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Brauer character degrees and Sylow normalizers

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Abstract
If p and q are primes, and G is a p-solvable finite group, it is possible to detect that a q-Sylow normalizer is contained in a p-Sylow normalizer using the character table of G. This is characterized in terms of the degrees of p-Brauer characters. Some consequences, which include yet another generalization of the Itô–Michler theorem, are also obtained.

Keywords Character degrees · Brauer characters · Sets of primes

Mathematics Subject Classification 20C15 · 20C20

1 Introduction

If G is a finite group and π is a set of primes, let $\mathrm{Irr}_\pi(G)$ be the subset of the irreducible complex characters χ of G such that all the primes dividing the degree $\chi(1)$ lie in π. It is fair to say that the interaction between $\mathrm{Irr}_\pi(G)$ and the structure of G is one of the

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recurrent problems in character theory. Recently, in [12], we have asked if it is possible to characterize group-theoretically when $\mathrm{Irr}_\pi(G) = \mathrm{Irr}_\rho(G)$ for sets of primes π and ρ.

In this paper, we fix a prime p, and we turn our attention to p-Brauer characters, within the universe of finite p-solvable groups. (Degrees of modular representations in characteristic p outside p-solvable groups are usually deemed an intractable subject.) If G is a p-solvable finite group and $\mathrm{IBr}(G)$ is the set of the irreducible p-Brauer characters of G, we let $\mathrm{IBr}_\pi(G)$ be the subset of $\varphi \in \mathrm{IBr}(G)$ such that all the primes dividing $\varphi(1)$ lie in π. As usual, if r is a prime, r' denotes the set of primes different from r.

The following is our first main result.

Theorem A *Let G be a finite p-solvable group, and let q be a prime. Then, $\mathrm{IBr}_{q'}(G) \subseteq \mathrm{IBr}_{p'}(G)$ if and only if there are $Q \in \mathrm{Syl}_q(G)$ and $P \in \mathrm{Syl}_p(G)$ such that $\mathbf{N}_G(Q) \subseteq \mathbf{N}_G(P)$.*

Theorem A was the main result of [1], assuming that G is both p-solvable *and* q-solvable. To better understand our this new situation, we invite the reader to consider, for instance, $G = \mathrm{PSL}_2(3^5).5$, for $p = 5$. If $q = 2$, then $\mathbf{N}_G(P) = \mathbf{N}_G(Q)$ for some $p \in \mathrm{Syl}_p(G)$ and $Q \in \mathrm{Syl}_q(G)$. Of course, G is p-solvable, but not q-solvable. There are families of almost-simple groups like this, and we are able to deal with this new situation thanks to the main result of [6]. In particular, the proof of Theorem A depends on the Classification of Finite Simple Groups.

From Theorem A, and using McKay bijections for Brauer characters in p-solvable groups together with the recent proof of the divisibility of degrees between Glauberman correspondents by M. Geck in [2], we can prove our second main result.

Theorem B *Let G be a finite p-solvable group, and let q be a prime different from p. Then $\mathrm{IBr}_{p'}(G) = \mathrm{IBr}_{q'}(G)$ if and only if there are $P \in \mathrm{Syl}_p(G)$ and $Q \in \mathrm{Syl}_q(G)$ such that $\mathbf{N}_G(P) = P\mathbf{N}_G(Q)$ and Q is abelian.*

As the reader can easily check, in the *trivial* case where p does not divide $|G|$, Theorem B is yet another restatement of the Itô–Michler theorem.

As happens with complex irreducible characters (see [12]), it does not seem easy to group-theoretically characterize when $\mathrm{IBr}_\pi(G) = \mathrm{IBr}_\rho(G)$ for arbitrary sets of primes π and ρ, even if G is p-solvable. In the case, for instance, where $\pi = \mathbb{P}$ is the set of all primes and $\rho = q'$, where q is a prime, this constitutes Problem 3.2 of [9]. (This problem was studied long before in [8] and more recently in [5].) It has now been conjectured that $\mathrm{IBr}_{q'}(G) = \mathrm{IBr}_{\mathbb{P}}(G)$ if and only if the number of p-regular classes of G is the number of p-regular classes of $\mathbf{N}_G(Q)/Q'$, where $Q \in \mathrm{Syl}_q(G)$. This is a consequence of the Inductive McKay conjecture, and it seems difficult to obtain a direct proof. (See Conjecture D in [7].)

