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A stochastic order for interval valued random mappings and applications

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ABSTRACT

This manuscript introduces a criterion to compare interval valued random mappings by means of a stochastic order for such random elements. The comparison criterion is based on the distance to the origin on both sides of the values that those random elements assume. The proposed stochastic order is studied, providing characterizations and properties. One of those characterizations is based on a new stochastic order for bivariate random vectors when it is applied to the endpoints of the random intervals. Some examples with applications of the criterion to weather and economic problems are developed.

1. Introduction

Most of the mathematical models to approach randomness and uncertainty assume that random magnitudes can be modelled by means of random variables or random vectors. However, there are numerous real-life problems in which random magnitudes are better described by other mathematical tools like functional data, random matrices, random sets, etc.

One of those problems is the case in which random magnitudes take on interval values. A great number of real situations involve characteristics which take on interval values instead of real or vectorial values, like for instance, those involving ranges, fluctuations or intervals given by the minimum and maximum of a set of real values. The relevance of the analysis of interval data in applied problems is apparent from the large amount of recent publications on the field addressing issues such as decision making, sensitivity analysis, data analysis, optimization, etc. Some examples of papers dealing with interval data are [1], where a smoothing technique for interval data based on poynomial models is introduced, [2], where techniques for the likelihood-based statistical inference for intervals are developed, [3], where interval data are used to deal with imprecise data in data envelopment analysis, or [4], which includes an optimization model and method for thermal structure design with random, interval and fuzzy quantities, as well as some of the references therein.

The following problem serves as a motivation of the aim of the manuscript. Two manufacturers produce certain item. Ideally, the measure of a quality characteristic of those items should assume a nominal value α. Each day, a sample of the production is taken in each factory and the characteristic is measured in each of the items. The daily interval of values with respect to the fixed α is $[\underline{x}-\alpha,\overline{x}-\alpha]$, $\underline{x}$ and $\overline{x}$ being the smallest and largest records of the corresponding day. Which

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manufacturer produces "better" items? Intuitively, if the endpoints of the interval of a manufacturer are closer to zero than those of the other maker, the former would produce items with a closer value to α of the quality characteristic. For instance, that could be the case of makers producing automobile piston rings whose inner diameters should be 75 mm. If the interval of values of a maker with respect to the desirable diameter 75 mm tends to take on "more narrow" values around 0, that would mean that such a manufacturer produces piston rings more accordant with the measurement requirements.

Motivated by the increasing number of problems in which the random elements take on interval values and the necessity of the comparison of those elements, we introduce and analyze a stochastic order to compare interval valued random mappings with respect to the distance to the origin of the values they assume.

A stochastic order aims to compare probabilities in accordance with an appropriate criterion. Most of the stochastic orders in mathematical literature are in relation to the distributions of random variables or of random vectors (see, for instance, [5], [6] and [7] for an introduction to the theory of stochastic orders). To the best of our knowledge, not many stochastic orders have been proposed for the comparison of the probabilities induced by other random elements. Among them, we have some stochastic orders for different kinds of set valued random mappings (see, for instance, [8], which introduces and analyzes some stochastic orders for the so-called random closed sets, [9], devoted to a stochastic order for random measures, [10], about the comparison of random closed sets in terms of coupling results, [11], analyzing some stochastic dominance criteria for random vectors by means of the Aumann expectations of some associated random sets, or [12] and [13], which study the comparison of the shapes of bidimensional closed curves and the variability of planar star-shaped sets, respectively, with applications in image analysis).

The structure of the paper is the following. In Section 2, we collect the concepts and basic results that we need for the development of the manuscript. Section 3 is devoted to the introduction of the new stochastic order for interval valued mappings and analyze its main characteristics and properties. In Section 4, we study relations of the new order with some other stochastic orders for interval valued random elements, random vectors and random variables. Finally, examples with applications of the new order to climatic and economic issues are developed in Section 5. Some concluding remarks are presented in Section 6.

2. Preliminaries

Firstly, we include some notions on ordered sets. Later on, we bring in the concept of interval valued random mapping. Finally, a relevant result on stochastic orders for probabilities on Polish spaces is included.

A binary relation $\preceq$ on a set $\mathcal{X}$ which satisfies the reflexive and transitive properties is said to be a pre-order. If in addition, the binary relation is antisymmetric, then it is called a partial order. In that case, the pair $(\mathcal{X}, \preceq)$ is said to be a poset (partially ordered set).

Given a poset $(\mathcal{X}, \preceq)$, a subset $U \subset \mathcal{X}$ is said to be an upper set if given $x_1, x_2 \in \mathcal{X}$ with $x_1 \in U$ and $x_1 \preceq x_2$, then $x_2 \in U$.

A mapping $f : \mathcal{X} \to \mathbb{R}$ is said to be $\preceq$-preserving if for any $x_1, x_2 \in \mathcal{X}$ with $x_1 \preceq x_2$, it holds that $f(x_1) \le f(x_2)$.

The reader is refereed, for instance, to the monographs [14] and [15] for an introduction to the theory of ordered sets.

Let $\mathcal{K}_c$ be the class of non-empty compact intervals of $\mathbb{R}$. Given $A \in \mathcal{K}_c$, we will denote by $\underline{A}$ and by $\overline{A}$ the minimum and the maximum of the interval A, respectively.

Consider the Hausdorff metric on $\mathcal{K}_c$, given by.

$$d_H\big(A, B\big) = \max\{|\overline{A} - \overline{B}|, |\underline{A} - \underline{B}|\} \tag{1}$$

for any $A, B \in \mathcal{K}_c$. It is well-known that $(\mathcal{K}_c, d_H)$ is a complete and separable metric space.

Denote by τ_H the topology induced by the Hausdorff metric d_H on $\mathcal{K}_c$ and by σ_H the Borel σ-algebra generated by τ_H on $\mathcal{K}_c$.

Given a probability space $(\Omega, \mathcal{A}, P)$, a mapping $X : \Omega \to \mathcal{K}_c$ is said to be an interval valued random mapping if it is measurable with respect to the σ-algebras $\mathcal{A}$ and σ_H (see, for instance, [16]). It is well-know that $X : \Omega \to \mathcal{K}_c$ is an interval valued random mapping if and only if the mappings $\underline{X}, \overline{X} : \Omega \to \mathbb{R}$, with $\underline{X}(\omega) = \underline{X(\omega)}$ and $\overline{X}(w) = \overline{X(\omega)}$ for all $\omega \in \Omega$, are random variables. Note that an interval valued random mapping X induces a probability P_X on the measurable space $(\mathcal{K}_c, \sigma_H)$ in the usual way.

The symbol $\sim_{st}$ between any kind of random elements will mean that such elements are equal in distribution.

We will denote by B_s the interval $[-s, s]$ with $s \ge 0$ and if $A \in \mathcal{K}_c$, A^s will stand for $A + B_s$, where $+$ denotes the Minkowki addition.

The following result will be key for the development of the manuscript. Consider a Polish space $\mathcal{S}$ with a partial order $\preceq$ such that $\{(x, y) \in \mathcal{S} \times \mathcal{S} \mid x \preceq y\}$ is closed in the product topology of $\mathcal{S}$ (closed partial order). For two Borel measurable mappings X_1 and X_2 into $\mathcal{S}$, with induced probabilities P_1 and P_2, respectively, we say that X_1 (or P_1) is smaller than X_2 (or P_2) in the stochastic order generated by $\preceq$, denoted by $X_1 \,\widetilde{\preceq}\, X_2$, if $E(f(X_1)) \le E(f(X_2))$ for any $\preceq$-preserving, measurable and bounded mapping $f : \mathcal{S} \to \mathbb{R}$.

The following proposition summarizes some results in [20] of stochastic orders for probabilities on Polish spaces.

Proposition 2.1. *Under the above framework, the following conditions are equivalent,*

i) $X_1 \,\widetilde{\preceq}\, X_2$,

ii) *there exists a probability space* $(\Omega, \mathcal{A}, P)$ *and random elements* $\widetilde{X}_1, \widetilde{X}_2 : \Omega \to \mathcal{S}$ *with* $\widetilde{X}_1 \sim_{st} X_1$ *and* $\widetilde{X}_2 \sim_{st} X_2$, *such that* $\widetilde{X}_1 \preceq \widetilde{Y}_2$ *a.s.* $[P]$,

iii) $E(f(X_1)) \leq E(f(X_2))$ *for any* $\preceq$*-preserving continuous and bounded mapping* $f : \mathcal{S} \to \mathbb{R}$,

iv) $P_{X_1}(U) \leq P_{X_2}(U)$ *for all closed upper sets* U *of* $\mathcal{S}$.

