Growing Hearts in Associative Systems [1]

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Dedicated to the memory of Professor Hyo Chul Myung

Abstract: We show that associative systems with a sufficiently good module structure imbed in a primitive system with simple primitive heart, spanned by the original system and the heart, so extending the results of *J. Pure Appl. Algebra* **181** (2003) 131-139 to systems over more general rings of scalars. We also study associative systems with involution.

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Introduction

The present work, though exclusively dealing with associative systems, has its origin outside a pure associative setting, and it is indeed motivated by the relationship between associative and the so called Jordan systems. In fact, some of the most important examples of Jordan systems are those coming from associative systems by symmetrization ($R^{(+)}$ with product $x \circ y := xy + yx$). Consequently, there is a long tradition of mathematical work aimed at linking Jordan notions with their corresponding associative ones. In [3], we studied how simple was a Jordan system having simple all of its local algebras. As a tool, it was proved that an associative system R with this condition had a big simple heart equal to the Jordan cube of R, i.e., $(R^{(+)})^3$. Then in [4] we faced the problem of expressing Jordan cubes in terms of associative powers $R^n = R \cdots R$. Few new things could be said and the paper consisted, basically, on giving counterexamples to every (apparently) reasonable statement which came to mind.

But, in the process of building the above counterexamples, we developed several processes to "paste" a simple heart to an arbitrary associative system. In these processes we dealt only with associative systems over fields, sometimes even finite

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dimensional. We will improve the results obtained in [4] in two senses: on the one hand we will consider associative systems over more general rings of scalars, and on the other hand we will also deal with systems with involution.

After a preliminary paragraph containing definitions and basic properties, we will prove in Section 1 some results concerning primeness and primitivity which will be needed in the sequel. Section 2 is devoted to obtaining optimal versions of the results of [4] for associative systems with underlying modules which are not necessarily vector spaces over fields. We will see that the nature of the results strongly restrict the underlying module structure, leading to necessary conditions under which we will prove our results. To do that, we will use the known results over fields together with the auxiliary results of the previous paragraph. The last section is devoted to studying analogues of the results of Section 2 for associative systems with involution. Rather than giving independent constructions, we will use the results of Section 2, moving them to a suitable setting by using duplicated systems with the exchange involution.

0. Preliminaries

0.1 We will deal with associative systems (algebras, pairs, and triple systems) over an arbitrary ring of scalars Φ. We remark that associative triple systems are those of first kind in the sense of [9]. When $\Phi = \mathbf{Z}$ is the ring of integers, associative triple systems are just ternary rings in the sense of Lister [8]. Associative pairs are defined accordingly. The reader is referred to [1, 2, 8, 9, 10, 11] for basic facts and notions not explicitly mentioned in this section.

0.2 Given an associative system R, the *heart* $\mathrm{Heart}(R)$ of R is the intersection of all nonzero ideals of R. It can be readily seen that, when R is semiprime, $\mathrm{Heart}(R)$ is the unique simple ideal of R when it is nonzero [5, 2.1, 3.1, 3.7]. Given an associative system R with involution $*$, the $*$-*heart* $*$-$\mathrm{Heart}(R)$ of R is the intersection of all nonzero $*$-ideals of R. Similarly, when R is semiprime and $*$-$\mathrm{Heart}(R) \neq 0$, then it is the unique $*$-simple $*$-ideal of R.

0.3 An associative algebra gives rise to an associative triple system by just restricting to odd length products. By doubling any associative triple system R one obtains the *double associative pair* $V(R) = (R, R)$ with products defined in the obvious manner. From an associative pair $R = (R^+, R^-)$ one can get a (*polarized*) associative triple system $T(R) = R^+ \oplus R^-$ by defining $(x^+ \oplus x^-)(y^+ \oplus y^-)(z^+ \oplus z^-) = x^+y^-z^+ \oplus x^-y^+z^-$ [10, 1.13, 1.14].

