On depth-based fuzzy trimmed means and a notion of depth specifically defined for fuzzy numbers

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Abstract

Empirical trimmed means have been studied in general spaces and, in particular, they have been applied to the one-dimensional fuzzy case. They provide a competing robust estimation procedure of the central tendency for fuzzy number-valued data, but they are not the only way to define a trimmed mean in this space. The aim is to adapt trimmed means defined on the basis of certain depth function to the framework of fuzzy number-valued data and compare their behaviour with that of empirical fuzzy trimmed means. The first idea for evaluating the depth of a fuzzy number-valued observation consists of applying an existing functional depth to the expression of such an observation as a function. The second alternative introduces a depth function specifically defined for fuzzy numbers. The empirical performance of both proposals is analyzed.

Keywords: Fuzzy numbers, Empirical fuzzy trimmed mean, Depth-based fuzzy trimmed mean, D_θ-depth

1. Introduction

In the last decades, numerous statistical techniques have been proposed in the fuzzy-valued data framework with the aim of providing both solid mathematical foundations and suitable and easy-to-use tools for their applications. In particular, the robust estimation of the location of fuzzy number-valued data has been analyzed and some successful alternatives, such as the empirical fuzzy trimmed mean, the median and M-estimators, have been extended

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(see e.g. Colubi and González-Rodríguez [1] and Sinova *et al.* [2, 3, 4]). All these measures have been empirically compared in Sinova and Van Aelst [5, 6] by means of the bias, the variance, the mean square error and the maximum asymptotic bias. In particular, the empirical fuzzy trimmed mean with trimming proportion equal to .5 and M-estimators have behaved particularly good. Concerning fuzzy-valued data, the trimmed mean has been proposed in [1] as an adaptation of the concept of trimmed mean in separable Hilbert spaces. However, nothing has been stated about other possible adaptations like trimmed means based on depth functions yet, although this is a very frequent approach in the literature (see Cuesta-Albertos and Nieto-Reyes [7], Cuesta-Albertos and Fraiman [8] and López-Pintado and Romo [9], among others).

For adapting trimmed means defined on the basis of certain depth function to the framework of fuzzy number-valued data, it is obvious that a depth to evaluate the 'centrality' of the fuzzy-valued observations is needed. As far as the authors know, depths for fuzzy numbers have not been tackled in the literature yet, so the first proposal takes advantage of the already existing depths for functional-valued data, by expressing the fuzzy-valued observations as functions with a common domain determined by the 0-level of all the sample data. The particularities of the space of fuzzy numbers explain the interest of studying the empirical performance of this notion since they may cause important inconveniences when applying functional depths directly.

The second proposal introduces a depth function specifically defined for fuzzy numbers, with potential interest and implications in, for instance, the study of the central tendency of fuzzy number-valued data. The empirical performance of the fuzzy trimmed means based on the depth functions from both proposals is analyzed.

The manuscript is structured as follows. First, Section 2 revises all the preliminary concepts and results involving fuzzy numbers that are necessary for the correct understanding of the remaining sections. The empirical fuzzy trimmed mean is recalled in Section 3, whereas the adaptation of (functional) depth-based trimmed means to the one-dimensional fuzzy setting is presented in Section 4. Their comparison is addressed in Section 5 by considering both the mean square error as a measure of their finite-sample behaviour and the maximum asymptotic bias. Section 6 provides the novel concept of depth expressly defined for fuzzy numbers (the D_θ-depth) and shows some of its basic properties. Furthermore, the empirical competitiveness of the corresponding fuzzy trimmed mean based on the D_θ-depth is also shown.

Finally, Section 7 contains some concluding remarks.

2. Preliminaries on the space of fuzzy numbers

Fuzzy numbers are a generalization of interval-valued data, frequently used to model characteristics such as perceptions or opinions. In this section, some preliminaries about their space will be recalled.

Definition 2.1. *A* ***fuzzy number*** *is a mapping* $\widetilde{U} : \mathbb{R} \to [0,1]$ *such that, for each* $\alpha \in (0,1]$*, the corresponding* α*-level* $\widetilde{U}_\alpha = \{x \in \mathbb{R} : \widetilde{U}(x) \geq \alpha\}$ *is a nonempty compact interval of* $\mathbb{R}$*, and the* 0*-level* $\widetilde{U}_0 = \mathrm{cl}\{x \in \mathbb{R} : \widetilde{U}(x) > 0\}$ *(*cl *denoting the closure) is a nonempty interval of* $\mathbb{R}$*.*

For each $x \in \mathbb{R}$*,* $\widetilde{U}(x)$ *represents the 'degree of membership' of* x *to* $\widetilde{U}$ *or the 'degree of compatibility' of* x *with an ill-defined property* $\widetilde{U}$*.*

Equivalently, a fuzzy number can be defined as a normal (i.e., having nonempty 1-level) upper semi-continuous and quasi-concave $[0,1]$-valued function defined on $\mathbb{R}$. The space of fuzzy numbers is denoted by $\mathcal{F}_c(\mathbb{R})$.

The arithmetic that will be considered along this paper for dealing with fuzzy numbers is the usual fuzzy arithmetic based on Zadeh's extension principle (see Zadeh [10]). The two most important operations from the statistical point of view, the sum and the product by a scalar, extend the corresponding ones for intervals. Given $\widetilde{U}, \widetilde{V} \in \mathcal{F}_c(\mathbb{R})$ and $\gamma \in \mathbb{R}$, the *fuzzy sum of* $\widetilde{U}$ and $\widetilde{V}$ is defined as $\widetilde{U} + \widetilde{V} \in \mathcal{F}_c(\mathbb{R})$ such that for each $\alpha \in [0,1]$

$$(\widetilde{U} + \widetilde{V})_\alpha = \widetilde{U}_\alpha + \widetilde{V}_\alpha = \{y + z : y \in \widetilde{U}_\alpha, z \in \widetilde{V}_\alpha\},$$

and the *fuzzy product of* $\widetilde{U}$ *by the scalar* γ is defined as $\gamma \cdot \widetilde{U} \in \mathcal{F}_c(\mathbb{R})$ such that for each $\alpha \in [0,1]$

$$(\gamma \cdot \widetilde{U})_\alpha = \gamma \cdot \widetilde{U}_\alpha = \{\gamma y : y \in \widetilde{U}_\alpha\}.$$

As the previous definition states, this 'level-wise' sum of fuzzy numbers always provides a fuzzy number. It is interesting to notice that the usual fuzzy arithmetic presented above does not coincide with the usual functional arithmetic. Figure 1 illustrates how the fuzzy sum and the functional sum do not coincide in general. Indeed the functional sum from Figure 1 cannot be a fuzzy number because it takes values outside the unit interval, and the meaning of fuzzy values could be usually lost.

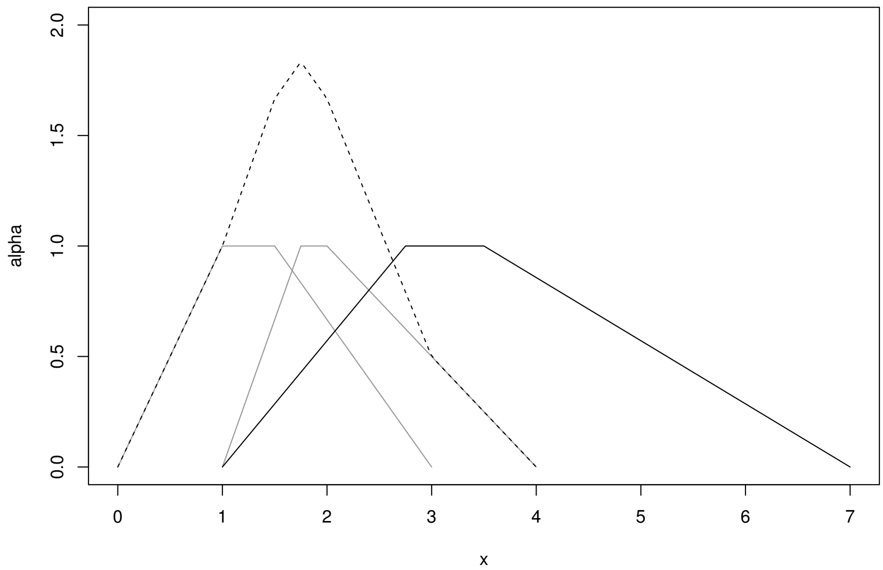

Figure 1: Two trapezoidal fuzzy numbers (in grey). In black, their fuzzy sum (solid line) and their functional sum (dashed line).

Actually, the space $\mathcal{F}_c(\mathbb{R})$ endowed with these two operations does not have a linear, but a semilinear-conical structure. Due to the previous reasons, fuzzy data should not be treated directly as functional data in the statistical developments, since the outputs may not belong to the space of fuzzy numbers. However, there exists a link with functional data in the following sense: fuzzy data can be identified with their support functions.

Definition 2.2. *The* ***support function*** *of* $\widetilde{U} \in \mathcal{F}_c(\mathbb{R})$ *(see Puri and Ralescu [11]) is given by the mapping*

$$\begin{array}{rccl} s_{\widetilde{U}}: & \{-1,1\} \times (0,1] & \longrightarrow & \mathbb{R} \\ & (u,\alpha) & \longmapsto & s_{\widetilde{U}}(u,\alpha) = \sup_{v \in \widetilde{U}_\alpha} \langle u, v \rangle, \end{array}$$

where $\langle \cdot, \cdot \rangle$ *denotes the inner product in* $\mathbb{R}$.

