] $[\underline{a}, \overline{a}]$.
- Given two intervals $\overline{\underline{a}}$ and $\overline{\underline{b}}$, $\overline{\underline{a}}$ dominates $\overline{\underline{b}}$ with respect to the interval dominance, denoted by $\overline{\underline{b}} \preceq \overline{\underline{a}}$, if $\underline{b} \leq \underline{a}$ and $\overline{b} \leq \overline{a}$. When at least one of the inequalities is strict, we say that there is strict interval dominance and we denote it by $\overline{\underline{b}} \prec \overline{\underline{a}}$.
- Given two intervals $\overline{\underline{a}}$ and $\overline{\underline{b}}$, we say that there is an interval dominance relationship between them if either $\overline{\underline{a}} \preceq \overline{\underline{b}}$ or $\overline{\underline{b}} \preceq \overline{\underline{a}}$. We say that there is an strict interval dominance relationship between them if $\overline{\underline{a}} \prec \overline{\underline{b}}$ or $\overline{\underline{b}} \prec \overline{\underline{a}}$.

Before giving the main result of this subsection, we prove a number of technical results. For this aim, we denote by $(\underline{F}_X, \overline{F}_X)$ and $(\underline{F}_Y, \overline{F}_Y)$ the marginal p-boxes, and by $(\underline{F}_{X,Y}, \overline{F}_{X,Y})$ the bivariate p-box they define using Eq. (13). Consider the following set:

$$S = \Big\{(x_i, y_j) \in \mathcal{X} \times \mathcal{Y} \mid \overline{\underline{F_{X,Y}}}(x_{i-1}, y_j) \prec \overline{\underline{F_{X,Y}}}(x_i, y_j) \text{ and } \overline{\underline{F_{X,Y}}}(x_i, y_{j-1}) \prec \overline{\underline{F_{X,Y}}}(x_i, y_j)\Big\}, \tag{14}$$

where of course we assume, for every $i = 1, \ldots, n$ and $j = 1, \ldots, m$, that:

$$\underline{F}_{X,Y}(x_i, y_0) = \overline{F}_{X,Y}(x_i, y_0) = \underline{F}_{X,Y}(x_0, y_i) = \overline{F}_{X,Y}(x_0, y_i) = 0.$$

Since we are assuming that the singletons have strictly positive upper probability, we know that $(x_1, y_1), (x_n, y_m) \in S$.

Our first technical result proves some interesting properties of this set.

Lemma 1 *Consider two marginal p-boxes $(\underline{F}_X, \overline{F}_X)$ and $(\underline{F}_Y, \overline{F}_Y)$, and the bivariate p-box $(\underline{F}_{X,Y}, \overline{F}_{X,Y})$ they define using Eq. (13), as well as the set S defined in Eq. (14). If for every $(x, y) \in \mathcal{X} \times \mathcal{Y}$ there is an interval dominance relation between $\overline{\underline{F}}_X(x)$ and $\overline{\underline{F}}_Y(y)$, then S satisfies the following properties:*

1. *S is an increasing set in $\mathcal{X} \times \mathcal{Y}$.*
2. *If $(x_i, y_j), (x_k, y_l) \in S$, $(x_i, y_j) \prec (x_k, y_l)$ and they are consecutive elements[2] in S, then any $(x, y) \notin S$ such that $(x_i, y_j) \prec (x, y) \prec (x_k, y_l)$ satisfies:*

$$\underline{F}_{X,Y}(x, y) = \underline{F}_{X,Y}(x_i, y_j), \quad \overline{F}_{X,Y}(x, y) = \overline{F}_{X,Y}(x_i, y_j). \tag{15}$$

[1] By interval we refer to the set $[\underline{a}, \overline{a}] = \{a \mid \underline{a} \leq a \leq \overline{a}\}$.

[2] This means that there is not other $(x^*, y^*) \in S$ such that $(x_i, y_j) \prec (x^*, y^*) \prec (x_k, y_l)$.

Proof 1. Let us first prove that S is increasing. Assume that there exist two non-comonotone elements $(x_i, y_j), (x_k, y_l) \in S$ with $k < i$ and $j < l$, and let us see that this leads to a contradiction. By hypothesis, there is an interval dominance relationship between $\underline{\overline{F}}_X(x_k)$ and $\underline{\overline{F}}_Y(y_j)$. Also, since

$$\underline{F}_{X,Y}(x_k, y_j) = \min\left\{\underline{F}_X(x_k), \underline{F}_Y(y_j)\right\}, \qquad \overline{F}_{X,Y}(x_k, y_j) = \min\left\{\overline{F}_X(x_k), \overline{F}_Y(y_j)\right\},$$

then $\underline{\overline{F}}_{X,Y}(x_k, y_j)$ coincides with either $\underline{\overline{F}}_X(x_k)$ or $\underline{\overline{F}}_Y(y_j)$. Assume that we are in the first case; the second follows by analogy. It holds that $\underline{F}_X(x_k) \leq \underline{F}_Y(y_j) \leq \underline{F}_Y(y_l)$, where the second inequality comes from the hypothesis $y_j < y_l$. Analogously, $\overline{F}_X(x_k) \leq \overline{F}_Y(y_j) \leq \overline{F}_Y(y_j)$. Thus, it holds that:

$$\underline{F}_{X,Y}(x_k, y_l) = \min\left\{\underline{F}_X(x_k), \underline{F}_Y(y_l)\right\} = \underline{F}_X(x_k).$$
$$\overline{F}_{X,Y}(x_k, y_l) = \min\left\{\overline{F}_X(x_k), \overline{F}_Y(y_l)\right\} = \overline{F}_X(x_k).$$

This implies $\underline{\overline{F}}_{X,Y}(x_k, y_l) = \underline{\overline{F}}_{X,Y}(x_k, y_j)$, so (x_k, y_l) does not belong to S, a contradiction. We conclude that S is an increasing set in $\mathcal{X} \times \mathcal{Y}$, so it can be expressed as $S = \{(u_1, v_1), \ldots, (u_s, v_s)\} \subseteq \mathcal{X} \times \mathcal{Y}$, where $(x_1, y_1) = (u_1, v_1) \prec (u_2, v_2) \prec \cdots \prec (u_s, v_s) = (x_n, y_m)$.

2. Take $(x_s, y_r) \notin S$ and $(x_i, y_j), (x_k, y_l) \in S$ such that $(x_i, y_j) \prec (x_s, y_r) \prec (x_k, y_l)$ and also such that (x_s, y_r) does not satisfy Eq. (15). Also, assume that (x_s, y_r) is the smallest element satisfying these properties, in the sense that any (x^*, y^*) satisfying $(x_i, y_j) \prec (x^*, y^*) \prec (x_s, y_r)$ does satisfy Eq. (15). We consider the following cases:

   Case 1: Assume that $x_i = x_s$. By definition of the set S, since $(x_i, y_j) \in S$, it holds that $\underline{\overline{F}}_{X,Y}(x_{i-1}, y_j) \prec \underline{\overline{F}}_{X,Y}(x_i, y_j)$. This implies that:

$$\underline{F}_{X,Y}(x_{i-1}, y_j) = \min\left\{\underline{F}_X(x_{i-1}), \underline{F}_Y(y_j)\right\} = \underline{F}_X(x_{i-1}),$$
$$\overline{F}_{X,Y}(x_{i-1}, y_j) = \min\left\{\overline{F}_X(x_{i-1}), \overline{F}_Y(y_j)\right\} = \overline{F}_X(x_{i-1}),$$

   because otherwise $\underline{\overline{F}}_{X,Y}(x_{i-1}, y_j)$ and $\underline{\overline{F}}_{X,Y}(x_i, y_j)$ would coincide. Hence $\underline{\overline{F}}_X(x_{i-1}) = \underline{\overline{F}}_{X,Y}(x_{i-1}, y_j) \prec \underline{\overline{F}}_{X,Y}(x_i, y_j)$. Also, since $\underline{\overline{F}}_{X,Y}(x_{i-1}, y_m) = \underline{\overline{F}}_X(x_{i-1})$, it holds that

$$\underline{\overline{F}}_X(x_{i-1}) = \underline{\overline{F}}_{X,Y}(x_{i-1}, y_j) = \cdots = \underline{\overline{F}}_{X,Y}(x_{i-1}, y_r).$$

   This means that $\underline{\overline{F}}_{X,Y}(x_{i-1}, y_r) \prec \underline{\overline{F}}_{X,Y}(x_i, y_r)$. But by hypothesis, $(x_i, y_s) \notin S$, which implies that Eq. (15) must hold.
   Case 2: Assume that $y_j = y_r$. This case follows by analogy to Case 1.
   Case 3: Assume that $x_i < x_s$ and $y_j < y_r$. By hypothesis, $\underline{\overline{F}}_{X,Y}(x_i, y_j) = \underline{\overline{F}}_{X,Y}(x_{s-1}, y_r) = \underline{\overline{F}}_{X,Y}(x_s, y_{r-1})$, and since (x_s, y_r) does not satisfy Eq. (15), $\underline{\overline{F}}_{X,Y}(x_i, y_j) \prec \underline{\overline{F}}_{X,Y}(x_s, y_r)$, meaning that $(x_s, y_r) \in S$, a contradiction.

   We therefore conclude that (x_s, y_r) must satisfy Eq. (15). □

From this lemma we conclude that S is an increasing set and also that $\underline{F}_{X,Y}$ and $\overline{F}_{X,Y}$ only increase in the elements in S, and between consecutive elements in S both $\underline{F}_{X,Y}$ and $\overline{F}_{X,Y}$ remain constant.