0.4 (i) Let R be a Φ-module (resp. $R = (R^+, R^-)$ be a pair of Φ-modules). The *annihilator of* R *in* Φ, $\operatorname{Ann}_\Phi(R) := \{\lambda \in \Phi \mid \lambda R = 0\}$ (resp. $\operatorname{Ann}_\Phi(R) := \{\lambda \in \Phi \mid \lambda R^\sigma = 0, \sigma = \pm\}$), is the kernel of the natural ring homomorphism of Φ in $\operatorname{End}_{\mathbf{Z}}(R)$ (resp. of Φ in $\operatorname{End}_{\mathbf{Z}}(R^+) \times \operatorname{End}_{\mathbf{Z}}(R^-)$). Let $\bar{\Phi}$ denote the quotient $\Phi/\operatorname{Ann}_\Phi(R)$. Notice that $\bar{\Phi}$ is isomorphic to the image of Φ in $\operatorname{End}_{\mathbf{Z}}(R)$ (resp. in $\operatorname{End}_{\mathbf{Z}}(R^+) \times \operatorname{End}_{\mathbf{Z}}(R^-)$) and R becomes a $\bar{\Phi}$-module (resp. a pair of $\bar{\Phi}$-modules). Moreover, if R is an associative system over Φ, then it is also an associative system over $\bar{\Phi}$ (cf. [7, Lemmas 1.1.1, 1.1.2]).

(ii) Conversely, if we take any proper ideal I of Φ, any Φ/I-module, or pair of Φ/I-modules, or associative system over Φ/I can be viewed as a Φ-module, or pair of Φ-modules, or associative system over Φ, respectively, in the obvious manner.

This paper is aimed at improving the following results.

0.5 Growing hearts in associative systems over a field [4, 2.3]. Let R be an associative system over a field Φ. There exists an associative system $\tilde{R}$ over Φ such that:

(i) R is isomorphic to a subsystem S of $\tilde{R}$,

(ii) $\tilde{R}$ is a left primitive system, hence it is prime,

(iii) $\operatorname{Heart}(\tilde{R})$ is simple and left primitive,

(iv) $\tilde{R} = S \oplus \operatorname{Heart}(\tilde{R})$, hence $\tilde{R}/\operatorname{Heart}(\tilde{R}) \cong R$.

Moreover,

(a) If R is an associative triple system, then $\operatorname{Heart}(\tilde{R}) = \operatorname{Heart}(\tilde{R}_1)$, for a left primitive associative algebra $\tilde{R}_1$ over Φ such that $\tilde{R}$ is a subsystem of the underlying triple system of $\tilde{R}_1$, and $\operatorname{Heart}(\tilde{R}_1)$ is simple and left primitive as an algebra.

(b) If R is an associative pair, then $\operatorname{Heart}(\tilde{R}) = V(\operatorname{Heart}(\tilde{R}_1))$, for a left primitive associative algebra $\tilde{R}_1$ over Φ such that $\tilde{R}$ is a subpair of $V(\tilde{R}_1)$, and $\operatorname{Heart}(\tilde{R}_1)$ is simple and left primitive as an algebra.

0.6 Growing hearts in finite-dimensional associative systems over a field [4, 2.4]. Let R be a finite dimensional associative system over a field Φ. There exist an associative system $\tilde{R}$ over Φ and an associative Φ-algebra $\tilde{R}_1$ ($\tilde{R} = \tilde{R}_1$ in the algebra case) satisfying (0.5), and there exist $\{f_i \mid i \in \mathbf{N}\} \subseteq \operatorname{Heart}(\tilde{R}_1)$, such that:

(i) $f_i f_j = f_j f_i = f_{\min\{i,j\}}$ for any $i, j \in \mathbf{N}$ and, in particular, f_i is an algebra idempotent for any $i \in \mathbf{N}$,

(ii) for any $h \in \operatorname{Heart}(\tilde{R}_1)$, there exists $i \in \mathbf{N}$ such that $h = hf_i = f_i h = f_i h f_i$,

(iii) $f_i s = s f_i$ for any $s \in S$ (for any $s \in S^\sigma$, $\sigma = \pm$ in the pair case), $i \in \mathbf{N}$.