Let $\mathcal{H}$ denote the space $\mathcal{L}^2(\{-1,1\} \times (0,1], \lambda_1 \times \lambda)$ of the L^2-type real-valued functions defined on the unit sphere of $\mathbb{R}$ times the interval $(0,1]$ with respect to the corresponding normalized Lebesgue measures denoted by λ_1 and λ; and let $\mathcal{F}_c^2(\mathbb{R}) = \{\widetilde{U} \in \mathcal{F}_c(\mathbb{R}) : s_{\widetilde{U}} \in \mathcal{H}\}$. A possible way to measure the distance between two fuzzy numbers is to define, for any $\theta \in (0, \infty)$, the mapping $D_\theta : \mathcal{F}^2_c(\mathbb{R}) \times \mathcal{F}^2_c(\mathbb{R}) \to [0, \infty)$ such that, for any $\widetilde{U}, \widetilde{V} \in \mathcal{F}^2_c(\mathbb{R})$,

$$D_\theta(\widetilde{U}, \widetilde{V}) = \sqrt{\int_{(0,1]} \left((\operatorname{mid} \widetilde{U}_\alpha - \operatorname{mid} \widetilde{V}_\alpha)^2 + \theta \, (\operatorname{spr} \widetilde{U}_\alpha - \operatorname{spr} \widetilde{V}_\alpha)^2 \right) d\alpha}.$$

The terms mid and spr denote the mid-point and the radius (or spread) of the corresponding α-level interval. This L^2-type metric (see Montenegro *et al.* [12] and Trutschnig *et al.* [13]) makes the identification between fuzzy numbers and functional data possible. The function

$$\begin{array}{rccl} s: & \mathcal{F}^2_c(\mathbb{R}) & \to & \mathcal{H} \\ & \widetilde{U} & \mapsto & s_{\widetilde{U}} \end{array}$$

states an isometrical embedding of $\mathcal{F}^2_c(\mathbb{R})$ onto a convex cone of $\mathcal{H}$ (in which the norm associated with the inner product, $\|s_{\widetilde{U}} - s_{\widetilde{V}}\|_\theta$, coincides with $D_\theta(\widetilde{U}, \widetilde{V})$). Apart from this identification, it is important to notice the true impact of metrics in the statistical developments with fuzzy data. One consequence of the lack of linearity is the absence of a difference operation that is always well-defined and preserves the connection it has with the sum in the real settings. Therefore, the use of distances will allow us to avoid the difference operator in many situations, and be able to cope with certain statistical techniques.

The metric D_θ is topologically equivalent to the distance d_2 given by (see Klement *et al.* [14])

$$d_2(\widetilde{U}, \widetilde{V}) = \sqrt{\int_{(0,1]} (d_H(\widetilde{U}_\alpha, \widetilde{V}_\alpha))^2 \, d\alpha},$$

where $d_H(K, K') = |\operatorname{mid} K - \operatorname{mid} K'| + |\operatorname{spr} K - \operatorname{spr} K'|$ is the well-known Hausdorff metric between two nonempty compact intervals of $\mathbb{R}$.

With respect to the mathematical model that formalizes the association of a fuzzy value with each outcome of a random experiment, *random fuzzy numbers* were introduced by Puri and Ralescu [15] as the level-wise extension of random intervals, but they can be equivalently formalized in different ways.

Definition 2.3. *Given a probability space* $(\Omega, \mathcal{A}, P)$*, a* ***random fuzzy number*** *is a Borel measurable mapping* $\mathcal{X} : \Omega \to \mathcal{F}^2_c(\mathbb{R})$ *w.r.t.* $\mathcal{A}$ *and the Borel* σ*-field generated by the topology induced by the metric* D_θ *on* $\mathcal{F}^2_c(\mathbb{R})$*.*

Equivalently, $\mathcal{X}$ is a random fuzzy number if and only if $s_{\mathcal{X}}$ is a Hilbert-valued random variable. One of the best-known measures to summarize the central tendency of a random fuzzy number is the Aumann-type mean value.

Definition 2.4. *If $\mathcal{X}$ is an integrably bounded random fuzzy number, that is, if $\max\{|\inf \mathcal{X}_0|, |\sup \mathcal{X}_0|\} \in L^1(\Omega, \mathcal{A}, P)$, the **Aumann-type mean value** has been defined by Puri and Ralescu [15] as the fuzzy number $\widetilde{E}(\mathcal{X}) \in \mathcal{F}_c^2(\mathbb{R})$ such that, for each $\alpha \in (0,1]$,*

$$(\widetilde{E}(\mathcal{X}))_\alpha = [E(\inf \mathcal{X}_\alpha), E(\sup \mathcal{X}_\alpha)].$$

Regarding the study of the dispersion of a random fuzzy number, its variance has been defined as a measure of the "error" (in terms of the D_θ distance) in estimating the values of the random fuzzy number through the Aumann-type mean value (see Lubiano *et al.* [16]).

Definition 2.5. *If $\mathcal{X}$ is a random fuzzy number associated with a probability space $(\Omega, \mathcal{A}, P)$, such that $\|s_{\mathcal{X}}\|_\theta \in L^2(\Omega, \mathcal{A}, P)$, the **Fréchet variance** has been defined as the real number $\sigma^2_{\mathcal{X}} = E\left[(D_\theta(\mathcal{X}, \widetilde{E}(\mathcal{X})))^2\right]$.*

Despite its nice statistical and probabilistic properties, the Aumann-type mean presents a clear and severe inconvenience. This location measure is too sensitive to data changes and outliers, which makes its use inadvisable in lots of real-life applications because data contamination is very common in those cases. Some robust location measures have already been introduced in the literature to tackle this problem, and one of the alternatives with best performance is the empirical (fuzzy) trimmed mean.

3. The empirical fuzzy trimmed mean and Cuesta-Albertos and Fraiman's approximation

Taking into account that every fuzzy number with support function belonging to $\mathcal{H}$ can be identified with an element of a Hilbert space (its own support function), Colubi and González-Rodríguez [1] adapted the concept of trimmed means in separable Hilbert spaces (see Cuesta-Albertos and Fraiman [8]) to the fuzzy-valued framework.

Definition 3.1. *For each trimming proportion $\beta \in (0,1)$, the corresponding **fuzzy trimmed expected value** of a random fuzzy number $\mathcal{X}$ is defined as $\widetilde{E}_{[\beta]}(\mathcal{X}) = \widetilde{E}(\mathcal{X}|A_{P_{\mathcal{X}}}) \in \mathcal{F}_c^2(\mathbb{R})$, with $A_{P_{\mathcal{X}}}$ denoting the **trimming region***

$$A_{P_{\mathcal{X}}} = \arg \min_{\substack{A \subset \mathcal{H} \\ P_{s_{\mathcal{X}}}(A) \geq 1-\beta}} \int_A \left(D_\theta(\tilde{x}, \widetilde{E}(\mathcal{X}|A)) \right)^2 \, dP_{s_{\mathcal{X}}}(s_{\tilde{x}})$$

and $P_{s_{\mathcal{X}}}$, the induced probability distribution on the Borel σ-algebra on $\mathcal{H}$.

Analogously, the sample trimming region is

$$\hat{A} = \arg \min_{\substack{A \subset \{1,\ldots,n\} \\ \#A = h}} \frac{1}{h} \sum_{i \in A} \| s_{\tilde{x}_i} - \frac{1}{h} \sum_{j \in A} s_{\tilde{x}_j} \|_\theta^2 = \arg \min_{\substack{A \subset \{1,\ldots,n\} \\ \#A = h}} \frac{1}{h} \sum_{i \in A} \| s_{\tilde{x}_i} - s_{\frac{1}{h}\sum_{j \in A} \tilde{x}_j} \|_\theta^2$$

$$= \arg \min_{\substack{A \subset \{1,\ldots,n\} \\ \#A = h}} \frac{1}{h} \sum_{i \in A} \left(D_\theta \left(\tilde{x}_i, \frac{1}{h} \sum_{j \in A} \tilde{x}_j \right) \right)^2 = \arg \min_{\substack{A \subset \{1,\ldots,n\} \\ \#A = h}} Var(\tilde{\mathbf{x}}|A),$$

where $\tilde{\mathbf{x}} = (\tilde{x}_1, \ldots, \tilde{x}_n)$ denotes the sample obtained from $\mathcal{X}$ and $h = n - \lfloor n\beta \rfloor$, with $\lfloor \cdot \rfloor$ the floor function. Then, the ***empirical fuzzy trimmed mean*** *becomes $\frac{1}{h}\sum_{i \in \hat{A}} \tilde{x}_i$.*

To compute trimmed means in separable Hilbert spaces, Cuesta-Albertos and Fraiman [8] proposed an algorithm that approximates the empirical trimmed mean by determining an element of the sample that converges to the trimmed mean whenever this value is in the support of the considered distribution. However, this algorithm is not applicable for large data sets given its computational complexity and is clearly theoretically oriented, whereas the interest could be on practical advantages and not only on theoretical properties. For this reason, Colubi and González-Rodríguez [1] presented a new algorithm to compute the empirical fuzzy trimmed mean that was inspired by the FAST-LTS algorithm (see Rousseeuw and Van Driessen [17]). Sinova *et al.* [2] adapted this algorithm in order to avoid, as much as possible, that any local minimum traps the iterative process. Both Cuesta-Albertos and Fraiman's and Sinova et al.'s algorithms will be considered in the simulation study in Section 5.

4. Depth-based fuzzy trimmed means: a first approach

An alternative that has been explored in functional Hilbert spaces is that of trimmed means based on depth functions (see [7, 8, 9], among others). This approach has not been considered for fuzzy-valued data yet. Despite the lack of depth functions for fuzzy-valued data in the literature, a straightforward option to adapt this notion to the fuzzy-valued framework would be to express fuzzy numbers as functions in order to compute the functional depth directly. Since a statistical depth evaluates the 'centrality' of an observation with respect to a data set, given the trimming proportion $\beta \in (0,1)$, depth-based trimmed means are defined as the means of the $n - \lfloor n\beta \rfloor$ deepest observations. Therefore, once the (functional) depth values have been obtained, the sample of fuzzy numbers is sorted from largest to smallest depth and the Aumann-type mean of the non-trimmed observations is computed. This means that the mean is coherent with the arithmetic on $\mathcal{F}_c(\mathbb{R})$ and the expression of fuzzy values as functions is only needed for the computation of the depth. Notice that, with the aim of expressing fuzzy numbers as functions for such computations, it is enough to consider a partition of the domain determined by the 0-level of all the sample fuzzy numbers.

Definition 4.1. *Let $\mathcal{X} : \Omega \to \mathcal{F}_c^2(\mathbb{R})$ be a random fuzzy number associated with a probability space $(\Omega, \mathcal{A}, P)$ and, for any $n \in \mathbb{N}$, let $(\mathcal{X}_1, \ldots, \mathcal{X}_n)$ be a simple random sample from $\mathcal{X}$. Given an empirical functional depth FD_n and a trimming proportion $\beta \in (0,1)$, the* ***depth-based fuzzy trimmed mean*** *estimator is defined as*

$$FD\text{-}\overline{\mathcal{X}}_{n,\beta} = \frac{\sum_{i=1}^{n} I_{[\gamma,\infty)}(FD_n(\mathcal{X}_i)) \cdot \mathcal{X}_i}{\sum_{i=1}^{n} I_{[\gamma,\infty)}(FD_n(\mathcal{X}_i))},$$

with I_A denoting the indicator function of the set $A \subset \mathbb{R}$ and $\gamma \in [0,\infty)$ chosen to guarantee that

$$\frac{1}{n} \sum_{i=1}^{n} I_{[\gamma,\infty)}(FD_n(\mathcal{X}_i)) \simeq 1 - \beta.$$

The following proposition shows that the depth-based fuzzy trimmed mean is indeed a statistic.

