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Nowhere Differentiability Conditions of Composites on Peano Curves

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Abstract

Sufficient conditions on a smooth, real-valued function g for the nowhere differentiability of $g \circ p$ are given, where p is Peano's curve. This generalizes Sagan's analytic proof on the nowhere differentiability of the coordinate functions of p. Most of the proofs are geometrically intuitive. The interest about the composites $g \circ p$ stems from their recent applications in technical branches.

Keywords Nowhere differentiability · Peano curve · Gradient

Mathematics Subject Classification 26A27 · 26A30 · 26B05

1 Introduction

Continuous, nowhere differentiable functions have been fascinating mathematicians since their discovery by Weierstrass in 1872. Almost each known example of these functions has been collected in [15]. A deep monograph about this subject can be found in [4], and more recent advances are given in [1,9].

Peano claimed without proof that the coordinate functions of his famous space-filling curve $p\colon [0,1] \longrightarrow [0,1]^2$ are also nowhere differentiable [12]. A proof for Peano's claim was given by Sagan [14] by means of analytical techniques, although in an earlier paper, Alsina proved the nowhere differentiability of Schoenberg space-filling curve [2].

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Section 3 is the core of the present paper and its main result is Theorem 1, which proves that $g \circ p$ has no finite derivative at any point if g is a real valued, Fréchet differentiable function with non-null gradient everywhere. Note that among these functions g are the coordinate functions $g(u_1, u_2) = u_i$, $i = 1, 2$.

The interest in the composites $g \circ p$ stems from their role in some technical branches like image compression [5,6,10,11,16,17].

The proof of Theorem 1 is geometrically intuitive in contrast with most of the known proofs of the existence of continuous, nowhere differentiable functions, which are mainly analytic. Let us remark that there are two examples built ad hoc to shed geometric light on the subject of nowhere differentiability: the Bolzano function [4,15] and Koch's curve, which resembles a beatiful *trace italienne* [3].

Section 3 ends with Propositions 1 and 2 which complement the information provided by Theorem 1.

The Peano curve considered in this article has been taken from [7] because of its simple geometric construction. This curve is briefly described in Sect. 2 and will be denoted by p from now on (see comments about the *geometric* Peano curve in Chapter 2 in [14]). However, it must be pointed out that the methods used here can be applied to most of continuous space-filling curves.

The main tools used are two: the first is the geometric properties of the gradient of g, denoted as ∇g as usual. If $\|\cdot\|$ is a fix norm on $\mathbb{R}^2$, the second tool is that there exist a pair of positive constants k and K such that $k|a-b| \leq \|p(a) - p(b)\|^2 \leq K|a-b|$ if a and b are numbers in $[0, 1]$ which are close enough (in particular, p is a Hölder function of class $\mathbb{H}^{1/2}$ as defined in [13]). This property of p can be guessed from the well-known fact that the function p borrowed from [7] preserves the Lebesgue measure, that is, if A is a Börel-Lebesgue subset of I then the value of the Lebesgue measures of A and $p(A)$ coincide. In particular, this implies $g \circ p$ and g have the same distribution function. Thus, although $g \circ p$ is a very bad function from the point of view of differentiability as shown in the results of Sect. 3, $g \circ p$ is as well behaved as g from the point of view of integrability which enables the implementation of simple techniques of integration like those of [8] if g sufficiently simple.

Notation: $\mathbb{N}$ represents the set of positive integers; $\mathbb{R}$ denotes the set of real numbers and $\overline{\mathbb{R}} := \mathbb{R} \cup \{-\infty, \infty\}$. The unit interval $[0, 1]$ is denoted by I. The set of all limit points of a subset A of $\mathbb{R}$ is denoted by A' as usual. Given $f : A \longrightarrow \mathbb{R}$ and $t_0 \in A \cap A'$, the derivative of f at t_0 is the limit

$$\lim_{t \to t_0} \frac{f(t) - f(t_0)}{t - t_0} \in \overline{\mathbb{R}}$$

if it exists, in which case is denoted by $f'(t_0)$. If $t_0 \in A \cap [A \cap (-\infty, t_0)]'$ (alt., $t_0 \in A \cap [A \cap (t_0, \infty)]'$), the left-sided (alt., right-sided) derivative of f at t_0 is the number $f'(t_0^-) := f\;|'_{A\cap(-\infty,t_0]}\;(t_0)$ if the derivative at t_0 of the restriction of f to $A \cap (-\infty, t_0]$ exists (alt., $f'(t_0^+) := f\;|'_{A\cap[t_0,\infty)}\;(t_0)$ if it exists). If f has a finite derivative at each $t \in A$,–that is, $f'(t) \in \mathbb{R}$– then, f is said to be *differentiable*.