1. Auxiliary Results on Regularity Conditions

1.1 LEMMA. *Let Φ be an integral domain, and $F = \Phi^{-1}\Phi$ its field of fractions. Let R be an associative system over F which then can also be seen as an associative system over Φ. If R is left (resp. right) primitive as an associative system over F then it is left (resp. right) primitive as an associative system over Φ.*

PROOF: Let us assume first that R is a triple system, which is left primitive at $b \in R$ over F, and let K be a primitizer with modulus $e \in R$, i.e., K is a left F-ideal of R such that it is modular at b with modulus e ($x - xbe \in K$, for any $x \in R$), and, for any nonzero F-ideal I of R, $R = K + I$ (cf. [2, 1.1, 1.2]). Thus K is a left Φ-ideal of R which is modular at b with modulus e, and we just need to check that it complements nonzero Φ-ideals of R: Let L be a nonzero Φ-ideal of R. Then $\Phi^{-1}L$ is a nonzero F-ideal and $R = K + \Phi^{-1}L$. Thus $e = k + \gamma^{-1}y$, for some $k \in K$, $\gamma \in \Phi$, and $y \in L$. Now, for any $x \in R$,

$$\begin{aligned} x &= x - xbe + xbe = x - xbe + xbk + xb(\gamma^{-1}y) \\ &= x - xbe + xbk + (\gamma^{-1}x)by \in K + RRK + RRL \subseteq K + L, \end{aligned}$$

which shows $R = K + L$.

When R is an associative pair, the above proof can be easily adapted. If R is an associative algebra, the proof for triple systems applies verbatim by simply deleting b. ■

If in the proof of the above result we replace ideals by $*$-ideals and one-sided primitivity by $*$-primitivity (cf. [2, 1.3]), we obtain an analogue for associative systems with involution.

1.2 LEMMA. *Let Φ be an integral domain, and $F = \Phi^{-1}\Phi$ its field of fractions. Let R be an associative system with involution $*$ over F which then can also be seen as an associative system with involution over Φ. If R is $*$-primitive as an associative system over F then it is $*$-primitive as an associative system over Φ.* ■

1.3 LEMMA. *Let R be a prime associative system (resp. $*$-prime associative system with involution $*$) and I be a nonzero ideal (resp. $*$-ideal) of R. If S is a subsystem (resp. $*$-subsystem) of R containing I, then S is prime (resp. $*$-prime).*

PROOF: The result without involution is just [4, 2.2]. Nevertheless we are recalling its proof since a major part of it will be used in the case with involution.

Let us assume first that R is a prime triple system.

(1) I is semiprime (equivalently, it is nondegenerate [1, 1.18]): for any $x \in I$, $xIx = 0$ implies $(xRx)R(xRx) = x(RxRxR)x \subseteq xIx = 0$, hence $xRx = 0$, and $x = 0$ by nondegeneracy of R.

(2) I is prime: For any nonzero ideal L of I, the ideal $\tilde{L}$ of R generated by $LLLLL$ is nonzero by (1), and is contained in L (see [3, 4.5]). If L_1 and L_2 are nonzero ideals of I, then $\tilde{L}_1$ and $\tilde{L}_2$ are nonzero ideals of R, hence $0 \neq \tilde{L}_1 \cap \tilde{L}_2$ by primeness of R, hence $0 \neq L_1 \cap L_2$.

(3) Any nonzero ideal K of S hits I ($K \cap I \neq 0$): For any $0 \neq a \in K$, $0 \neq aIa \subseteq K \cap I$ by [2, 1.7(ii)].

By (3), semiprimeness of I (1) implies semiprimeness of S. Now, for any nonzero ideals K_1, K_2 of S, $K_1 \cap I$ and $K_2 \cap I$ are nonzero ideals of I by (3), hence $(K_1 \cap I) \cap (K_2 \cap I) \neq 0$ by (2), which implies $K_1 \cap K_2 \neq 0$, hence S is prime.

To prove an analogue for $*$-prime associative triples systems, we can argue with $*$-ideals as in the above argument as soon as we have analogues $(1)^*$, $(2)^*$, and $(3)^*$ of (1), (2), and (3), respectively. Indeed $(1)^*$ is just (1) recalling the basic fact that semiprimeness is equivalent to $*$-semiprimeness, and the proof of (2) applies verbatim to $(2)^*$ simply replacing ideals by $*$-ideals. However, $(3)^*$ requires a little more effort.