Proposition 4.1. *Let $\mathcal{X} : \Omega \to \mathcal{F}_c^2(\mathbb{R})$ be a random fuzzy number associated with a probability space $(\Omega, \mathcal{A}, P)$ and, for any $n \in \mathbb{N}$, let $(\mathcal{X}_1, \ldots, \mathcal{X}_n)$ be a simple random sample from $\mathcal{X}$. Given an empirical functional depth FD_n that is upper semi-continuous and a trimming proportion $\beta \in (0,1)$, the depth-based fuzzy trimmed mean estimator is well-defined.*

Proof. The Borel measurability of $FD\text{-}\overline{\mathcal{X}}_{n,\beta}$ comes from:

- the Borel measurability of $\mathcal{X}_i$, $i \in \{1, \ldots, n\}$, due to the notion of random fuzzy numbers;
- the Borel measurability of the indicator function $I_{[\gamma,\infty)}$;
- the Borel measurability of the empirical functional depth FD_n, since FD_n is assumed to be an upper semi-continuous function. Notice that the upper semi-continuity of FD_n is one of the properties a reasonable depth function must fulfill (see Gijbels and Nagy [18] and Nieto-Reyes and Battey [19] for more details);
- the Borel measurability of the composition, product, sum and quotient (when the denominator is always different from 0) of Borel measurable functions.

Furthermore, it is guaranteed that the depth-based fuzzy trimmed mean estimator takes values on the space of fuzzy numbers as it is expressed as weighted means of the sample observations. □

Additionally, the observation that is closest to the 'centre', i.e., the observation that maximizes the depth, is known as median. In this work, the interest will be focused on the comparison of depth-based trimmed means with the empirical fuzzy trimmed mean, but, as we will point out in Section 7, we plan to address the comparative analysis of depth-based medians, fuzzy medians and M-estimators in the future.

Different proposals of depths for functional data can be found in the literature. Among them, we consider the following ones throughout this paper:

- Fraiman and Muniz's depth [20] is computed by integrating a univariate depth along the x-axis. The following options for the univariate depth have been considered for the simulation study: the original choice $1 - |\frac{1}{2} - F_n(x)|$, where F_n denotes the empirical distribution, and the well-known halfspace and simplicial depths (also known as Tukey and Liu depths, respectively). If the univariate depth is upper semi-continuous, the integrated depth is upper semi-continuous too (see Nagy *et al.* [21]).
- The random Tukey depth [7] approximates the Tukey depth by taking into account a finite number of one-dimensional projections selected at random. The upper semi-continuity of this depth has been shown in Nieto-Reyes and Battey [19].

- The random projection depth [22] computes the univariate depth of the projection of the data along a random direction. The depth of each datum is obtained by averaging such depths computed with a large number of different random directions (50 throughout this paper).

5. Comparative simulation study

In this section some simulations with fuzzy number-valued data are presented in order to compare the behaviour of the Aumann-type mean, the empirical fuzzy trimmed mean, Cuesta-Albertos and Fraiman's approximation and the depth-based fuzzy trimmed means listed in Section 4. The comparison of all these location measures is important to try to answer the following question: "which estimator should we choose to summarize the central tendency of a fuzzy number-valued data set?" Two perspectives are shown in Sections 5.1 and 5.2.

First, the comparison by means of the mean square error searches for the estimator that provides the nearest (in mean square error sense) fuzzy-valued estimates to the corresponding population value, and provides us with some information about the finite-sample behaviour of the estimators. Secondly, the comparison in terms of the maximum asymptotic bias is presented, with the aim of analyzing the impact of data contamination on the asymptotic value of the estimators.

The empirical study uses some distributions that are commonly considered when working with fuzzy data. Only trapezoidal-valued random fuzzy numbers have been considered, each of them characterized by four real-valued random variables:

$$X_1 = \operatorname{mid} \mathcal{X}_1, \quad X_2 = \operatorname{spr} \mathcal{X}_1, \quad X_3 = \inf \mathcal{X}_1 - \inf \mathcal{X}_0, \quad X_4 = \sup \mathcal{X}_0 - \sup \mathcal{X}_1,$$

so $\mathcal{X} = \operatorname{Tra}(X_1 - X_2 - X_3, X_1 - X_2, X_1 + X_2, X_1 + X_2 + X_4)$. Trapezoidal fuzzy numbers make the computation easier, and as previous sensitivity analysis have shown, the shape of fuzzy numbers scarcely affect statistical conclusions (see Lubiano *et al.* [23]).

5.1. Comparison in terms of the mean square error

A sample of size $n = 100$ is generated from $\mathcal{X}$ and divided into a contaminated subsample, of size nc_p ($c_p \in \{0, .1, .2, .4\}$ is the contamination proportion), and a noncontaminated subsample, distinguishing two cases.

In Case 1, $\{X_i\}_{i=1}^4$ are independent random variables with the following distributions:

- $X_1 \sim \mathcal{N}(0,1)$ and $X_2, X_3, X_4 \sim \chi_1^2$ for the noncontaminated subsample;
- $X_1 \sim \mathcal{N}(0,3) + C_D$ and/or $X_2, X_3, X_4 \sim \chi_4^2 + C_D$ for the contaminated subsample.

In Case 2, $\{X_i\}_{i=1}^4$ are dependent and their distributions are:

- $X_1 \sim \mathcal{N}(0,1)$ and $X_2, X_3, X_4 \sim 1/(X_1^2+1)^2 + .1\ \chi_1^2$ for the noncontaminated subsample;
- $X_1 \sim \mathcal{N}(0,3) + C_D$ and/or $X_2, X_3, X_4 \sim 1/(X_1^2+1)^2 + .1\ \chi_1^2 + C_D$ for the contaminated subsample.

In both cases, $C_D \in \{0, 1, 5, 10, 100\}$ is the contamination magnitude. The metric $D_{1/3}$ has been chosen and $NMC = 10000$ Monte Carlo replications have been used to approximate the population measures. The trimming proportion has been fixed depending on c_p ($\beta = .2$ if $c_p \leq .2$ and .45 otherwise). Finally, a partition of the domain $[\min\{\inf(\mathcal{X}_i)_0\}_{i=1}^n, \max\{\sup(\mathcal{X}_i)_0\}_{i=1}^n]$ into 500 equidistant points has allowed us to represent fuzzy numbers as functions and apply the depths given in Section 4, which are implemented in the R package `fda.usc` [24, 25]. In order to measure the performance of each location measure, the mean square error (MSE) and its standard deviation (s) have been computed through the following formulas by considering $N = 500$ replications

$$MSE = \frac{1}{N}\sum_{i=1}^{N}\left(D_\theta(\widehat{\widetilde{T}}_i, \widetilde{T})\right)^2, \quad s = \sqrt{\frac{1}{N}\sum_{i=1}^{N}\left(\left(D_\theta(\widehat{\widetilde{T}}_i, \widetilde{T})\right)^2 - MSE\right)^2},$$

where $\widehat{\widetilde{T}}_i$ denotes the estimator of the population value $\widetilde{T}$ obtained for the ith sample, $1 \leq i \leq N$. It is important to note that, although trimmed means are estimators of the population mean in the location-scale model, this does not hold in a general Hilbert space because it lacks symmetry, which is necessary for the population trimmed means to coincide with the population mean.

Results in Tables 1 and 2 show that in most situations the empirical fuzzy trimmed mean provides the minimum MSE (in bold for each situation), the differences being clearer in Case 1 than in Case 2. Among the depth-based fuzzy trimmed means, Fraiman and Muniz's proposal is the only one achieving the best performance in a few situations in Table 1, whereas the random Tukey depth and, more frequently, the random projection depth sometimes appear highlighted in Table 2 too. Overall, the adaptation of depth-based trimmed means to the fuzzy framework does not seem to be good enough and the reason is probably that the 0-level of the generated fuzzy numbers may be very different, opposite to what happens in the functional case, where it is assumed that functions share the domain.

5.2. Comparison in terms of the maxbias

In order to complete the comparison of the empirical fuzzy trimmed mean and the depth-based fuzzy trimmed means, we examine the maximum asymptotic bias of these estimators. The *maximum asymptotic D_θ-bias* (or *maxbias* for short) of a fuzzy number-valued estimator $\widehat{\widetilde{T}}$ at any distribution F_ϑ on $\mathcal{F}_c(\mathbb{R})$ is given by

$$MB_{\widehat{\widetilde{T}}}(\varepsilon, \vartheta) = \max_{F \in \mathcal{F}_{\varepsilon,\vartheta}} D_\theta(\widehat{\widetilde{T}}_\infty(F), \widehat{\widetilde{T}}_\infty(F_\vartheta)),$$

where $\mathcal{F}_{\varepsilon,\vartheta} = \{(1-\varepsilon)F_\vartheta + \varepsilon G : G \in \mathcal{G}\}$, with a family of distributions $\mathcal{G}$, represents an ε-neighbourhood of F_ϑ, and $\widehat{\widetilde{T}}_\infty(F)$ denotes the *asymptotic value* of the estimator (i.e., the limit in probability of the considered estimator). The maxbias is considered to be the most accurate measure of robustness for point estimators, since it informs about the behaviour of the estimator when the fraction of data contamination can be withstood.