3. A stochastic order for interval valued mappings: On the distance to the origin

In this section, we introduce and study a stochastic order for interval valued random mappings. That order aims to compare those mappings with respect to the distance to the origin on both sides of the values that those mappings take on.

For such a purpose, some partial orders on $\mathbb{R}$ and on $\mathcal{K}_c$ are defined.

Definition 3.1. Let $\preceq^1$ be the binary relation on $\mathbb{R}$ defined as follows, let $x, y \in \mathbb{R}$, then

$$x \preceq^1 y \qquad \text{if} \qquad \begin{cases} x \leq y & \text{when } x > 0, \\ y \leq x & \text{when } x < 0, \\ y \in \mathbb{R} & \text{when } x = 0. \end{cases}$$

Basically, $x \preceq^1 y$ means that y is further away from the origin than x in the same direction.

The proof of the following lemma is clear.

Lemma 3.2. *The relation* $\preceq^1$ *is a closed partial order on* $\mathbb{R}$, *when* $\mathbb{R}$ *is endowed with the usual topology.*

We introduce a binary relation on the class $\mathcal{K}_c$ by means of the above partial order on $\mathbb{R}$.

Definition 3.3. Let $\preceq_I$ be the binary relation on $\mathcal{K}_c$ defined as follows, let $A, B \in \mathcal{K}_c$, then $A \preceq_I B$ when for all $a \in A$ there exists $b \in B$ with $a \preceq^1 b$, and for all $b \in B$ there exists $a \in A$ with $a \preceq^1 b$.

The relation $A \preceq_I B$ means that given any point of A, there exists a point of B which is further away from the origin in the same direction as the point of A, and given any point of B, there is a point of A which closer to the origin in the same direction as the point of B. Roughly speaking, B is more distant from the origin than A, considering both sides, on the positive and the negative parts.

The binary relation $\preceq_I$ on $\mathcal{K}_c$ is a closed partial order when $\mathcal{K}_c$ is endowed with the topology τ_H. We state the following lemma to prove such a result.

Lemma 3.4. *Let* $A, B \in \mathcal{K}_c$. *Then,* $A \preceq_I B$ *if and only if* $\underline{A} \preceq^1 \underline{B}$ *and* $\overline{A} \preceq^1 \overline{B}$.

Proof. Suppose that $A \preceq_I B$. Thus, there exists $b \in B$ with $\underline{A} \preceq^1 b$. Moreover, there exists $a \in A$ with $a \preceq^1 \underline{B}$.

We distinguish the following possibilities,

i) $0 < \underline{A}$, in this case it holds that $0 < \underline{A} \preceq^1 a \preceq^1 \underline{B}$, and so $\underline{A} \preceq^1 \underline{B}$,

ii) $\underline{A} < 0$, because of $\underline{A} \preceq^1 b$, it is not possible that $\underline{B} \geq 0$, therefore, $b \leq \underline{A}$, and so $\underline{B} \leq \underline{A}$, that is, $\underline{A} \preceq^1 \underline{B}$,

iii) $\underline{A} = 0$, in this case it is obvious that $\underline{A} \preceq^1 \underline{B}$.

Therefore, $\underline{A} \preceq^1 \underline{B}$ when $A \preceq_I B$.

The analysis of $\overline{A} \preceq^1 \overline{B}$ is similar and so it is omitted.

Thus, $A \preceq_I B$ implies that $\underline{A} \preceq^1 \underline{B}$ and $\overline{A} \preceq^1 \overline{B}$.

Conversely, let us suppose that $\underline{A} \preceq^1 \underline{B}$ and $\overline{A} \preceq^1 \overline{B}$.

Let $a \in A$. At least one of the relations $a \preceq^1 \underline{A}$ and $a \preceq^1 \overline{A}$ is held. Because of the transitivity of $\preceq^1$, we obtain that at least one of the relations $a \preceq^1 \underline{B}$ and $a \preceq^1 \overline{B}$ is true. Therefore, for any $a \in A$ there exists $b \in B$ with $a \preceq^1 b$.

Now let $b \in B$. Consider the following cases,

i) $0 \leq \underline{B}$, we have that $\underline{A} \preceq^1 \underline{B} \preceq^1 b$, and so, $\underline{A} \preceq^1 b$,

ii) $\overline{B} \leq 0$, in this case $\overline{A} \preceq^1 \overline{B} \preceq^1 b$,

iii) if $\underline{B} < 0 < \overline{B}$, it holds that $\underline{A} \leq 0$ since $\underline{A} \preceq^1 \underline{B}$, and so, $\underline{B} \leq \underline{A} \leq 0$. Moreover, $\overline{A} \geq 0$ since $\overline{A} \preceq^1 \overline{B}$, and so $0 \leq \overline{A} \leq \overline{B}$. Thus, $0 \in A$ and $0 \preceq^1 b$.

Then, for any $b \in B$ there exists $a \in A$ with $a \preceq^1 b$. Hence, $A \preceq_I B$. □

Lemma 3.5. *The binary relation* $\preceq_I$ *on* $\mathcal{K}_c$ *is a closed partial order.*

Proof. The relation $\preceq_I$ is reflexive and transitive since $\preceq^1$ has the same properties on $\mathbb{R}$.

Suppose that $A \preceq_I B$ and $B \preceq_I A$. By Lemma 3.4, $\underline{A} \preceq^1 \underline{B}$, $\overline{A} \preceq^1 \overline{B}$, $\underline{B} \preceq^1 \underline{A}$ and $\overline{B} \preceq^1 \overline{A}$. Since $\preceq^1$ is a partial order on $\mathbb{R}$, $\underline{A} = \underline{B}$ and $\overline{A} = \overline{B}$, that is, $A = B$.

Now, let $\{A_n\}_n, \{B_n\}_n \subset \mathcal{K}_c$, with $A_n \preceq_I B_n$ for all $n \in \mathbb{N}$, such that $\lim_n d_H(A_n, A) = 0 = \lim_n d_H(B_n, B)$, where $A, B \in \mathcal{K}_c$. Let us see that $A \preceq_I B$. In accordance with formula (1),

$$\lim_n |\underline{A}_n - \underline{A}| = \lim_n |\underline{B}_n - \underline{B}| = \lim_n |\overline{A}_n - \overline{A}| = \lim_n |\overline{B}_n - \overline{B}| = 0.$$

By Lemma 3.4, $A_n \preceq_I B_n$ is equivalent to $\underline{A}_n \preceq^1 \underline{B}_n$ and $\overline{A}_n \preceq^1 \overline{B}_n$. Lemma 3.2 says that the partial order $\preceq^1$ on $\mathbb{R}$ is closed. Therefore, $\underline{A} \preceq^1 \underline{B}$ and $\overline{A} \preceq^1 \overline{B}$, which is equivalent to $A \preceq_I B$ by Lemma 3.4. Hence, $\preceq_I$ is closed. □

Recall that $\mathcal{K}_c$ endowed with the topology τ_H is a Polish space and $\preceq_I$ is a closed partial order on $\mathcal{K}_c$. Let $\mathcal{F}^{\preceq_I}$ be the class of mappings $f:\mathcal{K}_c\to\mathbb{R}$ which are $\preceq_I$-preserving, bounded and measurable with respect to the σ-algebras σ_H and $\mathcal{B}$, where $\mathcal{B}$ stands for the usual Borel σ-algebra on $\mathbb{R}$.

That mathematical framework permits to introduce a stochastic order for interval valued random mappings as follows.

Definition 3.6. Let X and Y be interval valued random mappings. It will be said that X is smaller than Y in the distance to the origin stochastic order, if $E(f(X))\le E(f(Y))$ for all $f\in\mathcal{F}^{\preceq_I}$. It will be denoted by $X\,\widetilde{\preceq}_I\,Y$.

The relation $X\,\widetilde{\preceq}_I\,Y$ means that the interval valued random mapping Y takes on more distant values from the origin on both sides than X.

Consider the example in relation to the inner diameter of piston rings described in the introduction of the manuscript. Let X be the interval valued random mapping 'daily interval of measurements with respect to value 75 mm' of the first manufacturer, that is, $X=[\underline{X}-75,\overline{X}-75]$, where $\underline{X}$ and $\overline{X}$ stand for the random variables smallest and largest diameters of the day, and let Y stand for the interval valued mapping of the second manufacturer. The relation $X\,\widetilde{\preceq}_I\,Y$ formalizes the idea that the former produces piston rings with diameters closer to 75 mm when we consider both excess and default with respect to 75 mm. Note that $X\,\widetilde{\preceq}_I\,Y$ means that $E(f(X))\le E(f(Y))$ for all $f:\mathcal{K}_c\to\mathbb{R}$ which preserves $\preceq_I$, that is, for any mapping on the set of compact intervals of $\mathbb{R}$ which increases as the endpoints of those intervals take on values further from the origin. Thus, Y assumes intervals with endpoints further from the origin than X does.