2 Proof of theorem A

Our notation for Brauer characters follows [9]. The deepest part of the proof of Theorem A comes from the main result in [MN].

Theorem 2.1 *Let G be a finite π-separable group. Let H be a Hall π-subgroup, let K be a π-complement of G, and let q be a prime. Then, every $\alpha \in \mathrm{Irr}_{q'}(H)$ extends to G if and only if there is $Q \in \mathrm{Syl}_q(H)$ such that $\mathbf{N}_G(Q) \subseteq \mathbf{N}_G(K)$.*

Proof This is Theorem A of [6]. □

A very useful result to deal with the hypotheses in this paper appears in Suzuki's book.

Theorem 2.2 *Let G be a p-solvable group, and let q be a prime. Let $\pi = \{p, q\}$. Then, G has a unique conjugacy class of Hall π-subgroups, and every π-subgroup of G is contained in one of them.*

Proof This follows from 5.3.13 of [14]. □

For the reader's convenience, let us prove the following standard result.

Lemma 2.3 *Let G be a finite group. Let Q be a Sylow q-subgroup of G and let N be a normal subgroup of G. If $\varphi \in \mathrm{IBr}_{q'}(G)$, then φ_N has a Q-invariant irreducible constituent and any two of them are $\mathbf{N}_G(Q)$-conjugate.*

Proof Let $\varphi \in \mathrm{IBr}_{q'}(G)$ and let $\theta \in \mathrm{IBr}(N)$ be an irreducible constituent of φ_N. Let G_θ be the stabilizer of θ in G. By Clifford's theorem (see Corollary 8.9 of [9], for instance), we have that $|G : G_\theta|$ divides $\varphi(1)$, and hence, it is a q'-number. It follows that there is an element $g \in G$ such that Q^g is contained in G_θ. Hence, $Q \subseteq G_\theta^{g^{-1}} = G_{\theta^{g^{-1}}}$ and $\theta^{g^{-1}}$ is a Q-invariant irreducible constituent of φ_N.

Now, suppose that θ and μ are two Q-invariant irreducible constituents of φ_N. Again by Clifford's theorem we have that there is an element $g \in G$ such that $\mu = \theta^g$. It follows that Q and Q^g are contained in G_μ, and hence, there is an element $x \in G_\mu$ such that $Q = (Q^g)^x$. Therefore $gx \in \mathbf{N}_G(Q)$ and $\theta^{gx} = (\theta^g)^x = \mu^x = \mu$, as wanted. □

The following result implies Theorem A. In its proof, we shall use Fong characters of Brauer characters, and we refer the reader to Chapter 10 of [9] for their main properties. (The term *Fong character* was coined by I. M. Isaacs after some results of P. Fong.) We also use the fact that if G is p-solvable, H is a p-complement of G, and $\varphi \in \mathrm{IBr}(G)$ has degree not divisible by p, then $\varphi_H \in \mathrm{Irr}(H)$. (See Theorem 10.9 of [9].) A complication when dealing with Brauer characters is that we do not have any form of Frobenius reciprocity, even in favorable conditions. For instance, in the previous situation where H is a p-complement of a p-solvable group G, if $\alpha \in \mathrm{Irr}(H)$, and φ is an irreducible constituent of p'-degree of the induced Brauer character α^G, then α needs not be the irreducible character φ_H; a fact that would simplify our proof below. (If $G = \mathsf{A}_4$, $p = 2$, and $H = \mathsf{C}_3$, then the Brauer character $(1_H)^G = 21_G + \lambda_1 + \lambda_2$, where λ_i are distinct linear Brauer characters. Now take $\alpha = 1_H$ and $\varphi = \lambda_i$.)