Notice that when the location measure is strongly consistent, the asymptotic value of the corresponding estimator coincides with the population location value, $\widetilde{T}_F$, and the formula for the maxbias becomes

$$MB_{\widehat{\widetilde{T}}}(\varepsilon, \vartheta) = \max_{F \in \mathcal{F}_{\varepsilon,\vartheta}} D_\theta(\widetilde{T}_F, \widetilde{T}_{F_\vartheta}).$$

The strong consistency of the empirical fuzzy trimmed mean is an immediate consequence of Cuesta-Albertos and Fraiman's results [8, 26] for general Hilbert-valued random elements when the metric space $(\mathcal{F}_c^2(\mathbb{R}), D_\theta)$ can be

Table 1: Independent case. Mean square error and, in brackets, its standard deviation for the Aumann-type mean, the empirical fuzzy trimmed mean (EFTM), Cuesta-Albertos and Fraiman's approximation of the empirical trimmed mean (C&F) and the depth-based fuzzy trimmed means w.r.t. the following depths: Fraiman and Muniz's depth computed with the original univariate depth (FM-DTM), with the Tukey depth (T-DTM) and with the Liu depth (L-DTM), the random Tukey depth (RT-DTM) and the random projection depth (RP-DTM) with trimming proportions .2 and .45. In bold, the minimum mean square error in each situation.

c_p	C_D	Mean	EFTM ($\beta = .2$)	C&F ($\beta = .2$)	FM-DTM (.2)	T-DTM (.2)	L-DTM (.2)	RT-DTM (.2)	RP-DTM (.2)
0	0	.01932 (.01999)	.02899 (.03512)	.14230 (.12906)	.01883 (.02190)	**.01860** (.02170)	.01878 (.02175)	.02639 (.02765)	.02224 (.02443)
.1	0	.07116 (.05494)	.03480 (.03664)	.17626 (.15142)	.03343 (.03381)	**.02829** (.02813)	.02833 (.02909)	.05890 (.05052)	.03550 (.03439)
.1	1	.11139 (.07926)	.03221 (.03179)	.19149 (.19687)	.03727 (.03612)	.02977 (.02846)	**.02941** (.02859)	.07708 (.06820)	.04032 (.03832)
.1	5	.40809 (.19119)	**.03035** (.03227)	.20719 (.28315)	.07573 (.09143)	.04293 (.04427)	.04348 (.04583)	.16597 (.13406)	.06572 (.07184)
.1	10	1.1781 (.48350)	**.03031** (.03443)	.21394 (.24947)	.21200 (.30702)	.08153 (.12188)	.08634 (.12868)	.31875 (.29063)	.13531 (.15874)
.1	100	86.809 (31.759)	**.02946** (.03247)	.21804 (.35167)	14.709 (22.695)	3.7855 (6.8818)	4.1689 (7.3246)	14.895 (17.135)	8.6391 (13.205)
.2	0	.19575 (.11730)	**.05338** (.04368)	.20626 (.17979)	.06572 (.05155)	.05451 (.04213)	.05455 (.04257)	.15126 (.10963)	.07962 (.06456)
.2	1	.35665 (.18357)	**.05661** (.04328)	.25335 (.23912)	.09377 (.06795)	.06866 (.05052)	.06835 (.05066)	.26242 (.17219)	.11395 (.08376)
.2	5	1.6121 (.60118)	**.06102** (.05012)	.78322 (.86716)	.27253 (.27552)	.16947 (.15573)	.17393 (.16335)	.99616 (.59201)	.31495 (.22904)
.2	10	4.5359 (1.5358)	**.06757** (.05050)	1.6113 (2.3199)	.77279 (1.0795)	.40063 (0.4958)	.41223 (.51539)	2.4813 (1.3632)	.75543 (.68496)
.2	100	349.42 (112.86)	**.07145** (.04878)	2.1312 (2.9408)	66.729 (100.72)	31.132 (43.922)	32.955 (46.439)	162.89 (95.130)	56.340 (67.794)
c_p	C_D	Mean	EFTM ($\beta = .45$)	C&F ($\beta = .45$)	FM-DTM (.45)	T-DTM (.45)	L-DTM (.45)	RT-DTM (.45)	RP-DTM (.45)
.4	0	.72295 (.41615)	.10218 (.08814)	.20845 (.21625)	**.09463** (.08125)	.10397 (.08813)	.09794 (.07290)	.47324 (.36000)	.29316 (.22318)
.4	1	1.2635 (.60222)	**.09680** (.08289)	.22700 (.18870)	.14111 (.15566)	.13658 (.11759)	.12213 (.08910)	.92687 (.62497)	.46562 (.35579)
.4	5	6.1007 (2.0100)	**.08227** (.06614)	.30997 (.38059)	.94421 (1.2542)	.34922 (.41431)	.40960 (.48241)	4.1953 (2.5185)	1.5439 (1.0464)
.4	10	18.294 (5.3148)	**.08301** (.06431)	.38450 (.46045)	3.7916 (5.6403)	1.0683 (1.8203)	1.3163 (2.0271)	11.841 (6.8894)	3.7453 (2.5111)
.4	100	1386.3 (436.59)	**.08165** (.06411)	.44090 (.57968)	401.28 (590.78)	97.889 (181.68)	125.19 (206.32)	803.70 (518.33)	258.19 (223.56)

Table 2: Dependent case. Mean square error and, in brackets, its standard deviation for the Aumann-type mean, the empirical fuzzy trimmed mean (EFTM), Cuesta-Albertos and Fraiman's approximation of the empirical trimmed mean (C&F) and the depth-based fuzzy trimmed means w.r.t. the following depths: Fraiman and Muniz's depth computed with the original univariate depth (FM-DTM), with the Tukey depth (T-DTM) and with the Liu depth (L-DTM), the random Tukey depth (RT-DTM) and the random projection depth (RP-DTM) with trimming proportions .2 and .45. In bold, the minimum mean square error in each situation.

c_p	C_D	Mean	EFTM ($\beta = .2$)	C&F ($\beta = .2$)	FM-DTM (.2)	T-DTM (.2)	L-DTM (.2)	RT-DTM (.2)	RP-DTM (.2)
0	0	.13469 (.02577)	.02498 (.03393)	.04398 (.05818)	.01686 (.02090)	.01839 (.02128)	.01860 (.02166)	**.01679** (.02137)	.01842 (.02082)
.1	0	.14729 (.03474)	.02510 (.03386)	.04695 (.07369)	.02619 (.03824)	.02273 (.03009)	.02312 (.03225)	.02114 (.02835)	**.01798** (.02190)
.1	1	.10717 (.03528)	.02578 (.03076)	.05056 (.09437)	.03043 (.03856)	.01727 (.01951)	.01828 (.02122)	.01801 (.02009)	**.01439** (.01700)
.1	5	.12696 (.12449)	.02269 (.02870)	.05581 (.14072)	.08812 (.12596)	**.01768** (.02220)	.01885 (.02487)	.03881 (.03769)	.01841 (.02794)
.1	10	.53518 (.35945)	**.02325** (.02731)	.05235 (.16734)	.19414 (.31319)	.02401 (.03323)	.02797 (.03944)	.14001 (.17918)	.04942 (.09031)
.1	100	82.235 (32.190)	**.02126** (.02469)	.06388 (.24001)	15.750 (24.218)	.35878 (1.2388)	.60516 (2.9270)	22.434 (28.483)	6.6321 (11.450)
.2	0	.15350 (.03760)	.02569 (.03286)	.04772 (.08292)	.02810 (.04135)	.02453 (.03054)	.02362 (.03026)	.02488 (.03287)	**.02147** (.02891)
.2	1	.08950 (.05302)	.03231 (.03018)	.04537 (.06438)	.05256 (.06461)	**.01634** (.01913)	.01816 (.01915)	.02661 (.02625)	.01988 (.02349)
.2	5	.52042 (.33895)	**.02357** (.02555)	.65345 (1.0095)	.23113 (.34695)	.02877 (.04889)	.03618 (.06072)	.34249 (.20532)	.09577 (.12742)
.2	10	2.5858 (1.1544)	**.02253** (.01948)	1.0909 (1.4781)	.70741 (1.1502)	.06771 (.12321)	.08375 (.15325)	1.4402 (.88760)	.35501 (.48249)
.2	100	325.98 (107.56)	**.02273** (.02098)	1.2198 (1.5773)	66.900 (105.01)	6.7141 (15.540)	7.6786 (16.755)	177.82 (119.22)	46.486 (61.052)

c_p	C_D	Mean	EFTM ($\beta = .45$)	C&F ($\beta = .45$)	FM-DTM (.45)	T-DTM (.45)	L-DTM (.45)	RT-DTM (.45)	RP-DTM (.45)
.4	0	.17274 (.05211)	**.04318** (.06289)	.05771 (.08141)	.09010 (.12065)	.07830 (.09726)	.07134 (.09582)	.06709 (.08271)	.06722 (.07044)
.4	1	.09708 (.11447)	.09791 (.07741)	.08943 (.13950)	.22337 (.28367)	**.03127** (.03475)	.04914 (.04365)	.10570 (.10339)	.09461 (.08136)
.4	5	2.5030 (1.1320)	**.06875** (.05150)	.13651 (.42402)	1.2336 (2.0305)	.07819 (.17334)	.14027 (.24784)	1.8025 (1.1410)	.48566 (.36616)
.4	10	10.867 (3.9967)	**.06543** (.04219)	.17340 (.40960)	4.7083 (7.9106)	.25901 (.80772)	.43607 (1.0792)	7.4399 (4.7562)	1.7444 (1.2851)
.4	100	1283.3 (405.77)	**.06456** (.04232)	.17365 (.42412)	517.81 (843.86)	30.274 (94.320)	45.300 (116.17)	753.51 (579.68)	164.71 (187.32)

isometrically embedded onto a closed convex cone of a Hilbert space (see Sinova *et al.* [2]). The closeness of the convex cone holds if the 0-level of the fuzzy values is allowed to be unbounded (see Gil *et al.* [27]), as in $\mathcal{F}_c(\mathbb{R})$.

With respect to the depth-based fuzzy trimmed means, Fraiman and Muniz [20] proved the strong consistency of the integrated data depth in functional spaces under some conditions. They consider that each group of functions to be analyzed in terms of the depth is obtained as a realization of n independent and identically distributed stochastic processes with continuous trajectories defined on an interval (which they assume to be $[0,1]$ without any loss of generality), $X_1(t), \ldots, X_n(t)$. For every t, F_t denotes the marginal univariate distribution function of $X_1(t)$ and $F_{n,t}$, the empirical distribution of the sample $x_1(t), \ldots, x_n(t)$. If $\mathcal{J}$ represents the space of functions defined on the interval that take values on an arbitrary space J, it is assumed that $\lim_{n\to\infty} \sup_{g\in\mathcal{J}} \left| \int_0^1 F_{n,t}(g(t))\, dt - \int_0^1 F_t(g(t))\, dt \right| = 0$ almost surely. Nagy [28] proved, in general, the uniform strong consistency of the integrated depths in the setup of Borel measurable functions. On the other hand, the random Tukey depth was extended to the functional case under the assumption that the sample space is a separable Hilbert space and its uniform strong consistency was shown in Cuesta-Albertos and Nieto-Reyes [7].

The following result proves the strong consistency of the depth-based fuzzy trimmed mean in case the empirical functional depth involved in its computation is uniformly strongly consistent.