Given an open set Ω in $\mathbb{R}^n$, $f : \Omega \longrightarrow \mathbb{R}^m$ $(n, m \in \mathbb{N})$ is said to be Fréchet differentiable at $x_0 \in \Omega$ if there exists a linear mapping $L : \mathbb{R}^n \longrightarrow \mathbb{R}^m$ such that

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$\lim_{x \to x_0} \| f(x) - f(x_0) - L(x - x_0)||/||x - x_0|| = 0$, in which case, L is denoted by $\mathrm{d}f(x_0)$ (let us remind that Fréchet differentiability does not depend on the chosen norms in $\mathbb{R}^n$ and $\mathbb{R}^m$).

The distance in $\mathbb{R}^2$ will be measured with the norm $\|(x, y)\|_\infty := \max\{|x|, |y|\}$, which will be simply denoted as $\|(\cdot, \cdot)\|$ for short. Let us recall that $\sqrt{2}^{-1} \|(\cdot, \cdot)\|_2 \leq \|(\cdot, \cdot)\|_\infty \leq \|(\cdot, \cdot)\|_2$, where $\|(\cdot, \cdot)\|_2$ is the Euclidean norm of $\mathbb{R}^2$.

2 Peano's Geometric Curve

Henceforth, a (plain) *curve* is a continuous function $f : [a, b] \longrightarrow \mathbb{R}^2$ where a and b are real numbers such that $a < b$; the points $f(a)$ and $f(b)$ are, respectively, termed as *initial* and *terminal* points of the curve f.

A brief geometric description of the space-filling Peano's curve $p : I \longrightarrow I^2$ defined in [7] is necessary. Given two bounded, closed intervals L and J in $\mathbb{R}$, $L \leq J$ means $\max I \leq \min J$.

The 9-adic subintervals of I of order $k \in \mathbb{N}$ have length equal to $1/9^k$ and are labeled as follows: the intervals of order 1 are $I_i := [i/9, (i+1)/9]$, $i = 0, 1, 2, \ldots, 8$. Each interval $I_{i_1 \ldots i_k}$ of order k ($i_l = 0, 1, \ldots, 8$) is divided into nine closed intervals $I_{i_1 \ldots i_k j}$, $j = 0, 1, \ldots, 8$, of length $1/9^{k+1}$ such that

(i) $I_{i_1 \ldots i_k} = \bigcup_{j=0}^{8} I_{i_1 \ldots i_k j}$
(ii) $I_{i_1 \ldots i_k 0} \leq I_{i_1 \ldots i_k 1} \leq \cdots \leq I_{i_1 \ldots i_k 8}$.

The interval I will be considered as the 9-adic interval of order 0.

For every $k \in \mathbb{N}$, the square I^2 is also divided into closed subsquares whose side length is equal to $1/3^k$; these subsquares are labeled in a Cartesian manner as follows:

$$C_i^j := \left[\frac{i-1}{3^k}, \frac{i}{3^k} \right] \times \left[\frac{j-1}{3^k}, \frac{j}{3^k} \right], \quad 1 \leq i, j \leq 3^k.$$

Henceforth, these subsquares will be called *k-subsquares*, and the subintervals of order k will be called *k-subintervals* for short. The square I^2 will be referred to as the 0-*square*.

Given $n \in \mathbb{N}$, two n-subintervals are said to be *adjacent* if their intersection is a singleton; two n-subsquares are *adjacent* if they share one and only one side; a finite collection $\{G_i\}_{i=1}^m$ of n-subintervals (alt., n-subsquares) is called a *chain* if they are pairwise different and G_i and G_{i+1} are adjacent for all $1 \leq i \leq m - 1$.

Peano's curve is the uniform limit of a sequence of polygonal curves $p_n : I \longrightarrow I^2$ which are defined as follows:

The curve p_0 linearly maps the interval I onto the NE diagonal of I^2, so that $p_0(0) = (0, 0)$ and $p_0(1) = (1, 1)$ (the graph of p_0 is depicted in Fig. 1, NE).