Theorem 2.4 *Let G be a p-solvable group, let $\mathrm{IBr}(G)$ be the set of irreducible Brauer characters of G, and let $q \neq p$ be a prime. Let $P \in \mathrm{Syl}_p(G)$ and $Q \in \mathrm{Syl}_q(G)$ such that $U = PQ$ is a subgroup of G. Suppose that H is a p-complement of G containing Q. Then, the following are equivalent.*

(a) $\mathrm{IBr}_{q'}(G) \subseteq \mathrm{IBr}_{p'}(G)$.
(b) $\mathbf{N}_G(Q) \subseteq \mathbf{N}_G(P)$.
(c) *Every* $\alpha \in \mathrm{Irr}_{q'}(H)$ *extends to* G.

Proof Set $\pi = \{p, q\}$. By Theorem 2.2, notice that we can find $P \in \mathrm{Syl}_p(G)$ and $Q \in \mathrm{Syl}_q(G)$ such that $U = PQ$ is a Hall π-subgroup of G. (In fact, given a Hall π-subgroup U of G, then $U = PQ$ for every $P \in \mathrm{Syl}_p(U)$ and $Q \in \mathrm{Syl}_q(U)$.)

Assume (a). We prove (b) by induction on $|G|$. If K is a minimal normal subgroup of G, then we have that $\mathbf{N}_G(Q) \subseteq \mathbf{N}_G(P)K$, by using induction in G/K. Since G is p-solvable, then K is either a p-group or a p'-group. In the first case, $\mathbf{N}_G(P)K = \mathbf{N}_G(P)$ and we are done. So we assume that K is a p'-group.

Let V be any subgroup of G containing KQ and let $\delta \in \mathrm{IBr}_{q'}(V)$. By Lemma 2.3, there exists a Q-invariant irreducible constituent $\tau \in \mathrm{IBr}(K)$ of the restriction δ_K (using that $Q \in \mathrm{Syl}_q(V)$). We claim that τ is also P-invariant. By Corollary 8.7 of [9], we have that δ is an irreducible constituent of the induced Brauer character τ^V. Now consider the Brauer character δ^G, which has degree $|G : V|\delta(1)$, which is not divisible by q. Hence, there exists an irreducible constituent $\varphi \in \mathrm{IBr}(G)$ of δ^G of degree not divisible by q. By hypothesis, we have that φ has degree not divisible by p. Now, φ is an irreducible constituent of τ^G, and therefore, τ is an irreducible constituent of the restriction φ_K, again by Corollary 8.7 of [9]. If $I = G_\tau$ is the stabilizer of τ in G, and $\mu \in \mathrm{IBr}(I)$ is the Clifford correspondent of φ over τ, we have that $|G : I|$ is a π'-number. Therefore, using Theorem 2.2, we have that $U \subseteq I^g$ for some $g \in G$. Then, τ and τ^g are Q-invariant constituents of φ, and thus, by Lemma 2.3, we have that $\tau^g = \tau^x$, for some $x \in \mathbf{N}_G(Q) \subseteq K\mathbf{N}_G(P)$. Hence, we may assume that $\tau^g = \tau^y$ for some $y \in \mathbf{N}_G(P)$. Then, $I^{gy^{-1}} = I$, $U^{y^{-1}} \subseteq I$, and we conclude that $P = P^{y^{-1}} \subseteq I$. In other words, τ is P-invariant, as claimed.

Now, let $V = \mathbf{N}_G(P)K$. We claim that $\mathrm{IBr}_{q'}(V) \subseteq \mathrm{IBr}_{p'}(V)$. Let $\delta \in \mathrm{IBr}_{q'}(V)$. By the claim in the previous paragraph, let $\tau \in \mathrm{IBr}(K)$ be PQ-invariant under δ. By Theorem 8.11 of [9], there is a unique $\hat{\tau} \in \mathrm{IBr}(KP)$ over τ, extending τ, and we conclude that δ lies over $\hat{\tau}$. Now, $KP \lhd V$, V/KP is a p'-group, and we have that $\delta(1)/\hat{\tau}(1)$ divides $|V : KP|$ by Theorem 8.30 of [9]. Since $\tau(1)$ is not divisible by p, we conclude that δ has p'-degree, as claimed. By induction, we may assume that $V = G$. That is, $KP \lhd G$. We want to use Theorem 2.1.