The interpretation of the order will be reinforced with the next result on $\widetilde{\preceq}_I$, which follows from Proposition 2.1.

Proposition 3.7. *Let X and Y be interval valued random mappings. The following conditions are equivalent,*

i) $X\,\widetilde{\preceq}_I\,Y$,
ii) *there exists a probability space* $(\Omega,\mathcal{A},P)$ *and interval valued random mappings* $\widetilde{X},\widetilde{Y}:\Omega\to\mathcal{K}_c$ *with* $\widetilde{X}\sim_{st}X$ *and* $\widetilde{Y}\sim_{st}Y$, *such that* $\widetilde{X}\preceq_I\widetilde{Y}$ *a.s.* $[P]$,
iii) $E(f(X))\le E(f(Y))$ *for all bounded and continuous* $\preceq_I$*-preserving mappings* $f:\mathcal{K}_c\to\mathbb{R}$ *(continuity in the metric* d_H*),*
iv) $P_X(U)\le P_Y(U)$ *for all closed upper sets* U *of* $\mathcal{K}_c$ *(upper set in the partial order* $\preceq_I$*, closed in the topology* τ_H *on* $\mathcal{K}_c$*).*

For the inner diameter piston rings example, $X\,\widetilde{\preceq}_I\,Y$ is equivalent to the existence of interval valued random mappings $\widetilde{X}$ and $\widetilde{Y}$ on the same probability space, with the same probabilistic behaviour as X and Y, respectively, satisfying that $\widetilde{X}\preceq_I\widetilde{Y}$ *a.s.* $[P]$. By Lemma 3.4, $\widetilde{X}\preceq_I\widetilde{Y}$ *a.s.* means that the endpoints of $\widetilde{X}$ are closer to the origin than those of $\widetilde{Y}$ *a.s.* That is, the intervals of measurements with respect to 75 mm of the first manufacturer are, from a probabilistic point of view, "more narrow around that value" than those of the second maker.

It is well-known that for any interval valued random mappings X and Y, $X\sim_{st}Y$ if and only if the bivariate random vectors $(\underline{X},\overline{X})$ and $(\underline{Y},\overline{Y})$ satisfy that $(\underline{X},\overline{X})\sim_{st}(\underline{Y},\overline{Y})$. That will be applied to give a characterization of the order $\widetilde{\preceq}_I$.

Next, we develop a characterization of the order $\widetilde{\preceq}_I$ for interval valued random mappings, by means of a new stochastic order for bivariate random vectors when it is applied to the endpoints of the intervals. Such a characterization permits to delve into the analysis of the order $\widetilde{\preceq}_I$.

Consider on $\mathbb{R}^2$ the binary relation $\preceq^2$ which is the componentwise order of $\preceq^1$, that is, $(x_1,x_2)\preceq^2(y_1,y_2)$ when $x_1\preceq^1 y_1$ and $x_2\preceq^1 y_2$, with $(x_1,x_2),(y_1,y_2)\in\mathbb{R}^2$.

By Lemma 3.2, it can be seen that $\preceq^2$ is a closed partial order on $\mathbb{R}^2$, when $\mathbb{R}^2$ is endowed with the usual topology.

That permits to introduce an integral stochastic order for bivariate random vectors whose generator is given by the class of real bounded measurable mappings which preserve the partial order $\preceq^2$ on $\mathbb{R}^2$.

Definition 3.8. Let W and Z be bivariate random vectors. It will be said that W is smaller than Z in the stochastic order generated by $\preceq^2$, if $E(f(W))\le E(f(Z))$ for any $\preceq^2$-preserving, measurable and bounded mapping $f:\mathbb{R}^2\to\mathbb{R}$. It will be denoted by $W\,\widetilde{\preceq}^2\,Z$.

The following result is a consequence of Proposition 2.1.

Proposition 3.9. *Let W and Z be bivariate random vectors. The following conditions are equivalent,*

i) $W\,\widetilde{\preceq}^2\,Z$,
ii) *there exists a probability space* $(\Omega,\mathcal{A},P)$ *and bivariate random vectors* $\widetilde{W},\widetilde{Z}:\Omega\to\mathbb{R}^2$ *with* $\widetilde{W}\sim_{st}W$ *and* $\widetilde{Z}\sim_{st}Z$, *such that* $\widetilde{W}\preceq^2\widetilde{Z}$ *a.s.* $[P]$,
iii) $E(f(W))\le E(f(Z))$ *for all bounded and continuous* $\preceq^2$*-preserving mappings* $f:\mathbb{R}^2\to\mathbb{R}$,
iv) $P_W(U)\le P_Z(U)$ *for all closed upper sets* U *of* $\mathbb{R}^2$ *(upper set in the partial order* $\preceq^2$*).*

The following result relates the stochastic order $\widetilde{\preceq}_I$ for interval valued random mappings and the stochastic order $\widetilde{\preceq}^2$ for bivariate random vectors.

Proposition 3.10. *Let X and Y be interval valued random mappings. The following statements are equivalent,*

i) $X\,\widetilde{\preceq}_I\,Y$,
ii) $(\underline{X},\overline{X})\,\widetilde{\preceq}^2\,(\underline{Y},\overline{Y})$.

Proof. Firstly, suppose that $X \tilde{\preceq}_I Y$. By *ii*) in Proposition 3.7, there exists a probability space $(\Omega, \mathcal{A}, P)$ and interval valued random mappings $\widetilde{X}, \widetilde{Y} : \Omega \to \mathcal{K}_c$ with $\widetilde{X} \sim_{st} X$ and $\widetilde{Y} \sim_{st} Y$, such that $\widetilde{X} \preceq_I \widetilde{Y}$ a.s. $[P]$.

Lemma 3.4 says that the condition $\widetilde{X} \preceq_I \widetilde{Y}$ a.s. $[P]$ is equivalent to $\underline{\widetilde{X}} \preceq^1 \underline{\widetilde{Y}}$ and $\overline{\widetilde{X}} \preceq^1 \overline{\widetilde{Y}}$ a.s. $[P]$, all the above random variables defined on the same probability space, and so we obtain that $(\underline{\widetilde{X}}, \overline{\widetilde{X}}) \preceq^2 (\underline{\widetilde{Y}}, \overline{\widetilde{Y}})$ a.s. $[P]$.

It holds that $(\underline{\widetilde{X}}, \overline{\widetilde{X}}) \sim_{st} (\underline{X}, \overline{X})$ and $(\underline{\widetilde{Y}}, \overline{\widetilde{Y}}) \sim_{st} (\underline{Y}, \overline{Y})$.

By *ii*) in Proposition 3.9, we obtain that $(\underline{X}, \overline{X}) \tilde{\preceq}^2 (\underline{Y}, \overline{Y})$.

Conversely, let us assume that $(\underline{X}, \overline{X}) \tilde{\preceq}^2 (\underline{Y}, \overline{Y})$. By *ii*) in Proposition 3.9, there exists a probability space $(\Omega, \mathcal{A}, P)$ and random vectors $(X_1, X_2),\ (Y_1, Y_2) : \Omega \to \mathbb{R}^2$, with the same distribution of $(\underline{X}, \overline{X})$ and $(\underline{Y}, \overline{Y})$, respectively, such that $(X_1, X_2) \preceq^2 (Y_1, Y_2)$ a.s. $[P]$.

Consider the interval valued random mappings $[X_1, X_2]$ and $[Y_1, Y_2]$. We have that $[X_1, X_2] \sim_{st} X$ and $[Y_1, Y_2] \sim_{st} Y$. They are defined on the same probability space and $[X_1, X_2] \preceq_I [Y_1, Y_2]$ a.s. $[P]$ by Lemma 3.4, which proves the result applying *ii*) in Proposition 3.7. □

Propostion 3.10 permits to derive some consequences on the new stochastic order $\tilde{\preceq}_I$.

Proposition 3.11. *The stochastic order $\tilde{\preceq}_I$ is a partial order where equality is in distribution.*

Proof. Clearly $\tilde{\preceq}_I$ is reflexive. Moreover, $\tilde{\preceq}_I$ is transitive since the mappings of $\mathcal{F}^{\preceq_I}$ are bounded.