$(3)^*$ Any nonzero $*$-ideal K of S hits I ($K \cap I \neq 0$): if we can find $0 \neq a \in K$ such that $0 \neq aIa$, then $K \cap I \neq 0$ since it contains aIa. In particular, this happens if there is $0 \neq a \in K \cap H(R,*)$ by [2, 1.7(ii)]. Thus, if $(3)^*$ does not hold, we can assume $bIb = 0$ for any $b \in K$, and $K \cap H(R,*) = 0$. Take any $0 \neq b \in K$, and notice that $b + b^* \in K \cap H(R,*)$ forces $b^* = -b$, so that the ideal J of R generated by b is indeed a $*$-ideal. On the other hand, we can proceed as in the proof of [2, 1.7(ii)] using semiprimeness of R:

$$(IbR)R(IbR) = Ib(RRI)bR \subseteq IbIbR = 0, \text{ hence} IbR = 0,$$
$$(RIb)R(RIb) = RIb(RRI)b \subseteq RIbIb = 0, \text{ hence} RIb = 0,$$
$$(IRb)R(IRb) = IRb(RIR)b \subseteq IRbIb = 0, \text{ hence} IRb = 0,$$

and therefore I and J are nonzero orthogonal $*$-ideals of R, which is a contradiction.

The above assertions hold (with analogous proofs) for algebras. The analogues for pairs follow from the corresponding results for triple systems by using the functor T and [2, 1.4(i)]. ■

2. Generalizing the Module Structure

Firstly, we will find necessary conditions on the Φ-module structure of an associative system R to satisfy an analogue of (0.5).

2.1 Proposition. *If R is an associative system over Φ, and there exists a prime associative system $\tilde{R}$ over Φ such that R is isomorphic to a subsystem S of $\tilde{R}$, then $\bar{\Phi}$ is an integral domain acting without torsion on R.*

Proof: Let $\lambda+\mathrm{Ann}_\Phi(R), \mu+\mathrm{Ann}_\Phi(R) \in \bar{\Phi}$. If $(\lambda+\mathrm{Ann}_\Phi(R))(\mu+\mathrm{Ann}_\Phi(R)) = 0$, then the ideals $Id_{\tilde{R}}(\lambda S)$, $Id_{\tilde{R}}(\mu S)$ of $\tilde{R}$ are orthogonal. By primeness of $\tilde{R}$, either $\lambda S = 0$ or $\mu S = 0$, i.e., either $\lambda R = 0$ or $\mu R = 0$, i.e., $\lambda + \mathrm{Ann}_\Phi(R) = 0$ or $\mu + \mathrm{Ann}_\Phi(R) = 0$.

If $\lambda + \mathrm{Ann}_\Phi(R) \in \bar{\Phi}$ and $r \in R$ ($r \in R^\sigma$, $\sigma \in \{+,-\}$, when we are dealing with pairs) satisfy $\lambda r = 0$, then $\lambda s = 0$, for the image s of r under the isomorphism $R \cong S$. Now, $Id_{\tilde{R}}(s)$ and $\lambda\tilde{R}$ are orthogonal ideals of $\tilde{R}$. By primeness of $\tilde{R}$, either $s = 0$, i.e., $r = 0$, or $\lambda\tilde{R} = 0$, which implies $\lambda S = 0$, i.e., $\lambda R = 0$, i.e., $\lambda + \mathrm{Ann}_\Phi(R) = 0$. ■

The converse of the above result is true, which gives the optimal version of (0.5).

2.2 Theorem. *Let R be an associative system over a ring of scalars Φ such that $\bar{\Phi}$ is an integral domain acting without torsion on R. There exists an associative system $\tilde{R}$ over Φ such that:*

(i) *R is isomorphic to a subsystem S of $\tilde{R}$,*

(ii) *$\tilde{R}$ is a left primitive system, hence it is prime,*

(iii) $\mathrm{Heart}(\tilde{R})$ *is simple and left primitive,*

(iv) *$\tilde{R} = S \oplus \mathrm{Heart}(\tilde{R})$, hence $\tilde{R}/\mathrm{Heart}(\tilde{R}) \cong R$.*

Moreover,

(a) *If R is an associative triple system, then* $\mathrm{Heart}(\tilde{R}) = \mathrm{Heart}(\tilde{R}_1)$*, for a left primitive associative algebra $\tilde{R}_1$ over Φ such that $\tilde{R}$ is a subsystem of the underlying triple system of $\tilde{R}_1$, and* $\mathrm{Heart}(\tilde{R}_1)$ *is simple and left primitive as an algebra.*