Next, we show that every $\alpha \in \mathrm{Irr}_{q'}(H)$ extends to G. By Lemma 2.3, let $\tau \in \mathrm{Irr}(K)$ be Q-invariant under α. By the claim in the third paragraph, we have that τ is P-invariant too. By Corollary 6.28 of [3], we have that τ has a canonical extension $\gamma \in \mathrm{Irr}(KP)$. Using the uniqueness of γ, we easily check that $G_\gamma \cap H = H_\tau$. By Isaacs restriction theorem (Lemma 6.8(d) of [11]), the Clifford correspondence and Mackey's theorem (Theorem 1.16 of [10]), we have that α extends to G. If $\pi_0 = p'$ is the set of primes dividing $|G|$ different from p, by Theorem 2.1 applied to π_0, we conclude that there is $Q_1 \in \mathrm{Syl}_q(H)$ such that $\mathbf{N}_G(Q_1) \subseteq \mathbf{N}_G(P)$. In particular, $P \lhd PQ_1$. By Theorem 2.2, PQ_1 and PQ are G-conjugate. Hence $P \lhd PQ$ and therefore $Q, Q_1 \in \mathrm{Syl}_q(\mathbf{N}_G(P))$. Hence, $Q_1^z = Q$ for some $z \in \mathbf{N}_G(P)$, and $\mathbf{N}_G(Q) = \mathbf{N}_G(Q_1)^z \subseteq \mathbf{N}_G(P)$.

We have that (b) implies (c), by Theorem 2.1 applied to $\pi_0 = p'$.

Finally, we prove that (c) implies (a) by using Fong characters. Suppose now that every $\alpha \in \mathrm{Irr}_{q'}(H)$ extends to G. We show that $\mathrm{IBr}_{q'}(G) \subseteq \mathrm{IBr}_{p'}(G)$. Let $\varphi \in \mathrm{IBr}_{q'}(G)$, and let $\alpha \in \mathrm{Irr}(H)$ be an irreducible constituent of φ_H such that $\alpha(1) = \varphi(1)_{p'}$ (using Theorem 10.18 of [9].) In other words, α is a Fong character for φ. Then, $\alpha(1)$ is not divisible by q, and by hypothesis, we have that α extends to some $\chi \in \mathrm{Irr}(G)$. Then, the Brauer character $\mu = \chi^0 \in \mathrm{IBr}(G)$ extends α. (Indeed, if $\chi^0 = \varphi_1 + \varphi_2$ for Brauer characters φ_i of G, then the irreducible character α would be written as $(\varphi_1)_H + (\varphi_2)_H$.) By Theorem 10.17 of [9], we have that $\varphi = \mu$, and $\varphi(1) = \mu(1) = \alpha(1)$ has degree not divisible by p. □

3 Proof of theorem B

If N is a normal subgroup of G and $\theta \in \mathrm{IBr}(N)$, we write $\mathrm{IBr}(G \mid \theta)$ to denote the set of irreducible Brauer characters φ of G such that θ is an irreducible constituent of φ_N. We write $\mathrm{IBr}_{p'}(G \mid \theta) = \mathrm{IBr}_{p'}(G) \cap \mathrm{IBr}(G \mid \theta)$.

Next is the version for Brauer characters and p-solvable groups of Theorem A of [13], which we shall need to prove Theorem B. It is worth mentioning that its proof uses the recent proof of the divisibility of the degrees of the Glauberman correspondence in [2].

Theorem 3.1 *Let G be a p-solvable group, and $P \in \mathrm{Syl}_p(G)$, then there is a bijection $f : \mathrm{IBr}_{p'}(G) \to \mathrm{IBr}_{p'}(\mathbf{N}_G(P))$ such that $f(\varphi)(1)$ divides $\varphi(1)$ for all $\varphi \in \mathrm{IBr}_{p'}(G)$. Furthermore, $\varphi(1)/f(\varphi)(1)$ divides $|G : \mathbf{N}_G(P)|$.*

Proof We argue by induction on $|G|$. Since $\mathbf{O}_p(G)$ is in the kernel of every $\varphi \in \mathrm{IBr}(G)$, by induction we may assume that $\mathbf{O}_p(G) = 1$ and hence $K = \mathbf{O}_{p'}(G) > 1$. Let $S/K = \mathbf{O}_p(G/K)$ and notice that $P_0 = P \cap S$ is a Sylow p-subgroup of S. By the Frattini argument, we have that $G = K\mathbf{N}_G(P_0)$. Notice also that $\mathbf{N}_G(P_0) < G$ since $\mathbf{O}_p(G) = 1$.