Now, suppose that $X \tilde{\preceq}_I Y$ and $Y \tilde{\preceq}_I X$. By Proposition 3.10, this is the same as $(\underline{X}, \overline{X}) \tilde{\preceq}^2 (\underline{Y}, \overline{Y})$ and $(\underline{Y}, \overline{Y}) \tilde{\preceq}^2 (\underline{X}, \overline{X})$. Applying Proposition 3.9, that is equivalent to $P_{(\underline{X}, \overline{X})}(U) = P_{(\underline{Y}, \overline{Y})}(U)$ for all closed upper sets in the order $\preceq^2$ on $\mathbb{R}^2$. It is clear that the class of closed upper sets in the order $\preceq^2$ is a π-system. Moreover, it can be seen that the σ-algebra generated by such a class is the usual Borel σ-algebra on $\mathbb{R}^2$. Therefore, $P_{(\underline{X}, \overline{X})} = P_{(\underline{Y}, \overline{Y})}$, that is, $(\underline{X}, \overline{X}) \sim_{st} (\underline{Y}, \overline{Y})$, and so, $X \sim_{st} Y$. □

Proposition 3.12. *The stochastic order $\tilde{\preceq}_I$ is closed under weak convergence.*

Proof. It follows from *iii*) in Proposition 3.7. □

Proposition 3.13. *Let X and Y be interval valued random mappings. Then, $X \tilde{\preceq}_I Y$ if and only if $-X \tilde{\preceq}_I -Y$.*

Proof. The partial order $\preceq^1$ on $\mathbb{R}$ satisfies that $a \preceq^1 b$ if and only if $-a \preceq^1 -b$. Let $A, B \in \mathcal{K}_c$. By Lemma 3.4, $A \preceq_I B$ is equivalent to $\underline{A} \preceq \underline{B}$ and $\overline{A} \preceq^1 \overline{B}$, which is the same as $-\underline{A} \preceq^1 -\underline{B}$ and $-\overline{A} \preceq^1 -\overline{B}$, that is, $-A \preceq_I -B$. Then, $A \preceq_I B$ if and only if $-A \preceq_I -B$.

By Proposition 3.7, the condition $X \tilde{\preceq}_I Y$ is equivalent to the existence of a probability space $(\Omega, \mathcal{A}, P)$ and interval valued random mappings $\widetilde{X}, \widetilde{Y} : \Omega \to \mathcal{K}_c$ with $\widetilde{X} \sim_{st} X$ and $\widetilde{Y} \sim_{st} Y$ such that $\widetilde{X} \preceq_I \widetilde{Y}$ a.s. $[P]$, equivalently, $-\widetilde{X} \preceq_I -\widetilde{Y}$ a.s. $[P]$, which is the same as $-X \tilde{\preceq}_I -Y$. □

The distance to the origin stochastic order for interval valued random mappings can be viewed as an order in concentration of probability outside the element $\{0\}$ of $\mathcal{K}_c$, as the following result shows.

Let $B_H(K, \varepsilon)$ be the closed ball in the Hausdorff distance, centred at $K \in \mathcal{K}_c$, and with radius equal to $\varepsilon \geq 0$. The superindex ^{c} will stand for the complementary set.

Proposition 3.14. *Let X and Y be interval valued random mappings such that $X \tilde{\preceq}_I Y$. For any $\varepsilon > 0$, it holds that $P(X \in B_H(\{0\}, \varepsilon)^c) \leq P(Y \in B_H(\{0\}, \varepsilon)^c)$.*

Proof. We will prove that the mapping $f_{\{0\},\varepsilon} : \mathcal{K}_c \to \mathbb{R}$, with $f_{\{0\},\varepsilon}(A) = I_{B_H(\{0\},\varepsilon)^c}(A)$ for any $A \in \mathcal{K}_c$, where I_* is the indicator function of the intervals in $*$, is $\preceq_I$-preserving and measurable. Note that this leads to the desired result after *iii*) in Proposition 3.7.

Let $A, B \in \mathcal{K}_c$ with $A \preceq_I B$, let us see that $f_{\{0\},\varepsilon}(A) \leq f_{\{0\},\varepsilon}(B)$.

The result is clear if $f_{\{0\},\varepsilon}(A) = 0$. Suppose that $f_{\{0\},\varepsilon}(A) = 1$. This means that $A \notin B_H(\{0\}, \varepsilon)$, that is, $d_H(\{0\}, A) = \max\{|\underline{A}|,\ |\overline{A}|\} > \varepsilon$. Let us see that $B \notin B_H(\{0\}, \varepsilon)$. Consider the following cases,

i) if $\overline{A} < 0$, then $d_H(\{0\}, A) = |\underline{A}| \leq |\underline{B}| = d_H(\{0\}, B)$,

ii) if $\underline{A} > 0$, then $d_H(\{0\}, A) = \overline{A} \leq \overline{B} = d_H(\{0\}, B)$,

iii) in any other case, we have that $|\underline{A}| \leq |\underline{B}|$ and $0 \leq \overline{A} \leq \overline{B}$, which implies that $d_H(\{0\}, A) = \max\{|\underline{A}|, \overline{A}\} \leq \max\{|\underline{B}|, \overline{B}\} = d_H(\{0\}, B)$.

Therefore, $f_{\{0\},\varepsilon}(B) = 1$, and so, $f_{\{0\},\varepsilon}$ is $\preceq_I$-preserving.

On the other hand, $B_H(K, \varepsilon) \in \tau_H \subset \sigma_H$, hence, the mapping $f_{\{0\},\varepsilon}$ is measurable. □

4. Connections with other stochastic orders

In this section, we state relations of the distance to the origin stochastic order with other stochastic orders for interval valued random mappings and with stochastic orders for random variables and vectors. For that, we include some stochastic orders for random variables, random vectors and interval valued random mappings.

Let X and Y be random variables, X is said to be smaller than Y in the *i*) usual stochastic order, denoted by $X \preceq_{st} Y$, if $E(f(X)) \leq E(f(Y))$ for all increasing mappings $f : \mathbb{R} \to \mathbb{R}$ such that the above expectations exist,

ii) bidirectional order, denoted by $X \preceq_{bd} Y$, if $X_+ \preceq_{st} Y_+$ and $X_- \preceq_{st} Y_-$ hold simultaneously, where given $a \in \mathbb{R}$, a_+ stands for the positive part of a, that is, $\max\{a, 0\}$, and a_- for the negative part, $\max\{-a, 0\}$ (see [17] and [18]).

Let X and Y be $\mathbb{R}^d$ valued random vectors, X is said to be smaller than Y in the

i) usual stochastic order, denoted by $X \preceq_{st} Y$, if $E(f(X)) \leq E(f(Y))$ for all increasing mappings $f : \mathbb{R}^d \to \mathbb{R}$ such that the above expectations exist,

ii) V-directional stochastic order, where $V = \{v_1, \ldots, v_l\}$ is a set of vectors in $\mathbb{R}^d$, if $E(f(X)) \leq E(f(Y))$ for any $f \in \mathcal{F}_V$ such that the above expectations exist, with $\mathcal{F}_V = \{f : \mathbb{R}^d \to \mathbb{R} \mid \ f(x + \epsilon v_i) \geq f(x) \text{ for all } x \in \mathbb{R}^d, \epsilon \geq 0, \ v_i \in V\}$, that is, the set of mappings which are increasing in the direction of the vectors in V. This relation will be denoted by $X \preceq_V Y$ (see [21]).

To introduce some stochastic orders for interval valued random mappings, the notion of expected value of those random elements is necessary.

Let $(\Omega, \mathcal{A}, P)$ be a probability space and $X : \Omega \to \mathcal{K}_c$ an interval valued random mapping. Take the mapping $\|X\| : \Omega \to \mathbb{R}$ given by $\|X\|(\omega) = \sup\{|x| \mid x \in X(\omega)\}$. If $\|X\| \in L^1(P)$, the Aumann expectation of X, denoted by $\mathbb{E}(X)$, is the set $\mathbb{E}(X) = [E(\underline{X}), E(\overline{X})]$.

Let X and Y be interval valued random mappings, then

i) X is said to be smaller than Y in the symmetric order, denoted by $X \preceq_{sym} Y$, if $\mathbb{E}(co(X \cup B_r)) \subset \mathbb{E}(co(Y \cup B_r))$ for any $r > 0$, where co stands for the convex hull (see [11]),

ii) X is said to be smaller than Y in the set linear convex order, denoted by $X \preceq_{slcx} Y$, if $\mathbb{E}(co(X^s \cup B_r)) \subset \mathbb{E}(co(Y^s \cup B_r))$ for any $s, r > 0$ (see [11]),

iii) X is said to be stochastically smaller than Y, denoted by $X \preceq_{strs} Y$, if there exist a probability space $(\Omega, \mathcal{A}, P)$ and interval valued random mappings $\widetilde{X}, \widetilde{Y} : \Omega \to \mathcal{K}_c$ with $\widetilde{X} \sim_{st} X$ and $\widetilde{Y} \sim_{st} Y$, such that $\widetilde{X} \subset \widetilde{Y}$ a.s. $[P]$ (see, for instance, [19]).