Example 2 Consider the possibility spaces $\mathcal{X} = \{x_1, x_2, x_3\}$ and $\mathcal{Y} = \{y_1, y_2, y_3\}$, and the p-boxes $(\underline{F}_X, \overline{F}_X)$ and $(\underline{F}_Y, \overline{F}_Y)$ given by:

$\mathcal{X}$	x_1	x_2	x_3
$\underline{F}_X(x_i)$	0	0.4	1
$\overline{F}_X(x_i)$	0.2	0.8	1

$\mathcal{Y}$	y_1	y_2	y_3
$\underline{F}_Y(y_j)$	0.1	0.4	1
$\overline{F}_Y(y_j)$	0.4	0.8	1

For every $(x, y) \in \mathcal{X} \times \mathcal{Y}$, there is an interval dominance relation between $\underline{\overline{F}}_X(x)$ and $\underline{\overline{F}}_Y(y)$:

$$\begin{aligned}\underline{\overline{F}}_X(x_1) &= [0, 0.2] \prec \underline{\overline{F}}_Y(y_1) = [0.1, 0.4] \\ &\prec \underline{\overline{F}}_X(x_2) = \underline{\overline{F}}_Y(y_2) = [0.4, 0.8] \prec \underline{\overline{F}}_X(x_3) = \underline{\overline{F}}_Y(y_3) = [1, 1].\end{aligned}$$

The bivariate p-box $(\underline{F}_{X,Y}, \overline{F}_{X,Y})$ they define through Eq. (13) is given by:

$\underline{\overline{F}}_X(x_i)$	[0, 0.2]	[0.4, 0.8]	[1, 1]	
y_3	[0, 0.2]	[0.4, 0.8]	[1, 1]	[1, 1]
y_2	[0, 0.2]	[0.4, 0.8]	[0.4, 0.8]	[0.4, 0.8]
y_1	[0, 0.2]	[0.1, 0.4]	[0.1, 0.4]	[0.1, 0.4]
$\underline{\overline{F}}_{X,Y}(x_i, y_j)$	x_1	x_2	x_3	$\underline{\overline{F}}_Y(y_j)$

Let us compute the set S given in Eq. (14). First of all, since $\overline{F}_{X,Y}(x_1, y_1) = 0.2 > 0$, $(x_1, y_1) \in S$. Next, $(x_2, y_1) \in S$ because:

$$\underline{\overline{F}}_{X,Y}(x_1, y_1) = [0, 0.2] \prec \underline{\overline{F}}_{X,Y}(x_2, y_1) = [0.1, 0.4].$$

Similarly, we can see that (x_2, y_2) and (x_3, y_3) also belong to S. This implies that the set S, highlighted in blue in the previous table, is given by:

$$S = \{(x_1, y_1), (x_2, y_1), (x_2, y_2), (x_3, y_3)\}. \tag{16}$$

From Lemma 1, this set is increasing, $(x_1, y_1) \prec (x_2, y_1) \prec (x_2, y_2) \prec (x_3, y_3)$, and between consecutive elements in S both $\underline{F}_{X,Y}$ and $\overline{F}_{X,Y}$ are constant. □

Consider now two p-boxes $(\underline{F}_X, \overline{F}_X)$ and $(\underline{F}_Y, \overline{F}_Y)$ with an interval dominance relationship between $\underline{\overline{F}}_X(x)$ and $\underline{\overline{F}}_Y(y)$ for every $(x, y) \in \mathcal{X} \times \mathcal{Y}$. Lemma 1 assures that the set S defined in Eq. (14) is increasing. Consider now an increasing superset S^* of S (for example, $S^* = S$), which can be expressed as $S^* = \{(u_1, v_1), \ldots, (u_s, v_s)\}$, where $(u_1, v_1) \prec \cdots \prec (u_s, v_s)$. Also, $(S^*, \prec)$ is a totally ordered set. We define a (univariate) possibility space $\mathcal{Z} = \{z_1, \ldots, z_s\}$, with the same cardinality as S^* and the order $z_1 < \cdots < z_s$, and we establish a correspondence between S^* and $\mathcal{Z}$:

$$\begin{aligned} S^* &\overset{g}{\longleftrightarrow} \mathcal{Z} \\ (u_i, v_i) &\longleftrightarrow z_i \qquad \forall i = 1, \ldots, s. \end{aligned} \tag{17}$$

This correspondence is order-preserving:

$$z_i = g(u_i, v_i) < z_j = g(u_j, u_j) \Leftrightarrow g^{-1}(z_i) = (u_i, v_i) \prec g^{-1}(z_j) = (u_j, v_j).$$

Using this correspondence, the bivariate p-box $(\underline{F}_{X,Y}, \overline{F}_{X,Y})$ defines a (univariate) p-box $(\underline{F}_Z, \overline{F}_Z)$ in $\mathcal{Z}$ as follows:

$$\underline{F}_Z(z_i) = \underline{F}_{X,Y}(u_i, v_i), \quad \overline{F}_Z(z_i) = \overline{F}_{X,Y}(u_i, v_i) \quad \forall i = 1, \ldots, n. \tag{18}$$

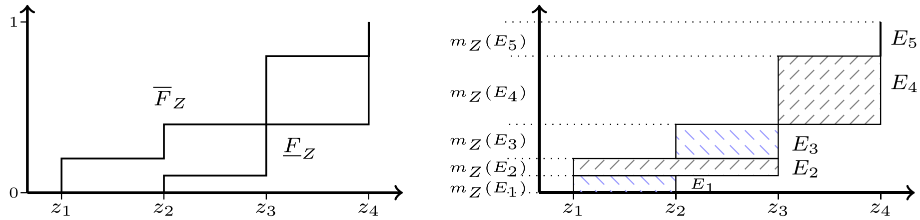

Fig. 3 P-box $(\underline{F}_Z, \overline{F}_Z)$ in Example 3 (left-hand side figure) and the focal events of its associated belief function (right-hand side figure)

Example 3 Let us continue with Example 2. The set S was given in Eq. (16), and according to the previous comments it can be expressed as:

$$S = \{(u_1, v_1), (u_2, v_2), (u_3, v_3), (u_4, v_4)\}.$$

Considering $S^* = S$, we define the possibility space $\mathcal{Z} = \{z_1, z_2, z_3, z_4\}$ with $z_1 < z_2 < z_3 < z_4$, and consider the correspondence in Eq. (17). We can now define the p-box $(\underline{F}_Z, \overline{F}_Z)$ as in Eq. (18), which is given by:

$\mathcal{Z}$	z_1	z_2	z_3	z_4
$\underline{F}_Z(z_i)$	0	0.1	0.4	1
$\overline{F}_Z(z_i)$	0.2	0.4	0.8	1

This p-box, as well as the focal events of its induced belief function (using Eq. (7)) has been graphically depicted in Fig. 3. □

Proposition 2 *Consider the marginal p-boxes $(\underline{F}_X, \overline{F}_X)$ and $(\underline{F}_Y, \overline{F}_Y)$ satisfying the condition in Lemma 1 and the bivariate p-box $(\underline{F}_{X,Y}, \overline{F}_{X,Y})$ they define through Eq. (13). Consider an increasing set $S^* \supseteq S$, the correspondence in Eq. (17) and the p-box $(\underline{F}_Z, \overline{F}_Z)$ defined in Eq. (18). Then, there is a one-to-one correspondence between the credal sets:*

$$\mathcal{M}\big(\underline{F}_Z, \overline{F}_Z\big) = \big\{P_Z \in \mathbb{P}(\mathcal{Z}) \mid \underline{F}_Z \leq F_{P_Z} \leq \overline{F}_Z\big\} \text{ and} \tag{19}$$

$$\mathcal{M} = \big\{P_{X,Y} \in \mathbb{P}(\mathcal{X} \times \mathcal{Y}) \mid P_{X,Y}(S^*) = 1,\ \underline{F}_{X,Y} \leq F_{P_{X,Y}} \leq \overline{F}_{X,Y}\big\}. \tag{20}$$

Proof For any $P_Z \in \mathcal{M}\big(\underline{F}_Z, \overline{F}_Z\big)$, we define $P_{X,Y} \in \mathbb{P}(\mathcal{X} \times \mathcal{Y})$ by:

$$P_{X,Y}(A) = P_Z\big(g(A \cap S^*)\big) \quad \forall A \subseteq \mathcal{X} \times \mathcal{Y}. \tag{21}$$

$P_{X,Y}$ satisfies $P_{X,Y}(S^*) = P_Z\big(g(S^*)\big) = P_Z(\mathcal{Z}) = 1$. Using the notation in Eqs. (1)–(3), for any $(u_i, v_i) \in S^*$ it holds that:

$$F_{P_{X,Y}}(u_i, v_i) = P_{X,Y}\big(A_{(u_i,v_i)}\big) = P_Z\big(g(A_{(u_i,v_i)} \cap S^*)\big) = P_Z(A_{z_i}) = F_{P_Z}(z_i)$$

which belongs to $\overline{\underline{F}}_Z(z_i)$. From Eq. (18), $F_{P_{X,Y}}(u_i, v_i) \in \overline{\underline{F}}_{X,Y}(u_i, v_i)$.

For any (x_i, y_j), consider $l = \max\{k = 1, \ldots, s \mid (u_k, v_k) \preceq (x_i, y_j)\}$. It holds that:

$$F_{P_{X,Y}}(x_i, y_j) = F_{X,Y}(u_l, v_l) = F_{P_Z}(z_l) \in \overline{\underline{F}}_Z(z_l) = \overline{\underline{F}}_{X,Y}(u_l, v_l).$$

The second item in Lemma 1 implies that $\underline{F}_{X,Y}(x_i, y_j) = \underline{F}_{X,Y}(u_l, v_l)$ and $\overline{F}_{X,Y}(x_i, y_j) = \overline{F}_{X,Y}(u_l, v_l)$, hence $\underline{F}_{X,Y}(x_i, y_j) \leq F_{P_{X,Y}}(x_i, y_j) \leq \overline{F}_{X,Y}(x_i, y_j)$. We conclude that $P_{X,Y} \in \mathcal{M}$.

Reciprocally, for any $P_{X,Y} \in \mathcal{M}$, we define $P_Z \in \mathbb{P}(\mathcal{Z})$ by:

$$P_Z(C) = P_{X,Y}\big(g^{-1}(C)\big) \quad \forall C \subseteq \mathcal{Z}. \tag{22}$$

P_Z satisfies $P_Z(\mathcal{Z}) = P_{X,Y}(S^*) = 1$ and also:

$$F_{P_Z}(z_i) = P_Z\big(A_{z_i}\big) = P_{X,Y}\big(A_{(u_i,v_i)}\big) = F_{P_{X,Y}}(u_i, v_i) \quad \forall i = 1, \ldots, s.$$

Using Eq. (18), we deduce that $\underline{F}_Z \leq F_{P_Z} \leq \overline{F}_Z$, hence $P_Z \in \mathcal{M}\big(\underline{F}_Z, \overline{F}_Z\big)$.