For $n \in \mathbb{N}$, assume that p_{n-1} linearly maps each $(n-1)$-subinterval $I_{i_1 \ldots i_{n-1}}$ onto one diagonal of certain $(n-1)$-subsquare C. If the graph of $p_{n-1}(I_{i_1 \ldots i_{n-1}})$ is of the form NE (alt., SE; SW; NW) as indicated in Fig. 1, then p_n is piecewise defined by linearly mapping each n-subinterval $I_{i_1 \ldots i_{n-1} j}$, $0 \leq j \leq 8$, onto one diagonal of

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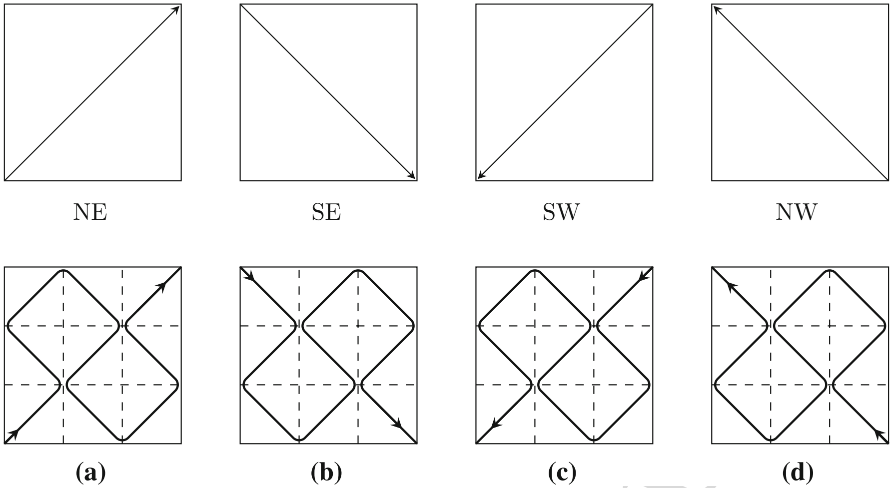

Fig. 1 Some corners have been intentionally rounded for clarity

certain n-subsquare contained in C so that the graph of $p_n\,|_{I_{i_1\ldots i_{n-1}j}}$ replicates the graph depicted in Fig. 1a (alt., b–d).

Clearly, $p_n\,|_{I_{i_1\ldots i_{n-1}}}$ is continuous, and if $i_{n-1} < 8$, the terminal point of $p_n\,|_{I_{i_1\ldots i_{n-1}}}$ coincides with the initial point of $p_n\,|_{I_{i_1\ldots(i_{n-1}+1)}}$. Hence, p_n is continuous on I.

The crucial properties of the curves p_n are:

(iii) if A and B are adjacent n-subintervals, then $p_n(A)$ and $p_n(B)$ are contained in adjacent n-subsquares;

(iv) if B is an n-subinterval and $p_n(B)$ is contained in a n-subsquare C, then $p_{n+k}(B) \subset C$ for all $k \in \mathbb{N}$;

(v) for each n-subinterval $I_{i_1\ldots i_n}$ there is a unique n-subsquare C such that $p_n(I_{i_1\ldots i_n}) \subset C$, and vice versa: for every n-subsquare C there exists a unique n-subinterval $I_{i_1\ldots i_n}$ such that $p_n(I_{i_1\ldots i_n}) \subset C$; the n-subsquare C will be labeled as $C = D_{i_1\ldots i_n}$.

It can be proved from (iii), (iv) and (v) that the curves p_n converge uniformly to a continuous curve $p\colon I \longrightarrow I^2$. Moreover, it is easy to see that for every $(x_0, y_0) \in I^2$, there exists a sequence (C_n) where each C_n is a n-subsquare, $C_n \supset C_{n+1}$ and $\{(x_0, y_0)\} = \cap_{n=1}^{\infty} C_n$. Let I_n be the only n-subinterval such that $p_n(I_n) \subset C_n$. Then, (I_n) is a decreasing sequence of compact intervals. Thus, $\{t_0\} = \cap_{n=1}^{\infty} I_n$ for some $t_0 \in I$. It is straightforward that $p(t_0) = (x_0, y_0)$ and therefore, $p(I) = I^2$. The same argument proves $p(I_n) = C_n$ for all n. As a consequence, p satisfies the following properties:

(iii') if A and B are adjacent n-subintervals, then $p(A)$ and $p(B)$ are contained in adjacent n-subsquares;

(iv') if B is a n-subinterval and $p_n(B)$ is contained in a n-subsquare C, then $p(B) \subset C$;

(v') $p(I_{i_1\ldots i_n}) = D_{i_1\ldots i_n}$ for all n and all $0 \leq i_k \leq 8$.