As we will see, the distance to the origin stochastic order is stronger than the symmetrically smaller stochastic order. Firstly, we develop the following result.

Lemma 4.1. *Let $A, B \in \mathcal{K}_c$ and let $r > 0$. If $A \preceq_I B$, then $co(A \cup B_r) \subset co(B \cup B_r)$.*

Proof. Let us consider the following cases,

i) $0 \leq \underline{A}$, in this case $co(A \cup B_r) = [-r, \max\{\overline{A}, r\}] \subset [-r, \max\{\overline{B}, r\}] = co(B \cup B_r)$, note that by Lemma 3.4 $\overline{A} \preceq^1 \overline{B}$, and so, $\overline{A} \leq \overline{B}$,

ii) $\overline{A} \leq 0$, now $co(A \cup B_r) = [\min\{\underline{A}, -r\}, r] \subset [\min\{\underline{B}, -r\}, r] = co(B \cup B_r)$, observe that Lemma 3.4 implies that $\underline{B} \leq \underline{A}$,

iii) $\underline{A} < 0 < \overline{A}$, note that $co(A \cup B_r) = [\min\{\underline{A}, -r\}, \max\{\overline{A}, r\}] \subset [\min\{\underline{B}, -r\}, \max\{\overline{B}, r\}] = co(B \cup B_r)$, in this case Lemma 3.4 assures that $\underline{B} \leq \underline{A}$ and $\overline{A} \leq \overline{B}$. □

Proposition 4.2. *Let X and Y be interval valued random mappings such that $X \precsim_I Y$. Then, $X \preceq_{sym} Y$.*

Proof. By Proposition 3.7, there exist a probability space $(\Omega, \mathcal{A}, P)$ and interval valued random mappings $\widetilde{X}, \widetilde{Y} : \Omega \to \mathcal{K}_c$ with $\widetilde{X} \sim_{st} X$ and $\widetilde{Y} \sim_{st} Y$, such that $\widetilde{X} \preceq_I \widetilde{Y}$ a.s. $[P]$.

Therefore, $co(X \cup B_r) \sim_{st} co(\widetilde{X} \cup B_r)$ and $co(Y \cup B_r) \sim_{st} co(\widetilde{Y} \cup B_r)$ for any $r > 0$. In accordance with Lemma 4.1, we obtain that $co(\widetilde{X} \cup B_r) \subset co(\widetilde{Y} \cup B_r)$ a.s. $[P]$, and so $\mathbb{E}(co(\widetilde{X} \cup B_r)) \subset \mathbb{E}(co(\widetilde{Y} \cup B_r))$, which proves the result. □

The converse of the above result is not true. Note that Remark 17 in [11] says that the symmetrically smaller order is not antisymmetric on the set of probabilities induced by interval valued random mappings.

In relation to the set linear convex order, there is not a general relation between this order and the distance to the origin order. The stochastic order $\preceq_{slcx}$ does not imply the order $\precsim_I$, note that $\preceq_{slcx}$ does not satisfy the antisymmetric property. On the other hand $\precsim_I$ does not imply $\preceq_{slcx}$, consider the constant interval valued random mappings $X = [-2, -1]$ and $Y = [-12, -10]$. It is clear that $X \precsim_I Y$. Take $s = 3$ and $r = 1$ in the definition of $\preceq_{slcx}$. It can be seen that $\mathbb{E}(co(X^3 \cup B_1)) = [-5, 2]$ and $\mathbb{E}(co(Y^3 \cup B_1)) = [-15, 1]$, thus $X \preceq_{slcx} Y$ is false.

Regarding the stochastic orders $\precsim_I$ and $\preceq_{strs}$, it can be seen that there is not a general relation between both. The following result states a connection under additional assumptions.

Proposition 4.3. *Let X and Y be interval valued random mappings such that $\underline{X} < 0 < \overline{X}$ a.s. Then, $X \precsim_I Y$ if and only if $X \preceq_{strs} Y$.*

Proof. If $X \precsim_I Y$, there exist a probability space $(\Omega, \mathcal{A}, P)$ and interval valued random mappings $\widetilde{X}, \widetilde{Y} : \Omega \to \mathcal{K}_c$ with $\widetilde{X} \sim_{st} X$ and $\widetilde{Y} \sim_{st} Y$, such that $\widetilde{X} \preceq_I \widetilde{Y}$ a.s. $[P]$.

We have that $\underline{X} \sim_{st} \underline{\widetilde{X}}$, $\overline{X} \sim_{st} \overline{\widetilde{X}}$ and so, $\underline{\widetilde{X}} < 0$ and $\overline{\widetilde{X}} > 0$ *a.s.* Since $\widetilde{X} \preceq_I \widetilde{Y}$ a.s., we obtain that $\widetilde{X} \subset \widetilde{Y}$ a.s. $[P]$, that is, $X \preceq_{strs} Y$.

The converse can be reasoned in a similar way. □

Next we relate the distance to the origin stochastic order with some stochastic orders for random vectors and random variables.

Firstly, we will see that under some conditions on the interval valued random mappings, the order $\precsim_I$ is equivalent to the directional order $\preceq_V$ applied to the endpoints of the intervals with $V = \{-e_1, e_2\}$.

The example below shows that, in general, the orders $\precsim^2$ and $\preceq_V$ are not the same.

Take the constant interval valued mappings $X=[0,3]$ and $Y=[5,7]$. It is immediate that $(\underline{X},\overline{X}) \preceq^2 (\underline{Y},\overline{Y})$, and so, $(\underline{X},\overline{X}) \tilde{\preceq}^2 (\underline{Y},\overline{Y})$.

Let $v=(5,7)-(0,3)=(5,4)$. Clearly enough, $v \notin C_V$ with C_V the set of all conical combinations of vectors in V, that is, $C_V=\{\alpha_1(-e_1)+\alpha_2 e_2 \mid \alpha_i \geq 0,\ 1 \leq i \leq 2\}$. By Proposition 3 in [21], there exists a linear mapping $\xi : \mathbb{R}^2 \to \mathbb{R}$ with $\xi(-e_1) \geq 0, \xi(e_2) \geq 0$ and $\xi(v) < 0$, that is, $\xi(5,7) < \xi(0,3)$. Thus, $\xi \in \mathcal{F}_V$, hence $(\underline{X},\overline{X}) \preceq_V (\underline{Y},\overline{Y})$ is false.

Proposition 4.4. *Let $(\underline{X},\overline{X}), (\underline{Y},\overline{Y})$ be random vectors such that $\underline{X},\underline{Y} \leq 0$ a.s. and $\overline{X},\overline{Y} \geq 0$ a.s, and consider $V=\{-e_1,e_2\}$. Then $(\underline{X},\overline{X}) \tilde{\preceq}^2 (\underline{Y},\overline{Y})$ if and only if $(\underline{X},\overline{X}) \preceq_V (\underline{Y},\overline{Y})$.*

Proof. Let $x=(\underline{x},\overline{x}), y=(\underline{y},\overline{y}) \in \mathbb{R}^2$. Define the relation $\leq_{C_V}$ on $\mathbb{R}^2$ given by $x \leq_{C_V} y$ when $y-x \in C_V$, which is equivalent to $\underline{y}-\underline{x} \leq 0$ and $\overline{y}-\overline{x} \geq 0$.

It is not hard to see that when $\underline{x},\underline{y} \leq 0$ and $\overline{x},\overline{y} \geq 0$, then $x \leq_{C_V} y$ if and only if $x \preceq^2 y$.

Assume that $(\underline{X},\overline{X}) \preceq_V (\underline{Y},\overline{Y})$. Note that C_V does not contain non-trivial subspaces of $\mathbb{R}^2$ and so $\leq_{C_V}$ is an order. By Proposition 9 in [21], that order is closed. Applying Corollary 1 of that reference, there are random vectors $(\underline{X}',\overline{X}'), (\underline{Y}',\overline{Y}')$ on the same probability space, with $(\underline{X},\overline{X}) \sim_{st} (\underline{X}',\overline{X}')$ and $(\underline{Y},\overline{Y}) \sim_{st} (\underline{Y}',\overline{Y}')$, such that $(\underline{X}',\overline{X}') \leq_{C_V} (\underline{Y}',\overline{Y}')$ *a.s.*, which is the same as $(\underline{X}',\overline{X}') \preceq^2 (\underline{Y}',\overline{Y}')$ *a.s.* and so, $(\underline{X},\overline{X}) \tilde{\preceq}^2 (\underline{X},\overline{X})$ by Proposition 3.9 of the present manuscript.