Finally, if $P_{X,Y} \in \mathcal{M}$, P_Z is the probability measure defined through Eq. (22), and $Q_{X,Y}$ is the probability measure that P_Z defines using Eq. (21):

$$P_{X,Y}(A) = P_Z\big(g^{-1}(A \cap S^*)\big) = P_Z\big(g^{-1}(A)\big) = Q_{X,Y}\big(g(g^{-1}(A)\big) = Q_{X,Y}(A),$$

for any $A \subseteq S^*$, using the correspondence g. Therefore, $P_{X,Y} = Q_{X,Y}$. □

As we said in Sect. 2.3, the lower probability associated with a univariate p-box, obtained as the lower envelope of its credal set, is a belief function. Using this fact and Proposition 2 we can easily deduce the following result.

Corollary 1 *In the conditions of Proposition 2, the coherent lower probability associated with the credal set $\mathcal{M}$ is a belief function.*

Proof Denote by $\underline{P}_{X,Y}$ and $\underline{P}_Z$ the coherent lower probabilities that are the lower envelopes of the credal sets $\mathcal{M}$ and $\mathcal{M}\big(\underline{F}_Z, \overline{F}_Z\big)$ defined in Eqs. (20) and (19), respectively. From Proposition 2, there is a one-to-one correspondence between $\mathcal{M}$ and $\mathcal{M}\big(\underline{F}_Z, \overline{F}_Z\big)$, which implies that:

$$\underline{P}_{X,Y}(A) = \underline{P}_Z\big(g(A \cap S^*)\big) \quad \forall A \subseteq \mathcal{X} \times \mathcal{Y}. \tag{23}$$

Also, $\underline{P}_Z$ is a belief function because it is the lower probability associated with a univariate p-box. Denote by m_Z its associated basic probability assignment by means of Eq. (5), and define the function $m : \mathcal{P}(\mathcal{X} \times \mathcal{Y}) \to [0, 1]$ by:

$$m(A) = \begin{cases} 0 & \text{if } A \not\subseteq S^*, \\ m_Z\big(g(A)\big) & \text{if } A \subseteq S^*, \end{cases} \tag{24}$$

for every $A \subseteq \mathcal{X} \times \mathcal{Y}$. m is also a basic probability assignment because by definition $m(A) \in [0, 1]$ for every $A \subseteq \mathcal{X} \times \mathcal{Y}$ and it satisfies:

$$\sum_{A \subseteq \mathcal{X} \times \mathcal{Y}} m(A) = \sum_{A \subseteq S^*} m(A) = \sum_{A \subseteq S^*} m_Z\big(g(A)\big) = \sum_{C \subseteq \mathcal{Z}} m_Z(C) = 1.$$

Furthermore, for every $A \subseteq \mathcal{X} \times \mathcal{Y}$ it holds that:

$$\begin{aligned} \underline{P}_{X,Y}(A) = \underline{P}_Z\big(g(A \cap S^*)\big) &= \sum_{C \subseteq g(A \cap S^*)} m_Z(C) = \sum_{g^{-1}(C) \subseteq A \cap S^*} m_Z(C) \\ &= \sum_{g^{-1}(C) \subseteq A} m_Z(C) = \sum_{g^{-1}(C) \subseteq A} m\big(g^{-1}(C)\big) = \sum_{B \subseteq A} m(B). \end{aligned}$$

Thus, m is the basic probability assignment associated with $\underline{P}_{X,Y}$, so it is a belief function. □

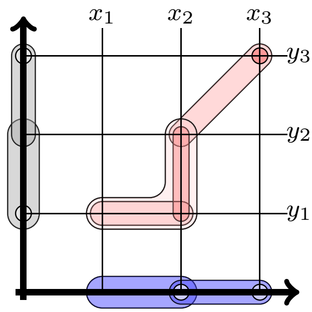

Fig. 4 Focal events of the belief function $\underline{P}_{X,Y}$ in Example 4

Example 4 Consider the same setting as in Example 2. Fig. 3 shows the p-box $(\underline{F}_Z, \overline{F}_Z)$ and the focal events of its associated belief function $\underline{P}_Z$, given by:

	E_1	E_2	E_3	E_4	E_5
E_i	$\{z_1, z_2\}$	$\{z_1, z_2, z_3\}$	$\{z_2, z_3\}$	$\{z_3, z_4\}$	$\{z_4\}$
$m_Z(E_i)$	0.1	0.1	0.2	0.4	0.2

Using Eq. (24), the focal events of $\underline{P}_{X,Y}$ are given by:

	F_1	F_2	F_3	F_4	F_5
$F_i = g^{-1}(E_i)$	(x_1, y_1) (x_2, y_1)	$(x_1, y_1), (x_2, y_2)$ (x_2, y_1)	(x_2, y_1) (x_2, y_2)	(x_2, y_2) (x_3, y_3)	(x_3, y_3)
$m(F_i) = m_Z(E_i)$	0.1	0.1	0.2	0.4	0.2

These focal events, together with the focal events of the marginal p-boxes, are graphically depicted in Fig. 4. □

After these preliminaries, we are ready to show the main result of this subsection, where we characterise the conditions under which there exists a comonotone extension of given marginal p-boxes.

Theorem 3 *Consider two marginal p-boxes* $(\underline{F}_X, \overline{F}_X)$ *and* $(\underline{F}_Y, \overline{F}_Y)$*. They have a comonotone extension* $\underline{P}_{X,Y}$ *if and only if for every* $(x, y) \in \mathcal{X} \times \mathcal{Y}$ *there is an interval dominance relation between* $\overline{\underline{F}}_X(x)$ *and* $\overline{\underline{F}}_Y(y)$*.*

Proof Assume that $\underline{P}_{X,Y}$ is a comonotone extension of $(\underline{F}_X, \overline{F}_X)$ and $(\underline{F}_Y, \overline{F}_Y)$, and let us prove that there is an interval dominance relation between $\overline{\underline{F}}_X(x)$ and $\overline{\underline{F}}_Y(y)$. Ex-absurdo, assume that there is $(x_i, y_j) \in \mathcal{X} \times \mathcal{Y}$ without an interval dominance relation between $\overline{\underline{F}}_X(x_i)$ and $\overline{\underline{F}}_Y(y_j)$. Assume for example:

$$\underline{F}_X(x_i) < \underline{F}_Y(y_j) \leq \overline{F}_Y(y_j) < \overline{F}_X(x_i). \tag{25}$$

Since $\underline{P}_{X,Y}$ is comonotone, its associated bivariate p-box is given by Eq. (13). From the coherence of $\underline{P}_{X,Y}$, there are $P_1, P_2 \in \mathcal{M}(\underline{P}_{X,Y})$ such that:

$$P_1(A_{x_i}) = \overline{P}_{X,Y}(A_{x_i}) = \overline{F}_{X,Y}(x_i, y_m) = \overline{F}_X(x_i),$$
$$P_2(A_{x_i,y_j}) = \underline{P}(A_{x_i,y_j}) = \underline{F}_{X,Y}(x_i, y_j) = \min\{\underline{F}_X(x_i), \underline{F}_Y(y_j)\} = \underline{F}_X(x_i),$$

where the last equality follows from our assumption in Eq. (25). Then:

$$P_1\left(D_2^{(x_i,y_j)}\right) = F_{P_1}(x_i, y_m) - F_{P_1}(x_i, y_j) = \overline{F}_X(x_i) - F_{P_1}(x_i, y_j)$$
$$\geq \overline{F}_X(x_i) - \overline{F}_{X,Y}(x_i, y_j) > \overline{F}_X(x_i) - \overline{F}_X(x_i) = 0, \tag{26}$$

where the last inequality follows from by our assumption in Eq. (25):

$$\overline{F}_{X,Y}(x_i, y_j) = \min\left\{\overline{F}_X(x_i), \overline{F}_Y(y_j)\right\} = \overline{F}_Y(y_j) < \overline{F}_X(x_i),$$

Using again Eq. (25), P_2 satisfies the following:

$$P_2\left(D_1^{(x_i,y_j)}\right) = F_{P_2}(x_n, y_j) - F_{P_2}(x_i, y_j) = F_{P_2}(x_n, y_j) - \min\left\{\underline{F}_X(x_i), \underline{F}_Y(y_j)\right\}$$
$$= F_{P_2}(x_n, y_j) - \underline{F}_X(x_i) \geq \underline{F}_Y(y_j) - \underline{F}_X(x_i) > 0. \tag{27}$$

From Eqs. (26) and (27), we obtain that:

$$\overline{P}_{X,Y}\left(D_2^{(x_i,y_j)}\right) \geq P_1\left(D_2^{(x_i,y_j)}\right) > 0, \quad \overline{P}_{X,Y}\left(D_1^{(x_i,y_j)}\right) \geq P_2\left(D_1^{(x_i,y_j)}\right) > 0.$$

From Theorem 2, this implies that $\underline{P}_{X,Y}$ cannot be comonotone, a contradiction.

Assume now that for every $(x, y) \in \mathcal{X} \times \mathcal{Y}$ there is an interval dominance relation between $\overline{\underline{F}}_X(x)$ and $\overline{\underline{F}}_Y(y)$, and denote by $(\underline{F}_{X,Y}, \overline{F}_{X,Y})$ the bivariate p-box they define using Eq. (13). Under this assumption, Lemma 1 assures that S is increasing in $\mathcal{X} \times \mathcal{Y}$, and that we can consider any increasing superset S^* of S. Consider the possibility space $\mathcal{Z}$ with the same cardinality as S^* and the correspondence g between S^* and $\mathcal{Z}$ in Eq. (17), as well as the p-box $(\underline{F}_Z, \overline{F}_Z)$ defined in Eq. (18). In Proposition 2 we have seen that there is a correspondence between the credal sets $\mathcal{M}$ and $\mathcal{M}(\underline{F}_Z, \overline{F}_Z)$ defined in Eqs. (19) and (20), and in Cor. 1 we have seen that the lower envelope of $\mathcal{M}$, $\underline{P}_{X,Y}$, is a belief function given by $\underline{P}_{X,Y}(A) = \underline{P}_Z\big(g(A \cap S^*)\big)$ for every $A \subseteq \mathcal{X} \times \mathcal{Y}$. Also, by definition of $\mathcal{M}$, the support of $\underline{P}_{X,Y}$ is S^*, which is increasing. Hence, from Theorem 2 we deduce that $\underline{P}_{X,Y}$ is comonotone.