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(vi') if t and τ belong to a n-subinterval J then $|t-\tau| \leq 1/3^{2n}$ and $\|p(t)-p(\tau)\| \leq 1/3^{n}$. Moreover, if m is the smallest positive integer for which there exists a chain $\{K_i\}_{i=1}^{m+2}$ of n-subsquares with $p(t) \in K_1$ and $p(\tau) \in K_{m+2}$, then $|t-\tau| \geq m/3^{2n}$.

Detailed proofs of these facts can be found in [7] (see also [12,14] or [15]) or can be done by the reader.

Given $n \in \mathbb{N} \cup \{0\}$, the ending points of the n-subintervals are called *n-nodes*. In particular, a n-node is a $(n+k)$-node for any non-positive integer k. The set of all n-nodes for any n is denoted $\mathcal{N}$ and its elements are called *nodes*, that is, $\mathcal{N} = \{k/9^n : n \in \mathbb{N} \cup \{0\},\ k = 0, 1, 2, \ldots, 9^n\}$.

The images of the nodes of every n-subinterval $J = [t_0, t_1]$ are a pair of opposite corners of the n-subsquare $p(J)$ and moreover, $p(t_i) = p_{n+k}(t_i)$ for $i = 0, 1$ and for $k = 0, 1, 2, \ldots$. It is very easy to check that the remaining pair of corners of $p(J)$ are $p(\xi_0)$ and $p(\xi_1)$ where $\xi_0 := t_0 + 9^{-n}\sum_{k=1}^{\infty} 2/9^k = t_0 + (1/9^n 4)$ and $\xi_1 := t_1 - (1/9^n 4)$. The numbers ξ_i are called *n-pseudonodes* or just *pseudonodes* if the order n is not specified. Note that $p(\xi_i) \notin p_n(I)$ for any $n \in \mathbb{N}$. The set of all pseudonodes of any order will be denoted as $\mathcal{F}$, each element of $p(\mathcal{N})$ will be called an *attainable corner* and each element of $p(\mathcal{F})$, a *non-attainable corner*.

3 Main Results

Theorem 1 *Let $g : \mathbb{R}^2 \longrightarrow \mathbb{R}$ be a Fréchet differentiable function such that $\nabla g(x, y) \neq (0, 0)$ at each $(x, y) \in \mathbb{R}^2$. Let $f := g \circ p$, where p is the Peano curve. Then f is continuous and moreover:*

(i) *f has no finite derivative at each $t \in I$;*
(ii) *additionally, if g is of class $\mathcal{C}^{(2}$ then f has no infinite derivative at any $t \in I \backslash (\mathcal{F} \cup \{0, 1\})$.*

Proof The continuity of f is immediate since g and p are continuous. In order to prove (i) and (ii), fix $t_0 \in I$ and let $(x_0, y_0) := p(t_0)$. As $\nabla g(x_0, y_0) \neq (0, 0)$, it will be assumed without loss of generality that $\frac{\partial g}{\partial y}(x_0, y_0) =: \alpha \neq 0$ (the case $\frac{\partial g}{\partial x}(x_0, y_0) \neq 0$ is similar).

For every $n \in \mathbb{N}$, let L_n be a n-subinterval such that

$$L_n \supset L_{n+1} \text{ and } \{t_0\} = \cap_{n=1}^{\infty} L_n. \tag{1}$$

Denote $C_n := p(L_n)$, so that $(x_0, y_0) = \cap_{n=1}^{\infty} C_n$.

Let s be the orthogonal line to $\nabla g(x_0, y_0)$ that passes through (x_0, y_0). In the case when $t_0 \in \mathcal{N} \backslash \{0, 1\}$, if k is the smallest nonnegative integer such that t_0 is a k-node, then two adjacent k-subintervals $I_{i_1 \ldots i_{k-1} j}$ and $I_{i_1 \ldots i_{k-1}, j+1}$ contain t_0; an appropriate choice of L_k among these two k-subintervals produces that $s \cap p(L_k)$ is infinite, and once L_k has been selected, the choice of the following intervals L_{k+l} is univocally determined by condition (1). Consequently,

$$s \cap C_n \text{is infinite for all} n \in \mathbb{N} \cup \{0\} \text{if} t_0 \in I \backslash (\mathcal{F} \cup \{0, 1\}). \tag{2}$$

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(i) Let r denote the parallel line to the Y-axis that passes through (x_0, y_0). Note that r cannot be orthogonal to $\nabla g(x_0, y_0)$ and $s \neq r$ because $\alpha \neq 0$.