The converse is analogous. □

The following connections of the order $\tilde{\preceq}_I$ with the usual multivariate stochastic order can be developed now.

Proposition 4.5. *Let X and Y be interval valued random mappings such that $\underline{X},\underline{Y} \leq 0$ a.s. and $\overline{X},\overline{Y} \geq 0$ a.s. Then $X \tilde{\preceq}_I Y$ if and only if $(-\underline{X},\overline{X}) \preceq_{st} (-\underline{Y},\overline{Y})$.*

Proof. Observe that $X \tilde{\preceq}_I Y$ is the same as $(\underline{X},\overline{X}) \tilde{\preceq}^2 (\underline{Y},\overline{Y})$ by Theorem 3.10. That is equivalent to $(\underline{X},\overline{X}) \preceq_V (\underline{Y},\overline{Y})$ with $V=\{-e_1,e_2\}$ by Proposition 4.4,

Take the linear mapping $h : \mathbb{R}^2 \to \mathbb{R}^2$ with $h(-e_1)=e_1$ and $h(e_2)=e_2$. By Proposition 7 in [21], $(\underline{X},\overline{X}) \preceq_V (\underline{Y},\overline{Y})$ if and only if $h(\underline{X},\overline{X}) \preceq_{\widehat{V}} h(\underline{Y},\overline{Y})$, with $\widehat{V}=\{e_1,e_2\}$. Note that $\preceq_{\widehat{V}}$ is the usual multivariate stochastic order $\preceq_{st}$, which proves the result. □

Proposition 4.6. *Let X and Y be interval valued random mappings such that $\underline{X},\underline{Y} \leq 0$ a.s., $\overline{X},\overline{Y} \geq 0$ a.s. and $(\underline{X},\overline{X})$ and $(\underline{Y},\overline{Y})$ have the same copula. Then, $X \tilde{\preceq}_I Y$ if and only if $\underline{Y} \preceq_{st} \underline{X}$, $\overline{X} \preceq_{st} \overline{Y}$.*

Proof. Note that $(-\underline{X},\overline{X})$ and $(-\underline{Y},\overline{Y})$ have the same copula. By Theorem 3.3.8 in [5], $(-\underline{X},\overline{X}) \preceq_{st} (-\underline{Y},\overline{Y})$, and so the result follows from Proposition 4.5. □

If W and Z are random variables, $\{W\}$ and $\{Z\}$ are interval valued random mappings. The following result shows that the order $\tilde{\preceq}_I$ is an extension of the order $\preceq_{bd}$ on the space of singleton valued random mappings.

Proposition 4.7. *Let W and Z be random variables. Then $\{W\} \tilde{\preceq}_I \{Z\}$ if and only if $W \preceq_{bd} Z$.*

Proof. By Proposition 3.10, $\{W\} \tilde{\preceq}_I \{Z\}$ is the same as $(W,W) \tilde{\preceq}^2 (Z,Z)$. By Proposition 3.9, this is equivalent to the existence of a probability space $(\Omega,\mathcal{A},P)$ and random variables $\widetilde{W},\widetilde{Z} : \Omega \to \mathbb{R}$ with $\widetilde{W} \sim_{st} W$ and $\widetilde{Z} \sim_{st} Z$ and $\widetilde{W} \preceq^1 \widetilde{Z}$ *a.s.* Note that $\widetilde{W} \preceq^1 \widetilde{Z}$ *a.s.* is the same as $\widetilde{W}_- \leq \widetilde{Z}_-$ *a.s.* and $\widetilde{W}_+ \leq \widetilde{Z}_+$ *a.s.* By Proposition 2 in [18], this is the equivalent to $W \preceq_{bd} Z$. □

5. Applications to climate and economic data analysis

The $\tilde{\preceq}_I$ order is illustrated below by means of two examples with weather and economic interval valued data. The weather example appears first and is explained in a greater detail, while only the parts that are new with respect to it are discussed in the economic example.

5.1. Comparison of continental and oceanic climates in terms of yearly temperature range

Continental climates are characterized by a substantial annual variation in temperature, having cold winters and hot summers, while oceanic climates have mild summers and cool winters, so their annual variation in temperature is somewhat narrow. We use the $\tilde{\preceq}_I$ order to compare the yearly temperatures in Madrid (central Spain, continental climate) and Oviedo (northern Spain, oceanic climate). Specifically, we compare the yearly temperature range centred by the yearly average temperature (average of the daily midpoints between absolute maximum and minimum temperatures) in both locations over the 25-year period comprised between 1996 and 2020. Data were taken from the information dissemination system of the Spanish Meteorological Agency (AEMET OpenData) https://opendata.aemet.es and correspond to weather stations 3195 in Madrid (Retiro Park) and 1249I in Oviedo.

In Fig. 1, we plot the raw collected data as a time-series. Madrid temperatures are in black, while those of Oviedo are in grey. The solid lines correspond to the extreme temperatures (yearly absolute maximum and minimum temperatures), while the dotted lines are the yearly average temperatures. The temperature variation range is the interval ranging between the absolute minimun minus the average temperature and the absolute maximum minus the average temperature.

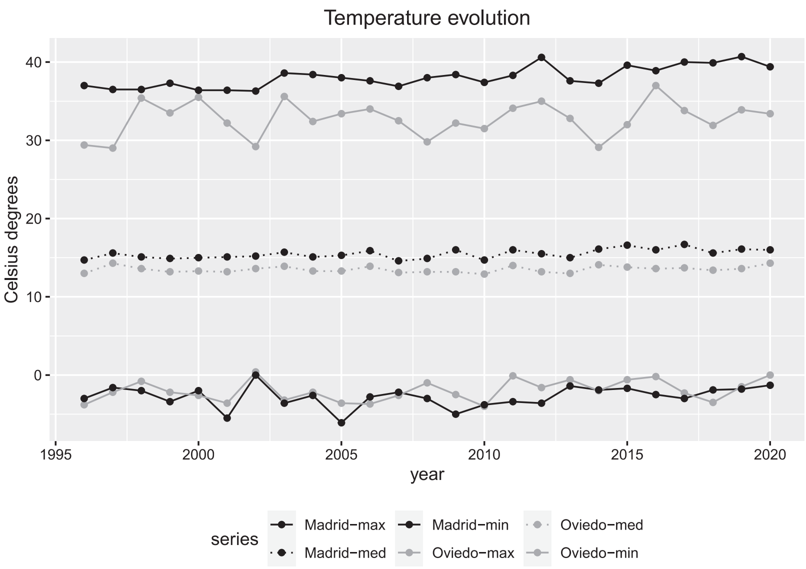

Fig. 1. Raw temperature data.

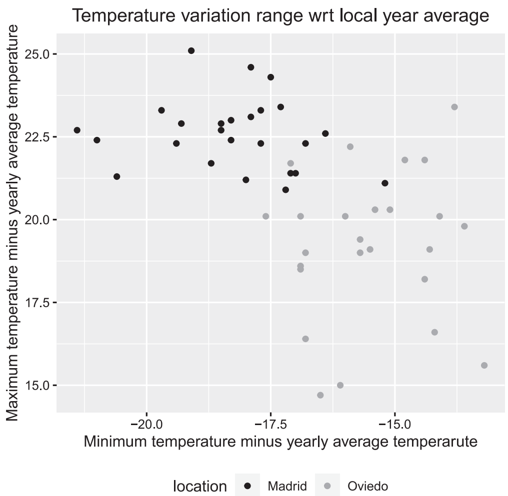

Fig. 2. Temperature variation range obtained as local extreme temperatures minus average temperature.

In Fig. 2, we present a scatterplot of the temperature variation range for each one of the 25 selected years with its left endpoint in the X axis and its right endpoint in the Y axis. As before, Madrid observations are in black, while those of Oviedo are in grey. We check that there is no structure in neither of the two time series of intervals by means of the global serial independence test for multivariate time series proposed by [22] which renders p-values of 0.5089 for the Madrid data and 0.7393 for the Oviedo data (all subsets up to lag 10 and 10^4 permutations were considered), so both can be assumed to be serially independent.

In order to compare the two series of temperatures by means of the $\tilde{\preceq}_I$ order, we make use of Proposition 4.6. First, we check that the bivariate random vectors given by the endpoints of the temperature variation ranges share a common copula. With such an objective, we have used the test proposed by [23] obtaining a p-value equal to 0.2698 (10^6 replicates), so we cannot reject that they indeed share the same copula. We have also run an independence test on each of the two datasets, and the endpoints of the variation range of both of them can be assumed to be independent (respective p-values equal to 0.2837 in Madrid and 0.5817 in Oviedo).