(b) *If R is an associative pair, then* $\mathrm{Heart}(\tilde{R}) = V(\mathrm{Heart}(\tilde{R}_1))$*, for a left primitive associative algebra $\tilde{R}_1$ over Φ such that $\tilde{R}$ is a subpair of $V(\tilde{R}_1)$, and* $\mathrm{Heart}(\tilde{R}_1)$ *is simple and left primitive as an algebra.*

Proof: First notice that we can replace Φ by $\bar{\Phi}$ and assume that Φ is an integral domain acting without torsion on R: We just need to work over $\bar{\Phi}$ and then read the result in terms of Φ-modules and Φ-systems (0.4)(ii). Isomorphisms, submodules, subsystems, ideals, one-sided ideals, primeness, primitivity (expressed in terms of maximal one-sided modular ideals of zero core [7, Th. 2.1.1], or in terms of primitizers [2, 1.1]), etc. over $\bar{\Phi}$ are the same over Φ, and R goes back to its initial Φ-module structure.

Now let $M := \Phi^{-1}\Phi \otimes_\Phi R$ (in the case of pairs $M = (M^+, M^-) := (\Phi^{-1}\Phi \otimes_\Phi R^+, \Phi^{-1}\Phi \otimes_\Phi R^-)$, which is an associative system over the field of fractions $F :=$

$\Phi^{-1}\Phi$ of Φ. If we apply (0.5) to M as an F-system, we obtain an F-system $\tilde{M}$ satisfying (0.5)(i—iv), and an F-algebra $\tilde{M}_1$ satisfying (0.5)(a,b). Let $H := \mathrm{Heart}(\tilde{M})$. It is a simple F-system which is, in particular a simple Φ-system (simplicity does not depend on the ring of scalars) and it is also left primitive as a Φ-system (1.1).

Let N be the isomorphic image of M in $\tilde{M}$. Since R is a Φ-subsystem of M due to the lack of torsion, its image S by the isomorphism $M \cong N$ is also a Φ-subsystem of N (hence of $\tilde{M}$) isomorphic to R.

Now we can define $\tilde{R} := S \oplus H$ ($S \cap H \subseteq N \cap H = 0$) which is a Φ-subsystem of $\tilde{M}$ and hence satisfies (i) for R. Since primeness does not depend on the ring of scalars, primeness of $\tilde{M}$ as an F-system implies its primeness as a Φ-system, hence $\tilde{R}$ is prime by (1.3). Since simplicity does not depend on the ring of scalars either, $\mathrm{Heart}(\tilde{R}) = H$ (H is a simple ideal of $\tilde{R}$, which is prime), so that we have (iii)(iv) for R and $\tilde{R}$. Finally, $\tilde{R}$ is left primitive since it is prime and it has the ideal H which is left primitive [2, 1.10], i.e., we have (ii).

We just need to show that $\tilde{R}_1 := \tilde{M}_1$ satisfies (a) or (b) when we deal with triple systems or pairs, respectively. Clearly $\tilde{R}_1$ is an associative Φ-algebra since it is an associative F-algebra. Moreover, $\tilde{R}_1$ is left primitive as a Φ-algebra by (1.1). Since $\mathrm{Heart}(\tilde{R}) = \mathrm{Heart}(\tilde{M})$, we just need to notice that the heart of $\tilde{R}_1$ as an F-algebra is the same as its heart as a Φ-algebra: $\mathrm{Heart}(\tilde{R}_1)$ is a simple F-ideal of $\tilde{R}_1$, hence it is a simple Φ-ideal of $\tilde{R}_1$, which is prime since it is left primitive. ∎

2.3 THEOREM. *Let R be an associative system over a ring of scalars Φ such that $\bar{\Phi}$ is an integral domain acting without torsion on R. Let us also assume that $\Phi^{-1}\Phi \otimes_\Phi R$ (in the case of pairs $(\Phi^{-1}\Phi \otimes_\Phi R^+, \Phi^{-1}\Phi \otimes_\Phi R^-)$) is finite dimensional over the field of fractions $F := \Phi^{-1}\Phi$ of Φ, for example, when R is a finitely generated Φ-module (a pair of finitely generated Φ-modules, in the pair case). There exist an associative system $\tilde{R}$ over Φ and an associative Φ-algebra $\tilde{R}_1$ ($\tilde{R} = \tilde{R}_1$ in the algebra case) satisfying (2.2), and there exist $\{f_i \mid i \in \mathbf{N}\} \subseteq \mathrm{Heart}(\tilde{R}_1)$, such that:*