Given $n \in \mathbb{N}$, let T and B, respectively, denote the top side and the bottom side of C_n. The set $(T \cup B) \cap r$ has exactly two elements. Among these two elements, choose the one is furthest from (x_0, y_0). Clearly, the element so chosen is of the form (x_0, y_n), and as the length of the sides of C_n is $1/3^n$, then

$$\frac{1}{2 \cdot 3^n} \leq \|(x_0, y_n) - (x_0, y_0)\| \leq \frac{1}{3^n}. \tag{3}$$

Assume $y_0 < y_n$ (if $y_n < y_0$, the actions to be done are similar). Let $\mathcal{C} := \{C_{i_0 j} : l \leq j \leq q\}$ be a collection of $(n+2)$-subsquares vertically stacked and contained in C_n such that $(x_0, y_0) \in C_{i_0 l}$ and $(x_0, y_n) \in C_{i_0 q}$ (note $1 \leq l \leq q \leq 3^{n+2}$). Let $J_l = p^{-1}(C_{i_0 l})$ and $J_q = p^{-1}(C_{i_0 q})$ and take $t_n \in q$ such that $p(t_n) = (x_0, y_n)$. Inequality (3) yields $3/3^{n+2} \leq \|(x_0, y_n) - (x_0, y_0)\|$, so l and q satisfy $|l - q| \geq 4$. In plain words, this means that there are at least three squares of $\mathcal{C}$ between $C_{i_0 l}$ and $C_{i_0 q}$ (see Fig. 2), and as $\mathcal{C}$ is the shortest chain of $(n+2)$-subsquares connecting $C_{i_0 l}$ with $C_{i_0 q}$, (vi') yields $|t_n - t_0| \geq 3/9^{n+2}$. Moreover, as both $p(t_n)$ and $p(t_0)$ belong to C_n, it is immediate that $|t_n - t_0| \leq 1/3^{2n}$, and this and (3) eventually gives

$$3^{n-1} \leq \frac{\|p(t_n) - p(t_0)\|}{|t_n - t_0|} \leq 3^{n+3}. \tag{4}$$

Since g is Fréchet differentiable, there is a function $\omega\colon \mathbb{R}^2 \longrightarrow \mathbb{R}$ such that $\omega(x, y) \to 0$ as $(x, y) \to (x_0, y_0)$ and

$$g(x, y) - g(x_0, y_0) = \mathrm{d}g(x_0, y_0)(x - x_0, y - y_0) + \|(x - x_0, y - y_0)\|\, \omega(x, y). \tag{5}$$

Plugging $(x, y) = p(t_n)$ and $(x_0, y_0) = p(t_0)$ into (5), we get

$$f(t_n) - f(t_0) = \mathrm{d}g\big(p(t_0)\big)\big(p(t_n) - p(t_0)\big) + \|p(t_n) - p(t_0)\|\, \omega\big(p(t_n)\big). \tag{6}$$

As $\mathrm{d}g\big(p(t_0)\big) \equiv (\frac{\partial g}{\partial x}(x_0, y_0), \alpha)$ and $p(t_n) - p(t_0) = (0, y_n - y_0)$, it follows

$$\big|\mathrm{d}g\big(p(t_0)\big)\big(p(t_n) - p(t_0)\big)\big| = |\alpha| \cdot \|p(t_n) - p(t_0)\|. \tag{7}$$

Moreover, as p is continuous and $t_n \to t_0$ as $n \to \infty$, there is $n_0 \in \mathbb{N}$ such that

$$|\omega\big(p(t_n)\big)| < |\alpha/2| \text{ for all} n \geq n_0. \tag{8}$$

Thus, dividing both sides of (6) by $t_n - t_0$ and taking their absolute values, a consecutive application of (7), (8) and (4) leads to

$$\left|\frac{f(t_n) - f(t_0)}{t_n - t_0}\right| \geq \left(|\alpha| - \frac{|\alpha|}{2}\right) \frac{\|p(t_n) - p(t_0)\|}{|t_n - t_0|} \geq \frac{|\alpha|}{2} 3^{n-1}. \tag{9}$$

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Since $t_n \to t_0$ as $n \to \infty$, formula (9) shows that f has not a finite derivative at t_0.