Let $\theta_1, \ldots, \theta_s$ be a complete set of representatives of the orbits of the action of $\mathbf{N}_G(P)$ on the P-invariant irreducible characters of K. By Lemma 2.3, we have that

$$\mathrm{IBr}_{p'}(G) = \mathrm{IBr}_{p'}(G \mid \theta_1) \cup \cdots \cup \mathrm{IBr}_{p'}(G \mid \theta_s)$$

is a disjoint union. Fix $\theta_i \in \mathrm{Irr}(K)$, P-invariant, and observe that θ_i is also P_0-invariant. Let $\theta_i^* \in \mathrm{Irr}(\mathbf{C}_K(P_0)) = \mathrm{IBr}(\mathbf{C}_K(P_0))$ be the Glauberman correspondent of θ_i and let $T_i = G_{\theta_i}$ be the stabilizer of θ_i in G. Since the Glauberman correspondence and the action of $\mathbf{N}_G(P_0)$ commute (see Lemma 2.10 of [10]), it follows that $\mathbf{N}_{T_i}(P_0) = T_i \cap \mathbf{N}_G(P_0)$ is the stabilizer of θ_i^* in $\mathbf{N}_G(P_0)$. By Dade's theorem (see Theorem (6.5) of [15]), we have that (T_i, K, θ_i) and $(\mathbf{N}_{T_i}(P_0), \mathbf{C}_K(P_0), \theta_i^*)$ are isomorphic character triples. In particular, there is a bijection $\Delta : \mathrm{Irr}(T_i|\theta_i) \to \mathrm{Irr}(\mathbf{N}_{T_i}(P_0)|\theta_i^*)$ such that $\chi(1)/\theta_i(1) = \Delta(\chi)(1)/\theta_i^*(1)$ for all $\chi \in \mathrm{Irr}(T_i|\theta_i)$. Now using Δ and Lemma 3.12 of [4] we can construct a bijection $^* : \mathrm{IBr}(T_i \mid \theta_i) \to \mathrm{IBr}(\mathbf{N}_{T_i}(P_0) \mid \theta_i^*)$ such that $\psi(1)/\theta_i(1) = \psi^*(1)/\theta_i^*(1)$ for all $\psi \in \mathrm{IBr}(T_i \mid \theta_i)$, and since $\theta_i(1)$ and $\theta_i^*(1)$ are p'-numbers we have that $^* : \mathrm{IBr}_{p'}(T_i \mid \theta_i) \to \mathrm{IBr}_{p'}(\mathbf{N}_{T_i}(P_0) \mid \theta_i^*)$ is a bijection.

Let $\chi \in \mathrm{IBr}_{p'}(G)$ and let $\theta_i \in \mathrm{IBr}(K) = \mathrm{Irr}(K)$ be such that $\chi \in \mathrm{IBr}_{p'}(G \mid \theta_i)$. By the Clifford correspondence (Theorem 8.9 of [9]), we have that there is $\psi \in \mathrm{IBr}_{p'}(T_i \mid \theta_i)$ such that $\psi^G = \chi$. Since $\mathbf{N}_{T_i}(P_0)$ is the stabilizer of θ_i^* in $\mathbf{N}_G(P_0)$, again by the Clifford correspondence we have that $(\psi^*)^{\mathbf{N}_G(P_0)} \in \mathrm{Irr}(\mathbf{N}_G(P_0))$. Furthermore, since $P \subseteq \mathbf{N}_{T_i}(P_0)$, we have that

$$(\psi^*)^{\mathbf{N}_G(P_0)}(1) = |\mathbf{N}_G(P_0) : \mathbf{N}_{T_i}(P_0)|\psi^*(1)$$

is a p'-number. Hence, we define

$$g : \mathrm{IBr}_{p'}(G) \to \mathrm{IBr}_{p'}(\mathbf{N}_G(P_0))$$

by $g(\chi) = (\psi^*)^{\mathbf{N}_G(P_0)}$.