Finally, by definition of the credal set $\mathcal{M}$, the bivariate p-box associated with $\underline{P}_{X,Y}$ coincides with $(\underline{F}_{X,Y}, \overline{F}_{X,Y})$. □

This theorem characterises the condition that the marginal p-boxes must satisfy to guarantee the existence of a comonotone extension. In fact, the sufficient and necessary condition is quite simple: it only requires interval dominance between $\overline{\underline{F}}_X(x)$ and $\overline{\underline{F}}_Y(y)$ for every $(x, y) \in \mathcal{X} \times \mathcal{Y}$.

Example 5 Let us conclude with our on-going Example 2. In Example 4 we have computed a comonotone extension $\underline{P}_{X,Y}$ of the marginal p-boxes $(\underline{F}_X, \overline{F}_X)$ and $(\underline{F}_Y, \overline{F}_Y)$. In fact, such lower probability is a belief function, hence its support coincides with the union of its focal events, which is the chosen set $S^* = S$. □

The results shown in this section not only characterise the existence of a comonotone extension of given marginal p-boxes but give also a constructive method for building such comonotone lower probability, which in fact is a belief function. The steps we have to follow to build it are summarised in Fig. 5.

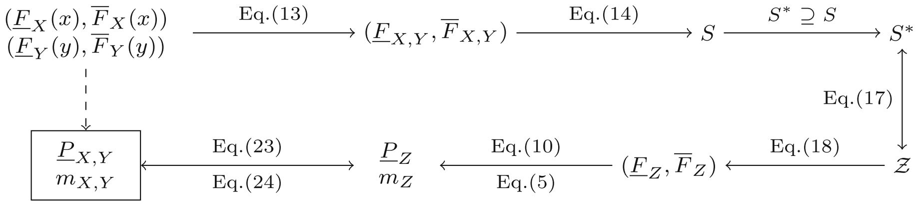

Fig. 5 Steps for building a comonotone extension $\underline{P}_{X,Y}$ of $(\underline{F}_X, \overline{F}_X)$ and $(\underline{F}_Y, \overline{F}_Y)$ satisfying the condition in Theorem 1

4.3 Uniqueness of a comonotone extension with given marginal p-boxes

We have already solved the problem of the existence and construction of a comonotone extension with given marginal p-boxes. However, our running example shows that the comonotone extension may not be unique.

Example 6 Consider again our on-going Example 2. In Example 4 we have shown a comonotone extension $\underline{P}_{X,Y}$ of the marginal p-boxes $(\underline{F}_X, \overline{F}_X)$ and $(\underline{F}_Y, \overline{F}_Y)$.

However, instead of considering the increasing set S, as we did in Example 3, we can consider the increasing supersets

$$S_1^* = \{(x_1, y_1), (x_2, y_1), (x_2, y_2), (x_2, y_3), (x_3, y_3)\}.$$
$$S_2^* = \{(x_1, y_1), (x_2, y_1), (x_2, y_2), (x_3, y_2), (x_3, y_3)\}.$$

Applying the procedure described in Fig. 5 to the S_1^* and S_2^*, we obtain two belief functions $\underline{P}_1$ and $\underline{P}_2$, which are also comonotone extensions of $(\underline{F}_X, \overline{F}_X)$ and $(\underline{F}_Y, \overline{F}_Y)$. Their basic probability assignments and focal events are:

	G_1	G_2	G_3	G_4	G_5
G_i	(x_1, y_1) (x_2, y_1)	$(x_1, y_1), (x_2, y_2)$ (x_2, y_1)	(x_2, y_1) (x_2, y_2)	$(x_2, y_2), (x_2, y_3)$ (x_3, y_3)	(x_3, y_3)
$m_1(G_i)$	0.1	0.1	0.2	0.4	0.2

	H_1	H_2	H_3	H_4	H_5
H_i	(x_1, y_1) (x_2, y_1)	$(x_1, y_1), (x_2, y_2)$ (x_2, y_1)	(x_2, y_1) (x_2, y_2)	$(x_2, y_2), (x_3, y_2)$ (x_3, y_3)	(x_3, y_3)
$m_2(H_i)$	0.1	0.1	0.2	0.4	0.2

These focal events are depicted in Fig. 6.

Both belief functions $\underline{P}_1$ and $\underline{P}_2$ are comonotone, their supports coincide with S_1^* and S_2^*, and their marginal p-boxes are $(\underline{F}_X, \overline{F}_X)$ and $(\underline{F}_Y, \overline{F}_Y)$. However, $\underline{P}_1$, $\underline{P}_2$, and the belief function $\underline{P}_{X,Y}$ defined in Example 4 are different comonotone extensions. □

This example shows that we cannot guarantee the uniqueness of a comonotone extension of the given marginal p-boxes. This lack of uniqueness encourages us to study the existence of a least-committal comonotone extension.

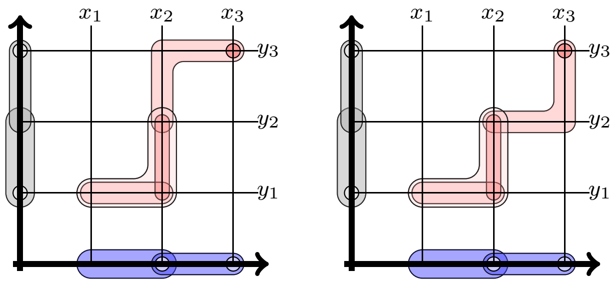

Fig. 6 Focal events of the comonotone extensions $\underline{P}_1$ and $\underline{P}_2$ in Example 6

5 Comonotone natural extension with given marginal p-boxes

Example 6 shows that the comonotone extension of two p-boxes, when it exists, may not be unique. In these cases, the usual procedure in the imprecise probability literature is to look for the least committal extension, which uses only the available information. This is done for example in the cases of independence, with the independent natural extension [5], conglomerability, with the conglomerable natural extension [16], or the extension of marginal models with no information about their dependence [23, Sec.3.2].

Definition 6 Consider two marginal p-boxes $(\underline{F}_X, \overline{F}_X)$ and $(\underline{F}_Y, \overline{F}_Y)$. A coherent lower probability $\underline{E}_{X,Y} : \mathcal{P}(\mathcal{X} \times \mathcal{Y}) \to [0, 1]$ is the *comonotone natural extension* of $(\underline{F}_X, \overline{F}_X)$ and $(\underline{F}_Y, \overline{F}_Y)$ if it is a comonotone extension and it satisfies $\underline{E}_{X,Y} \leq \underline{P}_{X,Y}$ for any other comonotone extension $\underline{P}_{X,Y}$ of $(\underline{F}_X, \overline{F}_X)$ and $(\underline{F}_Y, \overline{F}_Y)$.

Of course, one necessary condition for the existence of the comonotone natural extension is that $(\underline{F}_X, \overline{F}_X)$ and $(\underline{F}_Y, \overline{F}_Y)$ satisfy the condition in Theorem 3. However, the next example shows that this condition is not sufficient in general.

Example 7 Let us continue with our running Example 2. Example 6 shows two comonotone extensions $\underline{P}_1$ and $\underline{P}_2$ of $(\underline{F}_X, \overline{F}_X)$ and $(\underline{F}_Y, \overline{F}_Y)$. Ex-absurdo, assume that there exists a comonotone natural extension $\underline{E}_{X,Y}$. This implies that $\overline{E}_{X,Y} \geq \overline{P}$, where of course $\overline{E}_{X,Y}$ and $\overline{P}$ denote the conjugate upper probabilities of $\underline{E}_{X,Y}$ and $\underline{P}$, respectively. Comparing $\overline{E}_{X,Y}$ with $\overline{P}_1$ and $\overline{P}_2$, we deduce that:

$$\overline{E}_{X,Y}\big(\{(x_2, y_3)\}\big) \geq \overline{P}_1\big(\{(x_2, y_3)\}\big) > 0, \quad \overline{E}_{X,Y}\big(\{(x_3, y_2)\}\big) \geq \overline{P}_2\big(\{(x_3, y_2)\}\big) > 0$$

meaning that $(x_2, y_3), (x_3, y_2) \in \text{Supp}(\underline{E}_{X,Y})$, so $\underline{E}_{X,Y}$ is not comonotone. □

This example shows that the comonotone natural extension of two marginal p-boxes does not always exist, even when there are comonotone extensions. In fact, it also shows that the lower envelope of comonotone extensions is not a comonotone extension. Our aim now is to investigate which additional conditions must be imposed to guarantee the existence of the comonotone natural extension and, in such case, how to build it.

5.1 Properties of the comonotone extensions

We devote this subsection to show some technical properties of the comonotone extensions of given marginal p-boxes that will be useful later to characterise the existence of the comonotone natural extension.

The first result shows that a comonotone extension of given marginal p-boxes assigns strictly positive upper probability to all the elements in S.