(ii) One and only one of the following three cases may happen:

(a) $(x_0, y_0) \in \operatorname{Int} C_n$ for all n;
(b) there exists $n_0 \in \mathbb{N}$ such that $(x_0, y_0) \in \operatorname{Fr} C_n \backslash p(\mathcal{N} \cup \mathcal{F})$ for all $n \geq n_0$;
(c) (x_0, y_0) is an attainable corner of C_n for all $n \geq n_0$ but $(0, 0) \neq (x_0, y_0) \neq (1, 1)$.

Assume (a) holds. Then, for every $n \in \mathbb{N}$ there exists $m \geq n$ and a pair of positive integers k and l such that

$$(x_0, y_0) \in K_{33} \subset \bigcup_{1 \leq i, j \leq 5} K_{ij} =: K \subset C_n$$

$$K_{ij} = \left[\frac{k-1+i}{3^{m+2}}, \frac{k+i}{3^{m+2}}\right] \times \left[\frac{l-1+j}{3^{m+2}}, \frac{l+j}{3^{m+2}}\right]. \tag{10}$$

Let us label as J_{ij} the corresponding $(m+2)$-subinterval so that $p(J_{ij}) = K_{ij}$ and $t_0 \in J_{33}$.

Choose $(x_n, y_n) \in s \cap \operatorname{Fr} K$. Note that (x_n, y_n) belongs to some of the sixteen external $(m+2)$-subsquares that form K, that is, $(x_n, y_n) \in K_{uv}$ for some pair (u, v) with $u \in \{1, 5\}$ or $v \in \{1, 5\}$. Let $\tau_n \in J_{uv}$ such that $p(\tau_n) = (x_n, y_n)$. Clearly, $|\tau_n - t_0| \leq 1/3^{2m}$ and $\|p(\tau_n) - p(t_0)\| \leq 2/3^{m+2}$. Clearly, any chain of $(m+2)$-subsquares connecting K_{33} with K_{uv} must have at least three $(m+2)$-subsquares so, by virtue of (vi'), $1/3^{2(m+2)} \leq |\tau_n - t_0| \leq 3/3^{2(m+2)}$. Hence,

$$\frac{\|p(\tau_n) - p(t_0)\|^2}{|\tau_n - t_0|} \leq 4. \tag{11}$$

Next, since g is of class $\mathcal{C}^{(2}$, Young's formula provides $\psi : \mathbb{R}^2 \longrightarrow \mathbb{R}$ such that $\psi(x, y) \to 0$ as $(x, y) \to (x_0, y_0)$ and

$$\begin{aligned} g(x, y) - g(x_0, y_0) &= \mathrm{d}g(x_0, y_0)(x - x_0, y - y_0) \\ &\quad + \frac{1}{2}\mathrm{d}^2 g(x_0, y_0)(x - x_0, y - y_0)^{(2)} + \|(x - x_0, y - y_0)\|^2 \psi(x, y). \end{aligned} \tag{12}$$

where $\mathrm{d}^2 g(x_0, y_0)$ denotes the bilinear form associated with the hessian matrix of g.

Note that the orthogonality between the vectors $p(\tau_n) - p(t_0)$ and $\nabla g(x_0, y_0)$ yields

$$dg(x_0, y_0)\big(p(\tau_n) - p(t_0)\big) = 0. \tag{13}$$

Moreover, by virtue of (11) there is a constant M such that

$$\left| \mathrm{d}^2 g(x_0, y_0) \left(\frac{p(\tau_n) - p(t_0)}{\sqrt{|\tau_n - t_0|}} \right)^{(2)} \right| \leq M \text{ for all} n \in \mathbb{N} \tag{14}$$

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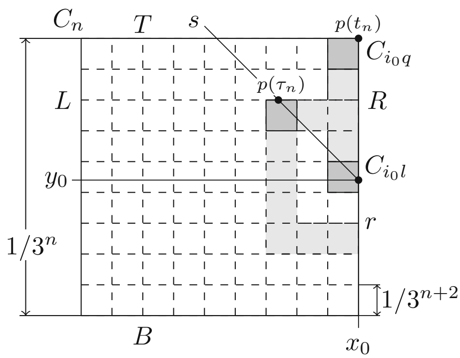
Fig. 2 .