Theorem 5.1. *Let $\mathcal{X} : \Omega \to \mathcal{F}_c^2(\mathbb{R})$ be a random fuzzy number associated with a probability space $(\Omega, \mathcal{A}, P)$ and assume that $\|s_{\mathcal{X}}\|_\theta \in L^2(\Omega, \mathcal{A}, P)$. For any $n \in \mathbb{N}$, let $(\mathcal{X}_1, \ldots, \mathcal{X}_n)$ be a simple random sample from $\mathcal{X}$. Consider an empirical functional depth FD_n which uniformly converges to its population version, FD, that is,*

$$\lim_{n\to\infty} \sup_{g\in\mathcal{J}} |FD_n(g) - FD(g)| = 0 \quad \textit{almost surely.}$$

Then, for any trimming proportion $\beta \in (0,1)$ and any $\theta \in (0,\infty)$, the depth-based fuzzy trimmed mean estimator strongly converges in D_θ-sense (and hence in the sense of all topologically equivalent metrics, like d_2) to the population measure

$$FD\text{-}\widetilde{E}_{[\beta]} = \frac{\widetilde{E}(I_{[\gamma,\infty)}(FD(\mathcal{X})) \cdot \mathcal{X})}{E(I_{[\gamma,\infty)}(FD(\mathcal{X})))},$$

with $\gamma \in (0,\infty)$ *chosen to guarantee that* $P(FD(\mathcal{X}) \in [\gamma,\infty)) = 1-\beta$ *and assuming that* FD *is Borel measurable and* $P(FD(\mathcal{X}) = \gamma) = 0$*. That is,*

$$\lim_{n\to\infty} D_\theta(FD\text{-}\overline{\mathcal{X}}_{n,\beta}, FD\text{-}\widetilde{E}_{[\beta]}) = 0 \ \text{ a.s. } [P].$$

Proof. First, $I_{[\gamma,\infty)}(FD(\mathcal{X}))\cdot\mathcal{X}$ is a random fuzzy number (see the proof of Proposition 4.1) which is integrably bounded because it is upper bounded by $\mathcal{X}$, so its Aumann-type mean is well-defined. Since $E(I_{[\gamma,\infty)}(FD(\mathcal{X}))) = P(FD(\mathcal{X}) \in [\gamma,\infty)) = 1-\beta > 0$, it holds that $FD\text{-}\widetilde{E}_{[\beta]} \in \mathcal{F}_c^2(\mathbb{R})$.

Let us see that $\lim_{n\to\infty} D_\theta(FD\text{-}\overline{\mathcal{X}}_{n,\beta}, FD\text{-}\widetilde{E}_{[\beta]}) = 0$ a.s. $[P]$, where

$$D_\theta(FD\text{-}\overline{\mathcal{X}}_{n,\beta}, FD\text{-}\widetilde{E}_{[\beta]}) = \left(\int_{(0,1]} \left[\left(\operatorname{mid}(FD\text{-}\overline{\mathcal{X}}_{n,\beta})_\alpha - \operatorname{mid}(FD\text{-}\widetilde{E}_{[\beta]})_\alpha\right)^2\right.\right.$$

$$\left.\left. +\theta\left(\operatorname{spr}(FD\text{-}\overline{\mathcal{X}}_{n,\beta})_\alpha - \operatorname{spr}(FD\text{-}\widetilde{E}_{[\beta]})_\alpha\right)^2\right] d\alpha\right)^{1/2}.$$

Due to the continuity of the square root function, it is enough to prove that

$$P\left(\lim_{n\to\infty}\int_{(0,1]} \left[\left(\operatorname{mid}(FD\text{-}\overline{\mathcal{X}}_{n,\beta})_\alpha - \operatorname{mid}(FD\text{-}\widetilde{E}_{[\beta]})_\alpha\right)^2\right.\right.$$

$$\left.\left. +\theta\left(\operatorname{spr}(FD\text{-}\overline{\mathcal{X}}_{n,\beta})_\alpha - \operatorname{spr}(FD\text{-}\widetilde{E}_{[\beta]})_\alpha\right)^2\right] d\alpha = 0\right) = 1.$$

The properties of limits and integrals allow us to alternatively write the previous expression as

$$P\left(\lim_{n\to\infty}\int_{(0,1]} \left(\operatorname{mid}(FD\text{-}\overline{\mathcal{X}}_{n,\beta})_\alpha - \operatorname{mid}(FD\text{-}\widetilde{E}_{[\beta]})_\alpha\right)^2 d\alpha\right.$$

$$\left. +\theta\lim_{n\to\infty}\int_{(0,1]} \left(\operatorname{spr}(FD\text{-}\overline{\mathcal{X}}_{n,\beta})_\alpha - \operatorname{spr}(FD\text{-}\widetilde{E}_{[\beta]})_\alpha\right)^2 d\alpha = 0\right)$$

$$= P\left(\left(\lim_{n\to\infty}\int_{(0,1]} \left(\operatorname{mid}(FD\text{-}\overline{\mathcal{X}}_{n,\beta})_\alpha - \operatorname{mid}(FD\text{-}\widetilde{E}_{[\beta]})_\alpha\right)^2 d\alpha = 0\right)\right.$$

$$\left.\bigcap\left(\lim_{n\to\infty}\int_{(0,1]} \left(\operatorname{spr}(FD\text{-}\overline{\mathcal{X}}_{n,\beta})_\alpha - \operatorname{spr}(FD\text{-}\widetilde{E}_{[\beta]})_\alpha\right)^2 d\alpha = 0\right)\right).$$

Let us see that the probability of these two intersected events is 1, which ends the proof. By using the properties of the mid-point and the spread, we get

$$\operatorname{mid}(FD\text{-}\overline{\mathcal{X}}_{n,\beta})_\alpha = \frac{\sum_{i=1}^n I_{[\gamma,\infty)}(FD_n(\mathcal{X}_i)) \operatorname{mid}(\mathcal{X}_i)_\alpha}{\sum_{i=1}^n I_{[\gamma,\infty)}(FD_n(\mathcal{X}_i))},$$

$$\operatorname{spr}(FD\text{-}\overline{\mathcal{X}}_{n,\beta})_\alpha = \frac{\sum_{i=1}^n I_{[\gamma,\infty)}(FD_n(\mathcal{X}_i)) \operatorname{spr}(\mathcal{X}_i)_\alpha}{\sum_{i=1}^n I_{[\gamma,\infty)}(FD_n(\mathcal{X}_i))},$$

$$\operatorname{mid}(FD\text{-}\widetilde{E}_{[\beta]})_\alpha = \frac{E[I_{[\gamma,\infty)}(FD(\mathcal{X})) \operatorname{mid}\mathcal{X}_\alpha]}{E[I_{[\gamma,\infty)}(FD(\mathcal{X}))]},$$

$$\operatorname{spr}(FD\text{-}\widetilde{E}_{[\beta]})_\alpha = \frac{E[I_{[\gamma,\infty)}(FD(\mathcal{X})) \operatorname{spr}\mathcal{X}_\alpha]}{E[I_{[\gamma,\infty)}(FD(\mathcal{X}))]}.$$

For any $\alpha \in (0,1]$,

$$P\left(\lim_{n\to\infty}\left(\operatorname{mid}(FD\text{-}\overline{\mathcal{X}}_{n,\beta})_\alpha - \operatorname{mid}(FD\text{-}\widetilde{E}_{[\beta]})_\alpha\right)^2 = 0\right)$$

$$= P\left(\lim_{n\to\infty}\left|\operatorname{mid}(FD\text{-}\overline{\mathcal{X}}_{n,\beta})_\alpha - \operatorname{mid}(FD\text{-}\widetilde{E}_{[\beta]})_\alpha\right| = 0\right)$$

$$= P\left(\lim_{n\to\infty}\left|\frac{\frac{1}{n}\sum_{i=1}^n I_{[\gamma,\infty)}(FD_n(\mathcal{X}_i)) \operatorname{mid}(\mathcal{X}_i)_\alpha}{\frac{1}{n}\sum_{i=1}^n I_{[\gamma,\infty)}(FD_n(\mathcal{X}_i))} - \frac{E[I_{[\gamma,\infty)}(FD(\mathcal{X})) \operatorname{mid}\mathcal{X}_\alpha]}{E[I_{[\gamma,\infty)}(FD(\mathcal{X}))]}\right| = 0\right).$$

Notice that [20] showed the almost sure convergence of the denominator $\frac{1}{n}\sum_{i=1}^n I_{[\gamma,\infty)}(FD_n(\mathcal{X}_i))$ to $E[I_{[\gamma,\infty)}(FD(\mathcal{X}))]$, so the problem reduces to check the almost sure convergence of the numerator, i.e.,

$$\lim_{n\to\infty}\left|\frac{1}{n}\sum_{i=1}^n I_{[\gamma,\infty)}(FD_n(\mathcal{X}_i)) \operatorname{mid}(\mathcal{X}_i)_\alpha - E[I_{[\gamma,\infty)}(FD(\mathcal{X})) \operatorname{mid}\mathcal{X}_\alpha]\right| = 0 \text{ a.s. } [P].$$

By the triangle inequality we have that

$$\lim_{n\to\infty}\left|\frac{1}{n}\sum_{i=1}^n I_{[\gamma,\infty)}(FD_n(\mathcal{X}_i)) \operatorname{mid}(\mathcal{X}_i)_\alpha - E[I_{[\gamma,\infty)}(FD(\mathcal{X})) \operatorname{mid}\mathcal{X}_\alpha]\right|$$

$$\leq \lim_{n\to\infty}\left|\frac{1}{n}\sum_{i=1}^n I_{[\gamma,\infty)}(FD_n(\mathcal{X}_i)) \operatorname{mid}(\mathcal{X}_i)_\alpha - \frac{1}{n}\sum_{i=1}^n I_{[\gamma,\infty)}(FD(\mathcal{X}_i)) \operatorname{mid}(\mathcal{X}_i)_\alpha\right|$$

$$+\lim_{n\to\infty}\left|\frac{1}{n}\sum_{i=1}^{n} I_{[\gamma,\infty)}(FD(\mathcal{X}_i))\ \mathrm{mid}\,(\mathcal{X}_i)_\alpha - E[I_{[\gamma,\infty)}(FD(\mathcal{X}))\ \mathrm{mid}\,\mathcal{X}_\alpha]\right|.$$