Proposition 3 *Consider two marginal p-boxes* $(\underline{F}_X, \overline{F}_X)$ *and* $(\underline{F}_Y, \overline{F}_Y)$ *with an interval dominance relation between* $\underline{\overline{F}}_X(x)$ *and* $\underline{\overline{F}}_Y(y)$ *for every* $(x, y) \in \mathcal{X} \times \mathcal{Y}$*, and let* $\underline{P}_{X,Y}$ *be a comonotone extension of* $(\underline{F}_X, \overline{F}_X)$ *and* $(\underline{F}_Y, \overline{F}_Y)$*. Then, its conjugate upper probability satisfies* $\overline{P}_{X,Y}\big(\{(x_i, y_j)\}\big) > 0$ *for every* $(x_i, y_j) \in S$*.*

Proof Take $(x_i, y_j) \in S$, and consider the events $B_1 = \{(x_i, y_1), \ldots, (x_i, y_j)\}$ and $B_2 = \{(x_1, y_j), \ldots, (x_i, y_j)\}$. The sub-additivity of $\overline{P}_{X,Y}$ and the inequalities for $\underline{P}_{X,Y}$ and $\overline{P}_{X,Y}$ implies:

$$\overline{P}_{X,Y}(B_1) \geq \underline{P}_{X,Y}(A_{x_i,y_j}) - \underline{P}_{X,Y}(A_{x_{i-1},y_j}) = \underline{F}_{X,Y}(x_i, y_j) - \underline{F}_{X,Y}(x_{i-1}, y_j) \geq 0.$$
$$\overline{P}_{X,Y}(B_1) \geq \overline{P}_{X,Y}(A_{x_i,y_j}) - \overline{P}_{X,Y}(A_{x_{i-1},y_j}) = \overline{F}_{X,Y}(x_i, y_j) - \overline{F}_{X,Y}(x_{i-1}, y_j) \geq 0.$$

Moreover, since $(x_i, y_j) \in S$, $\underline{\overline{F}}_{X,Y}(x_{i-1}, y_j) \prec \underline{\overline{F}}_{X,Y}(x_i, y_j)$, so at least one of the previous equalities is strictly positive. Using the same reasoning with B_2:

$$\overline{P}_{X,Y}(B_2) \geq \underline{P}_{X,Y}(A_{x_i,y_j}) - \underline{P}_{X,Y}(A_{x_i,y_{j-1}}) = \underline{F}_{X,Y}(x_i, y_j) - \underline{F}_{X,Y}(x_i, y_{j-1}) \geq 0.$$
$$\overline{P}_{X,Y}(B_2) \geq \overline{P}_{X,Y}(A_{x_i,y_j}) - \overline{P}_{X,Y}(A_{x_i,y_{j-1}}) = \overline{F}_{X,Y}(x_i, y_j) - \overline{F}_{X,Y}(x_i, y_{j-1}) \geq 0.$$

Again, since $\underline{\overline{F}}_{X,Y}(x_i, y_{j-1}) \prec \underline{\overline{F}}_{X,Y}(x_i, y_j)$, one of the previous inequalities is strict, hence $\overline{P}_{X,Y}(B_1) > 0$ and $\overline{P}_{X,Y}(B_2) > 0$. Using again the sub-additivity of $\overline{P}_{X,Y}$:

$$\begin{aligned} 0 < \overline{P}_{X,Y}(B_1) &\leq \overline{P}_{X,Y}\big(\{(x_i, y_j)\}\big) + \overline{P}_{X,Y}\big(\{(x_i, y_1), \ldots, (x_i, y_{j-1})\}\big) \\ &\leq \overline{P}_{X,Y}\big(\{(x_i, y_j)\}\big) + \overline{P}_{X,Y}\big(D_2^{(x_i,y_j)}\big), \end{aligned} \tag{28}$$
$$\begin{aligned} 0 < \overline{P}_{X,Y}(B_2) &\leq \overline{P}_{X,Y}\big(\{(x_i, y_j)\}\big) + \overline{P}_{X,Y}\big(\{(x_1, y_j), \ldots, (x_{i-1}, y_j)\}\big) \\ &\leq \overline{P}_{X,Y}\big(\{(x_i, y_j)\}\big) + \overline{P}_{X,Y}\big(D_1^{(x_i,y_j)}\big). \end{aligned} \tag{29}$$

Theorem 2 implies that either $\overline{P}_{X,Y}\big(D_1^{(x_i,y_j)}\big) = 0$ or $\overline{P}_{X,Y}\big(D_2^{(x_i,y_j)}\big) = 0$, and from Eqs. (28) and (29), we obtain that $\overline{P}_{X,Y}\big(\{(x_i, y_j)\}\big) > 0$. □

The second result is a technical property that will be useful in the next section.

Proposition 4 *Consider two marginal p-boxes* $(\underline{F}_X, \overline{F}_X)$ *and* $(\underline{F}_Y, \overline{F}_Y)$ *satisfying the condition in Theorem 3, and let* $\underline{P}_{X,Y}$ *be a comonotone extension of* $(\underline{F}_X, \overline{F}_X)$ *and* $(\underline{F}_Y, \overline{F}_Y)$*. If there exists* $(x_i, y_j) \in \mathcal{X} \times \mathcal{Y} \setminus \{(x_n, y_m)\}$ *such that* $\underline{F}_X(x_i) = \overline{F}_X(x_i) = \underline{F}_Y(y_j) = \overline{F}_Y(y_j)$*, then* $\overline{P}_{X,Y}\big(\{(x_{i+1}, y_j)\}\big) = \overline{P}_{X,Y}\big(\{(x_i, y_{j+1})\}\big) = 0$*.*

Proof By hypothesis, we know that

$$\underline{F}_{X,Y}(x_i, y_j) = \overline{F}_{X,Y}(x_i, y_j) = \underline{F}_X(x_i) = \overline{F}_X(x_i) = \underline{F}_Y(y_j) = \overline{F}_Y(y_j).$$

Also:

$$\underline{F}_{X,Y}(x_{i+1}, y_j) = \min\big\{\underline{F}_X(x_{i+1}), \underline{F}_Y(y_j)\big\} = \underline{F}_Y(y_j) = \underline{F}_{X,Y}(x_i, y_j),$$
$$\overline{F}_{X,Y}(x_{i+1}, y_j) = \min\big\{\overline{F}_X(x_{i+1}), \overline{F}_Y(y_j)\big\} = \overline{F}_Y(y_j) = \overline{F}_{X,Y}(x_i, y_j),$$

If $\underline{P}_{X,Y}$ is a comonotone extension of $(\underline{F}_X, \overline{F}_X)$ and $(\underline{F}_Y, \overline{F}_Y)$, any $P \in \mathcal{M}\big(\underline{P}_{X,Y}\big)$ satisfies $F_P(x_{i+1}, y_j) = \underline{F}_{X,Y}(x_{i+1}, y_j) = \underline{F}_{X,Y}(x_i, y_j) = F_P(x_i, y_j)$, hence:

$$P\big(\{(x_{i+1}, y_j)\}\big) = F_P(x_{i+1}, y_j) + F_P(x_i, y_{j-1}) - F_P(x_{i+1}, y_{j-1}) - F_P(x_i, y_j)$$

$$= F_P(x_i, y_{j-1}) - F_P(x_{i+1}, y_{j-1}) \leq 0.$$

Thus, any P in $\mathcal{M}(\underline{P}_{X,Y})$ satisfies $P(\{(x_{i+1}, y_j)\}) = 0$, and this implies that $\overline{P}_{X,Y}(\{(x_{i+1}, y_j)\}) = 0$. A similar reasoning leads to $\overline{P}_{X,Y}(\{(x_i, y_{j+1})\}) = 0$. □

5.2 Increasing set $S^* \supseteq S$ defining the comonotone natural extension

Section 4.1 shows that each increasing superset of the set S defined in Eq. (14) induces a comonotone extension. In this subsection, we look for the superset of S that induces the comonotone natural extension when it exists.

For this aim, we first proof a preliminary result:

Lemma 2 *Consider two marginal p-boxes $(\underline{F}_X, \overline{F}_X)$ and $(\underline{F}_Y, \overline{F}_Y)$ satisfying the condition in Theorem 3. Consider the bivariate p-box $(\underline{F}_{X,Y}, \overline{F}_{X,Y})$ they define using Eq. (13), as well as the set S defined in Eq. (14). Let $(u_l, v_l), (u_{l+1}, v_{l+1}) \in S$ be two consecutive elements in S, assume that $(u_l, v_l) = (x_i, y_j)$ and consider the following cases:*

1. *If $\underline{\overline{F}}_X(x_i) \prec \underline{\overline{F}}_Y(y_j)$ then $v_{l+1} = y_j$.*
2. *If $\underline{\overline{F}}_Y(y_j) \prec \underline{\overline{F}}_X(x_i)$ then $u_{l+1} = x_i$.*
3. *If $\underline{F}_X(x_i) = \overline{F}_X(x_i) = \underline{F}_Y(y_j) = \overline{F}_Y(y_j)$, then $(u_{l+1}, v_{l+1}) = (x_{i+1}, y_{j+1})$.*

Proof 1. Assume that $\underline{\overline{F}}_X(x_i) \prec \underline{\overline{F}}_Y(y_j)$. This implies that:

$$\underline{F}_{X,Y}(x_i, y_j) = \min\left\{\underline{F}_X(x_i), \underline{F}_Y(y_j)\right\} = \underline{F}_X(x_i) \leq \underline{F}_Y(y_j),$$
$$\overline{F}_{X,Y}(x_i, y_j) = \min\left\{\overline{F}_X(x_i), \overline{F}_Y(y_j)\right\} = \overline{F}_X(x_i) \leq \overline{F}_Y(y_j),$$

with at least one strict inequality. Also, since $(x_i, y_j) \in S$, $\underline{\overline{F}}_{X,Y}(x_i, y_{j-1}) \prec \underline{\overline{F}}_{X,Y}(x_i, y_j)$, which means that:

$$\underline{F}_{X,Y}(x_i, y_{j-1}) = \min\left\{\underline{F}_X(x_i), \underline{F}_Y(y_{j-1})\right\} \leq \underline{F}_X(x_i),$$
$$\overline{F}_{X,Y}(x_i, y_{j-1}) = \min\left\{\overline{F}_X(x_i), \overline{F}_Y(y_{j-1})\right\} \leq \overline{F}_X(x_i),$$

with at least one strict inequality, meaning that $\underline{\overline{F}}_{X,Y}(x_i, y_{j-1}) = \underline{\overline{F}}_Y(y_{j-1}) \prec \underline{\overline{F}}_X(x_i)$. Also, it holds that:

$$\underline{F}_{X,Y}(x_n, y_{j-1}) = \min\left\{\underline{F}_X(x_n), \underline{F}_Y(y_{j-1})\right\} = \underline{F}_Y(y_{j-1}) = \underline{F}_{X,Y}(x_i, y_{j-1}),$$
$$\overline{F}_{X,Y}(x_n, y_{j-1}) = \min\left\{\overline{F}_X(x_n), \overline{F}_Y(y_{j-1})\right\} = \overline{F}_Y(y_{j-1}) = \overline{F}_{X,Y}(x_i, y_{j-1}),$$

which implies that:

$$\begin{aligned}\underline{\overline{F}}_{X,Y}(x_{i+1}, y_{j-1}) = \cdots = \underline{\overline{F}}_{X,Y}(x_n, y_{j-1}) = \underline{\overline{F}}_Y(y_{j-1}) \\ \prec \underline{\overline{F}}_X(x_i) \preceq \underline{\overline{F}}_{X,Y}(x_{i+1}, y_j) \preceq \cdots \preceq \underline{\overline{F}}_{X,Y}(x_n, y_j).\end{aligned} \tag{30}$$

Finally, note that:

$$\underline{\overline{F}}_{X,Y}(x_i, y_j) = \underline{\overline{F}}_X(x_i) \prec \underline{\overline{F}}_Y(y_j) = \underline{\overline{F}}_{X,Y}(x_n, y_j),$$

which, together with Eq. (30), implies that $(x_k, y_j) \in S$, where $k = \min\left\{l = i, \ldots, n \mid \underline{\overline{F}}_X(x_{l-1}) \prec \underline{\overline{F}}_X(x_l)\right\}$.