(i) *$f_i f_j = f_j f_i = f_{\min\{i,j\}}$ for any $i, j \in \mathbf{N}$ and, in particular, f_i is an algebra idempotent for any $i \in \mathbf{N}$,*

(ii) *for any $h \in \mathrm{Heart}(\tilde{R}_1)$, there exists $i \in \mathbf{N}$ such that $h = hf_i = f_i h = f_i h f_i$,*

(iii) *$f_i s = s f_i$ for any $s \in S$ (for any $s \in S^\sigma$, $\sigma = \pm$ in the pair case), $i \in \mathbf{N}$.*

PROOF: The proof goes as that of (2.2), using (0.6). ∎

3. Growing Hearts in Associative Systems with Involution

The purpose of this section is establishing analogues with involution of the results of the previous section.

3.1 PROPOSITION. *If R is an associative system with involution $*$ over a ring of scalars Φ, and there exists a $*$-prime associative system $\tilde{R}$ over Φ with involution (also denoted $*$) such that R is $*$-isomorphic to a $*$-subsystem S of $\tilde{R}$, then $\bar{\Phi}$ is an integral domain acting without torsion on R.*

PROOF: Let $\lambda+\mathrm{Ann}_\Phi(R), \mu+\mathrm{Ann}_\Phi(R) \in \bar{\Phi}$. If $(\lambda+\mathrm{Ann}_\Phi(R))(\mu+\mathrm{Ann}_\Phi(R)) = 0$, then the ideals $Id_{\tilde{R}}(\lambda S)$, $Id_{\tilde{R}}(\mu S)$ are $*$-ideals (they are generated by subsets invariant by $*$) and are orthogonal. By $*$-primeness of $\tilde{R}$, either $\lambda S = 0$ or $\mu S = 0$, i.e., either $\lambda R = 0$ or $\mu R = 0$, i.e., $\lambda + \mathrm{Ann}_\Phi(R) = 0$ or $\mu + \mathrm{Ann}_\Phi(R) = 0$.

If $\lambda + \mathrm{Ann}_\Phi(R) \in \bar{\Phi}$ and $r \in R$ ($r \in R^\sigma$, $\sigma \in \{+,-\}$, when we are dealing with pairs) satisfy $\lambda r = 0$, then $\lambda r^* = (\lambda r)^* = 0$, and hence $\lambda s = \lambda s^* = 0$, for the image s of r under the $*$-isomorphism $R \cong S$. Now, $Id_{\tilde{R}}(\{s, s^*\})$ and $\lambda\tilde{R}$ are orthogonal $*$-ideals of $\tilde{R}$. By $*$-primeness of $\tilde{R}$, either $s = 0$, i.e., $r = 0$, or $\lambda\tilde{R} = 0$, which implies $\lambda S = 0$, i.e., $\lambda R = 0$, i.e., $\lambda + \mathrm{Ann}_\Phi(R) = 0$. ■

Every associative system with involution can be expressed in terms of the exchange involution, as shown in the following result with obvious proof.

3.2 LEMMA. *If R is an associative system with involution $*$, then the map $\varphi : (R,*) \longrightarrow (R \boxplus R^{op}, \mathrm{ex})$ (ex denotes the exchange involution), given by $\varphi(x) = (x, x^*)$, is a monomorphism of systems with involution . Thus $(R,*)$ is $*$-isomorphic to an ex-subalgebra of $(R \boxplus R^{op}, \mathrm{ex})$.* ■

In the next results, we will need a week regularity condition, namely the absence of nonzero *invisible elements* (an element r is invisible if every associative monomial of length > 1 containing r vanishes).

3.3 LEMMA. *Let R be an associative system. For any ideal I of R, $I \boxplus I$ is an ex-ideal of $(R \boxplus R^{op}, \mathrm{ex})$. Conversely, if R does not have invisible elements (for example if R is semiprime), then for any nonzero ex-ideal L of $(R \boxplus R^{op}, \mathrm{ex})$, there exists a nonzero ideal K of R such that $K \boxplus K \subseteq L$*

PROOF: The first assertion is clear. For the second, let $0 \neq (r,s) \in L$. Suppose, for example that $r \neq 0$. Since r is not invisible, the set K of sums of monomials of length bigger than one containing r is a nonzero ideal of R such that $K \boxplus 0 \subseteq L$. Since L is ex-invariant, $0 \boxplus K = (K \boxplus 0)^{\mathrm{ex}} \subseteq L$ too, and $K \boxplus K \subseteq L$ as desired. ■

3.4 REMARKS: (i) There are systems which are not semiprime but still do not have nonzero invisible elements: take the algebra $R = \Phi 1 \oplus \Phi x$ with $x^2 = 0$ which is not semiprime, $xRx = 0$, but does not have invisible elements since it is unital.