Next, we compare the lower and upper endpoints of the temperature variation ranges in Madrid and Oviedo (whose respective empirical cumulative distribution functions are plotted in Fig. 3) with respect to the usual stochastic order. In order to confirm that each endpoint of the temperature variation range in Madrid is, in absolute value, stochastically greater than the corresponding endpoint of the temperature variation range in Oviedo in the usual stochastic order, we run two one-

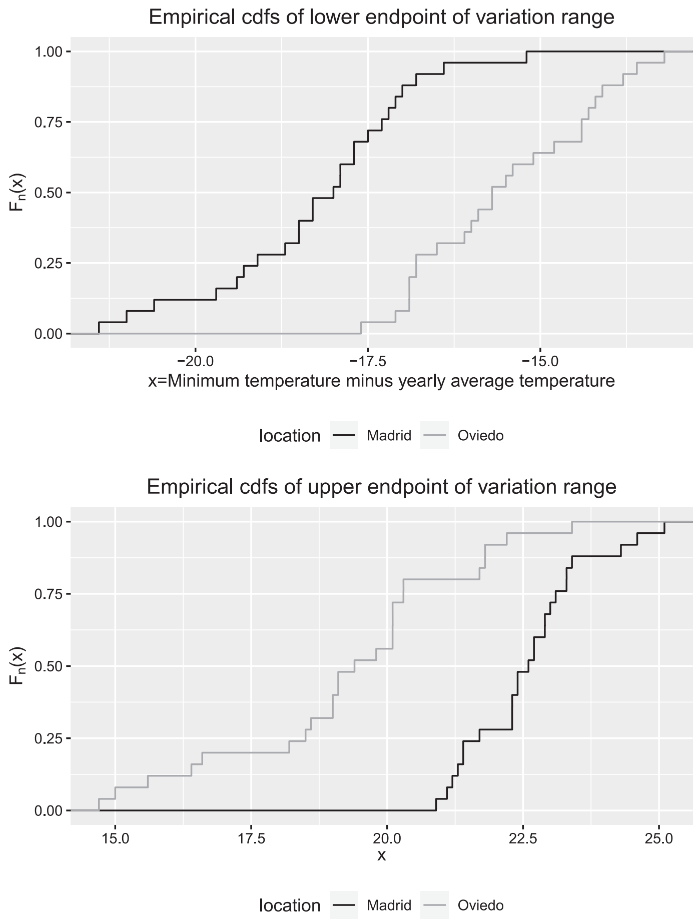

Fig. 3. Empirical cumulative distribution functions of lower (top) and upper (bottom) endpoints of temperature variation range.

sided Kolmogorov-Smirnov tests with the inequality that we want to prove in the alternative hypothesis, as suggested in [24, Sec. 3 - .Homogeneity against stochastic order]. Since observations come in pairs (each one corresponding to one specific year), both tests are run as permutation ones (with 10^6 permutations), obtaining p-values equal to 0, which shows strong evidence that the upper enpoint of the temperature variation range in Madrid is stochastically greater in the usual stochastic order than the one in Oviedo, while the lower endpoint of the temperature variation range in Madrid is stochastically smaller in the usual stochastic order than the one in Oviedo.

Finally, since the conditions of having a common copula and the usual stochastic order for the endpoints of the random intervals are fulfilled, after Proposition 4.6, we conclude that the temperature range in Madrid is stochastically greater than the one in Oviedo in the $\precsim_I$ order. In plain words, the extreme temperatures of Madrid are more distant to the yearly local average temperature than those of Oviedo.

5.2. Comparison of bitcoin and euro in terms of weekly log-returns

Since the volatility levels of digital currencies are usually much higher than the ones of classical currencies, we have decided to compare the Bitcoin and Euro weekly log-regurns over year 2020. Specifically, we have taken the weekly opening, closing, instant minimum, and instant maximum price of Bitcoin (BTC) and Euro (EUR) in United States Dollar (USD) over the 52 weeks of year 2020 from the dashboards for historical data BTC/USD - Bitcoin US Dollar[1] and EUR/USD - Euro US Dollar[2] With such prices, we have built the weekly log-return ranges of the two currencies and compared them in the $\precsim_I$ order.

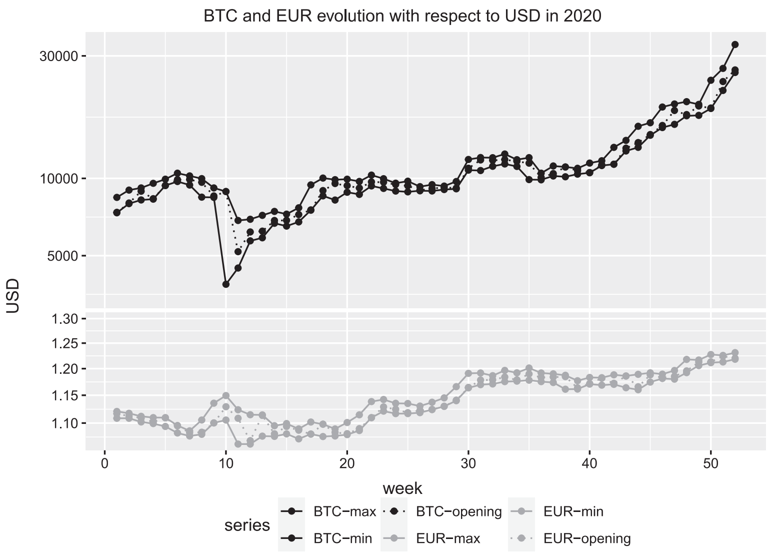

Fig. 4. Weekly maximum and minimum Bitcoin and Euro prices (solid line) and opening price (dotted line) throughout 2020.

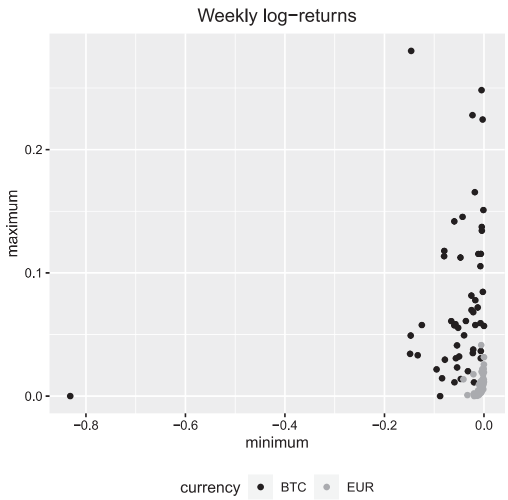

Fig. 5. Bivariate representation of the weekly log-return range.

In Fig. 4, we present the weekly maximum and minimum prices of Bitcoin and Euro (in USD) throughout the 52 weeks of 2020 in solid lines (black for Bitcoin and grey for Euro), together with their opening prices in a dotted line. A base-10 logarithmic scale is used for the Y axis, which is presented with a break between values 1.3 and 3500.

The weekly opening price is used together with the maximum and minimum prices to obtain the weekly log-return ranges, which are presented in Fig. 5 in a scatterplot with the minimum in the X axis and the maximum in the Y axis. We check that, with significance level 5%, both sequences of bivariate observations are serially independent, so it is possible to assume that there is no time structure. The respective p-values for the global serial independence test for multivariate time series proposed by [22] are 0.6369 for the Euro data and 0.0907 for the Bitcoin data (all subsets up to lag 10 and 10^4 permutations were considered).

As in the previous example, our goal is to confirm the $\tilde{\preceq}_I$ order between the considered random intervals by means of Proposition 4.6. Firstly, we check that the minimum and maximum log-returns of the two currencies share a common copula (p-value equal to 0.3279 with 10^6 replicates with the test proposed by [23]). Unlike in the previous example, the endpoints of the range (minimum and maximum log-returns) are not independent, showing a positive association (both endpoints of the range increase at the same time) that is already aparent in their scatterplots, see Fig. 5.

[1] https://www.investing.com/crypto/bitcoin/btc-usd-historical-data.
[2] https://www.investing.com/currencies/eur-usd-historical-data.