2. The case $\underline{\overline{F}}_Y(y_j) \prec \underline{\overline{F}}_X(x_i)$ follows by analogy with case 1.

3. Finally, assume that $\underline{F}_X(x_i) = \overline{F}_X(x_i) = \underline{F}_Y(y_j) = \overline{F}_Y(y_j)$. In this case, it holds that $\underline{\overline{F}}_{X,Y}(x_i, y_j) = \underline{\overline{F}}_X(x_i) = \underline{\overline{F}}_Y(y_j)$. Moreover:

$$\underline{F}_{X,Y}(x_{i+1}, y_j) = \min\left\{\underline{F}_X(x_{i+1}), \underline{F}_Y(y_j)\right\} = \underline{F}_Y(y_j) = \underline{F}_{X,Y}(x_i, y_j).$$
$$\overline{F}_{X,Y}(x_{i+1}, y_j) = \min\left\{\overline{F}_X(x_{i+1}), \overline{F}_Y(y_j)\right\} = \overline{F}_Y(y_j) = \overline{F}_{X,Y}(x_i, y_j).$$

With a similar reasoning, it follows that $\underline{\overline{F}}_{X,Y}(x_i, y_{j+1}) = \underline{\overline{F}}_{X,Y}(x_i, y_j)$.
Since we are assuming that $\overline{P}_{(\underline{F}_X,\overline{F}_X)}$ is strictly positive in the singletons and $\underline{F}_X(x_i) = \overline{F}_X(x_i)$, we deduce that $\overline{F}_X(x_{i+1}) > \underline{F}_X(x_i) = \overline{F}_X(x_i)$. Similarly, $\overline{F}_Y(y_{j+1}) > \underline{F}_Y(y_j) = \overline{F}_Y(y_j)$. This implies that:

$$\begin{aligned}\underline{F}_{X,Y}(x_{i+1}, y_{j+1}) &= \min\left\{\underline{F}_X(x_{i+1}), \underline{F}_Y(y_{j+1})\right\}\\ &\geq \min\left\{\underline{F}_X(x_i), \underline{F}_Y(y_j)\right\} = \underline{F}_{X,Y}(x_i, y_j).\\ \overline{F}_{X,Y}(x_{i+1}, y_{j+1}) &= \min\left\{\overline{F}_X(x_{i+1}), \overline{F}_Y(y_{j+1})\right\} > \min\left\{\overline{F}_X(x_i), \overline{F}_Y(y_j)\right\}\\ &= \overline{F}_{X,Y}(x_i, y_j) = \overline{F}_{X,Y}(x_{i+1}, y_j) = \overline{F}_{X,Y}(x_i, y_{j+1}).\end{aligned}$$

These two equations imply that $\underline{\overline{F}}_{X,Y}(x_{i+1}, y_j) \prec \underline{\overline{F}}_{X,Y}(x_{i+1}, y_{j+1})$ and $\underline{\overline{F}}_{X,Y}(x_i, y_{j+1}) \prec \underline{\overline{F}}_{X,Y}(x_{i+1}, y_{j+1})$, hence $(x_{i+1}, y_{j+1}) \in S$.

In the three cases we conclude that if $(x_i, y_j) = (u_l, v_l) \in S$, then either $u_{l+1} = x_{i+1}$ or $v_{l+1} = y_{k+1}$. □

In the conditions of the previous lemma, for every $(x, y) \in \mathcal{X} \times \mathcal{Y}$ either there is a strict interval dominance relationship between $\underline{\overline{F}}_X(x)$ and $\underline{\overline{F}}_Y(y)$ or $\underline{F}_X(x) = \overline{F}_X(x) = \underline{F}_Y(y) = \overline{F}_Y(y)$. For every $(u_l, v_l) \in S$, we define the set:

$$S_l = \begin{cases}\{(x, y) \in \mathcal{X} \times \mathcal{Y} \mid x = u_l,\ y \in [v_l, v_{l+1})\}, & \text{if } \underline{\overline{F}}_Y(v_l) \prec \underline{\overline{F}}_X(u_l).\\ \{(x, y) \in \mathcal{X} \times \mathcal{Y} \mid x \in [u_l, u_{l+1}),\ y = v_l\}, & \text{if } \underline{\overline{F}}_X(u_l) \prec \underline{\overline{F}}_Y(v_l).\\ \{(u_l, v_l)\}, \quad \text{if } \underline{F}_X(u_l) = \overline{F}_X(u_l) = \underline{F}_Y(v_l) = \overline{F}_Y(v_l). \end{cases}$$

Using the sets S_l, $l = 1, \ldots, s$, we define the set S^* given by $S^* = \cup_{l=1}^{s} S_l$. Let us see how to build this set in an example.

Example 8 Consider $\mathcal{X} = \{x_1, x_2, x_3\}$ and $\mathcal{Y} = \{y_1, y_2, y_3, y_4\}$, and the p-boxes $(\underline{F}_X, \overline{F}_X)$ and $(\underline{F}_Y, \overline{F}_Y)$ given by:

$\mathcal{X}$	x_1	x_2	x_3
$\underline{F}_X(x_i)$	0	0.8	1
$\overline{F}_X(x_i)$	0.5	0.8	1

$\mathcal{Y}$	y_1	y_2	y_3	y_4
$\underline{F}_Y(y_j)$	0.2	0.2	0.8	1
$\overline{F}_Y(y_j)$	0.8	0.8	0.8	1

Note that $\underline{F}_X(x_2) = \overline{F}_X(x_2) = \underline{F}_Y(y_3) = \overline{F}_Y(y_3) = 0.8$ and $\underline{F}_X(x_3) = \overline{F}_X(x_3) = \underline{F}_Y(y_4) = \overline{F}_Y(y_4) = 1$, and for any other i, j there is a strict interval dominance relation between $\underline{\overline{F}}_X(x_i)$ and $\underline{\overline{F}}_Y(y_j)$. In fact:

$$\begin{aligned}&\underline{\overline{F}}_X(x_1) = [0, 0.5] \prec \underline{\overline{F}}_Y(y_1) = \underline{\overline{F}}_Y(y_2) = [0.2, 0.8]\\ &\prec \underline{\overline{F}}_X(x_2) = \underline{\overline{F}}_Y(y_3) = [0.8, 0.8] \prec \underline{\overline{F}}_X(x_4) = \underline{\overline{F}}_Y(y_4) = [1, 1].\end{aligned}$$

The bivariate p-box $(\underline{F}_{X,Y}, \overline{F}_{X,Y})$ they define through Eq. (13) is given by:

$\overline{\underline{F}}_X(x_i)$	[0, 0.5]	[0.8, 0.8]	[1, 1]	
y_4	[0, 0.5]	[0.8, 0.8]	[1, 1]	[1, 1]
y_3	[0, 0.5]	[0.8, 0.8]	[0.8, 0.8]	[0.8, 0.8]
y_2	[0, 0.5]	[0.2, 0.8]	[0.2, 0.8]	[0.2, 0.8]
y_1	[0, 0.5]	[0.2, 0.8]	[0.2, 0.8]	[0.2, 0.8]
$\overline{\underline{F}}_{X,Y}(x_i, y_j)$	x_1	x_2	x_3	$\overline{\underline{F}}_Y(y_j)$

The set S defined in Eq. (14), represented in blue in the table, is given by:

$$S = \{(x_1, y_1), (x_2, y_1), (x_2, y_3), (x_3, y_4)\} = \{(u_1, v_1), \ldots, (u_4, v_4)\}.$$

Since $\overline{F}_X(u_1) \prec \underline{F}_Y(v_1)$, the set S_1 is given by:

$$S_1 = \{(x, y) \in \mathcal{X} \times \mathcal{Y} \mid x \in [u_1, u_2) = [x_1, x_2),\ y = v_1 = y_1\} = \{(x_1, y_1)\}.$$

Since $\overline{F}_Y(v_2) \prec \underline{F}_X(u_2)$, the set S_2 is given by:

$$S_2 = \{(x, y) \in \mathcal{X} \times \mathcal{Y} \mid x = u_2 = x_2,\ y \in [v_2, v_3) = [y_1, y_3)\} = \{(x_2, y_1), (x_2, y_2)\}.$$

Iterating the procedure, we obtain $S_3 = \{(x_2, y_3)\}$ and $S_4 = \{(x_3, y_4)\}$, hence the set S^* is given by:

$$S^* = \cup_{l=1}^{4} = \{(x_1, y_1), (x_2, y_1), (x_2, y_2), (x_2, y_3), (x_3, y_4)\}.\square \tag{31}$$

Let us now analyse some properties of the set $S^* = \cup_{l=1}^{s} S_l$.

Proposition 5 *In the conditions of Lemma 2, the set $S^* = \cup_{l=1}^{s} S_l$ satisfies the following properties:*

1. *S^* is a superset of S: $S \subseteq S^*$.*
2. *S^* is an increasing set in $\mathcal{X} \times \mathcal{Y}$.*
3. *The $\mathcal{X}$ and $\mathcal{Y}$ projections of S^* are $\mathcal{X}$ and $\mathcal{Y}$.*

Proof The fact that S^* is a superset of S follows by definition, since $(u_l, v_l) \in S_l \subseteq S^*$. To see that S^* is increasing, note that each S_l is increasing for every $l = 1, \ldots, s$. Also for every $(x, y) \in S_l$, $(u_l, v_l) \preceq (x, y) \prec (u_{l+1}, v_{l+1})$.