(ii) The condition on the absence of nonzero invisible elements is clearly necessary in (3.3): Take, any associative algebra R over a field Φ with a nonzero invisible element r. Then $\Phi(r,r)$ is an ex-ideal of $R\boxplus R^{op}$ of dimension one, hence it cannot contain $K\boxplus K$ for a nonzero ideal K of R.

The following results are direct consequences of (3.3).

3.5 Lemma. *An associative system R is simple if and only if $(R\boxplus R^{op}, \mathrm{ex})$ is* ex-*simple.* ∎

3.6 Lemma. *An associative system R is prime if and only if $(R\boxplus R^{op}, \mathrm{ex})$ is* ex-*prime.* ∎

3.7 Lemma. *Let R be an associative system without nonzero invisible elements. Then* ex-Heart$(R\boxplus R^{op}) = \mathrm{Heart}(R)\boxplus \mathrm{Heart}(R)^{op}$. ∎

3.8 Lemma. *An associative system R is one-sided primitive if and only if $(R\boxplus R^{op}, \mathrm{ex})$ is* ex-*primitive.*

Proof: We can make a short though very indirect proof by using some results on Jordan systems.

(1) R is one-sided primitive iff $R^{(+)} \cong H(R\boxplus R^{op}, \mathrm{ex})$ is a primitive Jordan system [2, 4.5(ii), 5.8(ii); 6, 4.2].

(2) Given an associative system A with involution $*$, A is $*$-prime and $H(A,*)$ is a primitive Jordan system iff A is $*$-primitive [2, 4.6(ii), 5.10(ii); 6, 4.4, 4.9].

Now, by (2), $(R\boxplus R^{op}, \mathrm{ex})$ is ex-primitive iff $(R\boxplus R^{op}, \mathrm{ex})$ is ex-prime and $H(R\boxplus R^{op}, \mathrm{ex})$ is a primitive Jordan system, which is equivalent to R being prime (3.6) and R being one-sided primitive (1). But that is equivalent to R just being one-sided primitive, since any primitive system is prime [2, 1.8; 7, Lemma 2.1.2]. ∎

3.9 Theorem. *Let R be an associative system with involution $*$ over a ring of scalars Φ such that $\bar{\Phi}$ is an integral domain acting without torsion on R. There exists an associative system $\tilde{R}$ with involution (also denoted $*$) over Φ such that:*

(i) *R is $*$-isomorphic to a $*$-subsystem S of $\tilde{R}$,*

(ii) *$\tilde{R}$ is a $*$-primitive system, hence it is $*$-prime,*

(iii) $*$-Heart$(\tilde{R})$ *is $*$-simple and $*$-primitive,*

(iv) $\tilde{R} = S \oplus *$-Heart$(\tilde{R})$*, hence* $\tilde{R}/*$-Heart$(\tilde{R}) \cong R$.

Moreover,

(a) If R is an associative triple system, then $*$-Heart$(\tilde{R}) = *$-Heart$(\tilde{R}_1)$, for a $*$-primitive associative algebra $\tilde{R}_1$ over Φ such that $\tilde{R}$ is a $*$-subsystem of the underlying triple system of $\tilde{R}_1$, and $*$-Heart$(\tilde{R}_1)$ is $*$-simple and $*$-primitive as an algebra.

(b) If R is an associative pair, then $*$-Heart$(\tilde{R}) = V(*\text{-Heart}(\tilde{R}_1))$, for a $*$-primitive associative algebra $\tilde{R}_1$ over Φ such that $\tilde{R}$ is a subpair of $V(\tilde{R}_1)$, and $*$-Heart$(\tilde{R}_1)$ is $*$-simple and $*$-primitive as an algebra.