and since $\tau_n \xrightarrow[n]{} t_0$, inequality (11) yields

$$\frac{\|p(\tau_n) - p(t_0)\|^2}{|\tau_n - t_0|} \psi\big(p(\tau_n)\big) \xrightarrow[n]{\qquad} 0. \tag{15}$$

Thus, plugging $(x, y) = p(\tau_n) - p(t_0)$ into (12) and dividing both sides of the resulting identity by $|\tau_n - t_0|$, an implementation of (13), (14) and (15) shows that the sequence $\big(gp(\tau_n) - gp(t_0))/(\tau_n - t_0)\big)$ is bounded. Therefore, since $\tau_n \xrightarrow[n]{} t_0$, f cannot have an infinite derivative at t_0 and this ends the proof for case (a).

If the case (b) holds then (x_0, y_0) belongs to one and only one side of C_{n_0}; if (x_0, y_0) belongs to the left side (alt. right, top, bottom side) then (x_0, y_0) belongs to the left side of C_n for all $n \geq n_0$ (alt. right, top, bottom side). Given $n \geq n_0$, denoted by L (alt. R, T, B) its left side (alt., right, top, bottom side). Then there exists $m \geq n$ and a pair of positive integers k and l such that

$$(x_0, y_0) \in K_{33} \subset \bigcup_{i=1}^{3} \bigcup_{j=1}^{5} K_{ij} =: K \subset C_n$$

$$K_{ij} = \left[\frac{k-1+i}{3^{m+2}}, \frac{k+i}{3^{m+2}}\right] \times \left[\frac{l-1+j}{3^{m+2}}, \frac{l+j}{3^{m+2}}\right] \tag{16}$$

and labeling as J_{ij} the corresponding $(m+2)$-subinterval so that $p(J_{ij}) = K_{ij}$ and $t_0 \in J_{33}$, and replacing (10) by (16), the same argument for proving (a) works now for case (b) when $(x_0, y_0) \in L$. The remaining cases when $(x_0, y_0) \in R$ (alt., T, B) are similar (the case $(x_0, y_0) \in L$ with $m = n$ is depicted in Fig. 2).

The case (c) admits a similar proof as case (b). Note that for case (c) is crucial that (2) holds true. □

The reason why each point $\xi \in \mathcal{F} \cup \{0, 1\}$ is excluded in Theorem 1 is that $s \cap p(J)$ may be finite, where J is a n-subinterval containing ξ such that $p(\xi)$ is a corner of $p(J)$. However, if $s \cap p(J)$ is infinite, the following result can be stated:

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Author Proof

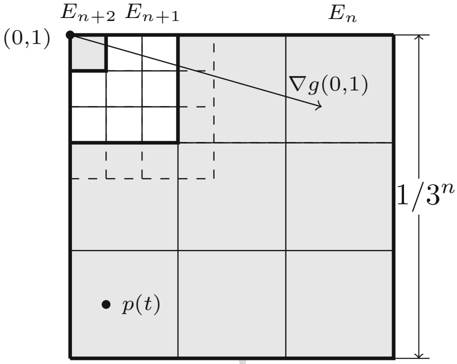

Fig. 3 .

Proposition 1 *Assume* $g\colon \mathbb{R}^2 \longrightarrow \mathbb{R}$ *is of class* $\mathcal{C}^{(2}$ *and let* $\xi \in J \cap (\mathcal{F} \cup \{0, 1\})$ *where* J *is a* n*-subinterval and* $p(\xi)$ *is a corner of* $p(J)$*. Let* s *be the line orthogonal to* $\nabla g\big(p(\xi)\big)$ *that passes through* $p(\xi)$ *and assume* $s \cap p(J)$ *is infinite. Then* $g \circ p$ *has no derivative at* ξ*.*

Proof As $s \cap p(J)$ is infinite, the same arguments of Theorem 1 (ii) work for ξ and therefore, if $f'(\xi)$ exists, it cannot be infinite. But Theorem 1 shows that $f'(\xi)$ cannot be finite either. This proves that f has no derivative at ξ. □

If the assumption that $s \cap p(J)$ is infinite is replaced in Proposition 1 by its negation then, $s \cap p(J)$ must be a singleton. Therefore, the following result completes the information.