Let us now see that the $\mathcal{X}$-projection of S^* coincides with $\mathcal{X}$. Take $x_i \in \mathcal{X}$ and consider two cases:

1. If $u_l = x_i$ for some $(u_l, v_l) \in S \subseteq S^*$, x_i belongs to the $\mathcal{X}$-projection of S^*.
2. Otherwise, denote by $l = \max\{k = 1, \ldots, s \mid (u_l, v_l) \in S,\ u_l < x_i\}$. This implies that $(u_{l+1}, v_{l+1}) \in S$ satisfies $u_l < x_i < u_{l+1}$. By definition of l and Lemma 2, it must hold $\overline{F}_X(u_l) \prec \underline{F}_Y(v_l)$, but by Eq. (31), $(x_i, v_l) \in S_l$.

In both cases, we deduce that the $\mathcal{X}$-projection of S^* is $\mathcal{X}$, and similarly the $\mathcal{Y}$-projection of S^* is $\mathcal{Y}$. $\square$

From this proposition we deduce that S^* is an increasing superset of S, hence it can be used to build a comonotone extension of $(\underline{F}_X, \overline{F}_X)$ and $(\underline{F}_Y, \overline{F}_Y)$ following the procedure described in Sect. 4.2.

Proposition 6 *Consider two marginal p-boxes $(\underline{F}_X, \overline{F}_X)$ and $(\underline{F}_Y, \overline{F}_Y)$ such that for every $(x, y) \in \mathcal{X} \times \mathcal{Y}$ either there is a strict interval dominance relationship between $\overline{\underline{F}}_X(x)$ and $\overline{\underline{F}}_Y(y)$ or $\underline{F}_X(x) = \overline{F}_X(x) = \underline{F}_Y(y) = \overline{F}_Y(y)$. Consider the sets S, defined in Eq. (14) and*

$S^* = \cup_{l=1}^{s} S'$, and let S' be an increasing superset of S^*, $S \subseteq S^* \subseteq S'$ and denote by $\underline{P}_{X,Y}$ and $\underline{P}'$ the comonotone extensions of $(\underline{F}_X, \overline{F}_X)$ and $(\underline{F}_Y, \overline{F}_Y)$ built using the procedure in Sect. 4.2 with the sets S^* and S', respectively. Then, $\underline{P}_{X,Y} = \underline{P}'$.

Proof As we have seen in Proposition 2 and Cor. 1, the credal set of $\underline{P}'$ and $\underline{P}_{X,Y}$ are given, respectively, by:

$$\mathcal{M}(\underline{P}') = \{P \in \mathbb{P}(\mathcal{X} \times \mathcal{Y}) \mid P(S') = 1,\ \underline{F}_{X,Y} \leq F_P \leq \overline{F}_{X,Y}\}.$$
$$\mathcal{M}(\underline{P}_{X,Y}) = \{P \in \mathbb{P}(\mathcal{X} \times \mathcal{Y}) \mid P(S^*) = 1,\ \underline{F}_{X,Y} \leq F_P \leq \overline{F}_{X,Y}\}.$$

If there is $(x_i, y_j) \in S' \setminus S^*$, from Lemma 2 and the definition of S^* it holds that either $\underline{F}_X(x_i) = \overline{F}_X(x_i) = \underline{F}_Y(y_{j-1}) = \overline{F}_Y(y_{j-1})$ or $\underline{F}_X(x_{i-1}) = \overline{F}_X(x_{i-1}) = \underline{F}_Y(y_j) = \overline{F}_Y(y_j)$, but in both cases it follows from Proposition 4 that $\overline{P}_{X,Y}(\{(x_i, y_j)\}) = 0$. This implies that any $P \in \mathcal{M}(\underline{P}')$ satisfies $P_1(S') = P_1(S^*) = 1$, hence $P \in \mathcal{M}(\underline{P}_{X,Y})$. This implies that $\mathcal{M}(\underline{P}') \subseteq \mathcal{M}(\underline{P}_{X,Y})$. However, since $S^* \subseteq S'$, the previous inequality is an inequality, and since the credal sets coincide, their lower envelopes coincide too: $\underline{P}' = \underline{P}_{X,Y}$. □

5.3 Existence and construction of the comonotone natural extension

Using these preliminary results, we can state the main result of this section that characterises under which conditions there exists a comonotone natural extension of $(\underline{F}_X, \overline{F}_X)$ and $(\underline{F}_Y, \overline{F}_Y)$ and explains how to compute it.

Theorem 4 *Consider the marginal p-boxes $(\underline{F}_X, \overline{F}_X)$ and $(\underline{F}_Y, \overline{F}_Y)$. There exists a comonotone natural extension of $(\underline{F}_X, \overline{F}_X)$ and $(\underline{F}_Y, \overline{F}_Y)$ if and only if for every $(x, y) \in \mathcal{X} \times \mathcal{Y}$, either there is a strict interval dominance relationship between $\underline{\overline{F}}_X(x)$ and $\underline{\overline{F}}_Y(y)$ or $\underline{F}_X(x) = \overline{F}_X(x) = \underline{F}_Y(y) = \overline{F}_Y(y)$.*

Moreover, when the condition holds, the comonotone natural extension $\underline{E}_{X,Y}$ is the belief function built in Sect. 4.2 using the set $S^* = \cup_{l=1}^{s} S_l$.

Proof Assume that there is a comonotone natural extension $\underline{E}_{X,Y}$ of $(\underline{F}_X, \overline{F}_X)$ and $(\underline{F}_Y, \overline{F}_Y)$. From Theorem 3 there is an interval dominance relationship between $\underline{\overline{F}}_X(x_i)$ and $\underline{\overline{F}}_Y(y_j)$ for every $(x_i, y_j) \in \mathcal{X} \times \mathcal{Y}$. Assume ex-absurdo that there is $(x_i, y_j) \in \mathcal{X} \times \mathcal{Y}$ such that $\underline{F}_X(x_i) = \underline{F}_Y(y_j) < \overline{F}_Y(y_j) = \overline{F}_X(x_i)$. Hence:

$$\underline{F}_{X,Y}(x_i, y_j) = \min\{\underline{F}_X(x_i), \underline{F}_Y(y_j)\} = \underline{F}_X(x_i) = \underline{F}_Y(y_j),$$
$$\overline{F}_{X,Y}(x_i, y_j) = \min\{\overline{F}_X(x_i), \overline{F}_Y(y_j)\} = \overline{F}_X(x_i) = \overline{F}_Y(y_j),$$

meaning that $\underline{\overline{F}}_{X,Y}(x_i, y_j) = \underline{\overline{F}}_X(x_i) = \underline{\overline{F}}_Y(y_j)$. Also, since $\underline{\overline{F}}_{X,Y}(x_i, y_m) = \underline{\overline{F}}_X(x_i)$ and $\underline{\overline{F}}_{X,Y}(x_n, y_j) = \underline{\overline{F}}_Y(y_j)$, it holds that $(x_{i+1}, y_j), \ldots, (x_{i+1}, y_m) \notin S$ and $(x_i, y_{j+1}), \ldots, (x_n, y_{j+1}) \notin S$. This implies that the sets $S' = S \cup \{(x_{i+1}, y_j)\}$ and $S'' = S \cup \{(x_i, y_{j+1})\}$ are increasing.

Denote by $\underline{P}'$ and $\underline{P}''$ the comonotone extensions defined using the sets S' and S'' and following the procedure in Sect. 4.2. Let $\mathcal{Z}$ be the possibility space in correspondence with S' with associated univariate p-box $(\underline{F}_Z, \overline{F}_Z)$ (Eqs. (17) and (18)). There is $z_l \in \mathcal{Z}$ such that there is a correspondence between (x_{i+1}, y_j) and z_l. Then:

$$\overline{P}'(\{(x_{i+1}, y_j)\}) = \overline{P}_Z(\{z_l\}) = \overline{F}_Z(z_l) - \underline{F}_Z(z_{l-1})$$

$$\geq \overline{F}_Z(z_l) - \underline{F}_Z(z_l) = \overline{F}_{X,Y}(x_{i+1}, y_j) - \underline{F}_{X,Y}(x_{i+1}, y_j) = \overline{F}_{X,Y}(x_i, y_j) - \underline{F}_{X,Y}(x_i, y_j) > 0,$$

where the second equality follows from Eq. (8). The same reasoning leads to $\overline{P}''\big(\{(x_i, y_{j+1})\}\big) > 0$.

Finally, since $\underline{E}_{X,Y}$ is the comonotone natural extension, its conjugate $\overline{E}_{X,Y}$ satisfies $\overline{E}_{X,Y} \geq \overline{P}'$ and $\overline{E}_{X,Y} \geq \overline{P}''$, which implies that:

$$\overline{E}_{X,Y}\big(\{(x_{i+1}, y_j)\}\big) \geq \underline{P}'\big(\{(x_{i+1}, y_j)\}\big) > 0.$$
$$\overline{E}_{X,Y}\big(\{(x_i, y_{j+1})\}\big) \geq \underline{P}''\big(\{(x_i, y_{j+1})\}\big) > 0.$$

These two equations imply that (x_i, y_{j+1}) and (x_{i+1}, y_j) belong to $\mathrm{Supp}\big(\underline{E}_{X,Y}\big)$, but these two elements are not comonotone, so $\mathrm{Supp}\big(\underline{E}_{X,Y}\big)$ would not be increasing.

Assume now that for every $(x, y) \in \mathcal{X} \times \mathcal{Y}$ either there is a strict interval dominance relationship between $\underline{\overline{F}}_X(x_i)$ and $\underline{\overline{F}}_Y(y_j)$ or $\underline{F}_X(x_i) = \overline{F}_X(x_i) = \underline{F}_Y(y_j) = \overline{F}_Y(y_j)$. Consider the set $S^* = \cup_{l=1}^{s}$, which by Proposition 5 is an increasing superset of S. Denote by $\underline{E}_{X,Y}$ the comonotone extension defined using S^* and the procedure in Sect. 4.2, and let us see that $\underline{E}_{X,Y}$ is the comonotone natural extension. For this aim, let $\underline{P}_{X,Y}$ be any other comonotone extension of the given p-boxes and let us see that $\underline{E}_{X,Y} \leq \underline{P}_{X,Y}$.