The Strong Law for Large Numbers guarantees the almost sure convergence of $\frac{1}{n}\sum_{i=1}^{n} I_{[\gamma,\infty)}(FD(\mathcal{X}_i))\ \mathrm{mid}\,(\mathcal{X}_i)_\alpha$ to $E[I_{[\gamma,\infty)}(FD(\mathcal{X}))\ \mathrm{mid}\,\mathcal{X}_\alpha]$, so the second term equals 0 a.s. $[P]$. For dealing with the first term, we follow the ideas in [20]. If δ is any positive number, the assumption

$$\lim_{n\to\infty}\sup_{g\in\mathcal{J}}|FD_n(g)-FD(g)|=0 \text{ almost surely}$$

establishes that there exists $n_0\in\mathbb{N}$ such that $S_n := \sup_{g\in\mathcal{J}}|FD_n(g)-FD(g)|<\delta$ for all $n\ge n_0$ almost surely. Then,

$$\lim_{n\to\infty}\left|\frac{1}{n}\sum_{i=1}^{n} I_{[\gamma,\infty)}(FD_n(\mathcal{X}_i))\ \mathrm{mid}\,(\mathcal{X}_i)_\alpha - \frac{1}{n}\sum_{i=1}^{n} I_{[\gamma,\infty)}(FD(\mathcal{X}_i))\ \mathrm{mid}\,(\mathcal{X}_i)_\alpha\right|$$

$$\le \lim_{n\to\infty}\frac{1}{n}\sum_{i=1}^{n}\left|I_{[\gamma,\infty)}(FD_n(\mathcal{X}_i))\ - I_{[\gamma,\infty)}(FD(\mathcal{X}_i))\right|\ |\mathrm{mid}\,(\mathcal{X}_i)_\alpha|$$

$$\le \lim_{n\to\infty}\frac{1}{n}\sum_{i=1}^{n}\left|I_{[\gamma,\infty)}(FD(\mathcal{X}_i)+\delta) - I_{[\gamma,\infty)}(FD(\mathcal{X}_i)-\delta)\right| |\mathrm{mid}\,(\mathcal{X}_i)_\alpha|\, I_{\{S_n<\delta\}}$$

$$+\lim_{n\to\infty}\frac{1}{n}\sum_{i=1}^{n}\left|I_{[\gamma,\infty)}(FD(\mathcal{X}_i)+S_n) - I_{[\gamma,\infty)}(FD(\mathcal{X}_i)-S_n)\right| |\mathrm{mid}\,(\mathcal{X}_i)_\alpha|\, I_{\{S_n\ge\delta\}}$$

by taking into account that the function $I_{[\gamma,\infty)}$ is non-decreasing. The second limit is upper bounded by $\lim_{n\to\infty}\frac{1}{n}\sum_{i=1}^{n}|\mathrm{mid}\,(\mathcal{X}_i)_\alpha|\, I_{\{S_n\ge\delta\}}=0$ a.s. $[P]$ because $\lim_{n\to\infty}S_n=0$ a.s. $[P]$. On the other hand, by applying the Strong Law for Large Numbers, we get

$$\lim_{n\to\infty}\frac{1}{n}\sum_{i=1}^{n}\left|I_{[\gamma,\infty)}(FD(\mathcal{X}_i)+\delta) - I_{[\gamma,\infty)}(FD(\mathcal{X}_i)-\delta)\right| |\mathrm{mid}\,(\mathcal{X}_i)_\alpha|\, I_{\{S_n<\delta\}}$$

$$= E\left[\left|I_{[\gamma,\infty)}(FD(\mathcal{X})+\delta) - I_{[\gamma,\infty)}(FD(\mathcal{X})-\delta)\right| |\mathrm{mid}\,\mathcal{X}_\alpha|\right] \text{ a.s. } [P].$$

This bound is obtained for any $\delta>0$, so

$$\lim_{n\to\infty}\left|\frac{1}{n}\sum_{i=1}^{n} I_{[\gamma,\infty)}(FD_n(\mathcal{X}_i))\ \mathrm{mid}\,(\mathcal{X}_i)_\alpha - \frac{1}{n}\sum_{i=1}^{n} I_{[\gamma,\infty)}(FD(\mathcal{X}_i))\ \mathrm{mid}\,(\mathcal{X}_i)_\alpha\right|$$

$$\leq \lim_{\delta\to 0} E\left[\left|I_{[\gamma,\infty)}(FD(\mathcal{X})+\delta) - I_{[\gamma,\infty)}(FD(\mathcal{X})-\delta)\right| |\mathrm{mid}\,\mathcal{X}_\alpha|\right].$$

By the dominated convergence theorem (the function $I_{[\gamma,\infty)}$ is bounded and monotone and $E[|\mathrm{mid}\,\mathcal{X}_\alpha|] < \infty$ due to the condition $\|s_{\mathcal{X}}\|_\theta \in L^2(\Omega, \mathcal{A}, P)$), the latter limit is equal to 0 almost surely, which completes the proof of

$$P\left(\lim_{n\to\infty}\left(\mathrm{mid}\,(FD\text{-}\overline{\mathcal{X}}_{n,\beta})_\alpha - \mathrm{mid}\,(FD\text{-}\widetilde{E}_{[\beta]})_\alpha\right)^2 = 0\right) = 1.$$

Finally, the sequence $\{\mathrm{mid}\,(FD\text{-}\overline{\mathcal{X}}_{n,\beta})_\alpha\}_n$ is uniformly integrable as function of α over $(0,1]$ because it is upper bounded by $\max\{|\inf \mathcal{X}_0|, |\sup \mathcal{X}_0|\}$. Thus, the conditions to apply Vitali's Convergence Theorem are fulfilled, whence

$$P\left(\lim_{n\to\infty}\int_{(0,1]}\left(\mathrm{mid}\,(FD\text{-}\overline{\mathcal{X}}_{n,\beta})_\alpha - \mathrm{mid}\,(FD\text{-}\widetilde{E}_{[\beta]})_\alpha\right)^2 d\alpha = 0\right) = 1.$$

Analogously, it could be proven that

$$P\left(\lim_{n\to\infty}\int_{(0,1]}\left(\mathrm{spr}\,(FD\text{-}\overline{\mathcal{X}}_{n,\beta})_\alpha - \mathrm{spr}\,(FD\text{-}\widetilde{E}_{[\beta]})_\alpha\right)^2 d\alpha = 0\right) = 1. \qquad \square$$

Figure 2 shows the maxbias curve of the empirical fuzzy trimmed mean and the depth-based fuzzy trimmed means listed in Section 4. The maxbias curve is a plot of $MB_{\widetilde{\widetilde{T}}}(\varepsilon, \vartheta)$ as a function of ε, that is, the maxbias an estimator can present when a fraction ε of the data is contaminated. The great difficulties that the exact computation of the maxbias implies are the reason why this problem has only been solved for a limited number of estimators in the classical settings. The usual way to handle the maxbias in the literature consists of giving an empirical approximation, which establishes a lower bound for the real value of the maxbias. Following the simulation design by Sinova and Van Aelst [5], the maxbias curves have been empirically approximated by the following steps:

Step 1. For each of the different situations, 1000 samples of 1000 trapezoidal fuzzy number-valued data have been simulated from a random fuzzy number $\mathcal{X}$ as in Case 1 (noncontaminated subsample) of Section 5.1.

Step 2. The population location measures (the empirical fuzzy trimmed mean and the depth-based fuzzy trimmed means w.r.t. Fraiman and Muniz's depth with the original univariate depth, the Tukey depth and the simplicial depth, the random Tukey depth and the random projection depth with trimming proportion .5) have been approximated by Monte Carlo simulation using the samples from *Step 1.*

Step 3. A fraction ε of contamination (also denoted by c_p by analogy with Section 5.1) has been generated. In this sense, we have distinguished four different scenarios:

Scenario 3.1 Contaminated data are real-valued and we study the effect of point contamination at any point C_D, where C_D ranges in $\{0, 100, 200, 300, 400, 500\}$.

Scenario 3.2 Contaminated data are the translation of the population Aumann-type mean, $\mathrm{Tra}(-2, -1, 1, 2)$, with C_D units.

Scenario 3.3 Contaminated data are the population Aumann-type mean times the factor C_D.

Scenario 3.4 Contaminated data are fuzzy numbers generated as in Case 1 (contaminated subsample) of Section 5.1.

Step 4. For each scenario of *Step 3* and each value of C_D, the population location measures have been approximated by Monte Carlo simulation using the contaminated samples from *Step 3*.

Step 5. For each scenario of *Step 3* and each value of C_D, the $D_{1/3}$ distances between the noncontaminated and the contaminated approximated population measures have been computed. Then, the maximum $D_{1/3}$ distance for each location measure has been computed over the four scenarios and all values of C_D.

Step 6. The interval $[0, .49]$ has been partitioned in 10 equidistant parts and for each of the resulting 11 equidistant points, c_p, *Steps 1-5* have been repeated. Finally, the maximum $D_{1/3}$ distances from *Step 5* have been plotted versus the corresponding value of c_p in Figure 2.

The main conclusion that we can draw from Figure 2 is that the maxbias curves of the depth-based fuzzy trimmed means and the empirical fuzzy trimmed mean with $\beta = .5$ completely differ. Whereas the latter maxbias is scarcely affected by the contamination and the empirical fuzzy trimmed mean with $\beta = .5$ can tolerate up to 50% of contamination, the behaviour of the considered depth-based fuzzy trimmed means seems not to withstand any