Proof: We can apply (2.2) to R to find a system $\tilde{M}$ having a subsystem N isomorphic to R, and an algebra $\tilde{M}_1$ satisfying (2.2)(i—iv)(a,b). The involution of R induces an involution $*$ in N through the isomorphism $R \cong N$, so that R and N are $*$-isomorphic.

By (3.2), N is $*$-isomorphic to a $*$-subsystem S of $N \boxplus N^{op}$ which is a $*$-subsystem of $\tilde{M} \boxplus \tilde{M}^{op}$ (here $*$ also denotes the exchange involution ex). Since $\tilde{M} = N \oplus \text{Heart}(\tilde{M})$,

$$\tilde{M} \boxplus \tilde{M}^{op} = (N \boxplus N^{op}) \oplus \big(\text{Heart}(\tilde{M}) \boxplus \text{Heart}(\tilde{M})^{op}\big) = (N \boxplus N^{op}) \oplus H,$$

where $H := \text{Heart}(\tilde{M}) \boxplus \text{Heart}(\tilde{M})^{op} = *\text{-Heart}(\tilde{M} \boxplus \tilde{M}^{op})$ by (3.7) ($\tilde{M}$ does not have invisible elements since it is $*$-prime) is a $*$-simple system by (3.5).

Thus we can take $\tilde{R} = S \oplus H$, which is a $*$-subsystem of $\tilde{M} \boxplus \tilde{M}^{op}$. Moreover, $\tilde{R}$ is $*$-prime by (1.3) since it is a $*$-subsystem of $\tilde{M} \boxplus \tilde{M}^{op}$ which is $*$-prime by (3.6) and contains one of its $*$-ideals (namely H). As a consequence, H (a $*$-simple ideal of a $*$-prime system) is the $*$-Heart of $\tilde{R}$.

Moreover, H is $*$-primitive by (3.8), hence $\tilde{R}$ is also $*$-primitive by [2, 1.10; 6, 4.7(b)] (it is $*$-prime and contains a nonzero $*$-primitive ideal).

We can define $\tilde{R}_1 := \tilde{M}_1 \boxplus \tilde{M}_1^{op}$, which is a $*$-primitive algebra by (3.8). Moreover, $*$-Heart$(\tilde{R}_1) = \text{Heart}(\tilde{M}_1) \boxplus \text{Heart}(\tilde{M}_1)^{op}$ by (3.7) since $\tilde{M}_1$ does not have invisible elements (it is $*$-primitive hence $*$-prime [6, 4.4]), and hence $*$-Heart$(\tilde{R}_1)$ is a $*$-simple algebra by (3.5) which is $*$-primitive by (3.8). The remaining assertions of (a) and (b) are straightforward. ∎

3.10 Theorem. *Let R be an associative system with involution $*$ over a ring of scalars Φ such that $\bar{\Phi}$ is an integral domain acting without torsion on R. Let us also assume that $\Phi^{-1}\Phi \otimes_\Phi R$ (in the case of pairs $(\Phi^{-1}\Phi \otimes_\Phi R^+, \Phi^{-1}\Phi \otimes_\Phi R^-)$) is finite dimensional over the field of fractions $F := \Phi^{-1}\Phi$ of Φ, for example, when R is a finitely generated Φ-module (a pair of finitely generated Φ-modules, in the pair case). There exist an associative system $\tilde{R}$ with involution (also denoted $*$) over Φ and an associative Φ-algebra $\tilde{R}_1$ ($\tilde{R} = \tilde{R}_1$ in the algebra case) with involution (also denoted $*$) satisfying (3.9), and there exist $\{f_i \mid i \in \mathbf{N}\} \subseteq *\text{-Heart}(\tilde{R}_1) \cap H(\tilde{R}_1, *)$, such that:*

(i) *$f_i f_j = f_j f_i = f_{\min\{i,j\}}$ for any $i, j \in \mathbf{N}$ and, in particular, f_i is an algebra idempotent for any $i \in \mathbf{N}$,*

(ii) *for any* $h \in *\text{-Heart}(\tilde{R}_1)$, *there exists* $i \in \mathbf{N}$ *such that* $h = hf_i = f_ih = f_ihf_i$,

(iii) $f_is = sf_i$ *for any* $s \in S$ *(for any* $s \in S^\sigma$, $\sigma = \pm$ *in the pair case),* $i \in \mathbf{N}$.

PROOF: The proof goes as that of (3.9), taking ex-symmetric "duplicated" idempotents. ■

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