Define the set $S' = \mathrm{Supp}\big(\underline{P}_{X,Y}\big) \cup S^*$, and let us see that it is increasing. On the one hand, from Proposition 3 $S \subseteq \mathrm{Supp}\big(\underline{P}_{X,Y}\big)$ and it is increasing, as well as S^*. Take $(x_i, y_j) \in \mathrm{Supp}\big(\underline{P}_{X,Y}\big)$. If $(x_i, y_j) \in S$, then $(x_i, y_j) \in S^*$ so it is comonotone with all the elements in S^*. Assume now that $(x_i, y_j) \notin S$, and let $(u_l, v_l) \in S$ be the maximum element in S such that $(u_l, v_l) \prec (x_i, y_j) \prec (u_{l+1}, v_{l+1})$. From Lemma 2 there are three cases:

Case 1: Assume that $\underline{\overline{F}}_X(u_l) \prec \underline{\overline{F}}_Y(v_l)$. In this case, $v_{l+1} = v_l = y_j$, hence $(x_i, y_j) \in S_l \subseteq S^*$.
Case 2: Assume that $\underline{\overline{F}}_Y(v_l) \prec \underline{\overline{F}}_X(u_l)$. In this case, $u_{l+1} = u_l = x_i$, hence $(x_i, y_j) \in S_l \subseteq S^*$.
Case 3: Assume that $\underline{F}_X(u_l) = \overline{F}_X(u_l) = \underline{F}_Y(v_l) = \overline{F}_Y(v_l)$. In this case, it holds that either

$$(u_l, v_l) = (x_i, y_{j-1}), \text{ and } (u_{l+1}, v_{l+1}) = (x_{i+1}, y_j), \quad \text{or}$$
$$(u_l, v_l) = (x_{i-1}, y_j), \text{ and } (u_{l+1}, v_{l+1}) = (x_i, y_{j+1}),$$

but in both cases (x_i, y_j) is comonotone with all the elements in S_l and S_{l+1}, and hence also with all the elements in S^*.

Since S' is an increasing superset of S^*, from Proposition 6 both S' and S^* induce the same comonotone extension $\underline{E}_{X,Y}$, whose credal set is given by (Proposition 2):

$$\mathcal{M}\big(\underline{E}_{X,Y}\big) = \big\{P \in \mathbb{P}(\mathcal{X} \times \mathcal{Y}) \mid P(S^*) = 1,\ \underline{F}_{X,Y} \leq F_P \leq \overline{F}_{X,Y}\big\}$$
$$= \big\{P \in \mathbb{P}(\mathcal{X} \times \mathcal{Y}) \mid P(S') = 1,\ \underline{F}_{X,Y} \leq F_P \leq \overline{F}_{X,Y}\big\}.$$

Finally, any $P \in \mathcal{M}\big(\underline{P}_{X,Y}\big)$ satisfies $\underline{F}_{X,Y} \leq F_P \leq \overline{F}_{X,Y}$ and also $1 = P\Big(\mathrm{Supp}\big(\underline{P}_{X,Y}\big)\Big) \leq P(S')$, hence $P \in \mathcal{M}\big(\underline{E}_{X,Y}\big)$. This implies $\mathcal{M}\big(\underline{P}_{X,Y}\big) \subseteq \mathcal{M}\big(\underline{E}_{X,Y}\big)$, hence their lower envelopes satisfy $\underline{E}_{X,Y} \leq \underline{P}_{X,Y}$. □

This theorem not only characterises the conditions that the marginal p-boxes must satisfy to guarantee the existence of their comonotone natural extension, but also gives the constructive method for building it and assures that it is a belief function. The method for building

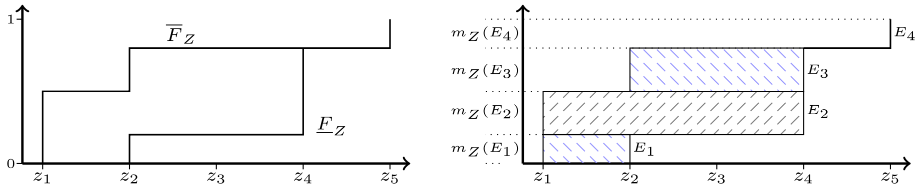

Fig. 7 P-box $(\underline{F}_Z, \overline{F}_Z)$ in Example 9 (left-hand side figure) and the focal events of its associated belief function (right-hand side figure)

the comonotone natural extension is just applying the procedure in Sect. 4.2 using the set $S^* = \cup_{l=1}^{s} S_l$.

Example 9 Let us continue with Example 8 applying the procedure in Sect. 4.2 with the increasing set $S^* = \cup_{l=1}^{s} S_l$. For this aim, consider the possibility space $\mathcal{Z} = \{z_1, z_2, z_3, z_4, z_5\}$ and the univariate p-box given by:

$\mathcal{Z}$	z_1	z_2	z_3	z_4	z_5
$\underline{F}_Z$	0	0.2	0.2	0.8	1
$\overline{F}_Z$	0.5	0.8	0.8	0.8	1

Its graphical representation, together with the focal events of its induced belief function, is shown in Fig. 7.

These focal events are given by

	E_1	E_2	E_3	E_4
E_i	$\{z_1, z_2\}$	$\{z_1, z_2, z_3, z_4\}$	$\{z_2, z_3, z_4\}$	$\{z_5\}$
$m_Z(E_i)$	0.2	0.3	0.3	0.2

Using Eq. (24), the focal events of the comonotone natural extension $\underline{E}_{X,Y}$ are given by:

	F_1	F_2	F_3	F_4
$F_i = g^{-1}(E_i)$	(x_1, y_1) (x_2, y_1)	$(x_1, y_1), (x_2, y_1)$ $(x_2, y_2), (x_2, y_3)$	$(x_2, y_1), (x_2, y_2)$ (x_2, y_3)	(x_3, y_4)
$m(F_i) = m_Z(E_i)$	0.2	0.3	0.3	0.2

These focal events have been graphically depicted in Fig. 8. □

6 Conclusions

This paper studies the problem of building a comonotone model with given marginals when these are given in terms of robust distribution functions, also known as p-boxes. For this aim, we have considered the definition of comonotonicity for coherent lower probabilities and their main properties given in [18]. We have considered that the probabilistic information about the marginal models is given in terms of p-boxes, and we have investigated the problem of the existence, construction and uniqueness of a comonotone lower probability with the given marginal p-boxes, called *comonotone extension*.

Even if the comonotone extension of the given marginals does not always exist, we have characterised the properties that the marginal p-boxes must satisfy to guarantee its existence. This necessary and sufficient condition imposes interval dominance between the values of the

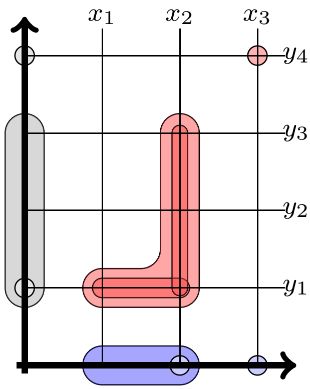

Fig. 8 Focal events of the comonotone natural extension $\underline{E}_{X,Y}$ in Example 8

marginal p-boxes (Theorem 3). Also, when this sufficient and necessary condition holds, we have given a constructive method to built a comonotone extension (Sect. 4.2). Nevertheless, we have also seen that this comonotone extension is not unique in general (Sect. 4.3).

This lack of uniqueness led us to investigate the existence of the comonotone *natural* extension. This follows the usual philosophy of the imprecise probability theory: we look for the least-committal extension adding the structural assumption of comonotonicity. Even if the comonotone natural extension does not always exist, we have proven that it exists if and only for every (x, y) either there is a strict interval dominance relationship between $\overline{\underline{F}}_X(x)$ and $\overline{\underline{F}}_Y(y)$ or the bounds given by the marginal p-boxes are trivial and coincide. In that case, the comonotone natural extension can be easily computed and, in fact, it is a belief function.

Throughout the paper we are doing two assumptions: (i) the possibility spaces $\mathcal{X}$ and $\mathcal{Y}$ are finite; and (ii) the upper probabilities associated with the marginal p-boxes are strictly positive in singletons. Assumption (ii) is not really restrictive, because we can always apply our results to $\mathcal{X}^* = \{x \in \mathcal{X} \mid \overline{P}_{(\underline{F}_X,\overline{F}_X)}(\{x\}) > 0\}$ and $\mathcal{Y}^* = \{y \in \mathcal{Y} \mid \overline{P}_{(\underline{F}_Y,\overline{F}_Y)}(\{y\}) > 0\}$, and once we obtain a comonotone extension $\underline{P}^*$ (when it exists), extend it to $\mathcal{X} \times \mathcal{Y}$ using $\underline{P}_{X,Y}(A) = \underline{P}^*\big(A \cap (\mathcal{X}^* \times \mathcal{Y}^*)\big)$ for every $A \subseteq \mathcal{X} \times \mathcal{Y}$. Removing condition (i) would make our problem much trickier. For example, if $\mathcal{X} = \mathcal{Y} = [0, 1]$, and $\underline{F}_X, \overline{F}_X, \underline{F}_Y$ and $\overline{F}_Y$ are continuous and strictly increasing, imposing the sufficient and necessary condition in Theorem 3 would lead to the trivial condition $(\underline{F}_X, \overline{F}_X) = (\underline{F}_Y, \overline{F}_Y)$. For this reason, we suspect that this problem would be difficult to solve in non-finite possibility spaces.

The research presented in this paper can be extended in different ways. First of all, in this paper we are modelling the available probabilistic information about the marginals using p-boxes. However, instead of considering p-boxes we could consider more general models such as lower probabilities or belief functions. Secondly, we could investigate what happens when the marginal p-boxes are built using some precise cdfs F_X^0 and F_Y^0 given with a degree of reliability. In this cases, we can build marginal p-boxes using the Kolmogorov model [13], one of the usual models in robust statistics and studied from the point of view of imprecise probabilities in [21, 22]. And thirdly, we could investigate some common applications of comonotonicity in choice under risk [38] or finance [7].

Acknowledgements The research reported in this paper has been supported by project PGC2018-098623-B-I00 from the Ministry of Science, Innovation and Universities of Spain.

Funding Open Access funding provided thanks to the CRUE-CSIC agreement with Springer Nature.

Declarations

Conflict of interest The author declares that he has no conflict of interest.

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