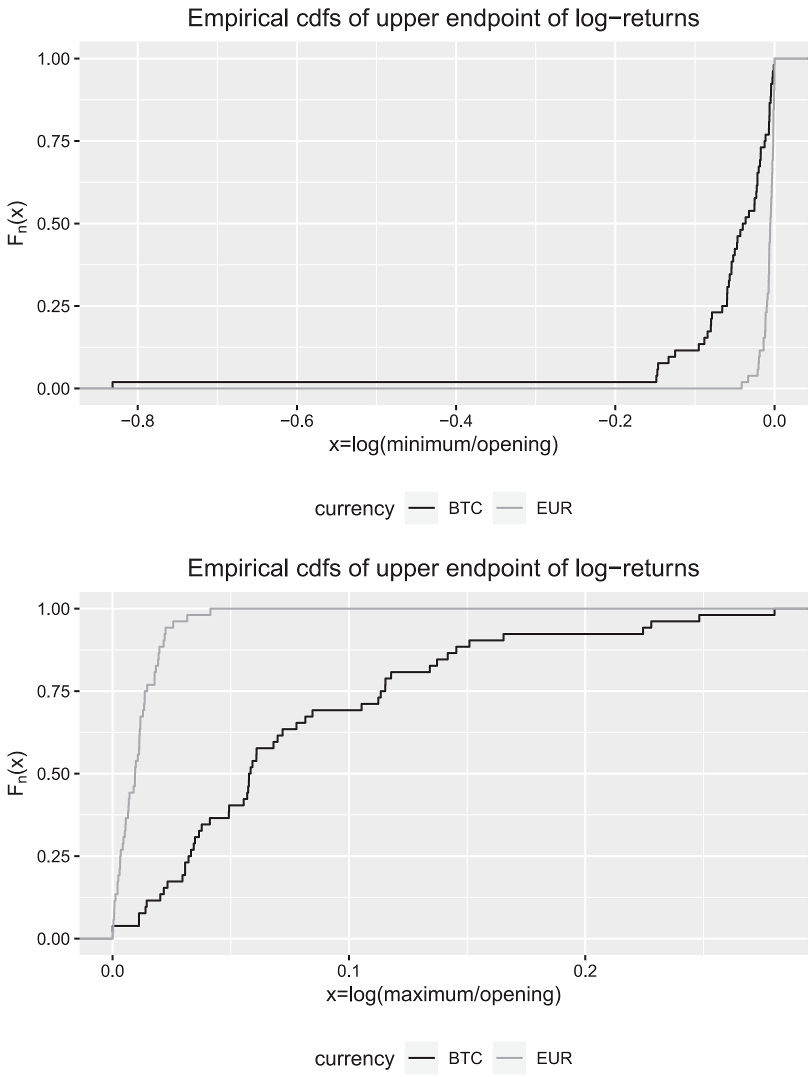

Fig. 6. Empirical cumulative distribution functions of the minimum (top) and maximum (bottom) of the log-return range.

Then, we check the orderings in terms of the usual stochastic order. The empirical cumulative distribution functions of the minimum and maximum log-returns are plotted in Fig. 6. A permutation version (with 10^6 permutations) of the one-sided Kolmogorov-Smirnov test was used to deal with the paired data situation that we face, rendering respective p-values of 0 for the minimum log-return and 0.0296 for the maximum log-return. In conclusion, there is enough evidence, at 5% significance level, to conclude that that the maximum log-return of the Bitcoin is stochastically greater in the usual stochastic order than the maximum log-return of the Euro and the minimun log-return of the Bitcoin is stochastically smaller in the usual stochastic order than the minimum log-return of the Euro.

Finally, after Proposition 4.6, we conclude that the log-return range of the Bitcoin is stochastically greater than the one of the Euro in the $\tilde{\preceq}_I$ order. The extreme weekly log-returns of the Bitcoin are thus more distant to zero than those of the Euro. Note that the log-return is positive (respectively negative) when the price increases (resp. decreases) from one week to the next, while it is zero when there is no variation. Our conclusion is coherent with the empirical observation that the Bitcoin has a higher volatility than the Euro.

Observe that, in the light of Proposition 4.4, our previous conclusion means that the random vectors $(\underline{X}, \overline{X})$ and $(\underline{Y}, \overline{Y})$, which respectively represent the endpoints of the log-return range of the Euro and Bitcoin, satisfy $E(f(\underline{X}, \overline{X})) \leq Ef(f(\underline{Y}, \overline{Y}))$ for every f increasing mapping in the direction of $-e_1$ and e_2 for which both expectations exist.

6. Conclusions

When working with interval data, the analysis of the closeness or distance of the endpoints of the interval with respect to a reference point, may be of great usefulness in many applied fields. This manuscript provides a mathematical model for such an analysis in terms of a stochastic for interval valued random mappings. A key advantage of the model is the characterization of that order by means of a new stochastic order for bivariate random vectors when this new order is applied to the endpoints of the interval valued mappings. Such a characterization has enabled us to obtain conditions on the random variables given by the endpoints of the intervals, which lead to the ordering of the corresponding interval mappings. These conditions can be analyzed from an inferential point of view, which permits the use of statistical inference techniques to test the proposed stochastic order. All of that has been illustrated in the manuscript with some examples regarding weather and economic interval valued data.

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References

[1] X. Han, J. Yang, A two-step method for interpolating interval data based on cubic hermite polynomial models, Appl. Math. Model. 81 (2020) 356–371.
[2] X. Zhang, B. Beranger, S.A. Sisson, Constructing likelihood functions for interval-valued random variables, Scand. J. Stat. 47 (2020) 1–35.
[3] A. Hatami-Marbini, A. Emrouznejad, P.J. Agrell, Interval data without sign restrictions in DEA, Appl. Math. Model. 38 (2014) 2028–2036.
[4] C. Wang, Z. Qiu, X. Zhiping, L. Menghui, Y. Li, Novel reliability-based optimization method for thermal structure with hybrid random, interval and fuzzy parameters, Appl. Math. Model. 47 (2017) 573–586.
[5] A. Müller, D. Stoyan, Comparison methods for stochastic models and risks, John Wiley & Sons, Chichester, 2002.
[6] M. Shaked, J.G. Shanthikumar, Stochastic order, Springer, New York, 2007.
[7] F. Belzunce, C. Martínez-Riquelme, J. Mulero, An introduction to stochastic orders, Elsevier/Academic Press, Amsterdam, 2016.
[8] H. Stoyan, D. Stoyan, On some partial orderings of random closed sets, Math. Operationsforsch, Statist. Ser. Optim. 11 (1980) 145–154.
[9] T. Rolski, R. Szekli, Stochastic ordering and thinning of point processes, Stochastic Process. Appl. 37 (1991) 299–312.
[10] T. Norberg, On the existence of ordered couplings of random sets with applications, Israel J. Math. 77 (1992) 241–264.
[11] I. Cascos, I. Molchanov, A stochastic order for random vectors and random sets based on the aumann expectation, Statist. Probab. Lett. 63 (2003) 295–305.
[12] C. Carleos, M.C. López-Díaz, M. López-Díaz, A stochastic order of shape variability with an application to cell nuclei involved in mastitis, J. Math. Image Vis. 38 (2010) 95–107.
[13] C. Carleos, M.C. López-Díaz, M. López-Díaz, Ranking star-shaped valued mappings with respect to shape variability, J. Math. Image Vis. 43 (2014) 1–12.
[14] J. Neggers, H.S. Kim, Basic posets, World Scientific Publishing, Singapore, 1998.
[15] B.S.W. Schröder, Ordered Sets, An introduction, Brikhäuser, Boston-Basel-Berlin, 2003.
[16] I. Molchanov, Limit Theorems for Unions of Random Closed Sets, Lecture Notes in Mathematics, 1561, Springer-Verlag, Berlin, 1993.
[17] A. Müller, Another tale of two tails: on characterizations of comparative risk, J. Risk Uncertain. 16 (1998) 187–197.
[18] M. López-Díaz, A stochastic order for random variables with applications, Aust. N. Z. J. Stat. 52 (2010) 1–16.
[19] I. Molchanov, Theory of random sets, Probability and its Applications (New York), Springer-Verlag, London, Ltd., London, 2005.
[20] T. Kamae, U. Krengel, G.L. O'Brien, Stochastic inequalities on partially ordered spaces, Ann. Probab. 5 (1977) 899–912.
[21] M.C. López-Díaz, M. López-Díaz, S. Martínez-Fernández, Directional stochastic orders with an application to financial mathematics, Mathematics 9 (380) (2021) 1–11.
[22] I. Kojadinovic, J. Yan, Tests of serial independence for continuous multivariate time series based on a möbius decomposition of the independence empirical copula process, Ann. Inst. Stat. Math. 63 (2011) 347–373.
[23] B. Remillard, O. Scaillet, Testing for equality between two copulas, J. Multivar. Anal. 100 (2009) 377–386.
[24] K. Mosler, Testing Whether Two Distributions Are Stochastically Ordered or Not, in: H. Rinne, B. Rüger, H. Strecker (Eds.), Grundlagen der Statistik und ihre Anwendungen: Festschrift für Kurt Weichselberger, Physica-Verlag HD, Heidelberg, 1995, pp. 149–155.