Proposition 2 *Let* $g\colon \mathbb{R}^2 \longrightarrow \mathbb{R}$ *be a differentiable function and let* $\xi \in J \cap (\mathcal{F} \cup \{0, 1\})$ *where* J *is a* n*-subinterval and* $p(\xi)$ *is a corner of* $p(J)$*. Let* s *be the line orthogonal to* $\nabla g\big(p(\xi)\big)$ *that passes through* $p(\xi)$ *and suppose* $s \cap p(J)$ *is a singleton. Let* $f = g \circ p$*. Then, the following statements hold:*

(i) *if* $\xi \in \{0, 1\}$ *then, there exists* $f'(\xi)$*;*
(ii) *if* $\xi \in \mathcal{F}$ *then, both sided derivatives* $f'(\xi^-)$ *and* $f'(\xi^+)$ *exist but* $f'(\xi^-) \neq f'(\xi^+)$.

Proof Only the case $\xi = 1/4$ will be demonstrated since the others admit a similar proof. Recall $p(\xi) = (0, 1)$ is a non-attainable corner. In order to avoid cumbersome notation, let us relabel

$$J_n := I_{3 \overset{n}{\ldots} 3},$$
$$E_n := C_{1,3^n} = p(J_n)$$
$$K_n := E_n \setminus E_{n+1} \; n \in \mathbb{N}.$$

Note that $\xi \in \operatorname{Int} J_n$ for all n. Consider Young's formula

$$g(x, y) - g(0, 1) = \mathrm{d}g(0, 1)(x, y - 1) + \|(x, y - 1)\|\omega(x, y) \qquad (17)$$

where $\omega(x, y) \longrightarrow 0$ as $(x, y) \to (0, 1)$.

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Without loss of generality, assume the bound vector $\nabla g(1,0)$ with initial point at $(1,0)$ points inwards the square I^2. A moment of reflection will show that the hypotheses on $\nabla g(1,0)$ give a real number $\beta > 0$ such that $\mathrm{d}g(0,1)(u) > \beta$ for all unitary vectors $u = (u_1, u_2)$ such that $u_1 \geq 0$ and $u_2 \leq 0$.

Take $m \in \mathbb{N}$ so that $|\omega(x, y)| < \beta/2$ for all $(x, y) \in E_m$. Hence,

$$\mathrm{d}g\big(p(\xi)\big)\left(\frac{p(t)-p(\xi)}{\|p(t)-p(\xi)\|}\right)+\omega\big(p(t)\big) \geq \frac{\beta}{2} > 0, \ \text{all} t \in J_m, \tag{18}$$

and in combination with (18),

$$\frac{g\big(p(t)\big)-g\big(p(\xi)\big)}{t-\xi}\begin{cases}\leq 0, \ t \in (-\infty, \xi) \cap J_m \\ \geq 0, \ t \in (\xi, \infty) \cap J_m.\end{cases} \tag{19}$$

Next, given any $t \in J_m \backslash \{\xi\} = \bigcup_{n=m}^{\infty}(J_n \backslash J_{n+1})$, let $n \geq m$ such that $t \in J_n \backslash J_{n+1}$, so $p(t) \in K_n$. Since E_{n+2} is separated from K_n by a double belt of $(n+2)$-subsquares (see Fig. 3), (vi') proves $1/3^{2n+4} \leq |t-\xi| \leq 1/3^{2n}$ and $1/3^{n+2} \leq \|p(t)-p(\xi)\| \leq 1/3^n$, so

$$\frac{\|p(t)-p(\xi)\|}{|t-\xi|} \geq 3^{n-2}. \tag{20}$$

Thus, from (17), (18) and (20),

$$\left|\frac{g\big(p(t)\big)-g\big(p(\xi)\big)}{t-\xi}\right| \geq \frac{\beta}{2}\frac{\|p(t)-p(\xi)\|}{|t-\xi|} \geq \frac{\beta}{2}3^{m-2}, \ t \in J_m \backslash \{\xi\}. \tag{21}$$

Since $\xi \in \operatorname{Int} J_m$, the length of J_m is $1/3^{2m}$ and m can be chosen as large as one pleases, it is straightforward from (19) and (21) that there exist $f'(\xi^-) = -\infty$ and $f'(\xi^+) = \infty$. □

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