Since $\theta_1, \ldots, \theta_s$ is a complete set of representatives of the action of $\mathbf{N}_G(P)$ on the P-invariant characters of K and the Glauberman correspondence commutes with the action of $\mathbf{N}_G(P) \subseteq \mathbf{N}_G(P_0)$, we have that $\theta_1^*, \ldots, \theta_s^*$ is a complete set of representatives of the action of $\mathbf{N}_G(P)$ on the P-invariant irreducible characters of $\mathrm{Irr}(\mathbf{C}_K(P_0)) = \mathrm{IBr}(\mathbf{C}_K(P_0))$. By Lemma 2.3, we have that

$$\mathrm{IBr}_{p'}(\mathbf{N}_G(P_0)) = \mathrm{IBr}_{p'}(\mathbf{N}_G(P_0) \mid \theta_1^*) \cup \cdots \cup \mathrm{IBr}_{p'}(\mathbf{N}_G(P_0) \mid \theta_s^*)$$

is a disjoint union and it follows that g is a bijection.

Let $\chi \in \mathrm{IBr}_{p'}(G|\theta_i)$ and let $\psi \in \mathrm{IBr}_{p'}(T_i|\theta_i)$ with $\psi^G = \chi$. Then $\psi(1) = \psi^*(1)\theta_i(1)/\theta_i^*(1)$ and, since $\theta_i^*(1)$ divides $\theta_i(1)$ (by the recent main theorem of [2]), we have that $\psi^*(1)$ divides $\psi(1)$. Since $G = \mathbf{N}_G(P_0)K$ we have that $g(\chi)(1) = |\mathbf{N}_G(P_0) : \mathbf{N}_{T_i}(P_0)|\psi^*(1)$ divides $|G : T_i|\psi(1) = \chi(1)$.

Finally, since $\mathbf{N}_G(P_0) < G$, we apply induction to obtain a bijection

$$h : \mathrm{IBr}_{p'}(\mathbf{N}_G(P_0)) \to \mathrm{IBr}_{p'}(\mathbf{N}_G(P))$$

such that $h(\psi)(1)$ divides $\psi(1)$ and $\chi(1)/h(\chi)(1)$ divides $|\mathbf{N}_G(P_0) : \mathbf{N}_G(P)|$ for all $\psi \in \mathrm{IBr}_{p'}(\mathbf{N}_G(P_0))$. Let $f = gh = h \circ g$. Clearly, f is a bijection and $f(\chi)(1)$ divides $\chi(1)$ for all $\chi \in \mathrm{IBr}_{p'}(G)$. Now, since $g(\chi)(1)/h(g(\chi))(1)$ divides $|\mathbf{N}_G(P_0) : \mathbf{N}_G(P)|$, we have that $\chi(1)/f(\chi)(1)$ divides $|\mathbf{N}_G(P_0) : \mathbf{N}_G(P)|\psi(1)/\psi^*(1) = |\mathbf{N}_G(P_0) : \mathbf{N}_G(P)|\theta_i(1)/\theta_i^*(1)$. By Problem 13.2 of [3], $\theta_i(1)/\theta_i^*(1)$ divides $|K : \mathbf{C}_K(P_0)| = |G : \mathbf{N}_G(P_0)|$. Hence, $\chi(1)/f(\chi)(1)$ divides $|G : \mathbf{N}_G(P)|$. □

The following is Theorem B.

Theorem 3.2 *Suppose that G is a p-solvable finite group and let q be a prime different from p. Then*

$$\mathrm{IBr}_{p'}(G) = \mathrm{IBr}_{q'}(G)$$

if and only if there is a Sylow p-subgroup P of G and a Sylow q-subgroup Q of G, such that $\mathbf{N}_G(P) = P\mathbf{N}_G(Q)$ and Q is abelian.