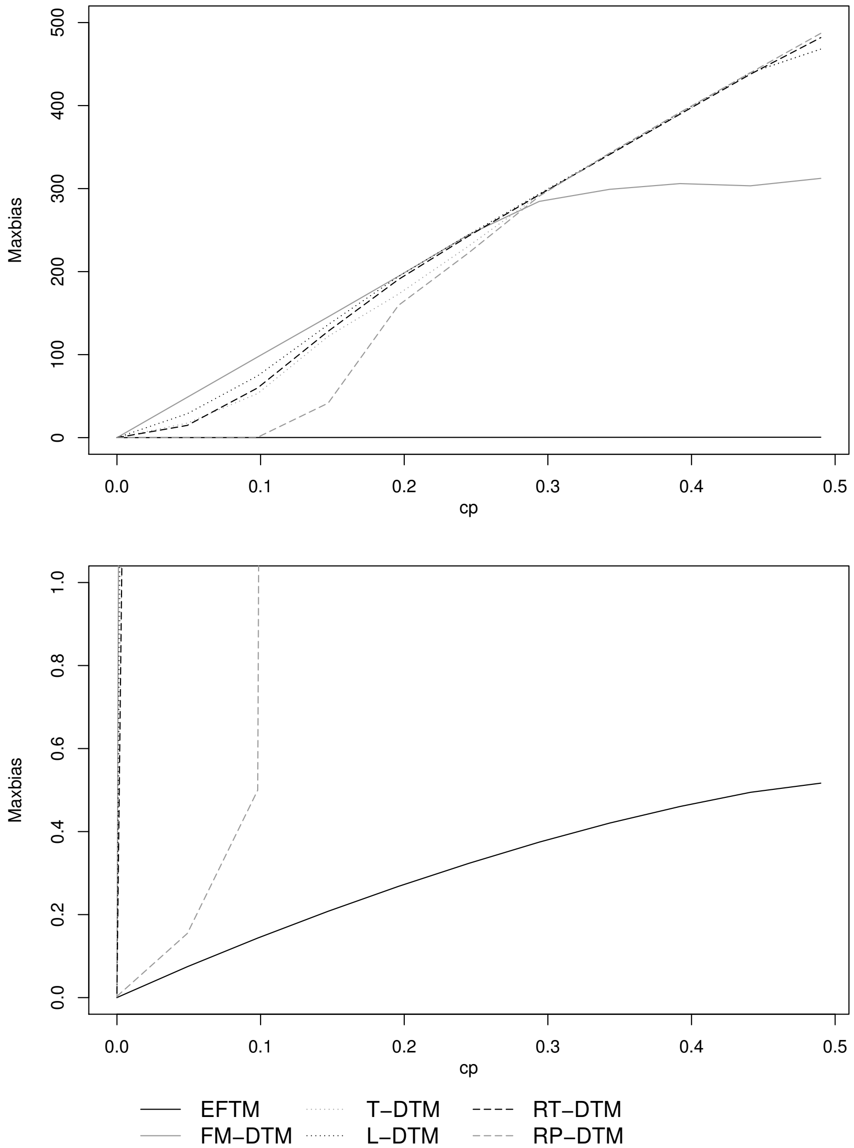

Figure 2: Maximum asymptotic biases of the empirical fuzzy trimmed mean (EFTM) and the depth-based fuzzy trimmed means w.r.t. Fraiman and Muniz's depth computed with the original univariate depth (FM-DTM), with the Tukey depth (T-DTM) and with the Liu depth (L-DTM), the random Tukey depth (RT-DTM) and the random projection depth (RP-DTM) with trimming proportion .5 as a function of c_p. The plot at the bottom shows the maxbias of the empirical fuzzy trimmed mean in detail.

contamination proportion. If we enlarge the graph, we will be able to detect some differences between the fuzzy trimmed mean based on the random projection depth and the empirical fuzzy trimmed mean even for contamination proportions smaller than .1 (see Figure 2 bottom).

Why is this approach working so poorly in practice? The answer is in the use of functional depths to measure the centrality of fuzzy numbers. Notice that two fuzzy numbers as different as the ones shown in Figure 3, Tra$(-.7383, -.2316, -.1678, .4500)$ (generated from the noncontaminated distribution) and Tra$(498, 499, 501, 502)$ (generated from the contaminated distribution of *Scenario 3.2*), are almost indistinguishable as real-valued functions defined on an interval that contains, at least, the interval $[-.7383, 502]$.

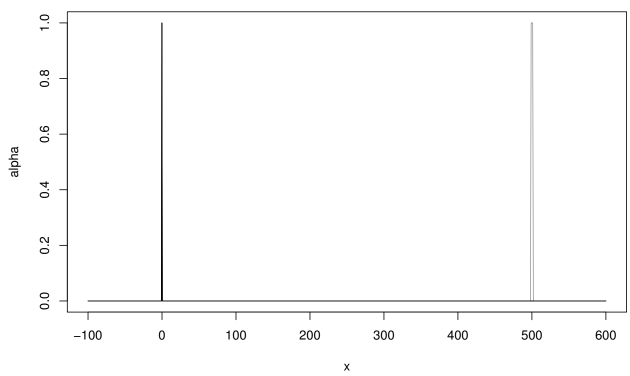

Figure 3: Fuzzy numbers Tra$(-.7383, -.2316, -.1678, .4500)$, in black, and Tra$(498, 499, 501, 502)$, in grey, as real-valued functions defined on $[-100, 600]$.

For this reason, functional depths fail to evaluate how deep a fuzzy number lies in the data cloud generated in the simulation study. To illustrate this with an example, let us remind that the idea of Fraiman and Muniz's depth is to measure how long a curve remains in the middle of a sample. When the noncontaminated and the contaminated fuzzy numbers lie as far as in Figure 3, all of them vanish at most of the domain determined by their 0-levels, which means that they coincide at many points and, therefore, they almost always remain in the middle of the sample. Consequently, their depth is going to be very similar. Figure 4 represents the depths of a data set obtained from *Scenario 3.1* with $c_p = .147$ and $C_D = 500$, and it can be seen that they are all concentrated in $[.500, .507]$.

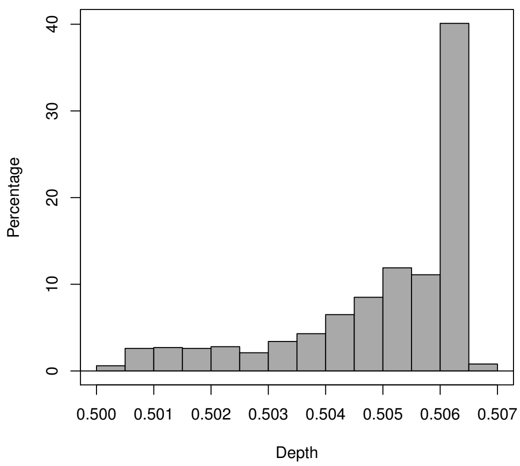

Figure 4: Fraiman and Muniz's depths for all the data from a sample generated from *Scenario 3.1* with $c_p = .147$ and $C_D = 500$.

The previous conclusions motivate the search for a depth function expressly defined for fuzzy numbers. As far as the authors know, this idea has not been considered in the literature yet, but its potential interest and implications in, for instance, the study of the central tendency of fuzzy number-valued data are irrefutable.

6. A depth for fuzzy numbers: the basis of the second approach

Now we will propose a depth function for fuzzy numbers, which will be used in the computation of a new depth-based fuzzy trimmed mean in order to avoid the issues explained in the previous section. As mentioned before, functional depths failed to detect which fuzzy numbers lie far from the 'centre' of the data set because those fuzzy numbers with a very different 0-level are indeed very similar as real-valued functions defined on a common domain. For this reason, it seems more natural to measure the depth in terms of a distance between fuzzy numbers, which really takes into account that fuzzy numbers with a very different 0-level are not close. The depth of a fuzzy number should become smaller as it lies further from the 'centre'. We consider the D_θ metric for this work, but other choices are, of course, plausible.

Definition 6.1. *Given a random fuzzy number* $\mathcal{X}$ *and a fixed value* $\theta \in (0, \infty)$*, the* $\boldsymbol{D_\theta}$***-depth*** *of* $\widetilde{V} \in \mathcal{F}_c^2(\mathbb{R})$ *with respect to the distribution of* $\mathcal{X}$ *is* *given by*

$$DD_\theta(\widetilde{V};\mathcal{X}) = \frac{1}{1+E[D_\theta(\mathcal{X},\widetilde{V})]}.$$

Given a simple random sample from $\mathcal{X}$, $(\mathcal{X}_1,\dots,\mathcal{X}_n)$, the ***empirical $\boldsymbol{D_\theta}$-depth*** *of $\widetilde{V}$ is given by*

$$DD_{\theta,n}(\widetilde{V};(\mathcal{X}_1,\dots,\mathcal{X}_n)) = \frac{1}{1+\frac{1}{n}\sum_{i=1}^{n} D_\theta(\mathcal{X}_i,\widetilde{V})}.$$

By simplicity, we will use the notation $DD_{\theta,n}(\widetilde{V})$ when no confusion is possible. This proposal is coherent with the general structure for the construction of statistical depth functions established in Zuo and Serfling [29] (see the information about the Type B depths and, in particular, the L^p depths). Let us see that the D_θ-depth fulfills some convenient properties.

Proposition 6.1. *Let $\mathcal{X}$ be a random fuzzy number and $\widetilde{V} \in \mathcal{F}_c^2(\mathbb{R})$. The D_θ-depth satisfies the following conditions:*

- *Nonnegativity and boundedness by 1: $0 \leq DD_\theta(\widetilde{V};\mathcal{X}) \leq 1$.*
- *Translation and rotation invariance: $DD_\theta(\widetilde{V}+\widetilde{U};\mathcal{X}+\widetilde{U}) = DD_\theta(\widetilde{V};\mathcal{X})$ for any $\widetilde{U} \in \mathcal{F}_c^2(\mathbb{R})$ and $DD_\theta((-1)\cdot\widetilde{V};(-1)\cdot\mathcal{X}) = DD_\theta(\widetilde{V};\mathcal{X})$.*
- *Centre: The centremost element is $\widetilde{V}_0 \in \mathcal{F}_c^2(\mathbb{R})$ such that $E[D_\theta(\mathcal{X},\widetilde{V}_0)] = \min_{\widetilde{V}\in\mathcal{F}_c^2(\mathbb{R})} E[D_\theta(\mathcal{X},\widetilde{V})]$ if this minimum exists. In particular, the centremost element of a random fuzzy number whose distribution is degenerate at $\widetilde{V}$ is $\widetilde{V}$, and its depth equals 1.*
- *Monotonicity relative to the deepest point: for any random fuzzy number $\mathcal{X}$ having a unique deepest value $\widetilde{V}_0$, it holds that $DD_\theta(\widetilde{V};\mathcal{X}) \leq DD_\theta(\widetilde{U};\mathcal{X})$ for any $\widetilde{U} \in \mathcal{F}_c^2(\mathbb{R})$ such that $E[D_\theta(\mathcal{X},\widetilde{V}_0)] \leq E[D_\theta(\mathcal{X},\widetilde{U})] \leq E[D_\theta(\mathcal{X},\widetilde{V})]$.*
- *Vanishing at infinity: $DD_\theta(\widetilde{V};\mathcal{X}) \to 0$ as $E[D_\theta(\mathcal{X},\widetilde{V})] \to \infty$, for each random fuzzy number $\mathcal{X}$.*

Proposition 6.1 can be proved straightforwardly thanks to the D_θ metric's properties. Notice that the centremost element with respect to the D_θ-depth is a notion of fuzzy median that is yet to be analyzed, and extends the d_θ-median of a random interval [30] to random fuzzy numbers. With respect to the second property of Proposition 6.1, the desirable condition mentioned in Zuo and Serfling [29] is stronger. Although the affine invariance does not hold for the D_θ-depth in general, it is possible to define a modified version that fulfills it. Let $\mathcal{X}$ be a random fuzzy number associated with a probability space $(\Omega, \mathcal{A}, P)$ such that $\|s_{\mathcal{X}}\|_\theta \in L^2(\Omega, \mathcal{A}, P)$. Taking into account that, for any $a \in \mathbb{R}$ and $\widetilde{U} \in \mathcal{F}_c^2(\mathbb{R})$, $\sigma^2_{a \cdot \mathcal{X} + \widetilde{U}} = a^2 \sigma^2_{\mathcal{X}}$ (see e.g. Blanco-Fernández *et al.* [31]), the modified version of the D_θ-depth would be

$$DD^*_\theta(\widetilde{V}; \mathcal{X}) = \frac{1}{1 + E\left[\frac{D_\theta(\mathcal{X}, \widetilde{V})}{\sigma_{\mathcal{X}}}\right]},$$

with $\sigma_{\mathcal{X}} = \sqrt{\sigma^2_{\mathcal{X}}}$ the standard deviation of $\mathcal{X}$.

The following result proves the strong consistency of the empirical D_θ-depth.

Proposition 6.2. *Let $\mathcal{X}$ be a random fuzzy number associated with a probability space $(\Omega, \mathcal{A}, P)$ such that $\|s_{\mathcal{X}}\|_\theta \in L^2(\Omega, \mathcal{A}, P)$. Let $(\mathcal{X}_1, \ldots, \mathcal{X}_n)$ be a simple random sample obtained from $\mathcal{X}$. For any $\widetilde{V} \in \mathcal{F}_c^2(\mathbb{R})$, it holds that*

$$DD_{\theta,n}(\widetilde{V}) \underset{n \to \infty}{\longrightarrow} DD_\theta(\widetilde{V}; \mathcal{X}) \ \textit{a.s. [P].}$$

Proof. Let $\widetilde{V}$ be any (fixed) element of $\mathcal{F}_c^2(\mathbb{R})$. Since $(\mathcal{F}_c^2(\mathbb{R}), D_\theta)$ is a metric space, the function

$$\begin{array}{rccl} Y: & \mathcal{F}_c^2(\mathbb{R}) & \longrightarrow & \mathbb{R} \\ & \widetilde{U} & \longmapsto & Y(\widetilde{U}) := D_\theta(\widetilde{U}, \widetilde{V}) \end{array}$$

is continuous. Consequently, the composition $Z = Y \circ \mathcal{X}$ is a real-valued random variable defined for each $\omega \in \Omega$ as follows

$$\begin{array}{rccl} Z: & \Omega & \longrightarrow & \mathbb{R} \\ & \omega & \longmapsto & Z(\omega) = Y(\mathcal{X}(\omega)) = D_\theta(\mathcal{X}(\omega), \widetilde{V}). \end{array}$$

Under the assumption $\|s_{\mathcal{X}}\|_\theta \in L^2(\Omega, \mathcal{A}, P)$, it is possible to apply the Strong Law for Large Numbers and conclude that

$$\frac{1}{n} \sum_{i=1}^n D_\theta(\mathcal{X}_i, \widetilde{V}) \underset{n \to \infty}{\longrightarrow} E[D_\theta(\mathcal{X}, \widetilde{V})] \text{ a.s. [P].}$$

Therefore, the denominator of $DD_{\theta,n}(\widetilde{V})$ converges almost surely to the denominator of $DD_\theta(\widetilde{V};\mathcal{X})$ and it is guaranteed that

$$DD_{\theta,n}(\widetilde{V}) \underset{n\to\infty}{\longrightarrow} DD_\theta(\widetilde{V};\mathcal{X}) \text{ a.s. [P]}.$$

□

A new notion of depth-based fuzzy trimmed mean can be stated in terms of the D_θ-depth.

Definition 6.2. *Let $\mathcal{X} : \Omega \to \mathcal{F}_c^2(\mathbb{R})$ be a random fuzzy number associated with a probability space $(\Omega, \mathcal{A}, P)$ and, for any $n \in \mathbb{N}$, let $(\mathcal{X}_1, \ldots, \mathcal{X}_n)$ be a simple random sample from $\mathcal{X}$. Given a trimming proportion $\beta \in (0,1)$, the* ***D_θ-depth-based fuzzy trimmed mean*** *estimator is defined as*

$$DD_\theta\text{-}\overline{\mathcal{X}}_{n,\beta} = \frac{\sum_{i=1}^n I_{[\gamma,\infty)}(DD_{\theta,n}(\mathcal{X}_i)) \cdot \mathcal{X}_i}{\sum_{i=1}^n I_{[\gamma,\infty)}(DD_{\theta,n}(\mathcal{X}_i))},$$

with

$$\frac{1}{n}\sum_{i=1}^n I_{[\gamma,\infty)}(DD_{\theta,n}(\mathcal{X}_i)) \simeq 1-\beta.$$

Analogous simulations to the ones considered in Section 5 have been developed for the comparison of the empirical fuzzy trimmed mean and the D_θ-depth-based fuzzy trimmed mean, in order to show that the new measure is a very competing alternative.

First, we conclude from Table 3 that in some situations the D_θ-depth-based fuzzy trimmed mean even improves the mean square error achieved by the empirical fuzzy trimmed mean. Regarding the maxbias, in Figure 5 we can see that the behaviour of both estimators is very similar except from the value $c_p = .49$ considered in the simulation study. In this latter case, there is a high percentage of contaminated data, which may be very concentrated (for instance, their variance equals zero in *Scenario 3.1*) and, therefore, present a lower depth than some noncontaminated data (whose variance never vanishes in this simulation study). Anyway, when the contamination proportion is .5, the empirical fuzzy trimmed mean is neither capable of distinguishing which half of the data is contaminated.

In summary, the empirical results are very promising for the D_θ-depth-based fuzzy trimmed mean. Additionally, the algorithmic computation of this new location measure is faster than that of the empirical fuzzy trimmed

Table 3: Mean square error and, in brackets, its standard deviation for the empirical fuzzy trimmed mean (EFTM) and the D_θ-depth-based fuzzy trimmed mean (D_θ-DTM) with trimming proportions .2 and .45. In bold, the minimum mean square error in each situation of the independent and the dependent cases.

		Independent		Dependent	
c_p	C_D	EFTM ($\beta = .2$)	D_θ-DTM ($\beta = .2$)	EFTM ($\beta = .2$)	D_θ-DTM ($\beta = .2$)
0	0	**.03104** (.03497)	.06123 (.03680)	**.02521** (.02990)	.04346 (.02411)
.1	0	**.03447** (.03724)	.04408 (.03603)	**.02664** (.03548)	.04135 (.02917)
.1	1	**.03130** (.03297)	.04450 (.03668)	**.02918** (.03664)	.05534 (.03386)
.1	5	**.02718** (.02599)	.04435 (.03553)	**.02657** (.03322)	.04118 (.03685)
.1	10	**.02801** (.03192)	.04717 (.03711)	**.02366** (.03215)	.03935 (.03326)
.1	100	**.02911** (.03061)	.04658 (.03644)	**.02314** (.02684)	.04023 (.03264)
.2	0	.05058 (.04346)	**.03316** (.03353)	**.02879** (.03432)	.03815 (.02814)
.2	1	.05343 (.04812)	**.03555** (.03858)	**.03717** (.03611)	.08332 (.04388)
.2	5	.05531 (.04653)	**.03791** (.03931)	**.02479** (.02288)	.02941 (.03252)
.2	10	.06294 (.05066)	**.03226** (.03200)	.02447 (.02323)	**.02379** (.02573)
.2	100	.06930 (.05076)	**.03188** (.03311)	.02449 (.02062)	**.02339** (.02342)
c_p	C_D	EFTM ($\beta = .45$)	D_θ-DTM ($\beta = .45$)	EFTM ($\beta = .45$)	D_θ-DTM ($\beta = .45$)
.4	0	.10444 (.08932)	**.06297** (.05776)	**.05398** (.08345)	.11968 (.06285)
.4	1	.09284 (.08053)	**.09192** (.09880)	**.10209** (.09797)	.26592 (.11718)
.4	5	**.07806** (.06275)	.13844 (.14114)	**.06880** (.05479)	.20188 (.17369)
.4	10	**.07730** (.05696)	.08953 (.09336)	**.06598** (.04740)	.15375 (.11830)
.4	100	.07668 (.05648)	**.07461** (.06846)	**.06796** (.04754)	.13631 (.09598)

mean considered in Sinova *et al.* [2], which uses some different starting points and repeats the initial steps several times in order to try to avoid that any local minimum traps the iterative process.

7. Concluding remarks

Trimmed means defined on the basis of a depth function have been adapted for dealing with fuzzy number-valued data by following two different approaches: the evaluation of a functional depth on the expression of fuzzy-valued observations as functions, and the introduction of a novel depth for fuzzy numbers, which has been motivated by the poor empirical performance of the depth-based fuzzy trimmed means obtained from the first approach. The fuzzy trimmed mean corresponding to the second approach has shown very promising empirical results and arises as a competing alternative when summarizing the central tendency of fuzzy number-valued data sets.

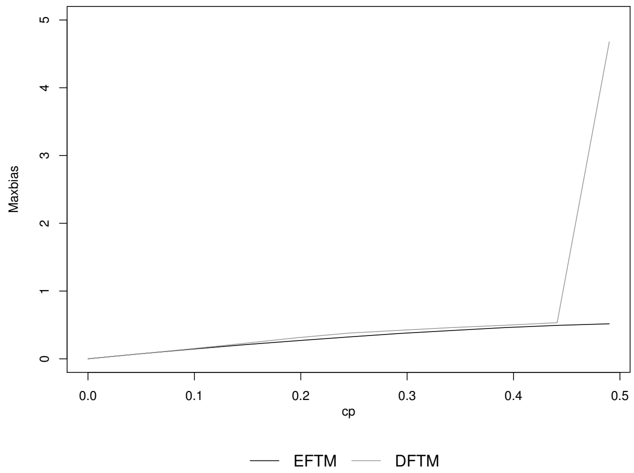

Figure 5: Maximum asymptotic biases of the empirical fuzzy trimmed mean (EFTM) and the D_θ-depth-based fuzzy trimmed mean with trimming proportion .5 as a function of c_p.

As future research lines, the simulation study could be extended to analyze fuzzy numbers from a given bounded referential (as happens with the fuzzy rating scale), to apply depths to support functions instead of fuzzy numbers directly and to compare depth-based medians with fuzzy medians and M-estimators. More *ad hoc* depth measures for fuzzy data could also be proposed and a careful study of which interesting properties they fulfill should be tackled, including a deeper view of the mathematical properties of the (functional) depth considered for the first approach as well. Regarding the D_θ-depth, some other properties like the uniform strong convergence should be analyzed.

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