Proof Suppose that $\mathrm{IBr}_{q'}(G) = \mathrm{IBr}_{p'}(G)$. By Theorem 2.4, we have that there is a Sylow p-subgroup of G and a Sylow q-subgroup of G such that $\mathbf{N}_G(Q) \subseteq \mathbf{N}_G(P)$. We claim that

$$\mathrm{IBr}_{p'}(\mathbf{N}_G(P)) = \mathrm{IBr}_{q'}(\mathbf{N}_G(P))\,.$$

First, we notice that Theorem 2.4 applied to $\mathbf{N}_G(P)$ shows $\mathrm{IBr}_{q'}(\mathbf{N}_G(P)) \subseteq \mathrm{IBr}_{p'}(\mathbf{N}_G(P))$. Let $\mu \in \mathrm{IBr}_{p'}(\mathbf{N}_G(P))$. By Theorem 3.1, there is $\varphi \in \mathrm{IBr}_{p'}(G)$ such that $\mu(1)$ divides $\varphi(1)$ which is a q'-number. Hence, we conclude $\mu \in \mathrm{IBr}_{q'}(\mathbf{N}_G(P))$. Then $\mathrm{IBr}_{q'}(\mathbf{N}_G(P)) = \mathrm{IBr}_{p'}(\mathbf{N}_G(P))$, as claimed.

Therefore, we may assume, arguing by induction that $P \triangleleft G$. Then, we have that $\mathrm{IBr}(G) = \mathrm{IBr}(G/P) = \mathrm{Irr}(G/P)$ and

$$\mathrm{Irr}_{q'}(G/P) = \mathrm{IBr}_{q'}(G) = \mathrm{IBr}_{p'}(G) = \mathrm{Irr}_{p'}(G/P) = \mathrm{Irr}(G/P).$$

By the Itô-Michler theorem, we know that G/P has a normal and abelian Sylow q-subgroup PQ/Q. Hence, Q is abelian and $PQ \lhd G$. Then $G = P\mathbf{N}_G(Q)$ by the Frattini's argument.

Conversely, suppose that there is $P \in \mathrm{Syl}_p(G)$ and $Q \in \mathrm{Syl}_q(G)$ such that $\mathbf{N}_G(P) = P\mathbf{N}_G(Q)$ and Q is abelian. Notice that since $\mathbf{N}_G(Q) \subseteq \mathbf{N}_G(P)$ by Theorem 2.4 we have that $\mathrm{IBr}_{q'}(G) \subseteq \mathrm{IBr}_{p'}(G)$. We only need to prove the reverse containment. If $\mathbf{N}_G(P) < G$, arguing by induction, we have that $\mathrm{IBr}_{q'}(\mathbf{N}_G(P)) = \mathrm{IBr}_{p'}(\mathbf{N}_G(P))$. Let $\varphi \in \mathrm{IBr}_{p'}(G)$ and let $f : \mathrm{IBr}_{p'}(G) \to \mathrm{IBr}_{p'}(\mathbf{N}_G(P))$ be the bijection given in Theorem 3.1. Hence $\varphi(1)/f(\varphi)(1)$ divides $|G : \mathbf{N}_G(P)|$ which is not divisible by q. Since $\mathrm{IBr}_{q'}(\mathbf{N}_G(P)) = \mathrm{IBr}_{p'}(\mathbf{N}_G(P))$, we have that $f(\varphi)(1)$ is not divisible by q and thus $\varphi \in \mathrm{IBr}_{q'}(G)$.

Hence, we may assume that P is a normal subgroup of G and then $\mathrm{IBr}(G) = \mathrm{Irr}(G/P)$. Also $G = \mathbf{N}_G(Q)P$ and hence PQ is normal in G. Then, PQ/P is an abelian normal Sylow q-subgroup of G/P. By Itô's theorem, we have that $\mathrm{Irr}(G/P) = \mathrm{Irr}_{q'}(G/P)$. It follows that

$$\mathrm{IBr}_{q'}(G) = \mathrm{Irr}_{q'}(G/P) = \mathrm{Irr}(G/P) = \mathrm{IBr}_{p'}(G),$$

and we are done. □

To prove the assertion in the Abstract, it is enough to notice that in p-solvable groups, the ordinary character table of G uniquely determines the Brauer characters of G, by using the Fong–Swan theorem. (See Theorem 10.1 and Corollary 10.4 of [9].)

Finally, using Isaacs π-characters, the Glauberman–Isaacs correspondence, and some ad hoc arguments, it is possible to replace in Theorems A and B of this paper p' by π, p-solvable groups by π-separable groups, and Sylow p-subgroups by Hall π-complements.

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